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A Limit Theorem for Solutions of Inequalities

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Abstract

Let $H(p)$ be the set $\{x \in X\colon\ h(x) \le p\}$, where h is a real-valued lower semicontinuous function on a locally compact second countable metric space X. A limit theorem is proved for the empirical counterpart of $H(p)$ obtained by replacing of h with its estimator.

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1. Introduction

Consider a certain lower semicontinuous real-valued function h defined on a locally compact second countable metric space (X, ρ). Then the set

$$H(p) = \{x \in X\colon\ h(x) \le p\} \tag{1.1}$$

is closed. The aim of this paper is to prove a limit theorem for the estimator $H_n(p)$ of the set $H(p)$ obtained by replacing $h(x)$ in (1.1) with its estimator $h_n(x)$:

$$H_n(x) = \{x \in X\colon\ h_n(x) \le p\}\ . \tag{1.2}$$

A simple problem of this kind originates in classical statistics.

Example **1.1.** Suppose that $h(x) = F(x)$, $x \in \mathbf{R}$, is the distribution function of a random variable ν. Then $H(p) = (-\infty, x_p]$, where x_p is the p-quantile of ν, and $H_n(p)$ is related to the corresponding empirical quantile if h_n is the empirical distribution function. A generalization for quantiles of random vectors and random closed sets was considered by Eddy [2] and Molchanov [7].

If h is a density, then the level set $H(p)$ appears in cluster analysis, see Hartigan [3]. An estimator of $H(p)$ based on minimization of the so-called excess mass was considered in [4, 9]. Similar problems appear also in the estimation of the support of a density, see [5].

Further we shall not discuss the nature of the estimator h_n. We only suppose that the estimator h_n is strongly consistent in the uniform metric and

$$\zeta_n = a_n(h_n - h) \tag{1.3}$$

admits a weak limit ζ as $n \to \infty$, i.e. each continuous in the uniform metric functional of ζ_n converges in distribution to its value on ζ. Here $a_n \to \infty$ as $n \to \infty$ is a sequence of norming constants.

Suppose also that each function h_n is almost surely lower semicontinuous. Then $H_n(p)$ is a *random closed set* as introduced in [6].

Space K of all compact subsets of X can be metrized by the Hausdorff distance:

$$\rho_{\mathrm{H}}(K, K_1) = \inf\{r > 0\colon\ K \subset K_1^r, K_1 \subset K^r\}\ ,$$

where $K, K_1 \in \mathcal{K}$,

$$K^r = \{x\colon\ b(x,r) \cap K \neq \emptyset\}$$

is the r-parallel set to K and $b(x,r)$ is the ball of radius r centered at x. For each set $M \subset X$ we shall write $\mathrm{cl}(M)$, $\mathrm{Int}\, M$ and ∂M for its closure, interior and boundary respectively.

2. Strong consistency

The estimator $H_n(p)$ is said to be *strongly consistent* if

$$\rho_{\mathrm{H}}(H_n(p) \cap K_0, H(p) \cap K_0) \to 0 \quad \text{a.s. as} \quad n \to \infty$$

for each compact K_0. The distance $\rho_{\mathrm{H}}(H_n(p) \cap K_0, H(p) \cap K_0)$ is a random variable, since $H_n(p) \cap K_0$ is a random closed set, and the function $\rho_{\mathrm{H}}(\cdot, K)$ is continuous.

Theorem 2.1. *Suppose that, for each compact K_0,*

$$\eta_n = \sup_{x \in K_0} |h_n(x) - h(x)| \to 0 \quad \text{a.s. as} \quad n \to \infty\,. \tag{2.1}$$

The estimator $H_n(p)$ is strongly consistent if

$$H(p) \subset \mathrm{cl}(H(p-))\,, \tag{2.2}$$

where $H(p-) = \{x\colon\ h(x) < p\}$. If for each x there exists a sequence $n(k)$ such that $h_{n(k)}(x) > h(x)$ a.s., then (2.2) is also a necessary condition. Moreover, if (2.2) is valid for each $p \in [c_1, c_2]$, then

$$\sup_{c_1 \le p \le c_2} \rho_{\mathrm{H}}(H_n(p) \cap K_0, H(p) \cap K_0) \to 0 \quad \text{a.s. as} \quad n \to \infty\,. \tag{2.3}$$

PROOF. To simplify notations suppose that X is compact, and $K_0 = X$.

