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Abstract

In analytic queueing theory, Rouché's theorem is frequently used, and when it can be applied, leads quickly to tangible results concerning ergodicity and performance analysis. For more complicated models it is sometimes difficult to verify the conditions needed to apply the theorem. The natural question that arises is: Can one dispense with this theorem, in particular when the ergodicity conditions are known? In the present study we consider an $M/G/1$-type queueing problem which can be modelled by N coupled random walks. It is shown that it can be fully analysed without using Rouché's theorem, once it is known that the relevant functional equation has a unique solution with prescribed regularity properties.

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1. Introduction.

In analytic queueing theory Rouché's theorem is one of the workhorses. For the basic queueing models it is usually not so difficult to verify the conditions which permit its application. When it can be applied it leads in a simple way to results concerning the ergodicity conditions and the construction of solutions of functional equations for generating functions or Laplace-Stieltjes transforms. Generally, the theorem is used to prove the existence of a certain number of zeros in a subdomain of the domain of regularity of a given function.

Presently, the queueing models encountered in performance analysis are often quite complicated and accordingly the verification of the conditions needed to apply Rouché's theorem is frequently quite difficult. Hence the question arises whether the use of this theorem cannot be avoided, the more so because the ergodicity conditions for the more complicated queueing models frequently can be derived from the available information concerning the basic models. Recent research has led to new approaches for the derivation of these conditions, see for example, Meyn and Tweedie [6] and Lindvall [4].

In the present study we analyse a relatively simple $M/G/1$-type queueing model and show that for the analysis of the time dependent case as well as that of the stationary case, the use of Rouché's theorem can indeed be avoided. The involved stochastic model consists of N coupled random walks, each with state space $\{0, 1, \ldots\}$; it is described in Section 2. In Section 3 the functional equations for the generating functions are derived; the time dependent case is considered here. By using the fact that these functional equations should have a unique solution that is regular in a known domain, it is shown that the construction of the solution may be performed without using Rouché's theorem. The analysis in Section 4 considers the case that the involved Markov chain is aperiodic and positive recurrent. The functional equations are formulated and by using the fact that the inherent Kolmogorov equations for the stationary state probabilities have a unique, absolutely convergent solution it again turns out that Rouché's theorem can be circumvented.

It seems plausible that the approach exposed in the present study is of wider applicability; it is certainly an approach to be considered whenever the conditions needed for the application of Rouché's theorem are difficult to verify. Examples of such cases are encountered in the studies [7] and [2]. In [7] Mitrani and Mitra also apply a stochastic argument to establish that the solution of their functional equation requires the existence of a certain number of zeros of a function in a prescribed domain. In [2] Gail et al. discuss extensively the zeros of determinants in a prescribed domain. The matrices considered in [2] are of a more general type than the one considered in the present study. The analysis in [2] is purely algebraic and the matrix relating to the time dependent case is analysed first. By a limiting procedure the results for the matrix encountered in the analysis of the stationary case are then derived. Similarly, in the present study, the results of Section 4 may be derived from those of Section 3. However, we derive the results in Section 4 independently, as the methods provided in that section are of independent interest. In [2] the set of linear equations needed to determine the stationary distribution is obtained without using Rouché's theorem. In

Section 4 of the present study this set of equations is directly derived from the Kolmogorov equations for the stationary state probabilities by using mainly probabilistic arguments.

Neuts [8] also presents an analysis of $M/G/1$-type queueing models without using Rouché's theorem. However, the problem setting in [8] differs from that in [2] and that of the present study. This will be addressed in Remark 4.2 below.

2. The model.

Let

$$p_{ij}, \quad i,j \in \{1,\dots,N\}, \tag{2.1}$$

be the one-step transition probabilities of a discrete time-parameter, aperiodic Markov chain

$$\mathbf{y}_n, \quad n = 0,1,2,\dots,$$

of which the state space is irreducible. Let

$$P = [p_{ij}], \tag{2.2}$$

be the one-step transition matrix and put

$$\boldsymbol{\eta}_n := \mathbf{y}_{n+1} - \mathbf{y}_n, \quad n = 0,1,2,\dots.$$

