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Abstract

The compensation approach has recently been introduced for the determination of the stationary distribution of two-dimensional random walks without one-step transitions to the North, the North-East and the East. It is claimed that this approach produces the exact representation for the stationary state probabilities. The present study shows that this claim is incorrect for the case of the symmetrical shortest queueing model.

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Introduction

The 'compensation approach' has recently been introduced for the determination of the stationary distribution of queueing processes which can be modelled by two-dimensional nearest-neighbour random walks for which one-step transitions to the North, North-East and East are not possible, cf. [1],[2],[3],[4]. This approach leads to a series representation for the stationary state probabilities of which the terms are obtained by a recursive algorithm. In [1] and [2] it is claimed that the compensation approach generates the exact solution, i.e. the unique absolutely convergent solution of the Kolmogorov equations for the inherent random walk, whenever it is positive recurrent. Recent research, cf.[6], suggested that the just mentioned statement does not seem to be true for the *asymmetrical* shortest queueing problem. In order to obtain a deeper insight concerning the compensation approach we compare the results obtained by the compensation approach with those resulting from the anaytic solution of the symmetrical shortest queue, cf.[5].

In Section 2 several results obtained in [5] are recapitulated. The two key functions $\Omega(r)$ and $\Phi(t)$ in the analysis of the symmetrical shortest queue are both meromorphic and have been represented in [5] as infinite product forms, see (2.8) below. For our present goal we need their partial fractions representation. From this representation the expressions for the state probabilities are derived in Section 3. With $\mathbf{x}_i$ the queue length of server i, $i=1,2$, p_{km}, $\Omega(r)$ and $\Phi(t)$ are defined by (note the symmetry): for $k,m \in \{0,1,2,\dots\}$,

$$p_{km} := \Pr\{\mathbf{x}_1 = k, \mathbf{x}_2 = k+m\} = \Pr\{\mathbf{x}_2 = k, \mathbf{x}_1 = k+m\}, \tag{1.1}$$

$$\Omega(r) := \Sigma_{m=0}^{\infty} p_{0m} r^m, \qquad |r| \le 1, \qquad \Phi(t) := \Sigma_{k=0}^{\infty} p_{k0} t^k, \qquad |t| \le 1.$$

Section 4 starts with the description of the compensation approach, and the derivation of expressions for the functions $\Omega^{(a)}(r)$ and $\Phi^{(w)}(t)$, the analogous functions of those in (1.1) but based on the compensation approach. It appears that $\Omega(r)$ and $\Omega^{(a)}(r)$, and, similarly $\Phi(t)$ and $\Phi^{(w)}(t)$, do not have the same structure, but they do have the same pole set. The expressions for the p_{km} calculated according to the compensation approach differ from those derived in Section 3. However, the leading terms in the asymptotic expression of p_{k0} for $k \to \infty$ derived from the results in Sections 3 and 4 do agree, similarly for p_{k1}, $k \to \infty$, and for p_{0m} for $m \to \infty$. In Section 4 it is further shown that the compensation approach cannot produce an absolutely convergent solution of all the Kolmogorov equations of the inherent random walk. Further it is explicitly shown that the p_{km} as calculated by the compensation approach, cf.[1] and [2], do not satisfy all the Kolmogorov equations. The available numerical results obtained by the compensation approach appear to be quite accurate. Presumable this is due to the fact that the first term in the asymptotic series for $p_{k0}^{(a)}$ is chosen in such a way in [1] and [2] that it agrees with the exact result obtained in [9] and to the fact that this leading asymptotic term yields already for moderate values of k a fairly sharp approximation, see appendix D.

2. The partial fractions representation

The bivariate generating function of the stationary distribution of the queue lengths $(\mathbf{x}_1, \mathbf{x}_2)$ can be expressed as a linear combination of the generating functions

$$\Omega(r) := \mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 = 0)\}, \quad |r| \le 1, \tag{2.1}$$

$$\Phi(t) := \mathrm{E}\{t^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2)\}, \quad |t| \le 1,$$

see [5]; here $\mathbf{x}_i$ is the queue length at server i, $i = 1, 2$. These functions are meromorphic functions, their infinite product form representations have been derived in [5]. For the purpose of the present study their partial fractions representation is needed. In the present section they will be derived.

Remark 2.1. The definitions of symbols used in the present text differs only in some minor points from those in [5]. □

Put, cf.(1.1),

$$F(r,t) := \mathrm{E}\{t^{\mathbf{x}_1} r^{\mathbf{x}_2 - \mathbf{x}_1}(\mathbf{x}_2 \ge \mathbf{x}_1)\} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} p_{km} t^k r^m, \qquad |t| \le 1,\ |r| \le 1. \tag{2.2}$$

From (2.3) of [5] it is seen that for $|r| \le 1$, $|t| \le 1$,

$$k_1(r,t)\mathrm{F}(r,t) = r[(r-t)\Omega(r) - \{r + \frac{1}{2}art - \frac{1}{2}(2+a)t\}\Phi(t)] + k_1(r,t)\Phi(t), \tag{2.3}$$

with

$$k_1(r,t) = at^2 + [1 - (2+a)r]t + r^2. \tag{2.4}$$

Denote by $t_1(r)$, $t_2(r)$ for fixed r the two zeros of $k_1(r,t)$, they may be so defined that, cf.(3.1) of [5],

$$|t_1(r)| < |r| < |t_2(r)| \quad \text{for} \quad |r| \ge 1,\ r \ne 1, \tag{2.5}$$

$$t_1(1) = \min(1, \frac{1}{a}), \qquad t_2(1) = \max(1, \frac{1}{a});$$

note that

$$0 < a < 2, \tag{2.6}$$

is the necessary and sufficient condition for the existence of a stationary distribution.

In [5] the sequences r_n, $n = 0, 1, 2, \ldots$; t_n, $n = 0, 1, 2, \ldots$, are defined by

$$r_0 = \frac{2}{a}, \quad t_0 = \frac{4}{a^2}, \tag{2.7}$$

$$k_1(r_n, t_{n-1}) = 0, \qquad k_1(r_n, t_n) = 0,$$

$$r_0 < t_0 < r_1 < t_1 < r_2 \quad \dots \quad < t_{n-1} < r_n < t_n \dots,$$

$$t_{n-1} = t_1(r_n),\ n = 1, 2, \dots, \qquad t_n = t_2(r_n),\ n = 0, 1, 2, \dots.$$

From (4.7) of [5] we have for $|r| \le 1$, $|t| \le 1$,

$$\Omega(r) = \frac{1}{2}(2-a)\{\prod_{n=0}^{\infty} \frac{r_n^- - r}{r_n^- - 1}\}\{\prod_{n=1}^{\infty} \frac{r_n - 1}{r_n - r}\}, \tag{2.8}$$

$$\Phi(t) = \frac{1}{1+a}\{\prod_{n=1}^{\infty} \frac{t_n^- - t}{t_n^- - 1}\}\{\prod_{n=0}^{\infty} \frac{t_n - 1}{t_n - t}\}.$$

These functions can be continued meromorphically, and

$r_n^-, n = 0, 1, 2, \dots,$ are the zeros of $\Omega(r)$, $r_n^- < 0$,
$t_n^-, n = 1, 2, \dots,$ are the zeros of $\Phi(t)$, $t_n^- < 0$,

$r_n, n = 1, 2, \dots$ are the poles of $\Omega(r)$, $r_n > 0$,
$t_n, n = 0, 1, \dots$ are the poles of $\Phi(t)$, $t_n > 0$,

all poles and zeros have multiplicity one, see (3.8) and (3.10) of [5].

