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The Complementary-slackness Class of Hybrid Systems

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Abstract

In this paper we understand by a 'hybrid system' one that combines features of continuous dynamical systems with characteristics of finite automata. We study a special class of such systems which we call the complementary-slackness class. We study existence and uniqueness of solutions in the special cases of *linear* and *Hamiltonian* complementary-slackness systems. For the latter class we also prove an energy inequality.

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1 Introduction

The idea of a system that interacts with its environment plays a role in computer science as well as it does in the theory of dynamical systems; in the computer science terminology, one speaks in this context of *reactive systems* [17]. Often a computer system will operate in an environment that is described by continuous rather than discrete variables, for instance in the control of chemical processes or of mechanical systems, and in these cases one deals with a combined system that incorporates features both of finite automata and of continuous dynamical systems. In a mirroring development, there is a trend in control theory [7] moving away from the standard paradigm that is formulated entirely in terms of differential and/or difference equations, towards formulations that also allow the presence of discrete elements, often described in this context as 'switching logic'. Actually this trend follows rather than precedes control engineering practice, since switching elements have already been used successfully although on an ad-hoc basis in many control applications. The incorporation of discrete elements in continuous dynamical systems has also attracted the interest of researchers in the area of 'classical' dynamical systems as a means for obtaining properties such as robust stability [13].

In this way, from the efforts of computer scientists, control theorists, and dynamicists, a new research field is emerging for which the name *hybrid systems* is now mostly used. Such hybrid systems arise for instance in rule-based control of continuous processes, in situations where continuous processes affect the operation of a discrete device (timing constraints, clock drift), and in general in any situation where 'logic' and 'dynamics' interact with each other.

Proposals for defining the general class of hybrid systems or languages for it have been made in computer science as well as in control theory and in the theory of dynamical systems (see for instance [12, 19, 4, 2]). In the present paper, the development of such general definitions will *not* be the main concern. Rather than a 'top-down' strategy we prefer to follow a 'bottom-up' approach in which we work with only a preliminary, not fully detailed definition of what a hybrid system is, and study special classes for which a good intuition is available about how they should behave. Even the study of quite restricted subclasses will be worthwile if these offer a strong basis for theoretical development, which may then serve as a guideline and a reference for other theories. In this paper we consider such a class of hybrid systems, which we have named the 'complementary-slackness class' after a term used in optimization theory. As we hope to demonstrate, there are indeed a number of nontrivial conclusions to draw from the study of this special class.

It is easy to give real-world examples of complementary-slackness systems. Electrical networks containing diodes, hydraulic systems containing one-way valves, and mechanical systems with stops can all be interpreted as complementary-slackness systems. An advantage of the fact that these systems occur in nature is that a strong intuition is available about their operation. A second advantage that will be employed below is the presence of the concept of *energy*, which allows making general statements about the trajectories of complementary-slackness systems. We also like to point out that 'natural' structures often serve as guidelines for artificial designs; in this context one may refer for instance to simulated annealing, or to the use of passivity in adaptive and nonlinear control.

The structure of this paper is as follows. We begin with two introductory sections, one on the concept of hybrid systems in general and one on constrained differential equations. In section 4 we introduce complementary-slackness systems and show how these describe a hybrid system in a very compact way. The next two sections are devoted to two special cases of special interest, namely *linear* and *Hamiltonian* complementary-slackness systems. We will be concerned with the most basic question that may be asked about a dynamical system: existence and uniqueness of solutions. A few simple examples will show the nontriviality of these issues in the context of complementary-slackness systems. In the case of Hamiltonian complementary-slackness systems, we prove an energy inequality and give an interpretation for the law of conservation of momentum. Conclusions follow in section 7.

2 Hybrid systems

It has already been noted in the introduction that there are several ways to think about hybrid systems, which are largely similar but may put more or less emphasis on particular aspects. As mentioned earlier, our purpose in this paper is not to set forth a general definition of any kind, but we do need some setting to work in. We shall look at a hybrid system as *a family of continuous-time dynamical systems parametrized by the nodes of a transition graph*. This is of course by no means a precise definition but it does provide a conceptual framework. The dynamical systems in the family may be described in a classical way by equations of the form

$$\dot{x}_i(t) = f_i(x_i(t), u(t)) \tag{2.1}$$

where $x_i(\cdot)$ is the *continous state* attached to node i (which represents the *discrete state*) and $u(\cdot)$ represents a *continuous input*. A *continuous output* $y(\cdot)$ can be defined by adding the equation

$$y(t) = h_i(x_i(t)), \tag{2.2}$$

and a *discrete output* can be associated with each transition. The *timing* of transitions emanating from node i is determined by conditions that can be expressed in the continuous state of the system at node i and a *discrete input*. The *effect* of a transition is given by conditions that involve again the states and the inputs; in particular a new discrete state j should be specified as well as a new continuous state, which serves as an initial condition for the continuous dynamical system at node j.

To illustrate the use of this framework, let us consider an example that is in many respects different from the systems that we shall consider in later sections. The example is the audio protocol that was suggested as a benchmark in [3]. Without going into the details of the actual implementation (see [3] for a more elaborate description), the protocol can be described as follows. The purpose of the protocol is to allow the components of a consumer audio system to exchange messages. A certain coding scheme is used which depends on time intervals between events, an 'event' being the voltage on a bus interface going from *low* to *high*. A basic time interval is chosen and the coding can be described as follows:

- the first event always signifies a 1 (messages always start with a 1);
- if a 1 has last been read, the next event signifies 1 if 4 basic time intervals have passed, 0 if 6 intervals have passed, and 01 if 8 intervals have passed;
- if a 0 has last been read, the next event signifies 0 if 4 intervals have passed, and 01 if 6 intervals have passed;
- if more than 9 intervals pass after a 1 has been read, or more than 7 after a 0 has been read, the message is assumed to have ended.

Due to clock drift and priority scheduling, the timing of events is uncertain and the design specifications call for a 5% tolerance. Clocks are reset with each event.

The protocol can be modeled as a hybrid system, whose discrete states correspond to the symbols last sent and received, and whose continuous states are given by the sender's and the receiver's clocks. The 5% tolerance in timing is incorporated by letting the dynamics of the clocks be given by

$$\begin{aligned} \dot{x}_{\mathrm{s}}(t) &= 1 + u_{\mathrm{s}}(t), \qquad & |u_{\mathrm{s}}(t)| \leq 0.05 \\ \dot{x}_{\mathrm{r}}(t) &= 1 + u_{\mathrm{r}}(t), \qquad & |u_{\mathrm{r}}(t)| \leq 0.05 \end{aligned}$$

where $u_{\mathrm{s}}(t)$ and $u_{\mathrm{r}}(t)$ denote disturbance inputs. The jump conditions at the discrete state (1,1) (both sender and receiver recorded 1 as the last transmitted symbol) can for instance be specified as follows. A jump takes place at times when one of the following conditions occurs:

(i) the input symbol is 1 and $x_{\mathrm{s}}(t) = 4$;

(ii) the input symbol is 0 and $x_{\mathrm{s}}(t) = 6$;

(iii) the input symbol is 01 and $x_{\mathrm{s}}(t) = 8$;

(iv) there is no input symbol and $x_{\mathrm{r}}(t) = 9$.

The effect of a jump can be described as follows: a discrete output value is produced which is 1 if $x_{\mathrm{r}}(t) < 5$, 0 if $5 \leq x_{\mathrm{r}}(t) < 7$, 01 if $7 \leq x_{\mathrm{r}}(t) < 9$, and *end-of-message* if $x_{\mathrm{r}}(t) = 9$; the new continuous state is $(0, 0)$ (resetting of the clocks), the new discrete state is (s_i, s_o) where s_i is the input symbol and s_o is the output symbol, and a new input symbol is read.