Further

$$\{\boldsymbol{\xi}_n^{(1)},\dots,\boldsymbol{\xi}_n^{(N)}\}, \quad n = 0,1,\dots,$$

is a sequence of i.i.d. stochastic vectors with state space $\{0,1,2,\dots\}^N$, and for each n the N components are independent, nonnegative stochastic variables such that for $j = 1,\dots,N$,

$$\mu_j := \mathsf{E}\{\boldsymbol{\xi}_n^{(j)}\} < \infty, \quad \phi_j(p) := \mathsf{E}\{p^{\boldsymbol{\xi}_n^{(j)}}\}, \quad |p| \le 1. \tag{2.3}$$

The sequence

$$\mathbf{x}_n, \quad n = 0, 1, \ldots,$$

is defined by

$$\mathbf{x}_0 = x_0, \quad \mathbf{y}_0 = y_0, \quad x_0 \in \{0, 1, 2, \ldots\}, \quad y_0 \in \{1, \ldots, N\}, \tag{2.4}$$

and for $\mathbf{x}_n \geq 0$,

$$\mathbf{x}_{n+1} = [\mathbf{x}_n - 1]^+ + \boldsymbol{\xi}_n^{(\mathbf{y}_n + \boldsymbol{\eta}_n)}. \tag{2.5}$$

Obviously

$$(\mathbf{x}_n, \mathbf{y}_n), \quad n = 0, 1, 2, \ldots,$$

is a discrete time-parameter Markov chain with state space $\{0, 1, 2, \ldots\} \times \{1, 2, ..., N\}$, cf. Remark 2.2 below.

For $|p| \leq 1$, $n = 0, 1, \ldots$; $j = 1, \ldots, N$, we define

$$\Phi_n(p, j) := \mathsf{E}\{p^{\mathbf{x}_n}(\mathbf{y}_n = j) | \mathbf{x}_0 = x_0, \mathbf{y}_0 = y_0\}.$$

From (2.4), (2.5), it is readily obtained that

$$\Phi_{n+1}(p, j) = \sum_{i=1}^{N} \Phi_n(p, i) p_{ij} \frac{\phi_j(p)}{p} + \frac{p-1}{p} \sum_{i=1}^{N} \Phi_n(0, i) p_{ij} \phi_j(p), \tag{2.6}$$

for $|p| \leq 1$, $j = 1, \ldots, N$, with

$$\Phi_0(p, j) = p^{x_0} \delta_{j y_0}, \tag{2.7}$$

and δ_{jk} Kronecker's symbol.

Remark 2.1 From (2.6) and (2.7) it is seen that the functions $\Phi_n(p, j)$ are uniquely, via recursion, defined once x_0 and y_0 are known.

Remark 2.2 From the structure of the $(\mathbf{x}_n, \mathbf{y}_n)$-process it is seen that the $\mathbf{x}_n$-process may be considered as an embedded Markov chain of queue lengths at departure epochs of an $M/G/1$ queueing model in which the arrival process is controlled by the Markov chain $\mathbf{y}_n$.

3. The time dependent case.

For $|r| < 1$, $|p| \leq 1$, $j = 1, \ldots, N$, let

$$\Phi(r,p,j) := \sum_{n=0}^{\infty} r^n \Phi_n(p,j).$$

From (2.6) and (2.7) it then follows that for $|r| < 1$, $|p| \leq 1$, $j = 1, \ldots, N$,

$$\Phi(r,p,j) - \frac{r}{p} \sum_{i=1}^{N} \Phi(r,p,i) p_{ij} \phi_j(p) = p^{x_0} \delta_{jy_0} + r \frac{p-1}{p} \sum_{i=1}^{N} \Phi(r,0,i) p_{ij} \phi_j(p). \tag{3.1}$$

Without restricting the generality, we may take as initial condition (cf. (2.4) and Remark 2.1),

$$x_0 = 0, \quad y_0 = 1.$$

Let

$$F_{0i;0j}(r), \quad |r| \leq 1, \quad i,j \in \{1, \ldots, N\},$$

be the generating function of the first entrance time into state $(0,j)$ from state $(0,i)$. From the theory of discrete time-parameter Markov chains it is known that

$$\Phi(r,0,j) = \delta_{1j} + \frac{F_{01;0j}(r)}{1 - F_{0j;0j}(r)}, \quad |r| < 1, \quad j = 1, \ldots, N. \tag{3.2}$$