For the construction of the partial fractions representations of $\Omega(r)$ and $\Phi(t)$ we need their residues ω_n and ϕ_n:

$$\begin{aligned} \phi_n &:= \lim_{t \to t_n} (t - t_n)\Phi(t), \qquad && n = 0, 1, 2, \dots, \\ \omega_n &:= \lim_{r \to r_n} (r - r_n)\Omega(r), \qquad && n = 1, 2, \dots. \end{aligned} \tag{2.9}$$

These residues can be obtained immediately from (2.8). However, since we need the recursive relations between these residues, it is more effective to start from the relations which have led in [5] to the expression (2.8), cf. (3.6) of [5]. These relations are: for $j = 1, 2$,

$$\begin{aligned} &\text{(i)} \quad \Omega(r) + k_2(r, t_j(r))\Phi(t_j(r)) = 0 \qquad && \text{for all} \quad r \ne r_n, \\ &\text{(ii)} \quad \lim_{r \to r_n} [\Omega(r) + k_2(r, t_j(r))\Phi(t_j(r))] = 0 \qquad && \text{for all} \quad n = 0, 1, 2, \dots, \end{aligned} \tag{2.10}$$

with

$$k_2(r, t) := -1 + \frac{1}{2}a^2 t - \frac{1}{2}ar. \tag{2.11}$$

The relations (2.10) follow from the condition that $\mathrm{F}(r, t)$, cf.(2.2), should be finite for all $|r| \le 1$, $|t| \le 1$, see [5].

To derive the recursive relations for the residues note that $k_1(r, t_j(r)) = 0$, $j = 1, 2$, implies that

$$\frac{\mathrm{d}t_j(r)}{\mathrm{d}r} = -\frac{2r - (2+a)t_j(r)}{2at_j(r) + 1 - (2+a)r}. \tag{2.12}$$

Because it follows from the properties of the roots of the quadratic equation $k_1(r, t) = 0$, cf. (b.4) of

[5]: that

$$r_n + r_{n+1} = (2+a)t_n, \quad t_n + t_{n+1} = \frac{1}{a}[(2+a)r_{n+1} - 1], \tag{2.13}$$

we obtain

$$\frac{\mathrm{d}t_2(r)}{\mathrm{d}r}|_{r=r_n} = \frac{r_n - r_{n+1}}{t_n - t_{n-1}}, \qquad \frac{\mathrm{d}t_1(r)}{\mathrm{d}r}|_{r=r_n} = \frac{r_n - r_{n-1}}{t_{n-1} - t_n}. \tag{2.14}$$

From (2.7) and (2.9) we obtain: for $j = 1, 2$,

$$\lim_{r\to r_n} (r - r_n)\Phi(t_2(r)) = \phi_n \left[\frac{\mathrm{d}t_2(r)}{\mathrm{d}r}\right]^{-1}_{r=r_n}, \quad n = 0, 1, 2, \ldots, \tag{2.15}$$

$$\lim_{r\to r_n} (r - r_n)\Phi(t_1(r)) = \phi_{n-1} \left[\frac{\mathrm{d}t_1(r)}{\mathrm{d}r}\right]^{-1}_{r=r_n}, \quad n = 1, 2, \ldots.$$

Hence from (2.9), (2.10)ii, (2.14) and (2.15), for $n = 1, 2, \ldots$,

$$\begin{aligned} \omega_n &= -k_2(r_n, t_n)\frac{t_n - t_{n-1}}{r_n - r_{n+1}}\phi_n, \\ &= -k_2(r_n, t_{n-1})\frac{t_{n-1} - t_n}{r_n - r_{n-1}}\phi_{n-1}, \end{aligned} \tag{2.16}$$

and so: for $n = 1, 2, \ldots$

$$\frac{\phi_n}{\phi_{n-1}} = \frac{-1 + \frac{1}{2}a^2 t_{n-1} - \frac{1}{2}ar_n}{-1 + \frac{1}{2}a^2 t_n - \frac{1}{2}ar_n}\ \frac{r_{n+1} - r_n}{r_n - r_{n-1}}, \tag{2.17}$$

$$\frac{\omega_{n+1}}{\omega_n} = \frac{-1 + \frac{1}{2}a^2 t_n - \frac{1}{2}ar_{n+1}}{-1 + \frac{1}{2}a^2 t_n - \frac{1}{2}ar_n}\ \frac{t_{n+1} - t_n}{t_n - t_{n-1}}.$$

From (2.16) and (2.17) it is seen that the ω_n and ϕ_n can be recursively calculated once ϕ_0 is known. For the determination of ϕ_0 see (2.29) below.

We consider the relations (2.17) for $n \to \infty$. It is readily seen from (2.4) and (2.7) that $r_n \to \infty$, $t_n \to \infty$ for $n \to \infty$ and that, cf. (4.2), (4.3), (b.7) of [5]: for $n \to \infty$,

$$\frac{t_n}{r_n} \to \delta, \qquad \frac{r_{n+1}}{t_n} \to a\delta, \qquad \frac{t_{n+1}}{t_n} \to a\delta^2, \qquad \frac{r_{n+1}}{r_n} \to a\delta^2, \tag{2.18}$$

$$\delta := \frac{1}{2a}[2 + a + \sqrt{a^2+4}] > \frac{2}{a} > 1.$$

From (2.17) and (2.18) it is readily seen that ϕ_n/ϕ_{n-1}, and similarly ω_{n+1}/ω_n, has a limit for $n \to \infty$ and

$$\begin{aligned} &\lim_{n\to\infty} \frac{\phi_n}{\phi_{n-1}} = -\mu_\phi, \qquad &&\lim_{n\to\infty} \frac{\omega_n}{\omega_{n-1}} = -\mu_\omega, \\ &\mu_\phi := a\delta\frac{\delta - 1}{a\delta - 1}, \qquad &&\mu_\omega := a^2\delta^3\frac{\delta - 1}{a\delta - 1}. \end{aligned} \tag{2.19}$$

Because δ, cf.(2.18), satisfies

$$a\delta^2 - (2+a)\delta + 1 = 0, \tag{2.20}$$

it follows readily from (2.18) that

$$\frac{1}{\delta}\frac{\delta-1}{a\delta-1} < 1, \qquad \mu_\phi > 1, \qquad \mu_\omega > 1. \tag{2.21}$$

Put

$$\begin{aligned} &\text{(i)} \quad \tilde{\Phi}(t) := \sum_{n=0}^{\infty} \frac{\phi_n}{t-t_n} \\ &\text{(ii)} \quad \tilde{\Omega}(t) := \sum_{n=1}^{\infty} \frac{\omega_n}{r-r_n}\frac{r}{r_n}. \end{aligned} \tag{2.22}$$

To justify the definitions note that for $n \to \infty$,

$$\begin{aligned} &\frac{\phi_{n+1}}{t-t_{n+1}} \Big/ \frac{\phi_n}{t-t_n} = \frac{\phi_{n+1}}{\phi_n}\frac{t_n}{t_{n+1}}\frac{1-\frac{t}{t_n}}{1-\frac{t}{t_{n+1}}} \to -\frac{1}{\delta}\frac{\delta-1}{a\delta-1}, \\ &\frac{\omega_{n+1}}{(r-r_{n+1})r_{n+1}} \Big/ \frac{\omega_n}{(r-r_n)r_n} = \frac{\omega_{n+1}}{\omega_n}\left(\frac{r_n}{r_{n+1}}\right)^2 \frac{1-\frac{r}{r_n}}{1-\frac{r}{r_{n+1}}} \to -\frac{1}{\delta}\frac{\delta-1}{a\delta-1}, \end{aligned} \tag{2.23}$$

for every finite $t \neq t_n,\ t_{n+1}$ and $r \neq r_n,\ r_{n+1}$, respectively. Hence, the sum in the righthand side of (2.22)i converges absolutely for every finite $t \neq t_n$, $n = 0,1,2,\ldots$, because of (2.21); similarly does the sum in (2.22)ii for every finite $r \neq r_n$, $n = 1,2,\ldots$. Consequently the functions. $\tilde{\Phi}(t)$ and $\tilde{\Omega}(r)$ are well-defined meromorphic functions.