The protocol is said to be *correct* if the string of output symbols is always equal to the string of input symbols. The correctness condition can be described as a (discrete) *reachability condition*: no discrete states (s_1, s_2) with $s_1 \neq s_2$ should be reachable. This means that transitions to such states should never be possible. The verification of this condition requires inspection of the jump conditions and the transition rules, and since these involve continuous dynamics, we have at each discrete state and for each discrete input value a *continuous* reachability problem. Actually in the present case the continuous dynamics is the same at each discrete state, so it suffices to draw a single picture. In Fig. 1 the reachable set of continuous states (x_1, x_2) is indicated (shaded) together with the set of points that should be avoided in order to prevent illicit transitions. The reachable set is a cone whose width is determined by the tolerance of the clocks; it is seen that the 5% tolerance is enough (although only barely so) to ensure the correctness of the protocol. An interesting feature

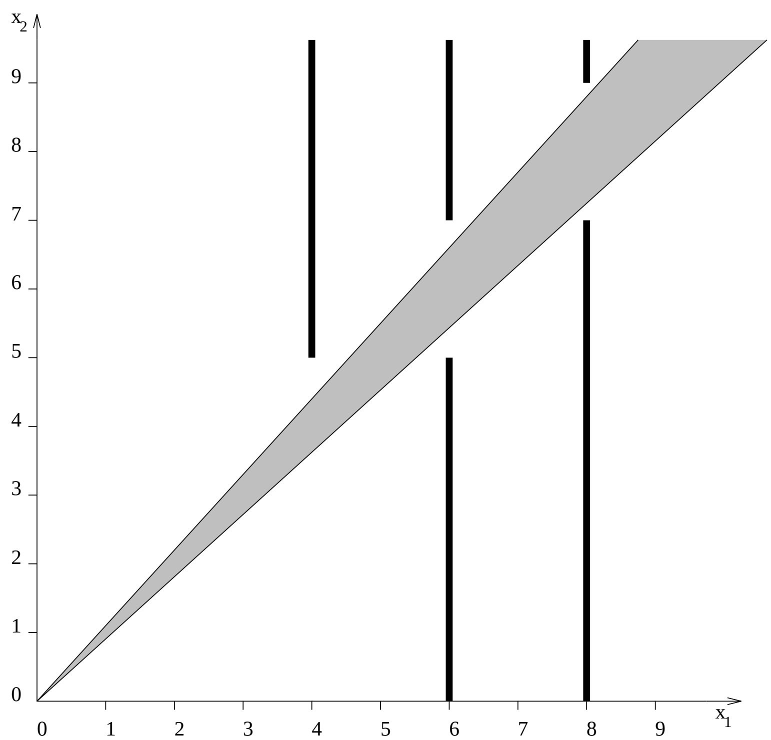

FIGURE 1. Reachable set and avoidance set for audio protocol

of the example is that although as far as the description of the dynamics is concerned there would be no need to distinguish for instance between the discrete states $(1,1)$ and $(1,01)$, this distinction does however become important if one wants to formulate correctness requirements. The example also shows that such requirements may take the form of reachability conditions that are ultimately formulated in the continuous state space.

3 Constrained differential equations

Complementary-slackness systems can be understood as dynamical systems that are subject to varying sets of constraints. The most natural description of such systems is by means of sets of differential and algebraic equations (DAEs). In this section we briefly review some properties of DAEs, without striving for the greatest possible generality. In particular we shall only consider equations with constant coefficients and without forcing functions.

A vector DAE in fully implicit form is a set of equations

$$f(z(t), \dot{z}(t)) = 0 \tag{3.1}$$

where f is a function from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N$. One also often encounters the so-called semi-explicit form

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t)) \\ 0 &= h(x(t), u(t)). \end{aligned} \tag{3.2}$$

where now f is a function from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$ and h maps $\mathbb{R}^{n+k}$ to $\mathbb{R}^k$. It is clear that (3.2) may be viewed as a special form of (3.1) by identifying $z(t)$ with the vector having components $x(t)$ and $u(t)$. From a system theory perspective, it can be useful to introduce an output $y(t) = h(x(t), u(t))$ and to look at (3.2) as the 'zero dynamics' of a system with state x, input u, and output y.

A *solution* of (3.1) is a continuously differentiable function $z : \mathbb{R} \to \mathbb{R}^N$ such that (3.1) holds for all t. We shall call $z_0 \in \mathbb{R}^N$ a *consistent point* of (3.1) if there is a solution $z(t)$ such that $z(0) = z_0$. It is not our purpose here to delve in the many singularities that may arise in the context of implicit systems, and in particular we shall always assume that the set of all consistent points forms a smooth manifold, which will be denoted by $\mathcal{V}(f)$. The system (3.1) will be called *autonomous* if for every point $z_0 \in \mathcal{V}(f)$ there is exactly one solution passing through z_0. (The terminology here follows that of system theory, in which an 'autonomous system' is thought of as 'a system with no inputs' [21], rather than that of the theory of ordinary differential equations. In computer science terminology, autonomous systems might be referred to as *deterministic*.) For a simple example of a non-autonomous system, consider

$$\begin{aligned} z_1(t)\dot{z}_1(t) + z_2(t)\dot{z}_2(t) &= 0 \\ z_1^2(t) + z_2^2(t) &= 1. \end{aligned} \tag{3.3}$$

Note that the first equation is implied by the second one. Therefore, the consistent manifold for (3.3) consists of the points $z \in \mathbb{R}^2$ for which $z_1^2 + z_2^2 = 1$. The solution set of the equations (3.3) consists of all trajectories $z(\cdot)$ of the form $z_1(t) = \sin(u(t) + \phi)$, $z_2(t) = \cos(u(t) + \phi)$ where $u(\cdot)$ is an arbitrary smooth function and ϕ is a constant; this representation explicitly shows that 'the system has an input'.

Sufficient conditions for the system (3.1) to be autonomous are provided by the various methods of reducing systems of the form (3.1) to a set of ODEs ('index reduction methods', see for instance [11, 5]). In the linear time-invariant case, necessary and sufficient conditions for (3.1) to be autonomous have been known for a long time and can be found for instance in [10, §XII.7]. We shall formulate these conditions in a way that will be convenient below. Linear systems of the form (3.1) can be written as

$$E\dot{z}(t) = Az(t) \tag{3.4}$$

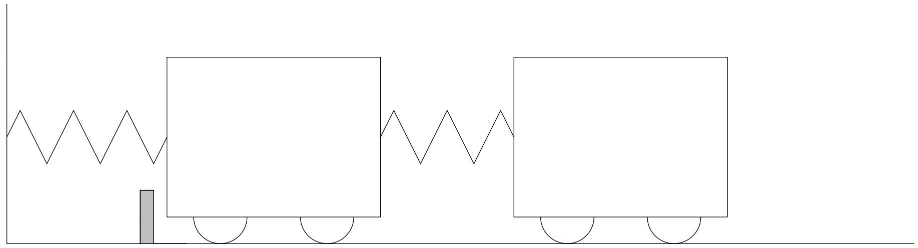

FIGURE 2. Example of a complementary-slackness system

where E and A are matrices of size $N \times N$. Now define two sequences of subspaces by the following iterations:

$$\mathcal{V}^0 = \mathbb{R}^N, \qquad \mathcal{V}^{j+1} = A^{-1}E\mathcal{V}^j \quad (j = 0, 1, \cdots) \tag{3.5}$$

$$\mathcal{T}^0 = \{0\}, \qquad \mathcal{T}^{j+1} = E^{-1}A\mathcal{T}^j \quad (j = 0, 1, \cdots). \tag{3.6}$$

The first sequence is nonincreasing and the second is nondecreasing, so both must have limits; we shall write

$$\mathcal{V}^*(E, A) \;=\; \lim \mathcal{V}^j(E, A) \tag{3.7}$$

and

$$\mathcal{T}^*(E, A) \;=\; \lim \mathcal{T}^j(E, A). \tag{3.8}$$

We then have the following proposition.

PROPOSITION 3.1 *Consider a system of linear algebraic and differential equations of the form (3.4). The subspace $\mathcal{V}^*(E, A)$ coincides with the set of consistent points of (3.4), and the system (3.4) is autonomous if and only if*

$$\mathcal{V}^*(E, A) \oplus \mathcal{T}^*(E, A) \;=\; \mathbb{R}^N. \tag{3.9}$$

PROOF For the first statement, see for instance [22, exc. 4.6]. The second statement follows by noting that the system (3.4) is autonomous if and only if $\det(sE - A) \neq 0$ [10, §XII.7], and by using the well-known geometric characterization of the regularity of the pencil $sE - A$ (see for instance [1]). □

The key importance of the direct-sum decomposition (3.9) will be clear later on, when a change of mode will call for a projection of vectors in the space $\mathbb{R}^N$ to the consistent manifold of a new dynamics. Without the presence of a guiding complementary subspace, this projection is not well-defined. A hint towards the intrinsic meaning of the subspace defined by (3.6) and (3.8) can be found in the literature on singular optimal control (see for instance [14]). Anticipating the generalization to the nonlinear case, we shall refer to $\mathcal{T}^*(E, A)$ as the *complementary foliation* that goes with the consistent manifold $\mathcal{V}(E, A)$ of (3.4). For consistency of notation, we shall drop the asterisk and write simply $\mathcal{T}(E, A)$.