Sufficiency. It is evident that the function

$$\phi(\varepsilon) = \rho_{\mathrm{H}}(H(p+\varepsilon), H(p)) .$$

is right-continuous, non-increasing for $\varepsilon < 0$ and non-decreasing for $\varepsilon > 0$. Note that $\phi(0) = 0$. Moreover, (2.2) yields $\rho_{\mathrm{H}}(H(p), H(p-)) = 0$, that is ϕ is continuous at zero.

Evidently, $H_n(p) \subset \{x\colon\ h(x) < p + \eta_n\}$. Similarly,

$$\begin{aligned} H(p) \subset H(p-\eta_n)^{\phi(\eta_n)} &= \cup\{b(x, \phi(\eta_n))\colon\ h(x) < p - \eta_n\} \\ &\subset H_n(p)^{\phi(\eta_n)} . \end{aligned}$$

Hence, (2.1) yields

$$\rho_{\mathrm{H}}(H_n(p), H_n) \le \phi(\eta_n) \to 0 \quad \text{a.s. as} \quad n \to \infty .$$

Furthermore, (2.3) follows from the monotonicity of H and its estimator.

Necessity. Let $H(p) \not\subset \mathrm{cl}(H(p-))$. Then there exists a point x, such that $h(x) = p$ and $\rho(x, \mathrm{cl}(H(p-)) = \delta > 0$. Then $h_{n(k)}(x) > h(x) = p$ by the condition of Theorem. Therefore, $x \notin H_{n(k)}(p)$, whence $\rho(H_{n(k)}(p), H(p)) > \delta$, contrary to the consistency of the estimator $H_n(p)$. $\square$

Heuristic, (2.2) for all p means that the function h is not constant on any open subset of X.

3. Limit Theorems

Let us proceed to find the asymptotic distribution of the Hausdorff distance between $H_n(p)$ and $H(p)$. First, for any function $f\colon X \mapsto \mathbf{R}$ and a compact set K_0 introduce the functional

$$\Phi(f) = \rho_{\mathrm{H}}(H(p; f) \cap K_0, H(p) \cap K_0) , \tag{3.1}$$

where

$$H(p; f) = \{x\colon\ h(x) \le p + f(x)\} .$$

Evidently, $\rho_{\mathrm{H}}(H_n(p), H(p)) = \Phi(\zeta_n(\cdot)/a_n)$. Let us put

$$\begin{aligned} K(p) &= \{x \in K_0\colon\ h(x) = p\} \\ K(p; \varepsilon) &= \{x \in K_0\colon\ |h(x) - p| \le \varepsilon\} , \quad \varepsilon > 0 . \end{aligned}$$

If $\sup_{x \in X} |f(x)| = \varepsilon$, then $H(p; f) = H(p; f|_\varepsilon)$, where

$$f|_\varepsilon(x) = \begin{cases} f(x) & , \quad x \in K(p; \varepsilon) \\ 0 & , \quad \text{otherwise} \end{cases} .$$

Therefore, in this case $\Phi(f) = \Phi(f|_\varepsilon)$.

