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# A Natural Differential Calculus on Lie Bialgebras with Dual of Triangular Type

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**Abstract**

We prove that for a specific class of Lie bialgebras, there exists a natural differential calculus. This class consists of the Lie bialgebras for which the dual Lie bialgebra is of triangular type. The differential calculus is explicitly constructed with the help of the $R$-matrix from the dual. The method is illustrated by several examples.

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## 1 Introduction

In [1] the authors discuss differential calculi of Poincaré-Birkhoff-Witt type on the universal enveloping algebra of a Lie algebra. It is shown in [2] that these differential algebras can be equipped with a differential Hopf algebra structure (for the definition see e.g. [3]) extending the standard Hopf algebra structure of the enveloping algebra. This Hopf algebra structure turns out to play a prominent role in the quantization of these differential calculi.

A Quantized Universal Enveloping algebra (QUE algebra) on a Lie algebra $\mathfrak{g}$ is a Hopf algebra deformation $U_h(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ (see e.g. [4]). Its classical limit determines a co-Poisson bracket on $U(\mathfrak{g})$ whose restriction to $\mathfrak{g}$ is called a cocommutator and defines a Lie bialgebra structure on $\mathfrak{g}$. In [5] the authors investigate deformations of De Rham complexes on $U(\mathfrak{g})$ in order to obtain a De Rham complex on the QUE algebra $U_h(\mathfrak{g})$. They show that a necessary condition for the existence of such deformations can be interpreted as a compatibility between the differential operator and the co-Poisson bracket.

One could say there are two main strategies to utilize this compatibility condition in order to construct De Rham complexes on QUE algebras. The first one is to start with a certain differential calculus on $U(\mathfrak{g})$ and compute a compatible cocommutator. This is the point of view presented in [5]. The second approach is to consider a certain quantization of the Lie algebra $\mathfrak{g}$, take its classical limit and try to construct a compatible diffential operator $d$. This is the approach we will use in this paper. In fact we will prove that, for a specific class of Lie bialgebras, there exists a natural compatible differential calculus. This specific class consists of the Lie bialgebras for which the dual Lie bialgebra is of triangular type.

## 2 The compatibility condition

Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra over the field of complex numbers $\mathbb{C}$ with commutator $[\ ,\ ]$ and cocommutator $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$. A cocommutator is a 1-cocycle with the additional property that its transpose $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ defines a commutator on $\mathfrak{g}^*$. We note that $\mathfrak{g}^*$ denotes the algebraic dual of $\mathfrak{g}$. With respect to a basis $\{X^p\}_p$ of $\mathfrak{g}$ one can write

$$(2.1) \qquad [X^p, X^q] = C_l^{pq} X^l \qquad \delta(X^p) = \alpha_{kl}^p X^k \otimes X^l$$

where $C_l^{pq}$ and $\alpha_{kl}^p$ are the so-called structure constants with respect to the given basis of $\mathfrak{g}$. Throughout this paper we will make use of the Einstein summation convention.

A differential calculus on $U(\mathfrak{g})$ of PBW type can be described by the universal enveloping algebra of an $\mathbb{N}$-graded colour Lie superalgebra $L$ (for the definition of colour Lie superalgebra see [6]) that extends $\mathfrak{g}$ in the following way (see [2]). A basis of $L$ is given by $\{X^p, \hat{X}^p\}_p$ where the elements $X^p$ have degree zero and the elements $\hat{X}^p$ have degree one. The commutator of $L$ is an extension of the commutator of $\mathfrak{g}$ of the form

$$(2.2) \qquad [X^p, \hat{X}^q] = A_l^{pq} \hat{X}^l \qquad [\hat{X}^p, \hat{X}^q] = 0,$$

the corresponding 2-cocycle is defined by

$$(2.3) \qquad \epsilon(m,n) = (-1)^{mn} \qquad m, n \in \mathbb{N}.$$

The commutator of $L$ is constructed such that the linear operator $d : L \to L$ defined by $d(X^p) = \hat{X}^p$ and $d(\hat{X}^p) = 0$ is a graded derivation of degree 1. For the structure constants $A_l^{pq}$ this yields the following conditions:

$$(2.4) \qquad A_l^{jk} A_m^{il} - A_p^{ik} A_m^{jp} = C_l^{ij} A_m^{lk}$$

$$(2.5) \qquad A_l^{pq} - A_l^{qp} = C_l^{pq}$$

Condition (2.4) expresses the Jacobi identity and condition (2.5) the derivation property of $d$. The differential operator is the graded derivation on $U(L)$ that uniquely extends $d$. In order to obtain a more intrinsic description of $L$ one can define the linear map $\rho : \mathfrak{g} \to gl(\mathfrak{g})$ by (see [1])

$$(2.6) \qquad \rho(X^p)(X^q) = A_l^{pq} X^l.$$

Conditions (2.4) and (2.5) are equivalent to

$$(2.7) \qquad \rho([x,y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x)$$

$$(2.8) \qquad [x,y] = \rho(x)(y) - \rho(y)(x)$$

for all $x, y \in \mathfrak{g}$. Such a map $\rho$ is therefore called a multiplicative representation of $\mathfrak{g}$ (see [7]), it completely determines the differential calculus on $U(\mathfrak{g})$.

In order to construct a De Rham complex on the QUE algebra $U_h(\mathfrak{g})$, which reduces to the previously described differential calculus on $U(\mathfrak{g})$ when $h$ is put equal to zero, the cocommutator $\delta$, which is the classical limit of $U_h(\mathfrak{g})$, and the differential operator $d$ must be compatible in a certain sense. This compatibility condition, which is introduced in [5], can be described as follows. Define the extension of $\delta$ from $\mathfrak{g}$ to $L$ by

$$(2.9) \qquad \delta \circ d = (d \otimes id + \tau \otimes d) \circ \delta = d_\otimes \circ \delta,$$

where $\tau : L \to L$ is defined as the linear map satisfying $\tau(x) = (-1)^p x$ for all $x \in L^p$. With respect to the given basis of $L$ the extension looks like

$$(2.10) \qquad \delta(\hat{X}^p) = \alpha_{kl}^p (\hat{X}^k \otimes X^l + X^k \otimes \hat{X}^l).$$

The differential $d$ and the cocommutator $\delta$ are compatible if and only if this extension defines a cocommutator on $L$. In that case we call $L$ a colour Lie bisuperalgebra. Hence, the problem of constructing a compatible differential calculus on the Lie bialgebra $\mathfrak{g}$ is to find a multiplicative representation of $\mathfrak{g}$ that defines a colour Lie superalgebra $L$ with the additional property that the induced extension of $\delta$ defined by (2.9) is a cocommutator. In order to study this problem we will look at it from the dual point of view.

## 3 The dual point of view

From the theory of Manin-triples (see e.g. [4]), we learn that a Lie bialgebra structure on $\mathfrak{g}$ induces a Lie bialgebra structure on $\mathfrak{g}^*$. The commutator of $\mathfrak{g}^*$ is the transpose of the cocommutator of $\mathfrak{g}$ and the cocommutator of $\mathfrak{g}^*$ is the transpose of the commutator of $\mathfrak{g}$. For instance, if the Lie bialgebra $\mathfrak{g}$ is described by (2.1) then the structure of $\mathfrak{g}^*$ is given by

$$[\Theta_p, \Theta_q] = \alpha^l_{pq}\Theta_l \qquad \delta^*(\Theta_p) = C^{kl}_p \Theta_k \otimes \Theta_l \tag{3.1}$$

where the elements $\Theta_p$ are defined to be dual to the basis elements $X^q$, i.e.

$$\Theta_p \in \mathfrak{g}^* \quad <\Theta_p, X^q>= \delta^q_p \tag{3.2}$$

and $\delta^q_p$ denotes the Kronecker delta symbol. Evidently this is also valid for colour Lie bisuperalgebras. We consider $L^*$ to be the dual of the colour Lie bisuperalgebra $L$ with basis $\{X^p, \hat{X}^p\}_p$ and structure maps as described by (2.1), (2.2) and (2.10), that represents the compatible differential calculus on the Lie bialgebra $\mathfrak{g}$. As basis of $L^*$ we choose $\{\Theta^p, \hat{\Theta}^p\}_p$ which is defined dual to the basis of $L$, i.e.

$$<\Theta_p, X^q>= \delta^q_p \quad <\Theta_p, \hat{X}^q>= 0 \quad <\hat{\Theta}_p, X^q>= 0 \quad <\hat{\Theta}_p, \hat{X}^q>= \delta^q_p. \tag{3.3}$$