4 COMPLEMENTARY-SLACKNESS SYSTEMS

In this section we introduce complementary-slackness systems and show how they specify a hybrid system. To motivate the development, we begin with a simple example (cf. [6]). Consider the physical system in Fig. 2. Two carts are connected to each other and to a fixed wall by springs. The motion of the left cart is restricted by a (purely non-elsatic) stop. There are two modes, corresponding to the constraint being active or not. For simplicity, we shall assume that the masses of the carts are normalized to 1, that the springs are linear with spring constants equal to 1, and that the stop is placed at the equilibrium position of the left cart. The equations of motion may then be written down as follows, where x_1 and x_2 denote the deviations of the left and the right cart respectively from their equilibrium positions, and $\lambda(t)$ represents the reaction force exerted by the stop when the constraint is active.

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_3(t) &= -2x_1(t) + x_2(t) + \lambda(t) \\ \dot{x}_4(t) &= x_1(t) - x_2(t) \\ y(t) &= x_1(t) \\ u(t) &= \lambda(t) \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)u(t) = 0. \end{aligned} \tag{4.1}$$

The two modes of the system correspond to situations in which either $y(t) = 0$ (active constraint) or $u(t) = 0$ (inactive constraint). By the physics of the system, the constraint force must be nonnegative and can only be positive if $y(t) = 0$, whereas the deviation of the left cart from its equilibrium position is always nonnegative, and the reaction force must be zero when this deviation is positive. These alternatives are expressed by the definitions of $y(t)$ and $u(t)$ and by the last line of (4.1).

Now in general, a complementary-slackness system with n states and k side constraints is given by equations of the form

$$\begin{aligned} f(z(t), \dot{z}(t)) &= 0 && (f : \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \to \mathbb{R}^n) \\ y(t) &= h_1(z(t)) && (y(t) \in \mathbb{R}^k) \\ u(t) &= h_2(z(t)) && (u(t) \in \mathbb{R}^k) \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)^T u(t) = 0. \end{aligned} \tag{4.2}$$

The inequalities are understood componentwise. The conditions on $y(t)$ and $u(t)$ imply that for each index i and at each time t we must have either $y_i(t) = 0$ and $u_i(t) \geq 0$ or $u_i(t) = 0$ and $y_i(t) \geq 0$. Paired conditions of this type occur in optimization problems with inequality constraints and are known in this context as 'complementary-slackness conditions'; the terminology is derived from saying that an inequality has 'slack' if it is strict. A special form of (4.2) related to the semi-explicit form of DAEs is the following:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) && (f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n) \\ y(t) &= h(x(t), u(t)) && (y(t) \in \mathbb{R}^k) \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)^T u(t) = 0. \end{aligned} \tag{4.3}$$

Whereas the general formulation (4.2) treats the dual variables $y(t)$ and $u(t)$ on an equal footing, this is no longer true in the formulation above. Note in the particular the double role played by $u(t)$. In (4.3) it is natural to think of $u(t)$ as an input and $y(t)$ as an output.

We now want to establish some terminology that will be used throughout the rest of the paper. Write K for the index set $\{1, \cdots, k\}$. To each subset of K corresponds a particular set of DAEs, to wit

$$\begin{aligned} f(z(t), \dot{z}(t)) &= 0 \\ h_{1i}(z(t)) &= 0 \quad (i \in I) \\ h_{2i}(z(t)) &= 0 \quad (i \in K \setminus I) \end{aligned} \tag{4.4}$$

This a differential-algebraic system of $n+k$ equations in $n+k$ unknowns. The *consistent manifold* of this system, in the sense of section 3, will be denoted by $\mathcal{V}_I$. The dynamics on $\mathcal{V}_I$ as defined by the equations (4.4) will be referred to as *mode* I of the system (4.2). The system (4.2) thus has 2^k modes.

Note that $\mathcal{V}_I$ is defined by only *equality* constraints. We also define the set of *feasible points of mode* I, denoted by $\mathcal{W}_I$, as the subset of $\mathcal{V}_I$ on which the *inequality* constraints corresponding to mode I are satisfied:

$$\mathcal{W}_I \;=\; \{z \in \mathcal{V}_I \mid h_{1i}(z) \geq 0 \;\; (i \in K \setminus I), \;\; h_{2i}(z) \geq 0 \;\; (i \in I)\}. \tag{4.5}$$

The possibility that $\mathcal{W}_I$ is empty for some index sets I is not *a priori* excluded. A point will be said to be *feasible* for the entire system (4.2) if it is feasible for at least one of the modes of (4.2). Consider now a point $z \in \mathcal{V}_I$, and assume that mode I represents a well-posed autonomous dynamics so that we have a unique solution of (4.4) starting from z. Denote the point reached from z after time t by $\theta(t, z; I)$. If there is an $\varepsilon > 0$ such that $\theta(t, z; I) \in \mathcal{W}_I$ for all $t \in [0, \varepsilon]$, then we say that *smooth continuation is possible from z in mode I*. If smooth continuation in mode I is not possible, then at least one of the following two index sets is nonempty:

$$\begin{aligned} \Gamma_1(z; I) \;&:=\; \{i \in I \mid \exists \varepsilon > 0 \text{ s.t. } \forall t \in (0, \varepsilon) \; (h_1(\theta(t, z; I)))_i < 0\} \\ \Gamma_2(z; I) \;&:=\; \{i \in K \setminus I \mid \exists \varepsilon > 0 \text{ s.t. } \forall t \in (0, \varepsilon) \; (h_2(\theta(t, z; I)))_i < 0\}. \end{aligned} \tag{4.6}$$

These index sets will play a role in the transition rules that we will define next.

The equations (4.2) as such do not yet define a hybrid system. According to the conceptual framework of section 2, we have to define a transition graph and associate a continuous dynamics to each node of this graph. An obvious choice is to let the nodes of the transition graph correspond to the modes of the system (4.2). We also need to define transitions between nodes, however, and this will call for an extra ingredient that is in general not automatically provided by the equations (4.2). This new ingredient is a *foliation* $\mathcal{T}_I$ associated with each mode I, which allows *projection* of points z onto the consistent manifold $\mathcal{V}_I$. The direct-sum decomposition (3.9) suggests that such a foliation is in some sense canonically given in the case of autonomous linear systems. We shall see below in section 6 that also in the case of Hamiltonian systems a canonical choice can be made.

So assume now that we have the equations (4.2) and that for each mode $I \subset K$ we have a foliation $\mathcal{T}_I$ that allows projection onto the consistent manifold $\mathcal{V}_I$ of mode I. We define the possible trajectories of the hybrid system associated to (4.2) and the foliations $\mathcal{T}_I$ by specifying all possible evolutions from an arbitrary initial point z. For each mode I such that $z \in \mathcal{V}_I$, the following evolutions are allowed:

- smooth continuation from z in mode I, if this is possible;
- otherwise, a jump from z along $\mathcal{T}_J$ onto $\mathcal{V}_J$, where J is the index set determined by

$$J \;=\; (I \setminus \Gamma_1(z; I)) \cup \Gamma_2(z; I). \tag{4.7}$$

If a jump occurs, then from the new point $z' \in \mathcal{V}_J$ the same alternatives as above are taken into consideration. In particular, we do not exclude the possibility that again a jump will occur. Also we allow in principle multiple solutions starting from points z that belong to the consistent manifolds of more than one mode. We shall say that the complementary-slackness system is *well-posed* (as a closed dynamical system) if from each feasible point there exists a unique solution path starting with at most a finite number of jumps followed by a smooth continuation.

REMARK 4.1 Our formulation here is motivated by the level of generality of the 'fully implicit' form (4.2). For systems written in the 'semi-explicit' form (4.3), one may be tempted to let jumps take place in the space of x-variables $\mathbb{R}^n$ rather than in the space of z-variables $\mathbb{R}^{n+k}$. The relation between these two formulations for the linear case will be discussed in more detail in the next section. One may note that the choice between the two alternatives is a classical one—it represents one of the differences between the setting used in the calculus of variations and the one used in optimal control theory.

