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A Limit Theorem for Solutions of Inequalities

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Abstract

Let \( H(p) \) be the set \( \{x \in X: h(x) \leq p\} \), where \( h \) is a real-valued lower semicontinuous function on a locally compact second countable metric space \( X \). A limit theorem is proved for the empirical counterpart of \( H(p) \) obtained by replacing of \( h \) with its estimator.

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1. Introduction

Consider a certain lower semicontinuous real-valued function \( h \) defined on a locally compact second countable metric space \((X, \rho)\). Then the set

\[ H(p) = \{x \in X: h(x) \leq p\} \quad (1.1) \]

is closed. The aim of this paper is to prove a limit theorem for the estimator \( H_n(p) \) of the set \( H(p) \) obtained by replacing \( h(x) \) in (1.1) with its estimator \( h_n(x) \):

\[ H_n(x) = \{x \in X: h_n(x) \leq p\} \quad (1.2) \]

A simple problem of this kind originates in classical statistics.

Example 1.1. Suppose that \( h(x) = F(x) \), \( x \in \mathbb{R} \), is the distribution function of a random variable \( \nu \). Then \( H(p) = (-\infty, x_p] \), where \( x_p \) is the \( p \)-quantile of \( \nu \), and \( H_n(p) \) is related to the corresponding empirical quantile if \( h_n \) is the empirical distribution function. A generalization for quantiles of random vectors and random closed sets was considered by Eddy [2] and Molchanov [7].
If \( h \) is a density, then the level set \( H(p) \) appears in cluster analysis, see Hartigan [3].
An estimator of \( H(p) \) based on minimization of the so-called excess mass was considered in [4, 9]. Similar problems appear also in the estimation of the support of a density, see [5].

Further we shall not discuss the nature of the estimator \( h_n \). We only suppose that the estimator \( h_n \) is strongly consistent in the uniform metric and

\[
\zeta_n = a_n(h_n - h)
\]

admits a weak limit \( \zeta \) as \( n \to \infty \), i.e. each continuous in the uniform metric functional of \( \zeta_n \) converges in distribution to its value on \( \zeta \). Here \( a_n \to \infty \) as \( n \to \infty \) is a sequence of norming constants.

Suppose also that each function \( h_n \) is almost surely lower semicontinuous. Then \( H_n(p) \) is a random closed set as introduced in [6].

Space \( K \) of all compact subsets of \( X \) can be metrized by the Hausdorff distance:

\[
\rho_H(K, K_1) = \inf \{ r > 0: K \subset K_1^r, K_1 \subset K^r \},
\]

where \( K, K_1 \in \mathcal{K} \),

\[
K^r = \{ x: b(x, r) \cap K \neq \emptyset \}
\]

is the \( r \)-parallel set to \( K \) and \( b(x, r) \) is the ball of radius \( r \) centered at \( x \). For each set \( M \subset X \) we shall write \( \text{cl}(M) \), \( \text{Int} M \) and \( \partial M \) for its closure, interior and boundary respectively.

2. STRONG CONSISTENCY

The estimator \( H_n(p) \) is said to be strongly consistent if

\[
\rho_H(H_n(p) \cap K_0, H(p) \cap K_0) \to 0 \quad \text{a.s. as } n \to \infty
\]

for each compact \( K_0 \). The distance \( \rho_H(H_n(p) \cap K_0, H(p) \cap K_0) \) is a random variable, since \( H_n(p) \cap K_0 \) is a random closed set, and the function \( \rho_H(\cdot, K) \) is continuous.

**Theorem 2.1.** Suppose that, for each compact \( K_0 \),

\[
\eta_n = \sup_{x \in K_0} |h_n(x) - h(x)| \to 0 \quad \text{a.s. as } n \to \infty .
\]

The estimator \( H_n(p) \) is strongly consistent if

\[
H(p) \subset \text{cl}(H(p-)),
\]

where \( H(p-) = \{ x: h(x) < p \} \). If for each \( x \) there exists a sequence \( n(k) \) such that \( h_n(k)(x) > h(x) \) a.s., then (2.2) is also a necessary condition. Moreover, if (2.2) is valid for each \( p \in [c_1, c_2] \), then

\[
\sup_{c_1 \leq p \leq c_2} \rho_H(H_n(p) \cap K_0, H(p) \cap K_0) \to 0 \quad \text{a.s. as } n \to \infty .
\]
PROOF. To simplify notations suppose that $X$ is compact, and $K_0 = X$.