From the duality between $L$ and $L^*$ we find the following expressions for the commutator and cocommutator of $L^*$:

$$[\Theta_p, \Theta_q] = \alpha^l_{pq}\Theta_l \qquad [\Theta_p, \hat{\Theta}_q] = \alpha^l_{pq}\hat{\Theta}_l \qquad [\hat{\Theta}_p, \hat{\Theta}_q] = 0, \tag{3.4}$$

$$\delta^*(\Theta_p) = C^{kl}_p\Theta_k \otimes \Theta_l \qquad \delta^*(\hat{\Theta}_p) = A^p_{kl}\Theta_k \otimes \hat{\Theta}_l - A^p_{lk}\hat{\Theta}_k \otimes \Theta_l. \tag{3.5}$$

The differential operator $d$ on $L$ satisfies

$$d \circ [\ ,\ ] = [\ ,\ ] \circ d_\otimes \qquad \delta \circ d = d_\otimes \circ \delta. \tag{3.6}$$

We define $\partial : \mathfrak{g}^* \to \mathfrak{g}^*$ to be the transpose of the differential $d$. It satisfies $\partial(\Theta_p) = 0$ and $\partial(\hat{\Theta}_p) = -\Theta_p$ and from (3.6) it follows that

$$\delta^* \circ \partial = \partial_\otimes \circ \delta^* \quad \partial \circ [\ ,\ ] = [\ ,\ ] \circ \partial_\otimes = [\ ,\ ] \circ (\partial \circ id + \tau \circ \partial). \tag{3.7}$$

If we look at the construction of a compatible differential calculus on $U(\mathfrak{g})$ from the point of $\mathfrak{g}$ then the problem is to find a multiplicative representation $\rho$ of $\mathfrak{g}$. This representation describes the commutator between elements of $L^0$ and $L^1$ and is such that the first condition of (3.6) is satisfied. The second condition of (3.6) is then used to extend the cocommutator and one needs to verify whether the extension meets the conditions of a cocommutator on $L$. However, dually we see that the colour Lie superalgebra structure of $L^*$ given by (3.4) is uniquely determined by the commutator of $\mathfrak{g}^*$ and the second property of $\partial$ from (3.7). Hence, in the dual formulation the problem is to find an extension of the cocommutator of $\mathfrak{g}^*$ which is a cocommutator on $L^*$ and satisfies the first condition of (3.7). Evidently, these problems are equivalent. However, in case the dual Lie bialgebra $\mathfrak{g}^*$ has a triangular structure there is a simple and natural solution which is evident from the dual point of view.

## 4 A canonical compatible differential calculus

Let us suppose that the dual Lie bialgebra $\mathfrak{g}^*$ has a triangular structure. This means that its cocommutator is a coboundary, i.e. $\delta^* : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ can be written by means of an $R$-matrix $R \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ as $\delta^*(\Theta) = \Theta \cdot R$, with a corresponding $R$-matrix which is antisymmetric and a solution of the Classical Yang Baxter Equation (CYBE). The dot denotes the action of the Lie algebra $\mathfrak{g}^*$ on the tensor product $\mathfrak{g}^* \otimes \mathfrak{g}^*$ induced by the adjoint representation. The antisymmetry of $R$ is reflected by the condition $\sigma(R) = -R$ where $\sigma : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ represents the flip operator on the tensor product. In terms of the dual basis $\{\Theta_p\}_p$ we can write

$$R = \gamma^{kl}\Theta_k \otimes \Theta_l \tag{4.1}$$

yielding

$$\delta^*(\Theta_p) = \gamma^{kl}(\alpha^q_{pk}\Theta_q \otimes \Theta_l + \alpha^q_{pl}\Theta_k \otimes \Theta_q). \tag{4.2}$$

The CYBE is usually written as

$$[|R,R|]=[R_{12},R_{13}]+[R_{12},R_{23}]+[R_{13},R_{23}]=0. \tag{4.3}$$

In this notation $R_{12}$ denotes the element $\gamma^{kl}\Theta_k\otimes\Theta_l\otimes 1$ in $U(\mathfrak{g}^*)\otimes U(\mathfrak{g}^*)\otimes U(\mathfrak{g}^*)$ and $[|R,R|]$ is considered as element of $\mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}^*\subset U(\mathfrak{g}^*)\otimes U(\mathfrak{g}^*)\otimes U(\mathfrak{g}^*)$.