Following Borovkov [1] and Molchanov [7] a functional Φ (in general not necessarily defined by (3.1)) is said to be continuously differentiable if there exists a functional Φ'

such that for each continuous function f and each sequence $\{f_\delta\}$, such that f_δ converges uniformly on K_0 to f as $\delta \downarrow 0$

$$\delta^{-1}\Phi(\delta f_\delta) \rightarrow \Phi'(f) \quad \text{as} \quad \delta \downarrow 0\,, \tag{3.2}$$

$$\Phi'(f_\delta) \rightarrow \Phi'(f) \quad \text{as} \quad \delta \downarrow 0\,, \tag{3.3}$$

$$\Phi'(f|_\varepsilon) \rightarrow \Phi'(f|_0) \quad \text{as} \quad \varepsilon \downarrow 0\,. \tag{3.4}$$

Theorem 3.1. *Let the random field (1.3) converge weakly to a continuous random field ζ. Suppose that the functional (3.1) is continuously differentiable. Then $a_n\rho_{\mathrm{H}}(H_n(p), H(p))$ converges in distribution to $\Phi'(\zeta|_0)$.*

PROOF. Evidently, $a_n\rho_{\mathrm{H}}(H_n(p), H(p)) = a_n\Phi((a_n^{-1}\zeta_n)|_{\eta_n})$, where η_n has been defined in (2.1). It is easy to show that the random variable $a_n\Phi((a_n^{-1}\zeta_n)|_\delta)$ converges in distribution to $\Phi'(\zeta|_\delta)$ for each sufficiently small δ. Now the statement of Theorem follows from (3.2)-(3.4). □

Let us now find a representation for the derivative Φ' of the functional (3.1). For this, define

$$\omega_h(x,t) = \inf\{h(y) - h(x)\colon\ \rho(x,y) \le t,\ y \in K_0\}\,, \quad x \in K_0\,. \tag{3.5}$$

Theorem 3.2. *Suppose that the following conditions are valid:*

(i) *For each x, $\omega_h(x,t)$ is continuous for t belonging to a certain neighborhood of the origin;*

(ii) *Function $\omega_h(x,t)$ is differentiable at $t = 0$ uniformly for $x \in K(p;\varepsilon)$ and its derivative $L(x) = \omega_h'(x,0)$ is upper semicontinuous and non-vanishing on $K(p;\varepsilon)$.*

Then the functional Φ is continuously differentiable,

$$\Phi'(f) = \sup_{x\in K(p)} |f(x)/L(x)|\,, \tag{3.6}$$

and $a_n\rho_{\mathrm{H}}(H_n(p)\cap K_0, H(p)\cap K_0)$ converges in distribution to $\sup_{x\in K(p)}|\zeta(x)/L(x)|$.

PROOF. Let us verify the differentiability of Φ and find its derivative. Let $M_+(\delta) = \{x \in K_0\colon\ f_\delta > 0\}$ and $M_-(\delta) = \{x \in K_0\colon\ f_\delta < 0\}$, where f_δ is the function from (3.2) and (3.3). Furthermore, put

$$S(\delta) = \Big\{x \in M_+(\delta)\colon\ h(x) \in (p, p+\delta f_\delta(x)]\Big\} \bigcup \Big\{x \in M_-(\delta)\colon\ h(x) \in (p+\delta f_\delta(x), p]\Big\}\,.$$

Then

$$\begin{aligned}\Phi(\delta f_\delta) &= \rho_{\mathrm{H}}(H(p;\delta f_\delta)\cap K_0, H(p)\cap K_0)\\ &= \max\left(\sup_{x\in M_+(\delta)\cap S(\delta)} \rho(x, H(p)\cap K_0),\ \sup_{x\in M_-(\delta)\cap S(\delta)} \rho(x, H(p;\delta f_\delta)\cap K_0)\right)\,.\end{aligned}$$

Put for any $\gamma < 0$

$$\bar\omega_h(x,\gamma) = \inf\{t \ge 0\colon\ \omega_h(x,t) = \gamma\}\,.$$

If $x \in M_+(\delta) \cap S(\delta)$, then $p = h(x) - \delta f_\delta(x) r_\delta(x)$, where $0 \le r_\delta \le 1$. Furthermore, $r_\delta(x) = 0$ for $x \in K(p)$.