Sufficiency. It is evident that the function

$$\phi(\varepsilon) = \rho_H(H(p + \varepsilon), H(p)),$$

is right-continuous, non-increasing for $\varepsilon < 0$ and non-decreasing for $\varepsilon > 0$. Note that $\phi(0) = 0$. Moreover, (2.2) yields $\rho_H(H(p), H(p - \varepsilon)) = 0$, that is $\phi$ is continuous at zero.

Evidently, $H_n(p) \subset \{x: h(x) < p + \eta_n\}$. Similarly,

$$H(p) \subset H(p - \eta_n)_{\phi(\eta_n)} = \cup \{b(x, \phi(\eta_n)) : h(x) < p - \eta_n\} \subset H_n(p)_{\phi(\eta_n)}.$$

Hence, (2.1) yields

$$\rho_H(H_n(p), H_n) \leq \phi(\eta_n) \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

Furthermore, (2.3) follows from the monotonicity of $H$ and its estimator.

Necessity. Let $H(p) \notin \text{cl}(H(p - \varepsilon))$. Then there exists a point $x$, such that $h(x) = p$ and $\rho(x, \text{cl}(H(p - \varepsilon)) = \delta > 0$. Then $h_n(x) > h(x) = p$ by the condition of Theorem. Therefore, $x \notin H_n(p)$, whence $\rho(H_n(p), H(p)) > \delta$, contrary to the consistency of the estimator $H_n(p)$. □

Heuristic, (2.2) for all $p$ means that the function $h$ is not constant on any open subset of $X$.

3. LIMIT THEOREMS

Let us proceed to find the asymptotic distribution of the Hausdorff distance between $H_n(p)$ and $H(p)$. First, for any function $f : X \mapsto \mathbb{R}$ and a compact set $K_0$ introduce the functional

$$\Phi(f) = \rho_H(H(p; f) \cap K_0, H(p) \cap K_0),$$

where

$$H(p; f) = \{x : h(x) \leq p + f(x)\}.$$

Evidently, $\rho_H(H_n(p), H(p)) = \Phi(\zeta_n(\cdot)/a_n)$. Let us put

$$K(p) = \{x \in K_0 : h(x) = p\},$$

$$K(p; \varepsilon) = \{x \in K_0 : |h(x) - p| \leq \varepsilon\}, \quad \varepsilon > 0.$$

If $\sup_{x \in X} |f(x)| = \varepsilon$, then $H(p; f) = H(p; f|_{\varepsilon})$, where

$$f|_{\varepsilon}(x) = \begin{cases} f(x), & x \in K(p; \varepsilon) \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, in this case $\Phi(f) = \Phi(f|_{\varepsilon})$.

Following Borovkov [1] and Molchanov [7] a functional $\Phi$ (in general not necessarily defined by (3.1)) is said to be continuously differentiable if there exists a functional $\Phi'$
such that for each continuous function $f$ and each sequence $\{f_\delta\}$, such that $f_\delta$ converges uniformly on $K_0$ to $f$ as $\delta \downarrow 0$

\begin{align*}
\delta^{-1} \Phi(\delta f_\delta) &\to \Phi(f) \quad \text{as} \quad \delta \downarrow 0, \quad (3.2) \\
\Phi'(f_\delta) &\to \Phi'(f) \quad \text{as} \quad \delta \downarrow 0, \quad (3.3) \\
\Phi'(f|_\varepsilon) &\to \Phi'(f|_0) \quad \text{as} \quad \varepsilon \downarrow 0. \quad (3.4)
\end{align*}

**Theorem 3.1.** Let the random field (1.3) converge weakly to a continuous random field $\zeta$. Suppose that the functional (3.1) is continuously differentiable. Then $a_n \rho_H(H_n(p), H(p))$ converges in distribution to $\Phi'(\zeta|_0)$.

**Proof.** Evidently, $a_n \rho_H(H_n(p), H(p)) = a_n \Phi'(a_n^{-1} \zeta_n|_{\eta_n})$, where $\eta_n$ has been defined in (2.1). It is easy to show that the random variable $a_n \Phi'(a_n^{-1} \zeta_n|_{\eta_n})$ converges in distribution to $\Phi'(\zeta|_\delta)$ for $\delta$ sufficiently small. Now the statement of Theorem follows from (3.2)-(3.4).