Obviously, for $j = 1, \ldots, N$,

$$F_{01;0j}(r) \textit{ is regular in } |r| < 1 \textit{ and } |F_{01;0j}(r)| \leq 1 \textit{ for } |r| \leq 1, \tag{3.3}$$

$$\Phi(r,p,j) \textit{ is regular in } |r| < 1 \textit{ for every fixed } p, \ |p| \leq 1.$$

For the remainder of this section, we will require the following row vectors and matrices. For $|r| < 1$, $|p| \leq 1$,

$$\left.\begin{array}{rcl}
\widetilde{\Phi}(r,p) & := & [\Phi(r,p,1), \ldots, \Phi(r,p,N)], \\
\delta_1 & := & [\delta_{11}, \ldots, \delta_{1N}], \quad \delta_{11} = 1, \quad \delta_{1j} = 0, \quad j = 2, \ldots, N, \\
\widetilde{F}(r) & := & \left[\delta_{11} + \frac{F_{01;01}(r)}{1-F_{01;01}(r)}, \cdots, \delta_{1N} + \frac{F_{01;0N}(r)}{1-F_{0N;0N}(r)}\right] = \widetilde{\Phi}(r,0), \\
\widetilde{P}(p) & := & [p_{ij}\phi_j(p)], \\
I & := & [\delta_{ij}].
\end{array}\right\} \tag{3.4}$$

It follows from (3.1) and (3.4) that for $|r| < 1$, $|p| \leq 1$,

$$\widetilde{\Phi}(r,p)[pI - r\widetilde{P}(p)] = p\delta_1 + r(p-1)\widetilde{\Phi}(r,0)\widetilde{P}(p), \tag{3.5}$$

or, equivalently, using (3.2)

$$[\widetilde{\Phi}(r,p) + (p-1)\widetilde{F}(r)][pI - r\widetilde{P}(p)] = p\delta_1 + p(p-1)\widetilde{F}(r). \tag{3.6}$$

Define

$$D(r,p) := \det[pI - r\widetilde{P}(p)], \quad |p| \leq 1.$$

It follows that $D(r,p)$ *is a polynomial in* r *of degree* N *and its coefficients are regular in* $|p| < 1$ *and continuous in* $|p| \leq 1$.

Lemma 3.1 *For* $|r| < 1$*:*

(i) $D(r,p) \neq 0$ *for* $|p| = 1$*,*

(ii) $D(r,p)$ *has* N *zeros in* $|p| < 1$ *for fixed* r*.*

Proof Suppose a p_0 exists such that for $|r| < 1$,

$$D(r,p_0) = 0, \quad |p_0| = 1,$$

i.e.,

$$\det[p_0 I - r\widetilde{P}(p_0)] = 0. \tag{3.7}$$

Obviously, this is impossible for $r = 0$. Hence consider $0 < |r| < 1$. Then (3.7) is equivalent with

$$\det[\lambda I - \frac{1}{p_0}\widetilde{P}(p_0)] = 0, \quad \lambda = \frac{1}{r}. \tag{3.8}$$

Hence λ is a characteristic value of the matrix $p_0^{-1}\widetilde{P}(p_0)$. There are N of these characteristic values $\lambda_k, k = 1, \ldots, N$, say, when counted according to their multiplicity. For these λ_k, using [5], p. 144,

$$\begin{aligned} |\lambda_k| &\leq \max_{1\leq i\leq N} \sum_{j=1}^{N} p_{ij} \left|\frac{\phi_j(p_0)}{p_0}\right| \\ &= \max_{1\leq i\leq N} \sum_{j=1}^{N} p_{ij}|\phi_j(p_0)| \\ &\leq \max_{1\leq i\leq N} \sum_{j=1}^{N} p_{ij} \\ &= 1, \quad k = 1, \ldots, N. \end{aligned} \tag{3.9}$$

Since in (3.8) $|\lambda| = |r^{-1}| > 1$, the first statement of the lemma follows.