Obviously $\Phi(t)$, cf.(2.8), and $\tilde{\Phi}(t)$ have the same pole set and their poles have the same multiplicity, similarly for $\Omega(r)$ and $\tilde{\Omega}(r)$. Consequently we may write

$$\begin{aligned} &\Phi(t) = \hat{\Phi}(t) + \tilde{\Phi}(t), \\ &\Omega(r) = \hat{\Omega}(r) + \tilde{\Omega}(r), \end{aligned} \tag{2.24}$$

with $\hat{\Phi}(t)$ and $\hat{\Omega}(r)$ entire functions. In appendix A it is shown that $\hat{\Phi}(t)$ is a constant and that $\hat{\Omega}(r)$ is a first degree polynomial in r. Hence from (2.22) and (2.24) we have

$$\begin{aligned} &\Phi(t) = \Phi(0) + \sum_{n=0}^{\infty}\frac{\phi_n}{t_n} + \sum_{n=0}^{\infty}\frac{\phi_n}{t-t_n}, \\ &\Omega(r) = \Omega(0) + [\Omega^{(1)}(0) + \sum_{n=1}^{\infty}\frac{\omega_n}{r_n^2}]r + \sum_{n=1}^{\infty}\frac{\omega_n}{r-r_n}\frac{r}{r_n}. \end{aligned} \tag{2.25}$$

From (4.8) and (4.9) of [5] we have

$$\Phi(1) = \frac{1}{1+a}, \qquad \Omega(1) = \Phi(\frac{1}{a}) = \frac{1}{2}(2-a), \tag{2.26}$$

and from (2.1),

$$\Omega(0) = \Phi(0) = -\sum_{n=0}^{\infty}\frac{\phi_n}{t_n}, \tag{2.27}$$

the second equality in (2.27) is proved in Section 3, see (3.16). In Section 3 it is also shown, see (3.15), that

$$\Omega^{(1)}(0) = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{(1+at_n)t_n}. \tag{2.28}$$

In appendix B it is derived that

$$\begin{aligned}\phi_0 &= -\frac{4}{a^2}\frac{(2-a)(4-a)}{4+a},\\ \omega_1 &= -\frac{2}{a^3}\frac{(2-a)(4-a)}{4+a}(a^2+4a+16).\end{aligned} \tag{2.29}$$

3. The expression for $\Pr\{\mathbf{x}_1 = k,\ \mathbf{x}_2 = k+m\}$

In this section we derive the explicit expressions for the stationary state probabilities, cf.(1.1),

$$p_{km} := \Pr\{\mathbf{x}_1 = k,\ \mathbf{x}_2 = k+m\}, \quad k = 0,1,2,\ldots;\ m = 0,1,2,\ldots. \tag{3.1}$$

From (2.3) we have for $|t| = 1$, $m = 1,2,\ldots$,

$$\sum_{k=0}^{\infty} p_{km}t^k = \frac{1}{2\pi\mathrm{i}}\int_{|r|=1}\frac{\mathrm{d}r}{r^m}\frac{(r-t)\Omega(r) - [r+\frac{1}{2}art - \frac{1}{2}(2+a)t]\Phi(t)}{k_1(r,t)} + \frac{1}{2\pi\mathrm{i}}\int_{|r|=1}\frac{\mathrm{d}r}{r^{m+1}}\Phi(t). \tag{3.2}$$

Note that

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{|r|=1}\frac{\mathrm{d}r}{r^{m+1}} &= 0 \quad \text{for} \quad m = 1,2,\ldots,\\ &= 1 \quad \text{for} \quad m = 0.\end{aligned} \tag{3.3}$$

To evaluate the integral in (3.2) it is noted that (2.10) implies that the zeros of $k_1(r,t)$ with $|t| = 1$ are no poles of the integrand so that its only poles are those of $\Omega(r)$. Put

$$R_N := \frac{1}{2}(r_{N+1} + r_N), \tag{3.4}$$

and note that, cf.(2.5),

$$k_1(r,t) = a[t - t_1(r)][t - t_2(r)]. \tag{3.5}$$

By applying Cauchy's theorem we obtain from (3.2) and (3.3) for $|t| = 1$, $m = 1,2,\ldots$,

$$\begin{aligned}\sum_{k=0}^{\infty} p_{km}t^k = &-\sum_{n=1}^{N}\frac{1}{r_n^m}\lim_{r\to r_n}(r-r_n)\frac{(r-t)\Omega(r) - [r+\frac{1}{2}art - \frac{1}{2}(2+a)t]\Phi(t)}{a(t-t_1(r))(t-t_2(r))}\\ &+\frac{1}{2\pi\mathrm{i}}\int_{|r|=R_N}\frac{\mathrm{d}r}{r^m}\frac{(r-t)\Omega(r) - [r+\frac{1}{2}art - \frac{1}{2}(2+a)t]\Phi(t)}{a(t-t_1(r))(t-t_2(r))}.\end{aligned} \tag{3.6}$$

In appendix C it is shown that for $N \to \infty$,

$$\Omega(R_N) = R_N\left[\Omega^{(1)}(0) + \sum_{n=1}^{\infty}\frac{\omega_n}{r_n^2}\right] + \mathrm{O}(1), \tag{3.7}$$

a result which can be also obtained from (2.8). Because $t_j(r)/r$, $j = 1, 2$, has a finite limit for $r \to \infty$, it is seen that the integrand of the integral in (3.6) behaves as R_N^{-m} for $N \to \infty$. Consequently the limit of this integral for $N \to \infty$ is zero for every $m = 2, 3, \ldots$.

Hence it follows from (3.6) that: for $|t| = 1$, $m \geq 2$,

$$\sum_{k=0}^{\infty} p_{km} t^k = \sum_{n=1}^{\infty} \frac{\omega_n}{r_n^m} \frac{t - r_n}{a[t - t_1(r_n)][t - t_2(r_n)]} = \frac{1}{a} \sum_{n=1}^{\infty} \frac{-\omega_n}{t_n - t_{n-1}} \left[\frac{t_n - r_n}{t_n - t} - \frac{t_{n-1} - r_n}{t_{n-1} - t} \right]. \tag{3.8}$$

From (3.8) it follows by expanding $[t_n - t]^{-1}$ and $[t_{n-1} - t]^{-1}$ in power series of t, note that $t_n > 1$, $n = 0, 1, 2, \ldots$, and by equating the coefficients of t^h, $h = 0, 1, 2, \ldots$, that: for $m = 2, 3, \ldots$; $k = 0, 1, 2, \ldots$,

$$p_{km} = \frac{1}{a} \sum_{n=1}^{\infty} \frac{-\omega_n}{r_n^m} \left[\frac{t_n - r_n}{t_n - t_{n-1}} t_n^{-(k+1)} + \frac{r_n - t_{n-1}}{t_n - t_{n-1}} t_{n-1}^{-(k+1)} \right]. \tag{3.9}$$

For $m = 1$ we note that the first integral in (3.2) has a simple pole in $r = 0$, and so we obtain for $|t| = 1$,

$$\sum_{k=0}^{\infty} p_{k1} t^k = \frac{-2\Omega(0) + (2 + a)\Phi(t)}{2(at + 1)}. \tag{3.10}$$

Because the lefthand side in (3.10) cannot have a pole in $t = -\frac{1}{a}$ if $2 > a \geq 1$, it follows that

$$\Phi(-\frac{1}{a}) = \frac{2}{2 + a} \Omega(0) \text{ for } 2 > a \geq 1. \tag{3.11}$$

We next show that (3.11) also holds for $0 < a < 1$. Consider, therefore, again (3.6) with $m = 1$, $|t| = 1$. For this case the limit of the integral in (3.6) is readily calculated, and it is found that for $m = 1$, $|t| = 1$,

$$\sum_{k=0}^{\infty} p_{k1} t^k = \frac{1}{a} \sum_{n=1}^{\infty} \frac{\omega_n}{r_n} \frac{1}{t_n - t_{n-1}} \left[\frac{t_{n-1} - r_n}{t_{n-1} - t} - \frac{t_n - r_n}{t_n - t} \right] + [\Omega^{(1)}(0) + \sum_{n=1}^{\infty} \frac{\omega_n}{r_n^2}]. \tag{3.12}$$

It is readily verified that the righthand side is a well-defined meromorphic function and that its only poles are t_n, $n = 0, 1, 2, \ldots$. Consequently $t = -\frac{1}{a}$, is not a pole of the lefthand side, and so (3.11) holds for $0 < a < 2$.