In order to extend $\delta^*$ from $\mathfrak{g}^*$ to $L^*$ it seems quite natural to preserve the coboundary property. Hence, we propose the following extension:

$$\delta^*(\hat\Theta)=\hat\Theta\cdot R \qquad \hat\Theta\in(L^*)^{-1}=(L^1)^*. \tag{4.4}$$

Evidently this extension $\delta^*$ defines a cocommutator: $\delta^*$ is a 1-cocycle since it is a 1-coboundary by construction and the commutator properties of the bracket on $L$ defined by its transpose are a direct consequence of the triangularity properties of the matrix $R$. Due to the fact that $\partial$ is a derivation on $L^*$ and that $\partial_\otimes(R)=0$ it follows that the first condition of (3.7) is satisfied. Hence this extension defines a compatible differential calculus on $U(\mathfrak{g})$. With respect to the basis $\{\Theta^p,\hat\Theta^p\}_p$ one finds that

$$\delta^*(\hat\Theta_p)=\hat\Theta_p\cdot R=\gamma^{kl}(\alpha^q_{pk}\hat\Theta_q\otimes\Theta_l+\alpha^q_{pl}\Theta_k\otimes\hat\Theta_q) \tag{4.5}$$

which in comparison with(3.5) yields

$$A^p_{kl}=\gamma^{kq}\alpha^l_{pq}. \tag{4.6}$$

In order to obtain a description of the corresponding multiplicative representation we study in more detail the coboundary property of $\delta^*$. By definition of $\delta^*$ we have

$$<\Theta\cdot R,x\otimes y>=<\delta^*(\Theta),x\otimes y>=<\Theta,[x,y]> \quad \Theta\in\mathfrak{g}^*,x,y\in\mathfrak{g} \tag{4.7}$$

and by (4.1)

$$\Theta\cdot R=\gamma^{st}([\Theta,\Theta_s]\otimes\Theta_t+\Theta_s\otimes[\Theta,\Theta_t]). \tag{4.8}$$

We focus on the first term and take $\Theta=\Theta_k$. Due to

$$<[\Theta_k,\Theta_s]\otimes\Theta_t,X^p\otimes X^q>=<\alpha^\mu_{ks}\Theta_\mu\otimes\Theta_t,X^p\otimes X^q>=\alpha^p_{ks}\delta^q_t \tag{4.9}$$

we can write the transpose of the action of the first component of $R$, which we will denote by $\phi_1:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$, as

$$\phi_1(X^p\otimes X^q)=-\gamma^{sq}\alpha^p_{sk}X^k \tag{4.10}$$

or equivalently

$$\phi_1=-R^{13}\circ(\delta\otimes id) \tag{4.11}$$

where $R^{13}$ denotes the map from $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ to $\mathfrak{g}$ which is defined by letting $R$ act on the first and third component of the tensor product. Similarly one derives that

$$\phi_2=-R^{12}\circ(id\otimes\delta) \tag{4.12}$$

which is defined as the transpose of the action of the second component of $R$. Hence, combining (4.7), (4.11) and (4.12) we obtain

$$[x,y]=-(R^{13}\circ(\delta\otimes id)+R^{12}\circ(id\otimes\delta))(x\otimes y) \quad x,y\in\mathfrak{g}. \tag{4.13}$$

In order to rewrite this expression we note that

$$id\otimes\delta=\sigma_{12}\circ\sigma_{23}\circ(\delta\otimes id)\circ\sigma_{12}. \tag{4.14}$$

From this we derive that

$$R^{12}\circ(id\otimes\delta)=R^{12}\circ\sigma_{12}\circ\sigma_{23}\circ(\delta\otimes id)\circ\sigma_{12}=-R^{12}\circ\sigma_{23}\circ(\delta\otimes id)\circ\sigma_{12}=-R^{13}\circ(\delta\otimes id)\circ\sigma_{12}$$

and by substiting this in (4.13) we obtain

$$[x,y]=R^{13}\circ(\delta\otimes id)\circ(\sigma_{12}-id)(x\otimes y) \quad x,y\in\mathfrak{g} \tag{4.15}$$

From the reasoning above we conclude that the mapping $\rho:\mathfrak{g}\to gl(\mathfrak{g})$ defined by

$$\rho(x)(y)=R^{13}\circ(\delta\otimes id)\circ\sigma_{12}(x\otimes y) \tag{4.16}$$

satisfies (2.7) and (2.8), i.e. it defines a multiplicative representation of $\mathfrak{g}$. From this reasoning we can extract the following result.