Obviously, the number of zeros of $D(r,p)$ in $|p| < 1$ is an integer valued continuous function of r, and from (i), $D(r,p)$ has for $|r| < 1$ no zeros on $|p| = 1$. Therefore, it follows that for $|r| < 1$ this number is independent of r. It equals N for $r = 0$ and so (ii) follows. □

The relations (3.5) represent a functional equation for $\widetilde{\Phi}(r,p)$, and as such this relation defines a class of functions of which the elements satisfy (3.5). Once $\widetilde{\Phi}(r,0)$ is known then (3.5) determines $\widetilde{\Phi}(r,p), |p| \leq 1$. Whenever an element of this class has the property that all of its components $\Phi(r,p,j), |p| \leq 1$ are regular functions of r with $|r| < 1$, then the coefficients of their series expansions in powers of r satisfy the set of equations (2.6) and (2.7). This set of equations has a unique solution, cf. Remark 2.1. Consequently, (3.5) can have only one solution $\widetilde{\Phi}(r,p)$ with $|p| \leq 1$ for which the $\Phi(r,p,j)$, $j = 1, \ldots, N$, are all regular functions of r in $|r| < 1$.

For $|r| < 1$ denote by

$$\pi_h(r), \quad h = 1, \ldots, N,$$

the zeros of $D(r,p)$ in $|p| \leq 1$. Below, cf. Remark 3.1, it will be shown that these zeros all have multiplicity one in $0 < |r| < 1$.

For $|r|<1$, $p\neq\pi_h(r), h=1,\dots,N$, and $|p|\leq 1$ we have $D(r,p)\neq 0$ and so we have from (3.6) that for $|r|<1, p\neq\pi_h(r), h=1,\dots,N, |p|\leq 1$,

$$\widetilde{\Phi}(r,p)+(p-1)\widetilde{F}(r)=[p\delta_1+p(p-1)\widetilde{F}(r)][pI-r\widetilde{P}(p)]^{-1}.$$

Because $\Phi(r,p,j), j=1,\dots,N$, is for every fixed r in $|r|<1$ regular in $|p|\leq 1$ and continuous in $|p|\leq 1$, it follows that for $|r|<1, h=1,\dots,N$,

$$[p[\delta_1+(p-1)\widetilde{F}(r)][pI-r\widetilde{P}(p)]^{-1}D(r,p)]_{p=\pi_h(r)}=0. \tag{3.10}$$

The relation (3.10) represents N inhomogeneous equations for the N components $\widetilde{F}_j(r)$ of the vector $\widetilde{F}(r), |r|<1$. These equations have a unique solution, as the functions $\Phi(r,p,j), |r|<1, |p|\leq 1, j=1,\dots,N$, are uniquely determined by the regularity conditions in $|r|<1$ for fixed $|p|\leq 1$ and those in $|p|<1$ for fixed $|r|<1$, and because the set of equations (2.6), (2.7) has a unique solution.

Remark 3.1 In the discussion above it has been assumed that the zeros $\pi_h(r), 0<r<1$ all have multiplicity one. We now show that this holds. Suppose $\pi_h(r)$ has multiplicity $n_h>1$ for $r=r_0, 0<|r_0|<1$. Then in addition to (3.10) we also have that the kth derivative with respect to p of the term between brackets in (3.10) should be zero for $p=\pi_h(r_0)$ and $k=1,\dots,n_h-1$. Again we obtain a set of N inhomogeneous equations for the $\widetilde{F}_j(r), |r|<1, j=1,\dots,N$, which should have a unique solution for the same reasons as given above. However, because $\pi_h(r)$ has multiplicity n_h for $r=r_0$, r_0 is a branch point of $\pi_h(r)$. Consequently, the equations which determine $\widetilde{F}_j(r)$ contain coefficients which are not regular functions of r in $0<|r|<1$, and so the $\widetilde{F}_j(r), j=1,\dots,N$, are not regular in $|r|<1$; this contradicts (3.3). Hence the $\pi_h(r), h=1,\dots,N$ all have multiplicity one.