From (2.25) and (3.10) we have for $|t| = 1$,

$$\begin{aligned} \sum_{k=0}^{\infty} p_{k1} t^k &= -\frac{\Omega(0)}{1 + at} + \frac{1}{2}(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{t - t_n} \frac{1}{1 + at} \\ &= \frac{-\Omega(0)}{1 + at} + \frac{1}{2}(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{1 + at_n} \{ \frac{1}{t - t_n} - \frac{a}{1 + at} \} = \\ &- \frac{1}{1 + at} \left[\Omega(0) + \frac{1}{2} a(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{1 + at_n} \right] + \frac{1}{2}(2 + a) \sum_{n=0}^{\infty} \frac{\phi}{1 + at_n} \frac{1}{t - t_n}. \end{aligned} \tag{3.13}$$

The relation (3.13) for $|t| = 1$ can be continued meromorphically, note that the last sum is a well-defined meromorphic function. Above it has been shown that $t = t_n$, $n = 0, 1, 2, \ldots$, are the only poles of the lefthand side of (3.13). So $t = -\frac{1}{a}$ is not a pole and consequently

$$\begin{aligned} &\text{(i)} \quad \Omega(0) = p_{00} = \frac{1}{2}a(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{1+at_n},\\ &\text{(ii)} \quad \sum_{k=0}^{\infty} p_{k1}t^k = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{\phi_n}{1+at_n}\frac{1}{t-t_n}. \end{aligned} \tag{3.14}$$

Because for $t=0$ the function $k_1(r,0)$ has $r=0$ as a zero of multiplicity two it follows from (2.10)i that

$$\left[\Omega^{(1)}(r) + [\frac{1}{2}a^2\frac{\mathrm{d}t_1(r)}{\mathrm{d}r} - \frac{1}{2}a]\Phi(r) - \Phi^{(1)}(t_1(r))\frac{\mathrm{d}t_1(r)}{\mathrm{d}r}\right]_{r=0} = 0,$$

so from (2.12),

$$p_{01} = \Omega^{(1)}(0) = \frac{1}{2}a\Phi(0) = \frac{1}{2}ap_{00} = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{(1+at_n)t_n}, \tag{3.15}$$

here the last relation follows from (3.14)ii. From (3.14)i and (3.15) we obtain

$$p_{00} = \Omega(0) = \Phi(0) = \sum_{n=0}^{\infty}\frac{-\phi_n}{t_0}. \tag{3.16}$$

From the results derived above we obtain

$$\text{(i)} \quad p_{km} = \frac{1}{a}\sum_{n=1}^{\infty}\frac{\omega_n}{r_n^m}\left[\frac{t_n-r_n}{t_n-t_{n-1}}t_n^{-(k+1)} + \frac{r_n-t_{n-1}}{t_n-t_{n-1}}t_{n-1}^{-(k+1)}\right], k\geq 0,\ m\geq 2, \tag{3.17}$$

$$\begin{aligned} &\text{(ii)} \quad p_{k1} = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{1+at_n}t_n^{-(k+1)}, \quad k\geq 0,\\ &\text{(iii)} \quad p_{k0} = \textstyle\sum_{n=0}^{\infty}(-\phi_n)t_n^{-(k+1)}, \quad k\geq 1,\\ &\text{(iv)} \quad p_{0m} = 2\sum_{n=1}^{\infty}(-\omega_n)r_n^{-(m+1)}, \quad m\geq 2,\\ &\text{(v)} \quad p_{00} = \Omega(0) = \Phi(0) = \sum_{n=0}^{\infty}\frac{-\phi_n}{t_n},\\ &\text{(vi)} \quad p_{01} = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{(1+at_n)t_n}. \end{aligned}$$

From (3.17) it is simple to obtain the asymptotic relations

$$p_{k0} = -\phi_0 t_0^{-(k+1)}\{1 + \mathrm{O}(\left(\frac{t_0}{t_1}\right)^{k+1})\} \qquad \text{for} \qquad k\to\infty, \tag{3.18}$$

$$p_{0m} = -\omega_1 r_1^{-(m+1)}\{1 + \mathrm{O}(\left(\frac{r_1}{r_2}\right)^{m+1})\} \qquad \text{for} \qquad m \to \infty,$$

$$p_{k1} = \frac{1}{2}(2+a)\frac{-\phi_0}{1+at_0}t_0^{-(k+1)}\{1 + \mathrm{O}(\left(\frac{t_0}{t_1}\right)^{k+1})\} \quad \text{for} \qquad k \to \infty.$$

4. The compensation approach

The 'compensation approach' applied to the symmetrical shortest queueing problem is exposed in the studies [1] and [2]. By $p_{km}^{(w)}$ and $p_{km}^{(a)}$ we shall indicate the stationary state probabilities when calculated by the compensation approach as applied in [1] and [2], respectively, analogously for $\Omega^{(a)}(r)$, $\Phi^{(a)}(t)$, $\Phi^{(w)}(t)$. Note that the notations in [1] and [2] differ slightly.

The solution claimed by the compensation approach as exposed in [1] and [2] reads: for $k = 0, 1, 2, \ldots$,

$$p_{km}^{(a)} = p_{km}^{(w)} = C^{-1}\sum_{i=0}^{\infty} d_i[c_i\alpha_i^k + c_{i+1}\alpha_{i+1}^k]\beta_i^k, \quad m = 1, 2, \ldots, \tag{4.1}$$

$$p_{k0}^{(a)} = C^{-1}\sum_{i=0}^{\infty} c_i f_i d_i^k, \tag{4.2}$$

for $p_{k0}^{(w)}$ see (4.8)iv, below.

The constants in (4.1) and (4.2) are defined by, cf.(3.20),…,(3.25) of [2]:

$$\alpha_n^{-1} = t_n, \quad \beta_n^{-1} = r_{n+1}, \quad n = 0, 1, 2, \ldots, \text{ cf.(2.8)}, \tag{4.3}$$

and for $i = 0, 1, 2, \ldots$,

$$c_{i+1} = \frac{\alpha_{i+1} - \beta_i}{\beta_i - \alpha_i}c_i, \qquad c_0 = 1, \tag{4.4}$$

$$d_{i+1} = -\frac{(\alpha_{i+1} + \frac{1}{2}a)/\beta_{i+1} - (1 + \frac{1}{2}a)}{(\alpha_{i+1} + \frac{1}{2}a)/\beta_i - (1 + \frac{1}{2}a)}d_i, \qquad d_0 = 1,$$

$$f_{i+1} = \frac{\alpha_{i+1}}{\alpha_{i+1} + \frac{1}{2}a}(d_i + d_{i+1}), \qquad f_0 = \frac{\alpha_0}{\alpha_0 + \frac{1}{2}a},$$

C is determined by the norming condition

$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} p_{km}^{(a)} + \sum_{k=0} p_{k0}^{(a)} = 1. \tag{4.5}$$

Remark 4.1. An explicit expression for $p_{k0}^{(w)}$ is not given in [1], its determination is only indicated; below we give the expression for $p_{k0}^{(w)}$ according to [1], see (4.8)iv. □

From the Kolmogorov equations for the stationary state probabilities the following equivalent set of equations is derived in [1]:

$$
\begin{aligned}
&\text{(i)} \quad (a+2)p_{km} = ap_{k-1,m+1} + p_{k,m+1} + p_{k+1,m-1}, && k \geq 1,\ m \geq 2;\\
&\text{(ii)} \quad (a+2)p_{k1} = ap_{k-1,2} + p_{k2} + \frac{2}{a+2}(ap_{k1} + p_{k+1,1}) + \frac{a}{a+2}(ap_{k-1,1} + p_{k1}), && k \geq 1;\\
&\text{(iii)} \quad (a+1)p_{0m} = p_{0,m+1} + p_{1,m-1}, && m \geq 2;\\
&\text{(iv)} \quad -(a+1)p_{01} + +p_{02} + \frac{2}{a+2}(ap_{01} + p_{11}) + p_{01} = 0,\\
&\text{(v)} \quad \frac{1}{2}(a+2)p_{k0} = ap_{k-1,1} + p_{k1}, && k \geq 1,\\
&\text{(vi)} \quad ap_{00} = p_{01}.
\end{aligned}
\tag{4.6}
$$

For our analysis of the compensation approach we need some asymptotic relations for the coefficients c_i, d_i and f_i, see [2]. They are easily derived from (4.4) by using (2.18) and (4.3). It results: for $i \to \infty$,

$$
\begin{aligned}
&\text{(i)} \quad 0 < \frac{c_{i+1}}{c_i} \to \frac{1}{\delta}\frac{\delta-1}{a\delta-1} < 1,\\
&\text{(ii)} \quad -\frac{d_{i+1}}{d_i} \to a\delta^2 > 1,\\
&\text{(iii)} \quad -\frac{d_{i+1}(c_{i+1}+c_{i+2})}{d_i(c_i+c_{i+1})} \to a\delta\frac{\delta-1}{a\delta-1} > 1,\\
&\text{(iv)} \quad -\frac{d_{i+1}(c_{i+1}+c_{i+2})\beta_{i+1}}{d_i(c_i+c_{i+1})\beta_i} \to \frac{1}{\delta}\frac{\delta-1}{a\delta-1} < 1,\\
&\text{(v)} \quad \frac{c_{i+1}f_{i+1}}{c_i f_i} \to \frac{1}{\delta^2}\left[\frac{\delta-1}{a\delta-1}\right]^2 < 1.
\end{aligned}
\tag{4.7}
$$