Let us now find a representation for the derivative $\Phi'$ of the functional (3.1). For this, define

$$
\omega_h(x, t) = \inf \{h(y) - h(x): \rho(x, y) \leq t, \ y \in K_0\}, \quad x \in K_0. \quad (3.5)
$$

**Theorem 3.2.** Suppose that the following conditions are valid:

(i) Each $x$, $\omega_h(x, t)$ is continuous for $t$ belonging to a certain neighborhood of the origin;

(ii) Function $\omega_h(x, t)$ is differentiable at $t = 0$ uniformly for $x \in K(p; \varepsilon)$ and its derivative $L(x) = \omega_h'(x, 0)$ is upper semicontinuous and non-vanishing on $K(p; \varepsilon)$.

Then the functional $\Phi$ is continuously differentiable,

$$
\Phi'(f) = \sup_{x \in K(p)} |f(x)/L(x)|, \quad (3.6)
$$

and $a_n \rho_H(H_n(p) \cap K_0, H(p) \cap K_0)$ converges in distribution to $\sup_{x \in K(p)} |\zeta(x)/L(x)|$.

**Proof.** Let us verify the differentiability of $\Phi$ and find its derivative. Let $M_+(\delta) = \{x \in K_0: f_\delta > 0\}$ and $M_-(\delta) = \{x \in K_0: f_\delta < 0\}$, where $f_\delta$ is the function from (3.2) and (3.3). Furthermore, put

$$
S(\delta) = \left\{x \in M_+(\delta): h(x) \in (p, p + \delta f_\delta(x))\right\} \cup \left\{x \in M_-(\delta): h(x) \in (p + \delta f_\delta(x), p)\right\}.
$$

Then

$$
\Phi(\delta f_\delta) = \rho_H(H(p; \delta f_\delta) \cap K_0, H(p) \cap K_0)
= \max \left( \sup_{x \in M_+(\delta) \cap S(\delta)} \rho(x, H(p) \cap K_0), \sup_{x \in M_-(\delta) \cap S(\delta)} \rho(x, H(p; \delta f_\delta) \cap K_0) \right).
$$

Put for any $\gamma < 0$

$$
\bar{\omega}_h(x, \gamma) = \inf \{t \geq 0: \omega_h(x, t) = \gamma\}.
$$
If \( x \in M_+(\delta) \cap S(\delta) \), then \( p = h(x) - \delta f_\delta(x)r_\delta(x) \), where \( 0 \leq r_\delta \leq 1 \). Furthermore, \( r_\delta(x) = 0 \) for \( x \in K(p) \).

Then, for \( x \in M_+(\delta) \cap S(\delta) \),
\[
\rho(x, H(p)) = \inf \left\{ t \geq 0: x \in (H(p) \cap K_0)^t \right\} \\
= \inf \left\{ t \geq 0: x \in b(y, t), h(y) \leq p, y \in K_0 \right\} \\
= \inf \{ \delta \geq 0: \omega_h(x, \delta) \leq -\delta f_\delta(x) r_\delta(x) \} \\
= \omega_h(x, -\delta f_\delta(x) r_\delta(x)).
\]

Similarly, for each \( x \in M_-(\delta) \cap S(\delta) \),
\[
\rho(x, H(p; \delta f_\delta) \cap K_0) = \omega_h(x, \delta f_\delta(x) r_\delta(x)).
\]

Thus, \( \Phi(\delta f_\delta) = \max(\phi_+(\delta), \phi_-(\delta)) \), where
\[
\phi_+(\delta) = \sup_{x \in M_+(\delta) \cap S(\delta)} \omega_h(x, -\delta f_\delta(x) r_\delta(x)), \\
\phi_-(\delta) = \sup_{x \in M_-(\delta) \cap S(\delta)} \omega_h(x, \delta f_\delta(x) r_\delta(x)).
\]

Let us show that the function \( \phi_+(\delta) \) is differentiable at zero and find \( \phi_+'(0) \). It follows from \( \text{(i)} \) and \( \text{(ii)} \) that \( \omega_h(x, \gamma) \) is differentiable at \( \gamma = 0 \) uniformly for \( x \in K(p; \varepsilon) \), and \( \omega_h(x, 0) = 1/L(x) \). Since the functions \( f \) and \( r_\delta \) are bounded and \( f_\delta \) converges to \( f \) uniformly, we get
\[
\omega_h(x, -\delta f_\delta(x) r_\delta(x)) = \omega_h'(x, 0)[-\delta f_\delta(x) r_\delta(x)] + \delta \kappa(x, \delta),
\]
where \( \sup_{x \in K(p; \varepsilon)} \kappa(x, \delta) \to 0 \) as \( \delta \to 0 \). Therefore,
\[
\phi_+'(0) = \lim_{\delta \to 0} \sup_{x \in M_+(\delta) \cap S(\delta)} \omega_h'(x, 0)[-\delta f_\delta(x) r_\delta(x)] \\
= \lim_{\delta \to 0} \sup_{x \in M_+(\delta) \cap S(\delta)} |f(x)/L(x)|.
\]