REMARK 4.2 It may seem contrived to allow a sequence of jumps, but already from simple examples it can be seen that correct physical modeling calls for this. Consider for instance the example of Fig. 2. We shall let the jumps be determined by the 'linear' foliation (3.9) (which in this case coincides with the 'Hamiltonian' foliation to be discussed in section 6). The consistent manifolds for the unconstrained mode and the constrained mode will be denoted by $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively, and the associated foliations (or complementary subspaces in the linear case) by $\mathcal{T}_0$ and $\mathcal{T}_1$. By computation using the algorithms (3.5) and (3.6) we find

$$\mathcal{V}_0 = \ker \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{V}_1 = \ker \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{T}_0 = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{T}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Take an initial point $z = (x_1, \ldots, x_4, \lambda)$ with $x_1 = 0$, $x_2 > 0$, $x_3 < 0$, $\lambda = 0$; this corresponds to a situation in which the left block hits the stop at a time when the right block is to the right of its equilibrium position. Note that z belongs to $\mathcal{V}_0$ but not to $\mathcal{V}_1$. Smooth continuation in the unconstrained mode is obviously not possible and so a jump will occur along $\mathcal{T}_1$ to $\mathcal{V}_1$. This produces a point z' with coordinates $z' = (0, x_2, 0, x_4, -x_2)$. In particular the λ-coordinate will be negative and so although z' belongs to $\mathcal{V}_1$ it is not even feasible for the constrained mode, and smooth continuation in this mode is therefore not possible. We now must jump from z' along $\mathcal{T}_0$ to $\mathcal{V}_0$; a new point z'' is produced with coordinates $z'' = (0, x_2, 0, x_4, 0)$. From this point, smooth continuation is possible in the unconstrained mode. The solution that is obtained in this way corresponds to the physical insight which tells us that, if the right block is to the right of its equilibrium position at the moment at which the left block hits the stop, it will immediately pull the left block away from the stop again. Another example of such a situation, in which certain constraints force a state jump but do not actually become active, will be provided in Example 6.3.

EXAMPLE 4.3 Consider an electrical network consisting of linear resistors, capacitors, inductors, transformers, and gyrators, and of k diodes. Replacing the diodes first by ports, one can write down equations for the network in the 'hybrid' (the term is overworked here) form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4.8}$$

where u_i denotes either current or voltage at the i-th port, and y_i denotes voltage or current at the i-th port accordingly. Connecting the diodes will produce equations $u_i = -V_i$ and $y_i = I_i$ for voltage-controlled ports and equations $u_i = I_i$, $y_i = -V_i$ for current-controlled ports. By finally adding the ideal diode characteristics

$$V_i \leq 0, \quad I_i \geq 0, \quad V_i I_i = 0 \tag{4.9}$$

one obtains a set of equations in the semi-explicit form (4.3). Complementary-slackness systems whose modes are linear, as in this example, will be studied further in the next section.

5 LINEAR COMPLEMENTARY-SLACKNESS SYSTEMS

In this section we shall study systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) & & x(t) \in \mathbb{R}^n \\ y(t) &= Cx(t) + Du(t) & & y(t) \in \mathbb{R}^k, \;\; u(t) \in \mathbb{R}^k \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)^T u(t) = 0. & & \end{aligned} \tag{5.1}$$

This is a linear version of the semi-explicit form (4.3). Of course it would also be possible to consider a linear version of the fully implicit form (4.2), but we shall not do that here. The vector $x(t)$ will be referred to as a *state vector*.

The system (5.1) has 2^k modes. Letting $K = \{1, 2, \ldots, k\}$ as usual, each mode corresponds to a subset $I \subset K$ by the requirements $y_i = 0$ $(i \in I)$, $u_i = 0$ $(i \notin I)$. The dynamics in mode I can be characterized as follows. Define matrices $\hat{C}_I$ and $\hat{D}_I$ by

$$\begin{array}{rcll} i\text{-th row of } \hat{C}_I & = & i\text{-th row of } C & \text{if } i \in I \\ & = & 0 & \text{if } i \notin I \end{array} \tag{5.2}$$

and

$$\begin{array}{rcll} i\text{-th row of } \hat{D}_I & = & i\text{-th row of } D & \text{if } i \in I \\ & = & i\text{-th row of } I_k & \text{if } i \notin I. \end{array} \tag{5.3}$$

Also define

$$E = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \qquad F_I = \begin{bmatrix} A & B \\ \hat{C}_I & \hat{D}_I \end{bmatrix}, \qquad z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{5.4}$$

The dynamics in mode I is now given by

$$E\dot{z}(t) = F_I z(t). \tag{5.5}$$

We shall always assume in this paper that all modes are autonomous. We then *define* an (autonomous) linear complementary-slackness system by the dynamics (5.1) together with the transition rules as specified in section 4 with projections determined by the following pairs of complementary subspaces (cf. Prop. 3.1):

$$\mathcal{V}_I := \mathcal{V}(E, F_I), \qquad \mathcal{T}_I := \mathcal{T}(E, F_I). \tag{5.6}$$

The special form of E in (5.4), which is of course due to the semi-explicit nature of (5.1), suggests a description of the consistent manifold and the complementary foliation based on subspace algorithms that take place in $\mathbb{R}^n$ rather than $\mathbb{R}^{n+k}$. For each $I \subset K$, define a sequence of subspaces of $\mathbb{R}^n$ by

$$\begin{array}{rcl} V_I^0 & = & \mathbb{R}^n \\ V_I^{k+1} & = & \{x \mid \exists u : Ax + Bu \in V_I^k,\ \hat{C}_I x + \hat{D}_I u = 0\}. \end{array} \tag{5.7}$$

This is a nonincreasing sequence of subspaces which therefore must reach a limit in a finite number of steps; the limit will be denoted by V_I^*. It is not difficult to verify that

$$\mathcal{V}^*(E, F_I) = \{[\begin{smallmatrix} x \\ u \end{smallmatrix}] \mid x \in V_I^*,\ \ Ax + Bu \in V_I^*,\ \ \hat{C}_I x + \hat{D}_I u = 0\} \tag{5.8}$$

where $\mathcal{V}^*(E, F_I)$ is as defined in (3.5) and (3.7). Likewise, we have

$$\mathcal{T}^*(E, F_I) = \{[\begin{smallmatrix} x \\ u \end{smallmatrix}] \mid x \in T_I^*\} \tag{5.9}$$

where T_I^* is the limit of the nondecreasing sequence of subspaces of $\mathbb{R}^n$ defined by

$$\begin{array}{rcl} T_I^0 & = & \{0\} \\ T_I^{k+1} & = & \{x \mid \exists \tilde{u}, \exists \tilde{x} \in T_I^k : x = A\tilde{x} + B\tilde{u},\ \hat{C}_I \tilde{x} + \hat{D}_I \tilde{u} = 0\}. \end{array} \tag{5.10}$$

The significance of the subspaces V_I^* and T_I^* can be described as follows. The dynamics in mode I is given by the differential-algebraic system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ 0 & = & \hat{C}_I x(t) + \hat{D}_I u(t). \end{array} \tag{5.11}$$

An element $x_0 \in \mathbb{R}^n$ will be called a *consistent state for mode I* if there exists a smooth solution $(x(\cdot), u(\cdot))$ of the above system such that $x(0) = x_0$. The set of consistent states of mode I will be denoted by V_I. It is not difficult to verify (cf. Prop. 3.1) that $V_I = V_I^*$; note in particular that by construction of V_I^*, there exists for every $x \in V_I^*$ a vector u such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{V}_I$. Again using Prop. 3.1, one verifies also that the dynamics in mode I is autonomous if and only if V_I^* and T_I^* are complementary; for this note that all vectors of the form $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right]$ belong to $\mathcal{T}^*(E, F_I)$. In this case it follows that to each $x \in V_I^*$ there exists a *unique* u such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{V}_I$, so the condition $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{V}_I^*$ implicitly defines u as a (linear) function of x on V_I^*. Moreover, projecting a vector $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ along $\mathcal{T}_I$ onto $\mathcal{V}_I$ comes down to projecting x along T_I^* onto V_I^*, and letting u be determined by the requirement $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{V}_I$. We conclude that T_I^* plays a role similar to that of $\mathcal{T}_I$ and so we shall write simply T_I instead of T_I^*.

Already in the case of linear complementary-slackness systems, solutions may be nonunique or may fail to exist. This is shown in the examples below.