4. The positive recurrent case.

The discrete time-parameter Markov chain $\{(\mathbf{x}_n,\mathbf{y}_n), n=0,1,\dots\}$ is obviously aperiodic, cf. (2.1) and (2.4), and has an irreducible state space. Hence

$$\lim_{n\to\infty} \mathsf{P}\{\mathbf{x}_n = i, \mathbf{y}_n = j\} \tag{4.1}$$

exists for all $j = 1, \ldots, N; i = 0, 1, 2, \ldots$; these limits are positive if and only if the $(\mathbf{x}_n, \mathbf{y}_n)$-process is positive recurrent.

From now on it is assumed that (cf. Remark 4.3),

$$\textit{the } (\mathbf{x}_n, \mathbf{y}_n)\textit{- process is positive recurrent;} \tag{4.2}$$

and it will be shown that without an appeal to Rouché's theorem the limiting values of (4.1) can be calculated.

It follows from (3.2) that for $j = 1, \ldots, N$,

$$0 < \lim_{r\to 1}(1-r)\widetilde{F}_j(r) = \lim_{r\to 1}\Phi(r, 0, j) = \mathsf{P}\{\mathbf{x} = 0, \mathbf{y} = j\},$$

with $(\mathbf{x}, \mathbf{y})$ a stochastic vector with distribution the stationary distribution of the $(\mathbf{x}_n, \mathbf{y}_n)$-process. A well-known Abelian theorem implies that for $j = 1, \ldots, N; |p| \le 1$,

$$\begin{aligned} \Omega(p, j) &:= \lim_{r\to 1}(1-r)\Phi(r, p, j) \\ &= \mathsf{E}\{p^{\mathbf{x}}(\mathbf{y} = j)\} \\ &= \sum_{i=0}^{\infty} p^i \mathsf{P}\{\mathbf{x} = i, \mathbf{y} = j\}, \end{aligned} \tag{4.3}$$

uniformly in $|p| \le 1$. Defining

$$\widetilde{\Omega}(p) := [\Omega(p, 1), \ldots, \Omega(p, N)],$$

it follows from (3.6) and (4.3) that for $|p| \le 1$,

$$[\widetilde{\Omega}(p) + (p-1)\widetilde{\Omega}(0)][pI - \widetilde{P}(p)] - p(1-p)\widetilde{\Omega}(0) = \mathbf{0}, \tag{4.4}$$

where $\mathbf{0}$ denotes the null vector.

From (4.4) it is seen that for every $j = 1, 2, \ldots, N$, $\Omega(p, j)$ is a regular function in $|p| < 1$, which is continuous in $|p| \le 1$. Insertion of the power series expansion in p of $\Omega(p, j)$ into

(4.4) and setting the coefficient of p^i equal to zero leads to the Kolmogorov equations for the stationary state probabilities $\mathsf{P}\{\mathbf{x}=i, \mathbf{y}=j\}$. These probabilities form the unique, absolutely convergent solution of this set of equations, apart from a constant factor.

Of these Kolmogorov equations exactly one equation is linearly dependent on the remainder, and in fact when these Kolmogorov equations for $\mathsf{P}\{\mathbf{x}=i, \mathbf{y}=j\}$ are summed over all $j=1,\ldots,N$, and all $i=0,1,\ldots$, then an identity results. This linear dependence is equivalent with the fact that $p=1$ is a zero of $D(1,p)$. Note that for $p=1$ the N relations (4.4) represent the Kolmogorov equations for the stationary state probabilities $\mathsf{P}\{\mathbf{y}=j\}, j=1,\ldots,N$, of the discrete time-parameter Markov chain with one-step transition matrix $P=\widetilde{P}(p)$, cf. (2.2), (3.4).

Denote by

$$\pi_h, \quad h=1,\ldots,M, \tag{4.5}$$

the zeros of

$$D(1,p)=\det[pI-\widetilde{P}(p)] \quad \text{in } |p|\leq 1. \tag{4.6}$$

Obviously $p=1$ is a simple zero; it is the only zero with $|p|=1$, note that the $\mathbf{y}_n$-process is aperiodic and $\widetilde{P}(1)=P$.

For the present it is assumed that

$$\textit{all zeros } \pi_h, h=1,\ldots,M, \textit{ have multiplicity one.} \tag{4.7}$$

Below, cf. Remark 4.1, we shall justify this assumption.