Note that \( \{ x: f(x) > \alpha_\delta \} \subset M_+(\delta) \subset \{ x: f(x) > -\alpha_\delta \} \), where
\[
\alpha_\delta = \sup_{x \in K_0} |f(x) - f_\delta(x)| \to 0.
\]

The continuity of \( f \) and \( \text{(ii)} \) yield the upper semicontinuity of the function \( |f(x)/L(x)| \). Hence
\[
\phi_+'(0) = \sup_{x \in K(p), f(x) \geq 0} |f(x)/L(x)|. \tag{3.7}
\]

Let us proceed to find the derivative \( \phi_-'(0) \). Clearly,
\[
\delta^{-1} \left| \omega_{h-\delta f_\delta}(x, t) - \omega_h(x, t) \right| \leq \sup \{ |f_\delta(x) - f_\delta(y)|; \rho(x, y) \leq t \}.
\]

Thus,
\[
|\omega_{h-\delta f_\delta}(x, t) - \omega_h'(x, 0) t| \leq h \Delta(x, \delta) + o(\delta) + o(t) \quad \text{as} \quad t \to 0,
\]
where $\Delta(x, \delta) \to 0$ as $\delta \to 0$ uniformly for $x \in X$. For $\gamma = \delta f_\delta(x) r_\delta(x)$ we get

$$
\delta^{-1} \omega_{h-\delta f_\delta}(x, \gamma) = \delta^{-1} \inf \{ t \geq 0 : \omega_{h-\delta f_\delta}(x, t) = \gamma \}
= \inf \{ t \geq 0 : \omega_{h-\delta f_\delta}(x, t\delta) = \gamma \}
\leq \inf \{ t \geq 0 : \omega'_h(x, 0)t\delta = \gamma + \delta \Delta(x, \delta) + o(\delta) + o(t\delta) \}
\leq \inf \{ t \geq 0 : \omega'_h(x, 0)t = f_\delta(x) r_\delta(x) + c(\delta) \},
$$

where $c(\delta) \to 0$ as $\delta \to 0$.

A similar bound from below is also true. From (ii) we get

$$
\phi'_-(0) = \lim_{\delta \downarrow 0} \sup_{x \in M_{-}(\delta) \cap s(\delta)} f_\delta(x) r_\delta(x)/L(x)
= \sup_{x \in K(p), f(x) \leq 0} |f(x)/L(x)|.
$$

From this and (3.7) it follows that the derivative $\Phi'(f)$ is determined by (3.6). The functional $\Phi'$ satisfies the conditions (3.3) and (3.4). Now the limit theorem for the Hausdorff distance follows directly from Theorem 3.1. □

If $X = \mathbb{R}^d$ and $h$ is continuously differentiable, then it is possible to find the corresponding derivative $L(x)$.

**Theorem 3.3.** Suppose that $K_0 = \text{cl}(\text{Int} K_0)$ ($K_0$ is regular closed) and $K_0$ has $C^1$ boundary $\partial K_0$. Let $n(x)$ be the unit outer normal vector to $K_0$ at $x \in \partial K_0$. Furthermore, let $h$ be continuously differentiable at a neighborhood of $K(p)$. Put

$$
v(x) = -\text{grad} h(x) = -\left( \frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_d} \right).
$$

Then the conditions (i) and (ii) are valid and

$$
L(x) = \begin{cases} 
\|v(x)\| & , \ x \in \text{Int} K_0 \\
\|v(x)\| & , \ x \in \partial K_0, \ \phi(x) \geq \frac{\pi}{2} \\
\|v(x)\| \sin \phi(x) & , \ x \in \partial K_0, \ \phi(x) < \frac{\pi}{2} 
\end{cases}
$$

where $\phi(x)$ is the angle between $v(x)$ and $n(x)$.