From the relations above the expressions for the following generating functions are readily derived. For $|r| \leq 1$, $|t| \leq 1$,

$$
\begin{aligned}
&\text{(i)} \quad \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} p_{km}^{(a)} t^k r^m = C^{-1}\sum_{i=0}^{\infty} d_i\left[\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1}t}\right]\frac{\beta_i r}{1-\beta_i r},\\
&\text{(ii)} \quad \Omega^{(a)}(r) := \sum_{m=0}^{\infty} p_{0m}^{(a)} r^m = p_{00}^{(a)} + C^{-1}\sum_{i=0}^{\infty} d_i(c_i+c_{i+1})\frac{\beta_i r}{1-\beta_i r},\\
&\text{(iii)} \quad \Phi^{(a)}(t) := \sum_{k=0}^{\infty} p_{k0}^{(a)} t^k = C^{-1}\sum_{i=0}^{\infty}\frac{c_i f_i}{1-\alpha_i t},\\
&\text{(iv)} \quad \Phi^{(w)}(t) := \sum_{k=0}^{\infty} p_{k0}^{(w)} t^k = \frac{2}{a+2}\left[p_{00}^{(w)} + (1+at)\sum_{k=0}^{\infty} p_{k1}^{(a)} t^k\right],\\
&\qquad = \frac{2}{a+2}\left[p_{00}^{(w)} + (1+at)C^{-1}\sum_{i=0}^{\infty} d_i\beta_i[\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1}t}]\right],
\end{aligned}
\tag{4.8}
$$

(v) $p_{00}^{(w)} = \frac{2}{a} C^{-1} \sum_{i=0}^{\infty} d_i \beta_i (c_i + c_{i+1}).$

The relation (4.8)i follows directly from (4.1), also (4.8)ii is obtained from (4.1); (4.8)iii is derived from (4.2)i. The relation for $\Phi^{(w)}(t)$ is obtained from (4.6)v and iv, and the second expression in (4.18)iv is obtained by using (4.8)i. By taking $t = 0$ in (4.8)iv and noting that $\Phi^{(w)}(0) = p_{00}^{(w)}$, the relation (4.8)v results. By using the asymptotic expressions (4.7) it is readily verified that the righthand sides in (4.8)i,. . .,iv, are for $|r| > 1$ and/or $|t| > 1$ well-defined meromorphic functions and so the lefthand sides in (4.8)i,. . .,iv, can be continued meromorphically into $|r| > 1$, and/or $|t| > 1$.

We next compare the results according to the compensation approach with those obtained in Sections 2 and 3. First we compare some asymptotic results.

From (4.8) the following asymptotic results are readily derived.

$$\begin{aligned}
&\text{(i)} && p_{k0}^{(a)} = C^{-1} c_0 f_0 \alpha_0^k [1 + \mathrm{O}(\left(\frac{\alpha_1}{\alpha_0}\right)^k)] && \text{for} \quad k \to \infty, \\
&\text{(ii)} && p_{k0}^{(w)} = \frac{2}{a+2} C^{-1} d_0 \beta_0 c_0 \alpha_0^k (1 + \frac{a}{\alpha_0})[1 + \mathrm{O}(\left(\frac{\alpha_1}{\alpha_0}\right)^k)] && \text{for} \quad k \to \infty, \\
&\text{(iii)} && p_{k1}^{(a)} = C^{-1} d_0 \beta_0 \alpha_0^k [1 + \mathrm{O}(\left(\frac{\alpha_1}{\alpha_0}\right)^k)] && \text{for} \quad k \to \infty, \\
&\text{(iv)} && p_{0m}^{(a)} = C^{-1} d_0 (c_0 + c_1) \beta_0^m [1 + \mathrm{O}(\left(\frac{\beta_1}{\beta_0}\right)^m)] && \text{for} \quad m \to \infty.
\end{aligned} \tag{4.9}$$

From (2.7), (4.3) and (4.4) it is seen that

$$\begin{aligned}
&\alpha_0^{-1} = t_0 = \frac{4}{a^2}, && \beta_0^{-1} = r_1 = \frac{2}{a^2}(a+4), \\
&\alpha_1^{-1} = t_1 = \frac{(a+4)^2}{a^3}, && \beta_1^{-1} = r_2 = \frac{a+4}{a^3}(a^2 + 4a + 8), \\
&c_1 + c_0 = \frac{a^2 + 4a + 16}{(2+a)(4+a)}, && f_0 = \frac{a}{a+2}.
\end{aligned} \tag{4.10}$$

It follows from (3.18) by using (2.19) and from (4.9) and (4.10),

$$\begin{aligned}
&\text{(i)} && p_{k0} \sim \tfrac{(2-a)(4-a)}{4+a} \left(\tfrac{a^2}{4}\right)^k, && p_{k0}^{(a)} \sim p_{k0}^{(w)} \sim C^{-1} \tfrac{a}{2+a} \left(\tfrac{a^2}{4}\right)^k, && k \to \infty, \\
&\text{(ii)} && p_{k1} \sim a \tfrac{(4-a^2)(4-a)}{2(4+a)^2} \left(\tfrac{a^2}{4}\right)^k, && p_{k1}^{(a)} \sim C^{-1} \tfrac{a^2}{2(4+a)} a \left(\tfrac{a^2}{4}\right)^k, && k \to \infty, \\
&\text{(iii)} && p_{0m} \sim \tfrac{(2-a)(4-a)}{a(4+a)^2} (a^2 + 4a + 16) \left[\tfrac{a^2}{2(a+4)}\right]^m, && p_{0m}^{(a)} \sim C^{-1} \tfrac{a^2+4a+16}{(2+a)(4+a)} \left[\tfrac{a^2}{2(a+4)}\right]^m, && m \to \infty.
\end{aligned} \tag{4.11}$$

In [1] and [2] the constant C is shown to be given by

$$C = \frac{a(4+a)}{(4-a^2)(4-a)}; \tag{4.12}$$

its derivation is based on $\Omega^{(a)}(1) = 1 - \frac{1}{2}a$, a result which follows from the norming condition, and the relation (2.3), cf.(3.34) of [2] or (32) of [1], and by considering (2.3) for $t = t_0 = 4/a^2$.

When inserting the expression (4.12) for C into (4.11) it is seen that the corresponding relations in (4.11) are identical.

Comparing the generating functions it is seen that $\Omega^{(a)}(r)$, cf.(4.8)ii, and $\Omega(r)$, cf.(2.25), have the same pole set but not the same structure because of the second term in the righthand side of (2.25), such a term does not occur in the righthand side of (4.8)ii, this term cannot be identically zero because of (3.7). Comparison of $\Phi^{(a)}(t)$, (4.8)iii, and $\Phi(t)$, cf.(2.25) and (2.27), show that they have the same pole set and structure. To compare $\Phi^{(w)}(t)$ and $\Phi(t)$ it suffices to compare, cf.(4.8)iv and (3.10), the function, cf.(4.1),

$$\sum_{k=0}^{\infty} p_{k1}^{(a)} t^k = C^{-1} \sum_{i=0}^{\infty} d_i \left[\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1} t} \right] \beta_i,$$

and $\Sigma_{k=0}^{\infty} p_{k1} t^k$ as given by (3.12); they have the same pole set, but not the same structure.

For a further comparison we consider the compensation approach in some more detail.

The compensation approach starts with the choice of α_0, i.e.

$$\alpha_0 = \frac{4}{a^2} > 1, \tag{4.13}$$

this choice is motived in [1] and [2] by experimental and heuristic arguments on the one hand and by a theoretical result obtained in [9] on the otherhand. This α_0 generates the infinite sequence

$$\alpha_0 > \beta_0 > \alpha_1 > \beta_1 > \ldots > \alpha_n > \beta_n > \alpha_{n+1}, \ldots; \tag{4.14}$$

β_n is uniquely defined by α_n, $n = 0, 1, 2, \ldots$, and similarly α_n by β_{n-1}, $n = 1, 2, \ldots$, for both cases as that zero of the biquadratic function $k_1(\frac{1}{\beta}, \frac{1}{\alpha})$ which satisfies the relevant inequality in (4.14).