The enumeration of these zeros is chosen so that

$$\pi_1=1, \quad |\pi_h|<1 \quad \text{for } h=2,\ldots,M.$$

From (4.4) it is seen that, apart from a constant factor, the functions $\Omega(p,j), j=1,\ldots,N, p\neq\pi_h, h=2,\ldots,M$, are determined once the values $\Omega(0,j), j=1,\ldots,N$ are known. The fact that the functions $\Omega(p, j)$ should be regular for $|p| < 1$, and continuous for $|p| \leq 1$, leads to the conditions:

$$[\tilde{\Omega}(0)[pI - \tilde{P}(p)]^{-1} D(1, p)]_{p=\pi_h} = \mathbf{0}, \tag{4.8}$$

for $h = 2, \ldots, M$. For $|p| \leq 1$, we define

$$[q_{ij}(p)] := [pI - \tilde{P}(p)]^{-1} D(1, p),$$

with

$$q_{ij}(\pi_h) := \lim_{p \to \pi_h, |p| \leq 1} q_{ij}(p), \quad h = 2, \ldots, M.$$

As a result of (4.6), the conditions (4.8) are equivalent with

$$\begin{vmatrix} \Omega(0,1) & q_{12}(\pi_h) & \cdots & q_{1N}(\pi_h) \\ \Omega(0,2) & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \Omega(0,N) & q_{N2}(\pi_h) & \cdots & q_{NN}(\pi_h) \end{vmatrix} = 0, \tag{4.9}$$

for $h = 2, \ldots, M$.

The relation (4.4) is equivalent with the Kolmogorov equations and (4.8) with the condition that a solution of these equations is absolutely convergent, so it follows that

$$M \geq N.$$

Otherwise, if $M < N$, the set of Kolmogorov equations would have more than one absolutely convergent solution, which is a contradiction to the assumption (4.2).

Consider the case $M > N$. Then the $M-1$ linear equations (4.9) for $\Omega(0, j), j = 1, \ldots, N$, have no solution when they are linearly independent. Because (4.2) implies that there should be a unique solution it follows that if $M > N$ then $M - N$ of these linear equations, those for $h = N + 1, \ldots, M$, say, should be linearly dependent on the other $M - 1$ equations, i.e. on those for $h = 2, \ldots, N$. The linear dependence of the equations for $h = N + 1, \ldots, M$

on those for $h = 2, \ldots, N$, leads to $M - N$ equations relating the values $q_{ij}(\pi_h)$ for $i, j = 1, \ldots, N; h = 2, \ldots, M$; i.e. $M - N$ equations between the values p_{ij}, $\phi_j(\pi_h)$ and π_h. These relations involve the probabilities $\mathsf{P}\{\boldsymbol{\xi}_n^{(j)} = k\}, k = 0, 1, 2, \ldots$ as, for $j = 1, \ldots, N$, $\phi_j(p)$ are their generating functions. The existence of such relations conflicts with the independence of the stochastic variables $\boldsymbol{\xi}_n^{(j)}, j = 1, \ldots, N$, cf. (2.3). Hence we have

$$M = N. \tag{4.10}$$

Note that the conclusion (4.10) has been reached without using the results of the previous section.

Remark 4.1 The $\mathbf{y}_n$-process of which P is the one step transition matrix is aperiodic and so by using (3.9) it is seen that $D(1, p)$ has no zeros with $|p| = 1, p \neq 1$. For $\pi_h(r)$, cf. (3.7), $|\pi_h(r)| < 1, h = 1, \ldots, N$, and so some of these zeros may have the same limiting value for $r \to 1$. However, of the $\pi_h, h = 1, \ldots, M$, cf. (4.5), one or more may be the limiting value for $r \uparrow 1$ of zeros $\pi_k(r)$, say, of $D(r, p)$ in $|p| < 1$. Whenever one or more of these zeros π_h have multiplicity larger than one then it follows as in Remark 3.1 that $r = 1$ is a branch point of $F_{01;01}(r)$. However, the positive recurrence of the $(\mathbf{x}_n, \mathbf{y}_n)$-process implies that

$$0 < \lim_{r \uparrow 1} \frac{1 - F_{01;01}(r)}{1 - r} < \infty,$$

and this cannot hold if $r = 1$ is a branch point of $F_{01;01}(r)$. Hence (4.7) is justified. (Note that the latter limit is the average return time to the state $(0, 1)$.)