**Theorem 1** *Let $\mathfrak{g}$ be a Lie bialgebra with corresponding dual Lie bialgebra $\mathfrak{g}^*$ of triangular type with R-matrix $R\in\mathfrak{g}^*\otimes\mathfrak{g}^*$. Then, the map $\rho:\mathfrak{g}\to gl(\mathfrak{g})$ defined by (4.16) is a multiplicative representation on $\mathfrak{g}$ and this multiplicative representation defines a differential calculus on $U(\mathfrak{g})$ which is compatible with the cocommutator $\delta$ of $\mathfrak{g}$.*

# 5 Examples

In this section we present some examples.

## 5.1 The 2-dimensional solvable Lie algebra

Consider the 2-dimensional Lie bialgebra $S$ over $\mathbb{C}$ with basis $\{a,b\}$ and structure maps

$$[a,b]=\lambda b \text{ and } \delta(a)=\lambda(a\otimes b-b\otimes a)=\lambda a\wedge b \quad \delta(b)=0 \tag{5.1}$$

We define the dual basis $\{\alpha,\beta\}$ of $S^*$, the structure maps of $S^*$ are given by

$$[\alpha,\beta]=\lambda\alpha \text{ and } \delta^*(\alpha)=0 \quad \delta^*(\beta)=\lambda\alpha\wedge\beta. \tag{5.2}$$

Evidently $S$ is triangular with $R$-matrix $R=\beta\wedge\alpha$. The extension of $\delta^*$ according to formula (4.4) is

$$\delta^*(\hat\alpha)=\hat\alpha\cdot(\beta\wedge\alpha)=\lambda\hat\alpha\wedge\alpha \qquad \delta^*(\hat\beta)=\hat\beta\cdot(\beta\wedge\alpha)=-\lambda\beta\wedge\hat\alpha \tag{5.3}$$

The corresponding multiplicative representation of $S$ is described by

$$\rho(a)=\begin{pmatrix} -\lambda & 0\\ 0 & 0\end{pmatrix} \quad \rho(b)=\begin{pmatrix} 0 & 0\\ -\lambda & 0\end{pmatrix} \tag{5.4}$$

and this defines the differential calculus on $U(S)$ given by

$$[a,\hat a]=-\lambda\hat a \quad [a,\hat b]=0 \quad [b,\hat a]=-\lambda\hat b \quad [b,\hat b]=0 \tag{5.5}$$

We remark that the Lie bialgebras $S$ and $(S^*)^{op}$ are isomorphic, an isomorphism $\varphi: S\to S^*$ is given by $\varphi(a)=-\beta, \varphi(b)=\alpha$.

## 5.2 The Heisenberg Lie algebra

Consider the $2n+1$-dimensional Heisenberg Lie bialgebra $H_n$ with basis $\{c,p_i,q_i\}_{1\le i\le n}$ and structure maps

$$[c,p_i]=[c,q_i]=0 \quad [p_i,q_j]=\delta^i_j c \quad \delta(c)=0 \quad \delta(p_i)=p_i\wedge c \quad \delta(q_i)=q_i\wedge c. \tag{5.6}$$

We define the dual basis $\{\gamma,\alpha_i,\beta_i\}_{1\le i\le n}$ of $H_n^*$ and obtain as structure maps

$$\begin{array}{llllll} [\gamma,\alpha_i] & =-\alpha_i & [\gamma,\beta_i] & =-\beta_i & [\alpha_i,\beta_j] & =0\\ \delta^*(\gamma) & =\sum\limits_{i=1}^n \alpha_i\wedge\beta_i & \delta^*(\alpha_i) & =0 & \delta^*(\beta_i) & =0\end{array} \tag{5.7}$$

One can easily verify that $H_n^*$ is triangular with $R$-matrix $R=\frac{1}{2}\sum_i\beta_i\wedge\alpha_i$. The extension of $\delta^*$ is given by

$$\delta^*(\hat\gamma)=\frac{1}{2}\sum_{i=1}^n(\hat\alpha_i\wedge\beta_i+\alpha_i\wedge\hat\beta_i) \quad \delta^*(\hat\alpha_i)=0 \quad \delta^*(\hat\beta_i)=0 \tag{5.8}$$

which gives the following differential calculus on $U(H_n)$

$$[p_i,\hat q_j]=\frac{1}{2}\delta^i_j\hat c \qquad [q_i,\hat p_j]=-\frac{1}{2}\delta^i_j\hat c \tag{5.9}$$

and all other commutators are equal to zero. For instance the multiplicative representation in case $n=1$ looks like

$$\rho(c)=0 \quad \rho(p)=\begin{pmatrix} 0 & 0 & \frac12\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \rho(q)=\begin{pmatrix} 0 & -\frac12 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{5.10}$$