**Proof** follows from the Taylor expansion of $h(y) - h(x)$ in (3.5).

In the analogous way also a system of inequalities $\{x : h_1(x) \leq p_1, \ldots, h_m(x) \leq p_m\}$ with $p_i > 0$ can be considered. This case can be reduced to the case (1.1) for the function $h(x) = \max_{1 \leq i \leq m} h_i(x)/p_i$.

It is possible also to consider analogs of the Hausdorff distance by

$$
\rho_B^K(K, K_1) = \inf \{ r > 0 : K \subset K_1 \oplus rB, \ K_1 \subset K \oplus rB \},
$$

where $\oplus$ is the Minkowski addition and $B$ is a convex set containing the origin as its interior point. In the usual definition of the Hausdorff distance $B$ is the unit ball. Then Theorem 3.2 is valid after replacing in (3.5) $\rho(x, y) \leq \delta$ by $y \in x + \delta B$. 
4. Examples

In the simplest case $h$ is a monotone (say increasing) function on the line. Then the estimator $H_n(p)$ is strongly consistent if $h$ is continuous at the point $x_p = \sup \{x : h(x) \leq p\}$. Furthermore, $a_n \rho_H(H_n(p), H(p))$ converges weakly to $|\zeta(x_p)/L(x_p)|$. If $h(x) = x$, then $L(x) = -1$ for $x \in (0,1)$. Thus, for $p \in (0,1)$, the limit is distributed as $|\zeta(p)|$. If $h(x) = x^2$, then $L(x) = -2|x|$, and $a_n \rho_H(H_n(p), H(p))$ converges in distribution to
\[
\max\{\zeta(\sqrt{p}), \zeta(-\sqrt{p})\}/2\sqrt{p}.
\]

Furthermore, in $\mathbb{R}^d$ put $h(x) = \|x\|^\alpha$, $\alpha > 0$. Then $L(x) = -\alpha\|x\|^{\alpha-1}$ inside $\text{Int} K_0$, and the weak limit of $a_n \rho(H_n(p), H(p))$ is equal to $\sup_{\|x\|=p} |\zeta(x)|/(\alpha p^{\alpha-1})$.

Let us consider also another example related to the theory of random closed sets. Let $\xi(x)$ be the support function of a random compact set $A$, that is
\[
\xi(x) = \sup \{(u \cdot x) : u \in A\},
\]
where $(u \cdot x)$ denotes the scalar product. We suppose that $\|A\| = \sup \{\|x\| : x \in A\}$ has a finite expectation. Then $h(x) = E \xi(x)$ is the support function of the Aumann expectation (mean body) $E A$ of $A$, see [12, 11]. For $p = 1$, the corresponding set $H(1)$ defined by (1.1) is a so-called polar set $(E A)^\circ$ of $E A$, see [10]. Suppose that $E A$ contains the origin as an interior point.

Let $\xi_1, \ldots, \xi_n$ be the support functions of iid copies of $A$. Then the set
\[
H_n(p) = \left\{x : h_n(x) = \frac{1}{n} \sum \xi_i(x) \leq p\right\} = p \left(\frac{1}{n} (A_1 \oplus \cdots \oplus A_n)\right)^\circ
\]
is a strongly consistent estimator of $H(p) = p(E A)^\circ$. Note that $h_n$ is the support function of $(A_1 \oplus \cdots \oplus A_n)/n$.

Pick compact $K_0$ such that $(E A)^\circ \subset \text{Int} K_0$. If the boundary of $E A$ is smooth ($C^1$), then the function (3.5) satisfies the conditions of Theorem 3.2 with $L(x) = \|x\|^{-1} h(x)$. It follows from Theorem 3.2 and the central limit theorem for Minkowski sums of random sets [12] that
\[
\sqrt{n} \rho_H(H_n(1) \cap K_0, H(1)) = \sqrt{n} \rho_H((A_1 \oplus \cdots \oplus A_n)/n)^\circ \cap K_0, (E A)^\circ
\]
converges weakly to
\[
\sup \{\zeta(x)\|x\| : x \in \partial (E A)^\circ\},
\]
where $\zeta$ is the centered Gaussian random field on $\mathbb{R}^d$ with the same covariance as the support function $\xi$ of $A$.

A solution of inequality was used in [8] to estimate the shape of a deterministic grain in a Boolean model. For this, the function $h$ is determined through the covariance function of the Boolean model. To avoid technicalities, we mention only that Theorem 3.2 can be applied to establish a limit theorem for the corresponding set-valued estimator.
References