For every $n = 0, 1, 2, \ldots$, and $k, m \in \{0, 1, 2, \ldots\}$

$$\alpha_n^k \beta_n^m \quad \text{and} \quad \alpha_{n+1}^k \beta_n^m \tag{4.15}$$

are solutions of the equation (4.6)i. Starting with the linear combination

$$\alpha_0^k \beta_0^m + c_1 \alpha_1^k \beta_0^m, \tag{4.16}$$

the coefficient c_1 is determined so that this linear combination satisfies (4.6)i and (4.6)iii. From (4.16) and $\alpha_1^k \beta_1^m$ the linear combination

$$\alpha_0^k \beta_0^m + c_1 \alpha_1^k \beta_0^m + d_1 \alpha_1^k \beta_1^m \tag{4.17}$$

is formed and d_1 is chosen in such a way that (4.17) satisfies (4.6)i and (4.6)ii. With d_1 so determined (4.17) violates (4.6)iii, therefore a term $c_2 \alpha_2^k \beta_1^m$ is added to (4.17) with c_2 chosen so that (4.6)i and (4.6)ii are satisfied. Again a term $d_2 \alpha_2^k \beta_2^m$ is added to satisfy (4.6)i and (4.6)iii. This process is iterated, it leads to the recursive relations for c_i and d_i, cf.(4.4), and in the limit to the expression (4.1) for $p_{km}^{(a)}$, which is an absolutely convergent series representation. Once (4.1) is known the $p_{k0}^{(a)}$ are calculated from (4.6)v,vi. The equation (4.6)iv is discarded because it is linearly dependent of the other equations in (4.6).

Obviously, the relation (4.1) so obtained depends only on the coefficient a in the biquadratic equation for α and β and the starting value α_0. For fixed a we can repeat the procedure above with another starting value $\tilde{\alpha}_0$, say, instead of α. This $\tilde{\alpha}_0$ generates the zeros $\tilde{\beta}_0$, $\tilde{\alpha}_1$, $\tilde{\beta}_1, \ldots$, and so on, a "$\sim$" is used to indicate that they stem from the initial $\tilde{\alpha}_0$; it is supposed that $\tilde{\alpha}_0 > 1$ and $\tilde{\alpha}_0 \neq \alpha_n$, $\tilde{\alpha}_0 \neq \beta_n$ for any n, say $1 < \tilde{\alpha}_0 < \alpha_0$. It is then readily seen that $\tilde{\alpha}_n \neq \alpha_m$, $\tilde{\beta}_n \neq \beta_m$ for all n and m. The recursive formulas for the coefficients $\tilde{c}_n$, $\tilde{d}_n$ keep the same structure with $\tilde{c}_0 = \tilde{d}_0 = 1$, but generally

$\tilde{c}_i \neq c_i$, $\tilde{d}_i \neq d_i$, $i = 1, 2, \ldots$.

The absolute convergence of the series in (4.1) follows from the structure of the recursive relations (4.4), from their asymptotics (4.7) and from the fact that

$$0 < \frac{1}{\delta}\,\frac{\delta - 1}{a\delta - 1} < 1. \tag{4.18}$$

Because δ is independent of α_0 and $\tilde{\alpha}_0$, it is seen that the sequences c_i and $\tilde{c}_i$, have the same asymptotic behaviour, similarly for the sequences d_i and $\tilde{d}_i$. Consequently, it follows that the series representation for $\tilde{p}_{km}^{(a)}$, i.e. (4.1) with c_i and d_i replaced by $\tilde{c}_i$ and $\tilde{d}_i$ is also absolutely convergent. In [1] and [2] it has been shown that

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p_{km}^{(a)}$$

converges absolutely, a result which follows readily from (4.8) with $r = t = 1$. Analogously it follows that

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \tilde{p}_{km}^{(a)}$$

converges absolutely. Consequently, if the compensation approach produces an absolute convergent solution of all the Kolmogorov equations then the $\tilde{p}_{km}^{(a)}$ and the $p_{km}^{(a)}$ are two of such solutions. The inherent Markov chain is irreducible and positive recurrent, note $0 < a < 2$, so it follows that the solutions $p_{km}^{(a)}$ and $\tilde{p}_{km}^{(a)}$ can only differ by a factor. However, they differ essentially, as it may be seen from their bivariate generating functions. The pole sets of these functions are $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$, and $\tilde{\alpha}_0, \tilde{\beta}_0, \tilde{\alpha}_1, \tilde{\beta}_1, \ldots$, respectively, and these are disjoint sets. Hence, we have reached a contradiction and have to conclude that neither $p_{km}^{(a)}$ nor $\tilde{p}_{km}^{(a)}$ is a solution of all the Kolmogorov equations, i.e. the $p_{km}^{(a)}$ as calculated by the compensation approach do not represent the stationary state probabilities.

A direct proof of the conclusion just reached follows from the structural differences between the generating functions in (2.25) on the one hand and those in (4.8) on the other hand. The following considerations provide further information.

The relation (2.3) between the generating functions is fully equivalent with the set of all the Kolmogorov equations, and the condition that the set of stationary state probabilities is an absolutely convergent solution of all the Kolmogorov equations is equivalent with the condition that any zero-tuple $(r, t(r))$ with $|r| \leq 1, |t(r)| \leq 1$, is a zero-tuple of the righthand side of (2.3). Rewrite (2.3) as: for $|r| \leq 1, |t| \leq 1$,

$$k_1(r,t)\mathrm{E}\{t^{\mathbf{y}} r^{\mathbf{z}}(\mathbf{z} \geq 1)\} = 2r[(r-t)\Omega(r) - \{r + \frac{1}{2}art - \frac{1}{2}(2+a)t\}\Phi(t)]. \tag{4.19}$$

In (4.19) we substitute the expression (2.4) for $k_1(r,t)$ and the generating functions $\Omega^{(a)}(r)$ and $\Phi^{(w)}(t)$ as given in (4.8). This leads to: for $|r| \leq 1, |t| \leq 1$,

$$\begin{aligned} &[at^2 + [1-(2+a)r]t + r^2]\sum_{i=0}^{\infty} d_i[\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1}t}]\frac{\beta_i}{1-\beta_i r} = \\ &(r-t)[p_{00}^{(a)}C + \sum_{i=0}^{\infty} d_i(c_i + c_{i+1})\frac{\beta_i r}{1-\beta_i r}] \\ &-\{r + \frac{1}{2}art - \frac{1}{2}(2+a)t\}\frac{2}{2+a}[p_{00}^{(a)}C + (1+at)\sum_{i=0}^{\infty} d_i\beta_i(\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1}t})]. \end{aligned} \tag{4.20}$$

The three series in (4.20) are all well-defined meromorphic functions and since (4.20) should be an identity for all $|r| \le 1, |t| \le 1$, if the compensation approach produces the correct solution, it follows that this identity can be continued analytically for all $|r| \ge 1$, all $|t| \ge 1$.

It is readily seen that (4.20) is indeed an identity for all t if $r = 0$, and also for all r if $t = 0$.

Next multiply (4.20) by $1 - \beta_j r$ and let $r \to \beta_j = r_{j+1}^{-1}$ then we obtain

$$\{at^2 + [1 - (2+a)r_{j+1}]t + r_{j+1}^2\} d_j [\frac{c_j}{1 - \alpha_j t} + \frac{c_{j+1}}{1 - \alpha_{j+1} t}] r_{j+1}^{-1} = d_j (c_j + c_{j+1}), \tag{4.21}$$

and this relation should hold for all t. Because, cf.(2.7),

$$\alpha_j = t_j^{-1}, \quad \alpha_{j+1} = t_{j+1}^{-1}$$

and

$$at^2 + [1 - (2+a)r_{j+1}]t + r_{j+1}^2 = a(t - t_1(r_{j+1}))(t - t_2(r_{j+1})) = a(t - t_j)(t - t_{j+1}),$$

we obtain from (4.21),

$$-[ac_j t_j (t - t_{j+1}) + ac_{j+1} t_{j+1} (t - t_j)] r_{j+1}^{-1} d_j = (c_j + c_{j+1}) d_j, \tag{4.22}$$

or

$$-[ac_j t_j + c_{j+1} t_{j+1}]t + a(c_j + c_{j+1}) t_j t_{j+1} = (c_j + c_{j+1}) r_{j+1}.$$

Since $d_j \ne 0$ and $at_j t_{j+1} = r_{j+1}^2$ we may rewrite (4.22) as

$$-a[c_j t_j + c_{j+1} t_{j+1}]t = r_{j+1}(c_j + c_{j+1})(1 - r_{j+1}), \tag{4.23}$$

and this should hold for all t.