### 5.3 The dual of the non-standard $sl_2$

We consider the 3-dimensional Lie bialgebra $K$ with basis $\{\gamma, \alpha, \beta\}$ and structure given by

$$(5.11)\qquad \begin{array}{llllll} [\gamma,\alpha] & = \gamma & [\gamma,\beta] & = 0 & [\alpha,\beta] & = -\beta \\ \delta(\gamma) & = \alpha\wedge\beta & \delta(\alpha) & = 2\gamma\wedge\alpha & \delta(\beta) & = -2\gamma\wedge\beta \end{array}$$

The corresponding dual basis of $K^*$ will be denoted by $\{h, e, f\}$ and its structure is given by

$$(5.12)\qquad \begin{array}{llllll} [h,e] & = 2e & [h,f] & = -2f & [e,f] & = h \\ \delta^*(h) & = h\wedge e & \delta^*(e) & = 0 & \delta^*(f) & = f\wedge e. \end{array}$$

From this we see that $K^*$ is isomorphic to the Lie algebra $sl_2$ equipped with the non-standard cocommutator determined by the $R$-matrix $R = \frac{1}{2}h \wedge e$. The extension of $\delta^*$ is given by

$$(5.13)\qquad \delta^*(\hat{h}) = h \wedge \hat{e} \quad \delta^*(\hat{e}) = e \wedge \hat{e} \quad \delta^*(\hat{f}) = \hat{f} \wedge e - \frac{1}{2} h \wedge \hat{h}$$

and the corresponding multiplicative representation of $K$ is described by

$$(5.14)\qquad \rho(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(\beta) = 0 \quad \rho(\gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

The differential calculus on $U(K)$ is given by

$$(5.15)\qquad \begin{array}{lll} [\alpha,\hat{\alpha}] = \hat{\alpha} & [\beta,\hat{\alpha}] = 0 & [\gamma,\hat{\alpha}] = \hat{\gamma} \\ [\alpha,\hat{\beta}] = -\hat{\beta} & [\beta,\hat{\beta}] = 0 & [\gamma,\hat{\beta}] = 0 \\ [\alpha,\hat{\gamma}] = 0 & [\beta,\hat{\gamma}] = 0 & [\gamma,\hat{\gamma}] = -\frac{1}{2}\hat{\beta} \end{array}$$

### 5.4 The dual of the Virasoro algebra

In this example we consider an infinite dimensional Lie algebra. By $V$ we denote the Virasoro algebra, this Lie algebra has a basis $\{e_p, c\}_p$ with commutator given by

$$(5.16)\qquad [e_p, e_q] = (q-p)e_{p+q} + \delta^0_{p+q}\frac{q^3-q}{12}c \quad [c, e_p] = 0 \quad p, q \in \mathbb{Z}.$$

The Virasoro algebra can be equipped with a triangular Lie bialgebra structure by means of the $R$-matrix $R = c \wedge e_0$, this yields the following cocommutator

$$(5.17)\qquad \delta(e_p) = pe_p \wedge c \quad \delta(c) = 0.$$

The dual Lie bialgebra $V^*$ consists of formal series in $\{f^p, \gamma\}_p$. These elements are defined to be dual with respect to the given basis of the Virasoro algebra. The commutator and cocommutator of $V^*$ are given by

$$[f^p, f^q] = 0 \quad [\gamma, f^p] = -pf^p \quad \delta^*(f^p) = \sum_q (2q-p) f^{p-q} \otimes f^q \quad \delta^*(\gamma) = \sum_q \frac{q^3-q}{12} f^{-q} \otimes f^q.$$

The extension of the cocommutator of $V$ to the colour Lie superalgebra with basis $\{e_p, \hat{e}_p, c, \hat{c}\}_p$ according to (4.4) is given by

$$(5.18)\qquad \delta(\hat{e}_p) = p(\hat{e}_p \otimes c - c \otimes \hat{e}_p) \quad \delta(\hat{c}) = 0.$$

The corresponding multiplicative representation on $V^*$ is described by

$$(5.19)\qquad \rho(f^p)(f^q) = 0 \quad \rho(f^p)(\gamma) = 0 \quad \rho(\gamma)(f^p) = -pf^p \quad \rho(\gamma)(\gamma) = 0.$$

We remark that in this example $V^*$ plays the role of $\mathfrak{g}$ and $V$ the role of $\mathfrak{g}^*$. So, in fact we used the triangular structure of $V \subset (V^*)^*$ to obtain a multiplicative representation of $V^*$.