From Remark 4.1 and the discussion above it is seen that the following theorem has been proved without an appeal to Rouché's theorem. (cf. also the results in [2]).

Theorem 4.1 *If the $(\mathbf{x}_n, \mathbf{y}_n)$-process is positive recurrent then $D(1, p)$ has in $|p| \leq 1$ exactly N zeros, $p = 1$ is the only zero in $|p| = 1$ and all of these zeros have multiplicity one.* □

Remark 4.2 In the present study, and also in [2], the main point is to show that $\det[pI - \tilde{P}(p)]$ has exactly N zeros in $|p| \le 1$. Once this is established, the linear equations for the components of the vector $\tilde{\Omega}(0)$ follow, cf. (4.9). In [8] this vector, apart from a scalar, is obtained as the characteristic vector belonging to the characteristic value one of a finite matrix K, which is a functional of a matrix $\tilde{G}$. (Theorem 3 of [8]). This $\tilde{G}$ is the minimal solution of a matrix equation, which is obtained by considering a class of first entrance time distributions. The approach in [8] thus does not need the zeros of $\det[pI - \tilde{P}(p)]$ in $|p| \le 1$ and so Rouché's theorem is avoided. Obviously, the tradeoff is the derivation of K and the matrix equation for $\tilde{G}$. From the numerical point of view, the iterative solution of the matrix equation for $\tilde{G}$ has to be compared with the determination of the N zeros of $\det[pI - \tilde{P}(p)]$ in $|p| \le 1$. The arguments used in [8] are all purely probabilistic, while those of the present study stem from the property that the stationary state probabilities form the unique absolutely convergent solution of the Kolmogorov equations.

Remark 4.3 Consider the $\mathbf{x}_n$-process. Obviously, we have, cf. (2.4) and (2.5),

$$\mathbf{x}_0 = x_0, \quad \mathbf{y}_0 = y_0,$$

$$\mathbf{x}_{n+1} = [\mathbf{x}_n - 1]^+ + \boldsymbol{\zeta}_n, \quad n = 0, 1, 2, \ldots,$$

with $\boldsymbol{\zeta}_n$, $n = 0, 1, 2, \ldots$, a sequence of independent stochastic variables satisfying

$$\mathsf{E}\{p^{\boldsymbol{\zeta}_n}\} = \sum_{j=1}^{N} p_{y_0 j}^{(n)} \phi_j(p), \quad |p| \le 1;$$

where $[p_{ij}^{(n)}]$ is the n-step transition matrix of the $\mathbf{y}_n$-process.

It follows, cf. (2.3),

$$\mathsf{E}\{\boldsymbol{\zeta}_n\} = \sum_{j=1}^{N} p_{y_0 j}^{(n)} \mu_j.$$

As the $\mathbf{y}_n$-process is positive recurrent, the following limits exist

$$\nu_j := \lim_{n\to\infty} p_{y_0 j}^{(n)} = \lim_{n\to\infty} \mathsf{P}\{\mathbf{y}_n = j | \mathbf{y}_0 = y_0\},$$

and so

$$\lim_{n\to\infty} \mathsf{E}\{\boldsymbol{\zeta}_n\} = \sum_{j=1}^{N} \nu_j \mu_j.$$

Obviously, the $\mathbf{x}_n$-process has ultimately a negative drift (given $|\mathbf{x}_n| > 0$) if $\mathsf{E}\{\boldsymbol{\zeta}_n\} < 1$ for $n \to \infty$, and so, the $(\mathbf{x}_n, \mathbf{y}_n)$-process is positive recurrent if and only if

$$\sum_{j=1}^{N} \nu_j \mu_j < 1,$$

which follows from a simple application of Foster's criterion.

5. Concluding remarks.

We have presented an approach where the analysis of a particular M/G/1 queueing system may be performed without the use of Rouché's theorem. It would be of great interest to determine whether the methods presented here are applicable to other random walks on semi-infinite strips. In particular, recent work on threshold-type queueing models [1, 3] provides examples which are similar to the model considered here and for which employing Rouché's theorem is extremely difficult. It appears that the work in this study may be readily adapted to those situations.

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