Then, for $x \in M_+(\delta) \cap S(\delta)$,

$$\begin{aligned}\rho(x, H(p)) &= \inf\left\{t \ge 0\colon\ x \in (H(p) \cap K_0)^t\right\}\\ &= \inf\{t \ge 0\colon\ x \in b(y,t),\ h(y) \le p,\ y \in K_0\}\\ &= \inf\{\delta \ge 0\colon\ \omega_h(x,\delta) \le -\delta f_\delta(x) r_\delta(x)\}\\ &= \bar\omega_h(x, -\delta f_\delta(x) r_\delta(x))\,.\end{aligned}$$

Similarly, for each $x \in M_-(\delta) \cap S(\delta)$,

$$\rho(x, H(p;\delta f_\delta) \cap K_0) = \bar\omega_{h-\delta f_\delta}(x, \delta f_\delta(x) r_\delta(x))\,.$$

Thus, $\Phi(\delta f_\delta) = \max(\phi_+(\delta), \phi_-(\delta))$, where

$$\begin{aligned}\phi_+(\delta) &= \sup_{x \in M_+(\delta) \cap S(\delta)} \bar\omega_h(x, -\delta f_\delta(x) r_\delta(x))\,,\\ \phi_-(\delta) &= \sup_{x \in M_-(\delta) \cap S(\delta)} \bar\omega_{h-\delta f_\delta}(x, \delta f_\delta(x) r_\delta(x))\,.\end{aligned}$$

Let us show that the function $\phi_+(\delta)$ is differentiable at zero and find $\phi'_+(0)$. It follows from **(i)** and **(ii)** that $\bar\omega_h(x,\gamma)$ is differentiable at $\gamma = 0$ uniformly for $x \in K(p;\varepsilon)$, and $\bar\omega'_h(x,0) = 1/L(x)$. Since the functions f and r_δ are bounded and f_δ converges to f uniformly, we get

$$\bar\omega_h(x, -\delta f_\delta(x) r_\delta(x)) = \bar\omega'_h(x,0)[-\delta f_\delta(x) r_\delta(x)] + \delta\kappa(x,\delta)\,,$$

where $\sup_{x \in K(p;\varepsilon)} \kappa(x,\delta) \to 0$ as $\delta \to 0$. Therefore,

$$\begin{aligned}\phi'_+(0) &= \lim_{\delta \downarrow 0} \sup_{x \in M_+(\delta) \cap S(\delta)} \bar\omega'_h(x,0)[-f_\delta(x) r_\delta(x)]\\ &= \lim_{\delta \downarrow 0} \sup_{x \in M_+(\delta) \cap S(\delta)} |f(x)/L(x)|\,.\end{aligned}$$

Note that $\{x\colon\ f(x) > \alpha_\delta\} \subset M_+(\delta) \subset \{x\colon\ f(x) > -\alpha_\delta\}$, where

$$\alpha_\delta = \sup_{x \in K_0} |f(x) - f_\delta(x)| \to 0\,.$$

The continuity of f and **(ii)** yield the upper semicontinuity of the function $|f(x)/L(x)|$. Hence

$$\phi'_+(0) = \sup_{x \in K(p),\, f(x) \ge 0} |f(x)/L(x)|\,. \tag{3.7}$$

Let us proceed to find the derivative $\phi'_-(0)$. Clearly,

$$\delta^{-1}|\omega_{h-\delta f_\delta}(x,t) - \omega_h(x,t)| \le \sup\{|f_\delta(x) - f_\delta(y)|\colon\ \rho(x,y) \le t\}\,.$$