EXAMPLE 5.1 Consider the complementary-slackness system given by the following equations:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= -u(t) \\ y(t) &= x_1(t) + x_2(t) + x_3(t) \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)u(t) = 0. \end{aligned} \tag{5.12}$$

This system has two modes, one in which $u(t) = 0$ (we shall call this mode 0) and one in which $y(t) = 0$ (mode 1). The set of consistent points for mode 0 is easily seen to consist of all points $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ whose u-component vanishes, and the dynamics in this mode is given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= 0 \\ y(t) &= x_1(t) + x_2(t) + x_3(t) \\ u(t) &= 0. \end{aligned} \tag{5.13}$$

In mode 1 we have the constraint $x_1 + x_2 + x_3 = 0$ which by differentiation leads to the constraint $x_2 + x_3 - u = 0$. These two equations define the set of consistent points for mode 1, as can be verified by means of Prop. 3.1. The consistent set can for instance be parametrized by the coordinates x_1 and x_2, and in terms of these coordinates the dynamics in mode 1 is given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t) \\ y(t) &= 0 \\ u(t) &= -x_1(t). \end{aligned} \tag{5.14}$$

Now consider the initial data $(x_1(0), x_2(0), x_3(0), u(0)) = (0, 1, -1, 0)$. This point is consistent in both modes, so we have to check the inequality constraints to see in which of these modes smooth continuation is possible. In mode 0, we have $y(0) = 0$,

$$\dot{y}(0) = x_2(0) + x_3(0) = 0,$$

and

$$\ddot{y}(0) = x_3(0) = -1$$

so that, if we follow the dynamics of mode 0, $y(t)$ would be negative for $t > 0$, which is not allowed. On the other hand, mode 1 produces $u(0) = 0$ and

$$\dot{u}(0) = -x_2(t) = -1$$

which again leads to a violation of the inequality constraints. We must conclude that smooth continuation is possible in neither of the two modes. On the other hand, since the given initial condition is consistent for both modes, the situation cannot be resolved by a jump. The conclusion from this example therefore is that complementary-slackness systems (even linear ones) may exhibit *deadlock*.

EXAMPLE 5.2 Consider now the same example as above, but with the initial condition $(x_1(0), x_2(0), x_3(0), u(0)) = (0, -1, 1, 0)$, which is sign reversed with respect to the one considered above. Doing the same calculations as in the previous example, we now find that smooth continuation is possible *both* in mode 0 and in mode 1; moreover, these modes clearly produce different solutions. So whereas the previous example showed a case of *nonexistence* of solutions, here we have a case of *nonuniqueness* of solutions.

REMARK 5.3 It is of interest to consider the above examples also in reverse time. One easily calculates that from the initial condition of Example 5.1 there are *two* solutions for $t < 0$, while there is *no* solution emanating in negative time from the initial condition in Example 5.2.

So the question arises: under what conditions is a linear complementary-slackness system well-posed in the sense that there is a unique solution starting from each feasible point? We shall provide an answer to this question only for the case of systems with a single constraint ($k = 1$). We shall come back to the case of systems with multiple constraints at the end of the section. We now first present some preparatory material.

For brevity, complementary-slackness systems with a single constraint will be referred to as *bimodal* systems. With an obvious terminology, the two modes of a system of the specific form (4.3) will be referred to as the *input-constrained mode* ($u(t) = 0$) and the *output-constrained mode* ($y(t) = 0$). These modes will also be denoted as mode 0 and mode 1 respectively. A bimodal system will be said to be *degenerate* if the state/input/output trajectories of one of the modes form a subset of the state/input/output trajectories of the other mode. In other words, a degenerate bimodal system is one that does not really have two distinct modes. For an example of such a system, consider the equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= x_1(t) \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)u(t) = 0. \end{aligned} \tag{5.15}$$

The output-constrained mode allows only the zero trajectory for states, inputs, and outputs; but this trajectory is of course also allowed by the input-constrained mode.

In the following lemma we record some basic facts about linear bimodal systems.

LEMMA 5.4 *A bimodal system of the form (5.1) is autonomous if and only if the transfer function* $g(s) := D + C(sI - A)^{-1}B$ *is nonzero, or, equivalently, if the Markov parameters defined by*

$$g_0 := D, \qquad g_j := CA^{j-1}B \ \ (j \geq 1) \tag{5.16}$$

do not all vanish. Assuming this, let κ *be the 'relative degree' defined by*

$$g_j = 0 \ \ (0 \leq j < \kappa), \quad g_\kappa \neq 0. \tag{5.17}$$

We then have

$$V_0 = \mathbb{R}^n, \qquad T_0 = \{0\} \tag{5.18}$$

and

$$V_1 = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\kappa-1} \end{bmatrix}, \qquad T_1 = \operatorname{im}[\, B \;\; AB \;\; \cdots \;\; A^{\kappa-1}B \,] \tag{5.19}$$

where the indices 0 and 1 refer to the input-constrained mode and the output-constrained mode respectively. The dynamics in mode 0 and mode 1 are given by

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \qquad u(t) = 0 \tag{5.20}$$

and

$$\dot{x}(t) = (A - g_\kappa^{-1}BCA^\kappa)x(t), \qquad u(t) = -g_\kappa^{-1}CA^\kappa x(t), \qquad y(t) = 0 \tag{5.21}$$

respectively.

PROOF As is well-known, in a neighborhood of infinity we have

$$g(s) = \sum_{j=0}^{\infty} g_j s^{-j} \tag{5.22}$$

so that the transfer function is indeed nonzero if and only if not all Markov parameters vanish. First assume that this holds; then one readily verifies (5.18–5.21) and in particular it is clear that the system is autonomous. On the other hand if the transfer function is zero, then starting from the initial state $x(0) = 0$ one may apply any input while the output $y(t)$ will remain at zero for all time, and so the output-constrained mode is nonautonomous. □

The coefficient g_κ appearing in (5.17) will be referred to as the *leading Markov parameter* of the system (5.1). By a standard convention, in case $\kappa = 0$ the formulas in (5.19) mean that $V_1 = \mathbb{R}^n$ and $T_1 = \{0\}$. For $\kappa \geq 1$, we may write $A - g_\kappa^{-1}BCA^\kappa = PA$ where

$$P = I - g_\kappa^{-1}BCA^{\kappa-1} \tag{5.23}$$

is the projection along $\operatorname{im} B$ onto $\ker CA^{\kappa-1}$.

We next give a characterization of nondegeneracy for bimodal systems.

LEMMA 5.5 *An autonomous bimodal system of the form (5.1) is degenerate if and only if the set of consistent states for the output-constrained mode coincides with the set of unobservable points for the pair (C, A).*

PROOF Degeneracy will occur either when $y(\cdot) = 0$ implies $u(\cdot) = 0$ or vice versa. The latter case is a rather trivial one; it can occur only when $C = 0$ and the transfer function is a nonzero constant. In this case V_1 is the whole state space $\mathbb{R}^n$ and it does coincide with the space of unobservable points. Let us now consider the other possibility, that $y(\cdot) = 0$ implies $u(\cdot) = 0$. From (5.19) and (5.21) we see that degeneracy occurs if and only if $Cx = 0, \ldots, CA^{\kappa-1}x = 0$ implies that also $CA^\kappa x = 0$. An equivalent formulation is that $x \in V_1$ should imply that also $Ax \in V_1$. If this holds, then V_1 is an A-invariant subspace of $\ker C$ and so all points in V_1 are unobservable. Since the reverse inclusion holds in general, the proof of the lemma is complete. □

REMARK 5.6 In the situation in which the pair (C, A) is observable, the condition for nondegeneracy comes down to $V_1 \neq \{0\}$. The triviality of V_1 corresponds to the situation in which the transfer function $g(s)$ is of the form $g(s) = (p(s))^{-1}$ where $p(s)$ is a polynomial. For instance, in the example (5.15) one has $g(s) = s^{-2}$.

In the proof of the theorem below we use the 'lexicographic inequality' for vectors which is defined as follows: if $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then $x \prec y$ if there is an index $i \in \{1, \cdots, n\}$ such that $x_j = y_j$ for all $j < i$, and $x_i < y_i$. We use $x \preceq y$ for the situation in which either $x \prec y$ or $x = y$; the expressions $x \succ y$ and $x \succeq y$ have the obvious meanings. We shall also need the following lemma; the simple proof is omitted.

LEMMA 5.7 *If L is a lower triangular matrix with positive diagonal elements, then $x \succ 0$ if and only if $Lx \succ 0$ and $x \prec 0$ if and only if $Lx \prec 0$.*

In other words, the lemma states that lower triangular matrices with positive diagonal elements are lexicographically sign-preserving. We now come to the main result of this section.

THEOREM 5.8 *An autonomous bimodal system of the form (5.1) is well-posed as a closed dynamical system if its leading Markov parameter is positive. For nondegenerate systems without feedthrough ($D = 0$), this condition is also necessary.*

PROOF Let κ be the index of the leading Markov parameter. For brevity of notation, define

$$C_1 \;=\; -g_\kappa^{-1} C A^\kappa, \qquad A_1 \;=\; A - g_\kappa^{-1} B C A^\kappa. \tag{5.24}$$

Introduce the observability matrices of mode 0 and mode 1:

$$\mathcal{O}_0 \;=\; \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad \mathcal{O}_1 \;=\; \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix}. \tag{5.25}$$

Note that, by the Cayley-Hamilton theorem, $\mathcal{O}_0 x = 0$ implies $CA^j x = 0$ for all $j \geq 0$ and likewise for mode 1. The sets of consistent points of mode 0 and of mode 1 will be denoted by $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively, whereas the sets of consistent states will be written as V_1 and V_0.