It is konwn that $r_{j+1} > 1$, and from (4.4) it is readily seen that

$$c_j + c_{j+1} \ne 0, \qquad c_j t_j + c_{j+1} t_{j+1} \ne 0.$$

and so (4.23) cannot hold for all t.

Consequently (4.20) does not hold for all r and t, i.e. we reach again the conclusion that the compensation approach does not produce the stationary state probabilities. It is readily seen that the same conclusion is reached if we substitute $\Phi^{(a)}(t)$, instead of $\Phi^{(w)}(t)$ in (4.19).

Appendix A

In this appendix it will be shown that, cf.(2.24), $\hat{\Phi}(t)$ is a constant and $\hat{\Omega}(r)$ a first degree polynomial in r.

Consider the sequence

$$\rho_0 < \tau_0 < \rho_1 < \tau_1 \ldots < \tau_{n-1} < \rho_n < \tau_n < \ldots, \tag{a.1}$$

with: for $n = 1, 2, \ldots$,

$$1 \le \rho_0 < \frac{2}{a}, \tag{a.2}$$

$$\tau_{n-1} := t_1(\rho_n), \qquad \tau_n := t_2(\rho_n).$$

Because $\rho_0 \neq 2/a$ it is readily seen that the τ_n are neither poles nor zeros of $\Phi(t)$, similarly the ρ_n are neither zeros nor poles of $\Omega(r)$, so

$$\rho_m \neq r_n, \qquad \tau_m \neq \tau_n \text{ for all } m, n \in \{0, 1, 2, \ldots\}. \tag{a.3}$$

Next note that

$$k_1(r,t) = 0 \text{ and } k_2(r,t) = 0 \Longleftrightarrow r = \frac{2}{a},\ t = \frac{4}{a^2} \text{ or } r = -1 - \frac{2}{a},\ t = -\frac{1}{a}.$$

Hence from (2.10), (2.11) and (a.2) we have for $n = 1, 2, \ldots$,

$$\frac{\Phi(\tau_n)}{\Phi(\tau_{n-1})} = \frac{k_2(\rho_n, \tau_{n-1})}{k_2(\rho_n, \tau_n)} = \frac{-1 + \frac{1}{2}a^2\tau_{n-1} - \frac{1}{2}a\rho_n}{-1 + \frac{1}{2}a^2\tau_n - \frac{1}{2}a\rho_n}. \tag{a.4}$$

From (a.4) it is seen that $\Phi(\tau_n)$ is bounded for every finite n. Obviously $\rho_n \to \infty$, $\tau_n \to \infty$, for $n \to \infty$ and it is readily verified that: for $n \to \infty$,

$$\frac{\tau_{n-1}}{\rho_n} \to \frac{1}{a\delta} < 1, \qquad \frac{\tau_n}{\rho_n} \to \delta > 1, \qquad \frac{\rho_{n+1}}{\rho_n} = a\delta^2 > 1. \tag{a.5}$$

From (a.4) we have for $m = 1, 2, \ldots$;

$$\frac{\Phi(\tau_{N+m})}{\Phi(\tau_N)} = \prod_{i=1}^{m} \frac{-1 + \frac{1}{2}a^2\tau_{N-1+i} - \frac{1}{2}a\rho_{N+i}}{-1 + \frac{1}{2}a^2\tau_{N+i} - \frac{1}{2}a\rho_{N+i}}. \tag{a.6}$$

For n large we have from (a.5), by using (2.20),

$$\frac{-1 + \frac{1}{2}a^2\tau_{n-1} + \frac{1}{2}a\rho_n}{-1 + \frac{1}{2}a^2\tau_n + \frac{1}{2}a\rho_n} = -\frac{\delta - 1}{\delta(a\delta - 1)}\left[1 - \frac{1}{\rho_n}\,\frac{2}{a}\,\frac{-2 + (1+a)\delta}{a\delta - 1}\right] + \mathrm{o}(\frac{1}{\rho_n}). \tag{a.7}$$

From (a.5) it is seen that

$$0 < \sum_{n=1}^{\infty} \frac{1}{\rho_n} < \infty, \tag{a.8}$$

so by using (2.21) it is seen from (a.6) that for sufficiently large N and $m = 1, 2, \ldots$,

$$|\frac{\Phi(\tau_{N+m})}{\Phi(\tau_N)}| \sim \left[\frac{\delta-1}{\delta(a\delta-1)}\right]^m \left[1-\frac{2}{a}\frac{(1+a)\delta-2}{a\delta-1}\sum_{k=1}^{m}\frac{1}{\rho_k}\right], \tag{a.9}$$

note that the righthand side of (a.9) tends to zero for $m \to \infty$.
From (2.10) we have for $m = 1, 2, \ldots$,

$$\frac{1}{\rho_{N+m}}\Omega(\rho_{N+m}) = -\frac{k_2(\rho_{N+m}, \tau_{N+m})}{\rho_{N+m}}\frac{\Phi(\tau_{N+m})}{\Phi(\tau_N)}\Phi(\tau_N). \tag{a.10}$$

For sufficiently large N it is readily seen that $k_2(\rho_{N+m}, \tau_{N+m})/\rho_{N+m}$ is bounded in $m = 1, 2, \ldots$, and so we obtain from (a.9) and (a.10), since $\Phi(\tau_N)$ is finite for finite N that

$$\begin{aligned} &|\Phi(\tau_{N+m})| \le \epsilon^{(\phi)}_{N+m} \quad \text{with} \quad \epsilon^{(\phi)}_{N+m} \to 0 \quad \text{for} \quad m \to \infty, \\ &|\Omega(\rho_{N+m})|/\rho_{N+m} \le \epsilon^{(\omega)}_{N+m} \quad \text{with} \quad \epsilon^{(\omega)}_{N+m} \to 0 \quad \text{for} \quad m \to \infty, \end{aligned} \tag{a.11}$$

for every finite N and every ρ_0 satisfying (a.2).
From (2.10)i we have for every r and $j = 1, 2$,

$$\frac{1}{r}\Omega(r) + \frac{1}{r}k_2(r, t_j)\Phi(t_j(r)) = 0. \tag{a.12}$$

For $r = r_n + \epsilon$ with $|\epsilon| << 1$ it is seen from (2.16), since $\hat{\Phi}(t)$ and $\hat{\Omega}(r)$ are entire functions and $t_j(r_n)$ and t_n are simple poles of $\tilde{\Phi}(t)$ and $\tilde{\Omega}(r)$, respectively, that

$$\frac{1}{r}\hat{\Omega}(r) + \frac{1}{r}\tilde{\Omega}(t_j(r)) + \frac{1}{r}\mathrm{O}(r - r_n) = 0 \text{ for } r \to r_n. \tag{a.13}$$

Consequently we obtain from (2.24), (a.11), (a.12) and (a.13) that for $j = 1, 2$, and $n \to \infty$,

$$\frac{1}{\rho_n}\hat{\Omega}(\rho_n) + \frac{2}{\rho_n}k_2(\rho_n, t_j(\rho_n))\hat{\Phi}(t_j(\rho_n)) \to 0. \tag{a.14}$$

Because

$$\lim_{n\to\infty} |\frac{1}{\rho_n}k_2(\rho_n, t_j(\rho_n))| \ne 0,$$

it follows that

$$\hat{\Phi}(t_j(\rho_n)) \to 0 \text{ for } n \to \infty,$$

and so $\hat{\Phi}(\rho_n) \to 0$ for $n \to \infty$, further (a.14) implies that $\hat{\Omega}(\rho_n)/\rho_n \to 0$ for $n \to \infty$. By noting that $\hat{\Phi}(t)$ and $\hat{\Omega}(r)$ are both entire functions, Liouville's theorem implies that

$$\begin{aligned} &\hat{\Phi}(t) = \hat{\Phi}(0), \\ &\hat{\Omega}(r) = \hat{\Omega}(0) + r\frac{\mathrm{d}}{\mathrm{dr}}\hat{\Omega}(r)|_{r=0}, \end{aligned} \tag{a.15}$$

because for varying ρ_0 with $1 \le \rho_0 < \frac{2}{a}$ the set of $\rho_n = \rho_n(\rho_0)$ will be dense in (r, ∞) for some $r > 0$.
A slightly different approach for the derivation of the results in this appendix from the product form representation (2.8) is Cauchy's method of decomposing meromorphic functions, see [8] Section VIII.4.