# 6 The quasi triangular case

In this section we discuss the possibility of extending the result of Theorem I to the case where $\mathfrak{g}^*$ is of quasi triangular type. A Lie bialgebra is said to be quasi triangular if its cocommutator is a coboundary determined by an $R$-matrix that satisfies the CYBE. In contrast to the triangular case, the $R$-matrix does not need to be antisymmetric. In general one can write $R = R_a + R_s$ where $R_a$ and $R_s$ denote the antisymmetric and the symmetric part of $R$ respectively, i.e. $\sigma(R_a) = -R_a$ and $\sigma(R_s) = R_s$. Since the cocommutator $\delta^*$ satisfies $\sigma \circ \delta^* = -\delta^*$, the symmetric part of $R$ is $\mathfrak{g}$-invariant. Suppose we extend $\delta^*$ to $L^*$ as described by (4.4), i.e. $\delta^*(\hat{\Theta}) = \hat{\Theta} \cdot R = \hat{\Theta} \cdot R_a + \hat{\Theta} \cdot R_s$. The antisymmetry condition for the extension $\delta^*$ implies that $R_s$ needs to be invariant under the adjoint action of the odd part of $L^*$. We can write

$$R_s = \sum_i \phi_i \otimes \psi_i \qquad \sigma(R_s) = R_s \tag{6.1}$$

where both $\{\phi_i\}_i$ and $\{\psi_i\}_i$ are linearly independent sets in $\mathfrak{g}^*$. Then

$$\hat{\Theta} \cdot R_s = \sum_i ([\hat{\Theta}, \phi_i] \otimes \psi_i + \phi_i \otimes [\hat{\Theta}, \psi_i]) = \sum_i ([\Theta, \phi_i]\hat{} \otimes \psi_i + \phi_i \otimes [\Theta, \psi_i]\hat{}) = 0 \tag{6.2}$$

implies that both sets are invariant under the adjoint action of $\Theta \in \mathfrak{g}^*$. Hence, the extension will only be well defined if all terms in $R_s$ consist of central elements of $\mathfrak{g}^*$. However, in that case

$$0 = [|R, R|] = [|R_a + R_s, R_a + R_s|] = [|R_a, R_a|] \tag{6.3}$$

which implies that $\mathfrak{g}^*$ is of triangular type with $R$-matrix $R_a$. The conclusion is therefore that the construction of section 3 can not be applied to 'proper' quasi triangular Lie bialgebras such as for instance doubles of Lie bialgebras (see e.g. [4]).

# References


[1] R. Martini, G.F. Post and P.H.M. Kersten (1995). *Differential Calculus on Universal Enveloping Algebras of Lie Algebras.* Memorandum 1261, University of Twente, Enschede.

[2] N. van den Hijligenberg and R. Martini (1995). *Differential Hopf Algebra Structures on the Universal Enveloping Algebra of a Lie algebra.* CWI-report R9515, Center for Mathematics and Computer Science, Amsterdam.

[3] G. Maltsiniotis (1990). Groupes quantiques et structures differentielles (Quantum groups and differential structures). *C.R. Acad. Sci. Paris* Sér I. **311**, 831-834.

[4] V. Chari and A. Pressley (1994). *Quantum Groups.* Cambridge University Press, Cambridge. p. 170-213.

[5] N. van den Hijligenberg, R. Martini and G.F. Post (1995). *Quantization of Differential Calculi on Universal Enveloping Algebras*, Preprint.

[6] Yu.A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky and M.V. Zaicev (1992). *Infinite Dimensional Lie Superalgebras.* Walter de Gruyter, Berlin-New York. p. 13.

[7] G.F. Post and R. Martini (1994). *Multiplicative representations of Lie algebras.* Preprint, University of Twente, Enschede.