Now assume that the leading Markov parameter is positive; we have to show that there exists a unique solution from each feasible point, consisting of smooth continuation after at most a finite number of jumps. We shall distinguish a number of cases, starting from a given point $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$.

1. $x \notin V_1$. This condition excludes smooth continuation in mode 1. We distinguish between two possibilities:

1.1. $\mathcal{O}_0 x \succeq 0$. Smooth continuation in mode 0 is the only option and leads to a unique solution.

1.2. $\mathcal{O}_0 x \prec 0$. In this case a jump to mode 1 must occur. After the jump we shall have a new initial point which will belong to $\mathcal{V}_1$ and so we end up in case 2 which will be treated next.

2. $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{V}_1$. Now we consider three possible situations.

2.1. $\mathcal{O}_0 x \prec 0$. Under this condition smooth continuation in mode 0 is excluded. We have to determine the lexicographic sign of $\mathcal{O}_1 x$ to see whether we can have continuation in mode 1. One easily shows by induction that, for all $j \geq 0$,

$$C_1 A_1^j \;=\; -g_\kappa^{-1} C A^{\kappa+j} + \sum_{i=0}^{j-1} \gamma_{ji} C A^{\kappa+i} \tag{5.26}$$

where the γ_{ji} are real coefficients whose exact values are irrelevant to the proof. It follows that

$$\mathcal{O}_1 x \;=\; \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} x \;=\; -g_\kappa^{-1} L \begin{bmatrix} CA^\kappa \\ CA^{\kappa+1} \\ \vdots \\ CA^{\kappa+n-1} \end{bmatrix} x \tag{5.27}$$

where L is a lower triangular matrix having 1's on its diagonal. Because $x \in V_1^*$ we have $CA^j x = 0$ for $j = 0, \ldots, \kappa - 1$, and so it follows from the above relation and Lemma 5.7, together with the assumption $g_\kappa > 0$, that $\mathcal{O}_1 x \succ 0$. Therefore smooth continuation in mode 1 is the only available option and it provides a unique solution.

2.2. $\mathcal{O}_0 x \succ 0$. By the same reasoning as above we find that $\mathcal{O}_1 x$ must be lexicographically negative so that smooth continuation in mode 1 is not possible. Distinguish two cases:

2.2.1. $u = 0$. We are in a point that is consistent for mode 0 and smooth continuation in this mode provides the unique solution.

2.2.2. $u \neq 0$. In this case we must jump along $\mathcal{T}_0$ to $\mathcal{V}_0$. Because $\mathcal{T}_0$ consists of all vectors of the form $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right]$, the jump does not affect the value of x and we end up in case 2.2.1.

2.3. $\mathcal{O}_0 x = 0$. In this case it follows from the relation (5.27) that also $\mathcal{O}_1 x = 0$ and so smooth continuation is possible *both* in mode 0 and in mode 1. Note however that $\ker \mathcal{O}_0 \cap \ker \mathcal{O}_1 = \ker \mathcal{O}_0$ is both A- and A_1-invariant, and that the restrictions of A and A_1 to this subspace are identical. This implies that, for initial points in $\ker \mathcal{O}_0$, the trajectory according to mode 0 as produced by (5.20) coincides with the trajectory according to mode 1 following (5.21). We conclude that even in this case there is a unique solution.

This concludes the sufficiency part of the proof. For the necessity part, assume that the leading Markov parameter is negative. By the assumption of nondegeneracy and by Lemma 5.4 there exists a vector $x \in V_1$ such that $\mathcal{O}_0 x \neq 0$, and so (by changing sign if necessary) we can find a vector $x \in V_1$ such that $\mathcal{O}_0 x \prec 0$. Because $g_\kappa < 0$, the reasoning applied under case 2.1 of the sufficiency part implies here that also $\mathcal{O}_1 x \prec 0$. Suppose now that we consider the initial point $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}_0$; this point is feasible for mode 0 because $x \in V_1 \subset \ker C$ by the assumption $D = 0$. Projecting the given point to $\mathcal{V}_1$ and back will only change u_0 to $-g_\kappa^{-1} CA^\kappa x$ and back to 0, whereas the state component will remain the same. Because both $\mathcal{O}_0 x$ and $\mathcal{O}_1 x$ are lexicographically negative, smooth continuation never becomes possible. □

REMARK 5.9 An example of a nondegenerate bimodal system that is well-posed in the sense that we defined although its leading Markov parameter is negative is provided by the following equations: $\dot{x}(t) = x(t) + u(t)$, $y(t) = x(t) - u(t)$.

EXAMPLE 5.10 Consider again the electrical network equations of Example 4.3. If the representation is minimal and the network is passive, then there exists a symmetric matrix $Q > 0$ such that

$$\begin{bmatrix} A^T Q + QA & QB - C^T \\ B^T Q - C & -(D + D^T) \end{bmatrix} \leq 0 \tag{5.28}$$

(see [20]). Consider now a passive network with one diode, so that in particular D is a scalar. It follows from the above equation that $D \geq 0$. Moreover, if $D = 0$ the equation implies that $C = B^T Q$ and so in this case we have $CB = B^T QB > 0$. In either case the leading Markov parameter is positive and hence the above theorem guarantees that the complementary-slackness equations are well-posed. Clearly this is only a partly satisfactory result since one should be able to prove that also passive networks with more than one diode give rise to well-posed equations.

The technique of the above proof can in principle be applied to the analysis of systems with more than one constraint, but the organization of the multitude of cases is a major problem that will not be attacked here. Instead we prove another proposition for general systems of the form (5.1), which gives a necessary and sufficient condition for the effective state space dimension to be the same in all modes. Recall that the *principal minors* of a square matrix M are the determinants of the submatrices of M that are obtained by selecting rows and columns with the same indices [9, p. 2].

PROPOSITION 5.11 *Consider a complementary-slackness system given by (5.1). We have $V_I^* = \mathbb{R}^n$ for all $I \subset K$ (i. e. all states are consistent in all modes) if and only if all principal minors of the matrix D are nonzero.*

PROOF It follows immediately from (5.7) that $V_I^* = \mathbb{R}^n$ if and only if the matrix $\hat{D}_I$ defined in (5.3) is nonsingular. Requiring that $\hat{D}_I$ is nonsingular for all $I \subset K$ is, by a standard matrix argument, the same as requiring that all principal minors of D are nonzero. □

Comparing this proposition with Lemma 5.4 and Thm. 5.8, one may conjecture (rather boldly) that an autonomous and nondegenerate linear complementary-slackness system of the form (5.1) will be well-posed if all principal minors of the transfer matrix $G(s) = C(sI - A)^{-1}B + D$ have a positive leading Markov parameter.

6 HAMILTONIAN COMPLEMENTARY-SLACKNESS SYSTEMS

In this section we consider complementary-slackness systems arising from Hamiltonian systems with geometric inequality constraints. Consider a conservative mechanical system, with n degrees of freedom $q_1, \cdots, q_n$, and total energy

$$\tfrac{1}{2}\dot{q}^T M(q)\dot{q} + V(q), \quad M(q) > 0 \tag{6.1}$$

where $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy corresponding the generalized mass matrix $M(q)$, and where $V(q)$ denotes the potential energy. In general, $q = (q_1, \cdots, q_n)$ gives local coordinates for an n-dimensional configuration manifold Q. The Hamiltonian equations of motion are obtained by defining the generalized momenta

$$p = M(q)\dot{q} \tag{6.2}$$

and are given as (with $\frac{\partial H}{\partial p}$ and $\frac{\partial H}{\partial q}$ denoting column vectors of partial derivatives)

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q,p) \end{aligned} \tag{6.3}$$

where the Hamiltonian $H(q,p)$ is the total energy expressed in the state variables $x = (q,p)$, i.e.

$$H(q,p) = \tfrac{1}{2}p^T P(q)p + V(q), \quad P(q) = M^{-1}(q) > 0. \tag{6.4}$$

The vector $x = (q,p)$ gives canonical local coordinates for the cotangent bundle T^*Q, with the classical Poisson bracket (F and G being functions on T^*Q)

$$\{F(q,p), G(q,p)\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} \right)(q,p). \tag{6.5}$$

(Coordinates are called 'canonical' if $\{p_i, q_j\} = \delta_{ij}$, $\{p_i, p_j\} = \{q_i, q_j\} = 0, i, j = 1, \cdots, n$.) The Hamiltonian equations of motion (6.3) on $\mathcal{X} = T^*Q$ will also be succinctly written as

$$\dot{x} = X_H(x), \quad x = (q,p) \in T^*Q \tag{6.6}$$

where X_H is called a Hamiltonian vectorfield on T^*Q. For any function $F : T^*Q \to \mathbb{R}$ we have the following identity concerning the derivative of F along the Hamiltonian dynamics (6.6):

$$\frac{dF}{dt} = L_{X_H}F = \{H, F\} \tag{6.7}$$

where L_{X_H} denotes the Lie derivative along X_H.