Thus,

$$|\omega_{h-\delta f_\delta}(x,t) - \omega'_h(x,0)t| \le h\Delta(x,\delta) + o(\delta) + o(t) \quad \text{as} \quad t \to 0\,,$$

where $\Delta(x,\delta) \to 0$ as $\delta \to 0$ uniformly for $x \in X$. For $\gamma = \delta f_\delta(x) r_\delta(x)$ we get

$$\begin{aligned}
\delta^{-1}\bar{\omega}_{h-\delta f_\delta}(x,\gamma) &= \delta^{-1}\inf\{t \ge 0\colon\ \omega_{h-\delta f_\delta}(x,t) = \gamma\} \\
&= \inf\{t \ge 0\colon\ \omega_{h-\delta f_\delta}(x,t\delta) = \gamma\} \\
&\le \inf\{t \ge 0\colon\ \omega_h'(x,0)t\delta = \gamma + \delta\Delta(x,\delta) + o(\delta) + o(t\delta)\} \\
&\le \inf\{t \ge 0\colon\ \omega_h'(x,0)t = f_\delta(x) r_\delta(x) + c(\delta)\}\ ,
\end{aligned}$$

where $c(\delta) \to 0$ as $\delta \to 0$.

A similar bound from below is also true. From **(ii)** we get

$$\begin{aligned}
\phi_-'(0) &= \lim_{\delta\downarrow 0} \sup_{x\in M_-(\delta)\cap S(\delta)} f_\delta(x) r_\delta(x)/L(x) \\
&= \sup_{x\in K(p), f(x)\le 0} |f(x)/L(x)|\,.
\end{aligned}$$

From this and (3.7) it follows that the derivative $\Phi'(f)$ is determined by (3.6). The functional Φ' satisfies the conditions (3.3) and (3.4). Now the limit theorem for the Hausdorff distance follows directly from Theorem 3.1. □

If $X = \mathbf{R}^d$ and h is continuously differentiable, then it is possible to find the corresponding derivative $L(x)$.

Theorem 3.3. *Suppose that* $K_0 = \mathrm{cl}(\mathrm{Int}\, K_0)$ *(K_0 is regular closed) and K_0 has C^1 boundary ∂K_0. Let $\mathbf{n}(x)$ be the unit outer normal vector to K_0 at $x \in \partial K_0$. Furthermore, let h be continuously differentiable at a neighborhood of $K(p)$. Put*

$$\mathbf{v}(x) = -\mathrm{grad}\, h(x) = -\left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_d}\right)\,.$$

Then the conditions **(i)** *and* **(ii)** *are valid and*

$$L(x) = \begin{cases} \|\mathbf{v}(x)\| & ,\quad x \in \mathrm{Int}\, K_0 \\ \|\mathbf{v}(x)\| & ,\quad x \in \partial K_0,\ \phi(x) \ge \frac{\pi}{2} \\ \|\mathbf{v}(x)\| \sin\phi(x) & ,\quad x \in \partial K_0,\ \phi(x) < \frac{\pi}{2} \end{cases}\,.$$

where $\phi(x)$ is the angle between $\mathbf{v}(x)$ and $\mathbf{n}(x)$.

Proof follows from the Taylor expansion of $h(y) - h(x)$ in (3.5).

In the analogous way also a system of inequalities $\{x\colon\ h_1(x) \le p_1, \ldots, h_m(x) \le p_m\}$ with $p_i > 0$ can be considered. This case can be reduced to the case (1.1) for the function $h(x) = \max_{1\le i\le m} h_i(x)/p_i$.

It is possible also to consider analogs of the Hausdorff distance by

$$\rho_{\mathrm{H}}^B(K, K_1) = \inf\{r > 0\colon\ K \subset K_1 \oplus rB,\ K_1 \subset K \oplus rB\}\ ,$$

where $\oplus$ is the Minkowski addition and B is a convex set containing the origin as its interior point. In the usual definition of the Hausdorff distance B is the unit ball. Then Theorem 3.2 is valid after replacing in (3.5) $\rho(x,y) \le \delta$ by $y \in x + \delta B$.