Appendix B

In this appendix we derive explicit expressions for ϕ_0 and ω_1.

Consider first the case

$$\frac{1}{a} \geq 1. \tag{b.1}$$

It follows, cf.(2.5),

$$t_2(1) = \frac{1}{a}, \qquad t_1(1) = 1, \qquad t_1(\frac{2}{a}) = \frac{1}{a}, \qquad t_2(\frac{2}{a}) = \frac{4}{a^2} = t_0. \tag{b.2}$$

Hence from, cf.(2.10),

$$\Omega(\frac{2}{a}) + \left[-1 + \frac{1}{2}a^2 t_1(\frac{2}{a}) - \frac{1}{2}a.\frac{2}{a}\right] \Phi(t_1(\frac{2}{a})) = 0,$$

and (2.26) we obtain

$$\Omega(\frac{2}{a}) = \frac{1}{4}(2-a)(4-a). \tag{b.3}$$

So from, cf.(2.10),

$$\Omega(\frac{2}{a}) + \lim_{r \to \frac{2}{a}} \frac{-1 + \frac{1}{2}a^2 t_2(\frac{2}{a}) - \frac{1}{2}ar}{t_2(r) - 4/a^2}(t_2(r) - \frac{4}{a^2})\Phi(t_2(r)) = 0,$$

the relation (2.29) for ϕ_0 follows.

Next consider the case.

$$\frac{1}{a} < 1.$$

Then

$$t_1(\frac{1}{a}) = \frac{1}{a}, \qquad t_2(1) = 1, \qquad k_1(\frac{2}{a}, \frac{1}{a}) = 0, \qquad t_0 = t_2(\frac{2}{a}) = \frac{4}{a^2},$$

and (b.3) follows again by using (2.10). As before we obtain again the relation (2.29) for ϕ_0.

To determine ω_1 note that $k_1(\frac{2}{a}, \frac{4}{a^2}) = 0$, hence

$$r_1 = \frac{at^2 + t}{r_0}\Big|_{t_0 = \frac{4}{a^2}} = \frac{2}{a^2}(a+4). \tag{b.4}$$

From (2.9) and (2.10) we have

$$\omega_1 + \lim_{r \to r_1} \left[-1 + \frac{1}{2}a^2 t_0 - \frac{1}{2}ar_1\right] \frac{r - r_1}{t_1(r) - 4/a^2}(t_1(r) - 4/a^2)\Phi(t_1(r)) = 0.$$

So

$$\omega_1 + \phi_0 \left[-1 + 2 - \frac{a+4}{a}\right] \left[\frac{\mathrm{d}t_1(r)}{\mathrm{d}r}\right]^{-1}_{r=r_1} = 0,$$

and by using (2.12) the expression (2.29) for ω_1 results.

Appendix C

In this appendix we show that: for $N \to \infty$,

$$|\Omega(R_N)| = R_N[\Omega^{(1)}(0) + \sum_{n=1}^{\infty} \frac{\omega_n}{r_n^2}] + \mathrm{O}(1).$$

From (2.25) we have

$$\begin{aligned} \{\Omega(R_N) - \Omega(0) - R_N[\Omega^{(1)}(0) + \sum_{n=1}^{\infty} \frac{\omega_n}{r_n^2}]\}R_N^{-1} &= \sum_{n=1}^{\infty} \frac{\omega_n}{R_N - r_n} \frac{1}{r_n} \\ &= \sum_{n=1}^{N} \frac{\omega_n}{R_N - r_n} \frac{1}{r_n} + \sum_{m=1}^{\infty} \frac{\omega_{N+m}}{R_N - r_{N+m}} \frac{1}{r_{N+m}}. \end{aligned} \tag{c.2}$$

For N sufficiently large we have, cf.(2.18), (2.19),

$$\begin{aligned} -\frac{\omega_{N+m}}{R_N - r_{N+m}} \frac{1}{r_{N+m}} &= \frac{\omega_{N+m}}{\omega_N r_N^2} \left[\frac{r_N}{r_{N+m}}\right]^2 \frac{\omega_n}{1 - \frac{R_N}{r_{N+m}}} \\ &\sim \frac{\omega_n}{r_N^2}(a\delta^2)^{-2m} \left[-a^2\delta^3 \frac{\delta - 1}{a\delta - 1}\right]^m \frac{1}{1 - \frac{R_N}{r_{N+1}}(a\delta^2)^{-m+1}} . \end{aligned}$$

From (2.18), (2.21) and (c.3) it is readily seen that the righthand side of (c.2) is uniformly bounded in N and so (c.1) follows.

Appendix D

In this appendix we shall compare numerically $p_{k0}^{(a)}$ and p_{k0} for the case $a = 1$ in order to obtain an insight concerning the numerical results obtained by the compensation approach.

From (2.25) and (2.27) and from (4.8)iii we have: for $k \to \infty$,

$$p_{k0} = \phi_0 t_0^{-(k+1)}\{1 + \frac{\phi_1}{\phi_0}\left(\frac{t_0}{t_1}\right)^{k+1} + \mathrm{O}(\left(\frac{t_0}{t_2}\right)^{k+1})\}, \tag{d.1}$$

$$p_{k0}^{(a)} = C^{-1}\frac{a}{2+a}t_0^{-k}\{1 + \frac{a+2}{a}c_1 f_1\left(\frac{t_0}{t_1}\right)^{k} + \mathrm{O}(\left(\frac{t_0}{t_2}\right)^{k})\}.$$

For $a = 1$ we have, cf.(4.10),

$$t_0 = 4, \qquad t_1 = 25, \qquad r_1 = 10, \qquad r_2 = 65,$$

$$\frac{\phi_1}{\phi_0} = \frac{-1 + \frac{1}{2}a^2 t_0 - \frac{1}{2}ar_1}{-1 + \frac{1}{2}a^2 t_1 - \frac{1}{2}ar_1}\,\frac{r_2 - r_1}{r_1 - r_0} = -\frac{55}{13}, \qquad \frac{\phi_1}{\phi_0}\,\frac{t_0}{t_1} = -\frac{44}{65}, \qquad \text{cf.(2.17)},$$

$$c_1 = \frac{r_1 - t_1}{t_0 - r_1}\,\frac{t_0}{t_1} = \frac{2}{5}, \qquad \text{cf.(4.4)},$$

$$\mathrm{d}_1 = \frac{(t_1^{-1} + \frac{1}{2}a)r_2 - (1 + \frac{1}{2}a)}{(t_1^{-1} + \frac{1}{2}a)r_1 - (1 + \frac{1}{2}a)} = -\frac{112}{13},$$

$$f_1 = \frac{1}{1 + \frac{1}{2}at_0} = \frac{1}{3}, \qquad \frac{a+2}{a}c_1 f_1 = -\frac{44}{195}.$$

Hence for $k \to \infty$,

$$\text{i.}\quad p_{k0} \sim \phi_0 t_0^{-(k+1)}\{1 - \frac{44}{65}(0.16)^k\},$$

$$\text{ii}\quad p_{k0}^{(a)} \sim C^{-1}\frac{a}{2+a}t_0^{-k}\{1 - \frac{44}{195}(0.16)^k\}. \tag{d.2}$$

With C given by (4.12) the first factors in (d.2)i and ii are equal, cf.(4.11), and so it is seen that for not too small values of k, the difference between p_{k0} and $p_{k0}^{(a)}$ will be very small. This is obviously due to the choice of C and the rapid increase of t_n for $n = 1, 2, \ldots$.

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