Now suppose k geometric constraints are imposed on the system, that is

$$C(q) = 0, \quad C : Q \to \mathbb{R}^k. \tag{6.8}$$

Then the constrained mechanical system is described by the constrained Hamiltonian equations of motion, see e.g. [16]:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \frac{\partial C}{\partial q}(q)\lambda, \quad \lambda \in \mathbb{R}^k \\ C(q) &= 0 \end{aligned} \tag{6.9}$$

where $\lambda(t) \in \mathbb{R}^k$ are the *constraint forces* needed to satisfy the geometric constraints $C(q(t)) = 0$ for all t (or the constraint forces *resulting* from imposing the geometric constraints).

By considering geometric *inequality* constraints $C_i(q) \geq 0, i = 1, \cdots, k$ (as in the first example considered in section 4) this leads to the equations of a *Hamiltonian complementary-slackness system*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \frac{\partial C}{\partial q}(q)\lambda \\ y &= C(q) \\ u &= \lambda \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y^T(t)u(t) = 0 \end{aligned} \tag{6.10}$$

where the inequalities are understood componentwise as in (4.2). In fact, the inequalities $u_i = \lambda_i \geq 0,\ i \in K$, have the following physical interpretation. The constraint force λ_i is the generalized force corresponding to the generalized velocity $\dot{y}_i$. Since in the case of inequality constraints $y_i \geq 0$ the constraint forces will be always pushing in the direction of rendering y_i non-negative this implies that $\lambda_i \geq 0$. For brevity, see (6.6), we will also write (6.10) as

$$\begin{aligned} \dot{x} &= X_H(x) - X_C(q)\lambda, \quad x = (q,p) \in T^*Q \\ y &= C(q) \\ u &= \lambda \\ y &\geq 0, \quad u \geq 0, \quad y^T \cdot u = 0. \end{aligned} \tag{6.11}$$

Also for brevity we will denote for every $I \subset K$ the vector with components $y_i, i \in I$, by y_I, and the map with components C_i for $i \in I$, by C_I. Furthermore, we will denote the vector with components $\lambda_i, i \in I$, by λ_I.

Throughout we will assume that the geometric constraints $C_i(q), i \in K$, are *independent*, i.e. the following assumption holds.

ASSUMPTION 6.1 We have

$$\text{rank}\ \frac{\partial C_I}{\partial q}(q) = |I| \tag{6.12}$$

for all q with $C_I(q) = 0$.

(Note that (6.12) is implied by, but not equivalent to, the condition rank $\frac{\partial C}{\partial q}(q) = k$ for all q.)

Each subset I of $K = \{1, \cdots, k\}$ corresponds to a "mode" of the Hamiltonian complementary-slackness system, given by (6.10) with $y_i = 0, i \in I$, and $u_i = 0, i \in K \backslash I$. So for any $I \subset K$ we can consider the constrained system

$$\begin{aligned} \dot{x} &= X_H(x) - X_{C_I}(q)\lambda_I \\ 0 = y_I &= C_I(q). \end{aligned} \tag{6.13}$$

The set of consistent states and the corresponding constraint forces are computed by differentiating y_I along (6.13):

$$\begin{aligned} y_I &= C_I(q) = 0 \\ \dot{y}_I &= L_{X_H - X_{C_I}\lambda_I}(C_I) = \{H, C_I\}(q,p) = 0 \\ \ddot{y}_I &= L_{X_H - X_{C_I}\lambda_I}\{H, C_I\} = \{H, \{H, C_I\}\}(q,p) - \{C_I, \{H, C_I\}\}\lambda_I = 0. \end{aligned} \tag{6.14}$$

By Assumption 6.1 and the fact that $P(q) > 0$, we have

$$-\{C_I, \{H, C_I\}\}(q,p) = \frac{\partial^T C_I}{\partial q}(q) P(q) \frac{\partial C_I}{\partial q}(q) =: R_I(q) > 0 \tag{6.15}$$

for all q with $C_I(q) = 0$. So the constrained state space (set of consistent states) for mode I is given as

$$V_I = \{(q,p) \in T^*Q \mid C_I(q) = 0, \{H, C_I\}(q,p) = \frac{\partial^T C_I}{\partial q}(q) P(q) p = 0\} \tag{6.16}$$

and the following constraint force, uniquely determined as

$$\lambda_I^c(q,p) := -R_I^{-1}(q)\{H, \{H, C_I\}\}(q,p), \tag{6.17}$$

renders V_I invariant (since $\ddot{y}_I$ is kept equal to zero). (Note that for $I = \emptyset$ we have $V_\emptyset = \mathcal{X} = T^*Q$ and $\lambda_\emptyset^c = 0$.) We thus arrive at the following conclusion.

PROPOSITION 6.2 *Let Assumption 6.1 be satisfied. Then for all $I \subset K$ the differential-algebraic system (6.13) is autonomous and*

$$\mathcal{V}_I = \{(q,p,\lambda) \mid (q,p) \in V_I, \quad \lambda_{K\setminus I} = 0, \quad \lambda_I = \lambda_I^c(q,p)\} \tag{6.18}$$

$$\mathcal{W}_I = \{(q,p,\lambda) \in \mathcal{V}_I \mid C_{K\setminus I} \geq 0, \quad \lambda_I^c(q,p) \geq 0\} \tag{6.19}$$

where V_I is given by (6.16) and $\lambda_I^c(q,p)$ by (6.17).

EXAMPLE 6.3 Consider a pendulum with massless rope of length 1, with a unit mass attached to the end of the rope. Set $g = 1$. The Hamiltonian (total energy) is

$$H(q_1, q_2, p_1, p_2) = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + q_2 \tag{6.20}$$

where $(q_1, q_2) \in Q = \mathbb{R}^2$ are the cartesian coordinates of the unit mass. The (single) geometric inequality constraint is

$$C(q) = 1 - q_1^2 - q_2^2 \geq 0. \tag{6.21}$$

Thus I is either $\emptyset$ or the singleton $K = \{1\}$. Trivially $V_\emptyset = T^*Q$. Clearly, Assumption 6.1 is satisfied, implying that

$$V_K = \{(q_1, q_2, p_1, p_2) \mid q_1^2 + q_2^2 = 1, \quad p_1 q_1 + p_2 q_2 = 0\}. \tag{6.22}$$

Furthermore, for all (q_1, q_2) such that $C(q) = 0$ we have $R_K(q) = 4(q_1^2 + q_2^2) = 4 > 0$. Finally, the constraint force which renders V_K invariant is given as

$$\lambda_K^c = \tfrac{1}{2}(p_1^2 + p_2^2 - q_2). \tag{6.23}$$

□

It is an important fact (provided Assumption 6.1 holds) that the dynamics of the autonomous differential algebraic systems (6.13) is *Hamiltonian* for every $I \subset K$. So not only the mode corresponding to $I = \emptyset$ is Hamiltonian, but every other mode is Hamiltonian as well. Indeed we recall, for example from [18, Ch. 12], that the dynamics (6.13) for any I is also given as the dynamics

$$\dot{x} = \{H - \lambda_I^c C_I, x\} \tag{6.24}$$

restricted to V_I; in fact, the Hamiltonian dynamics (6.24) leaves V_I invariant. Furthermore, the dynamics (6.13) for $I \neq \emptyset$ on V_I is also given as the Hamiltonian dynamics

$$\dot{x}_I = \{H_I, x_I\}_I, \quad x_I \in V_I, \tag{6.25}$$

where x_I are local coordinates for V_I, $H_I : V_I \to \mathbb{R}$ is the Hamiltonian $H : T^*Q \to \mathbb{R}$ *restricted* to V_I, and $\{\,,\,\}_I$ is the so-called *Dirac bracket* on V_I, given as

$$\{F,G\}_I = \{F,G\} + \{F,C_I\}R_I^{-1}(q)\{\{H,C_I\},G\} - \{F,\{H,C_I\}\}R_I^{-1}(q)\{C_I,G\} \tag{6.26}$$

for any functions $F, G : V_I \to \mathbb{R}$, arbitrarily extended to functions on T^*Q. (Note that if we define for completeness $\{\,,\,\}_\emptyset$ as the original Poisson bracket on $V_\emptyset = T^*Q$, then in fact (6.25) describes the Hamiltonian mode I for *every* subset $I \subset K$.)