4. Examples

In the simplest case h is a monotone (say increasing) function on the line. Then the estimator $H_n(p)$ is strongly consistent if h is continuous at the point $x_p = \sup\{x\colon\ h(x) \le p\}$. Furthermore, $a_n \rho_{\mathrm{H}}(H_n(p), H(p))$ converges weakly to $|\zeta(x_p)/L(x_p)|$. If $h(x) = x$, then $L(x) = -1$ for $x \in (0,1)$. Thus, for $p \in (0,1)$, the limit is distributed as $|\zeta(p)|$. If $h(x) = x^2$, then $L(x) = -2|x|$, and $a_n \rho_{\mathrm{H}}(H_n(p), H(p))$ converges in distribution to $\max(\left|\zeta(\sqrt{p})\right|, \left|\zeta(-\sqrt{p})\right|)/2\sqrt{p}$.

Furthermore, in $\mathbf{R}^d$ put $h(x) = \|x\|^\alpha$, $\alpha > 0$. Then $L(x) = -\alpha\|x\|^{\alpha-1}$ inside $\operatorname{Int} K_0$, and the weak limit of $a_n \rho_{\mathrm{H}}(H_n(p), H(p))$ is equal to $\sup_{\|x\|=p} |\zeta(x)|/(\alpha p^{\alpha-1})$.

Let us consider also another example related to the theory of random closed sets. Let $\xi(x)$ be the support function of a random compact set A, that is

$$\xi(x) = \sup\{(u \cdot x)\colon\ u \in A\}\ ,$$

where $(u \cdot x)$ denotes the scalar product. We suppose that $\|A\| = \sup\{\|x\|\colon\ x \in A\}$ has a finite expectation. Then $h(x) = \mathbf{E}\xi(x)$ is the support function of the Aumann expectation (mean body) $\mathbf{E}A$ of A, see [12, 11]. For $p = 1$, the corresponding set $H(1)$ defined by (1.1) is a so-called *polar set* $(\mathbf{E}A)^\circ$ of $\mathbf{E}A$, see [10]. Suppose that $\mathbf{E}A$ contains the origin as an interior point.

Let $\xi_1, \dots, \xi_n$ be the support functions of iid copies of A. Then the set

$$H_n(p) = \left\{x\colon\ h_n(x) = \frac{1}{n}\sum \xi_i(x) \le p\right\} = p\left(\frac{1}{n}(A_1 \oplus \cdots \oplus A_n)\right)^\circ$$

is a strongly consistent estimator of $H(p) = p(\mathbf{E}A)^\circ$. Note that h_n is the support function of $(A_1 \oplus \cdots \oplus A_n)/n$.

Pick compact K_0 such that $(\mathbf{E}A)^\circ \subset \operatorname{Int} K_0$. If the boundary of $\mathbf{E}A$ is smooth ($\mathcal{C}^1$), then the function (3.5) satisfies the conditions of Theorem 3.2 with $L(x) = \|x\|^{-1}h(x)$. It follows from Theorem 3.2 and the central limit theorem for Minkowski sums of random sets [12] that

$$\sqrt{n}\rho_{\mathrm{H}}(H_n(1) \cap K_0, H(1)) = \sqrt{n}\rho_{\mathrm{H}}((A_1 \oplus \cdots \oplus A_n)/n)^\circ \cap K_0, (\mathbf{E}A)^\circ)$$

converges weakly to

$$\sup\{\zeta(x)\|x\|\colon\ x \in \partial(\mathbf{E}A)^\circ\}\ ,$$

where ζ is the centered Gaussian random field on $\mathbf{R}^d$ with the same covariance as the support function ξ of A.

A solution of inequality was used in [8] to estimate the shape of a deterministic grain in a Boolean model. For this, the function h is determined through the covariance function of the Boolean model. To avoid technicalities, we mention only that Theorem 3.2 can be applied to establish a limit theorem for the corresponding set-valued estimator.

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