Later on we will recall that for every mode I the submanifold V_I can be identified with the cotangent bundle T^*Q_I, where $Q_I \subset Q$ is the constrained configuration space, and the Dirac bracket $\{\,,\,\}_I$ with the natural Poisson bracket on T^*Q_I.

EXAMPLE 6.3 (continued). The Dirac bracket on V_K is given as (F and G being arbitrary smooth functions on V_K)

$$\begin{aligned} \{F,G\}_K = \{F,G\} &+ \tfrac{1}{4}\{F, 1 - q_1^2 - q_2^2\}\{-2q_1p_1 - 2q_2p_2, G\} \\ &- \tfrac{1}{4}\{F, -2q_1p_1 - 2q_2p_2\}\{1 - q_1^2 - q_2^2, G\}. \end{aligned} \tag{6.27}$$

Later on, see (6.78), we will compute canonical coordinates with regard to the Dirac bracket as

$$q := \arctan \frac{q_2}{q_1}, \quad p := q_1p_2 - q_2p_1. \tag{6.28}$$

The constrained dynamics in these coordinates is then given as

$$\begin{aligned} \dot{q} &= \{H_K, q\}_K = \tfrac{\partial H_K}{\partial p} \\ \dot{p} &= \{H_K, p\}_K = -\tfrac{\partial H_K}{\partial q} \end{aligned} \tag{6.29}$$

where

$$H_K(q,p) = \tfrac{1}{2}p^2 + \sin q, \tag{6.30}$$

which are the Hamiltonian equations of motion for the mathematical pendulum with *rigid* link. □

Now we come to the full description of the dynamics of the Hamiltonian complementary-slackness system as a hybrid system. The Hamiltonian modes (6.13) will correspond to the nodes of the transition graph of the hybrid system; and we have to specify the transition rules as in section 4. So, consider a point $(q,p,\lambda) \in \mathcal{W}_I$. By Proposition 6.2, it is possible to integrate forward from this point in $\mathcal{V}_I$ according to the Hamiltonian dynamics of mode I. Denote the point reached from (q,p,λ) after time t by $(q(t), p(t), \lambda(t))$. Performing the same analysis as in section 4 we obtain for every jump point $(q,p,\lambda) \in \mathcal{W}_I$ the critical index set

$$\Gamma(q,p,\lambda;I) = \Gamma_1(q,p,\lambda;I) \cup \Gamma_2(q,p,\lambda;I) \tag{6.31}$$

with

$$\begin{aligned} \Gamma_1(q,p,\lambda;I) &= \{i \in K\backslash I \mid \exists \varepsilon > 0 \text{ s.t. } C_i(q(t)) < 0 \text{ for all } t \in (0,\varepsilon)\} \\ \Gamma_2(q,p,\lambda;I) &= \{i \in I \mid \exists \varepsilon > 0 \text{ s.t. } \lambda_{I,i}^c(q(t),p(t)) < 0 \text{ for all } t \in (0,\varepsilon)\} \end{aligned} \tag{6.32}$$

where $\lambda^c_{I,i}$ denotes the component of λ^c_I corresponding to $i \in I$, with λ^c_I given by (6.17). We define the new index set

$$J := (I \setminus \Gamma_2(q,p,\lambda;I)) \cup \Gamma_1(q,p,\lambda;I) \tag{6.33}$$

as in (4.7). Note that $\Gamma_1(q,p,\lambda;I)$ can be interpreted as the set of those indices for which the inequality constraints are going to be violated, while on the other hand $\Gamma_2(q,p,\lambda;I)$ denotes the indices corresponding to active (equality) constraints for which the constraint forces are going to be negative.

Before specifying the projection rule to $\mathcal{V}_J$, we will describe the projection rule to V_J, the set of consistent *states*. This projection will be specified by the distribution $\bar{D}_J$ on T^*Q spanned by the constraint force vector fields, that is

$$\bar{D}_J(q,p) := \text{span}\,\{X_{C_j}(q,p) \mid j \in J\}, \quad (q,p) \in T^*Q \tag{6.34}$$

Since the Lie bracket $[X_{C_i}, X_{C_j}]$ is zero for every $i,j \in K$, the distribution $\bar{D}_J$ is *involutive*, and moreover, because of the standing Assumption 6.1, the distribution $\bar{D}_J$ has constant dimension equal to $|J|$ on the submanifold

$$\ker C_J := \{(q,p) \in T^*Q \mid C_j(q) = 0, \;\; j \in J\}. \tag{6.35}$$

Since $L_{X_{C_i}} C_j = 0$ for every $i,j \in K$, it follows that we may *restrict* the distribution $\bar{D}_J$ to a distribution D_J on $\ker C_J$, which is involutive and of constant dimension. By Frobenius' theorem (see e.g. [18]) D_J therefore defines a *foliation* of $\ker C_J$, with leaves given by the integral manifolds of D_J, and the projection rule to V_J is *to project* (q,p) *along this foliation to a point* $(\bar{q},\bar{p}) \in V_J$. Actually this projection turns out to be linear, as stated in the following proposition.

PROPOSITION 6.4 *Let* $J \subset K$. *Then for every* $(q,p) \in \ker C_J$

$$D_J(q,p) = \text{span}\left\{\frac{\partial C_j}{\partial q}(q) \mid j \in J\right\}, \tag{6.36}$$

and consequently $D_J(q,p)$ *does not depend on* p. *Moreover, projection along the foliation corresponding to* D_J *is the linear projection along the linear subspace of* T^*_qQ *given by the right hand side of (6.36). Furthermore, for every* $(q,p) \in V_J$

$$T_{(q,p)}(\ker C_J) = T_{(q,p)}V_J \oplus D_J(q,p), \tag{6.37}$$

and so every $(q,p) \in \ker C_J$ *projects along the foliation corresponding to* D_J *to a unique point* $(q, \pi^q_J(p)) \in V_J$, *where* $\pi^q_J : T^*_qQ \to T^*_qQ$ *denotes the linear projection along the right hand side of (6.36) onto the subspace* $\{p \in T^*_qQ \mid \frac{\partial^T C_J}{\partial q}(q)P(q)p = 0\} \subset T^*_qQ$.

The above projection rule to V_J yields a well-defined projection rule to $\mathcal{V}_J$ as follows. Let $(q,p,\lambda) \in \mathcal{W}_I$ be a jump point leading to the new index set J. Necessarily $(q,p) \in \ker C_J$, and by Proposition 6.4 (q,p) is projected in a unique manner to $(q,\pi^q_J(p)) \in V_J$. Then (q,p,λ) will be projected to the point $(q,\pi^q_J(p),\bar{\lambda}) \in \mathcal{V}_J$, where $\bar{\lambda}_{K\setminus J} := 0$ and $\bar{\lambda}_J := \lambda^c_J(q,\pi^q_J(p))$. (In particular, if $\Gamma_1(q,p,\lambda;I)$ is empty, then the projection will only consist of setting $\bar{\lambda}_{K\setminus J} := 0$ and $\bar{\lambda}_J := \lambda^c_J(q,\pi^q_J(p))$.) This projection rule to $\mathcal{V}_J$ corresponds to the foliation of $\ker \mathcal{C}_J \subset \{(q,p,\lambda) \mid (q,p) \in T^*Q, \;\lambda \in \mathbb{R}^k\}$, with $\mathcal{C}_J(q,p,\lambda) := C_J(q)$, given by the leaves of the product distribution $D_J \times \text{span}\,\{\frac{\partial}{\partial\lambda_1}, \cdots, \frac{\partial}{\partial\lambda_k}\}$ on (q,p,λ)-space. This foliation plays a role analogous to that of $\mathcal{T}_J$ in the linear case; although actually only $\ker \mathcal{C}_J$ is foliated rather than the entire (q,p,λ)-space, this is enough since jumps to $\mathcal{V}_J$ will only occur from points already in $\ker \mathcal{C}_J$.

The projection rule for Hamiltonian complementary-slackness systems has the following important property, which we call the *energy inequality*.

Proposition 6.5 *Let $(q,p,\lambda) \in \mathcal{W}_I$ be a jump point leading to the new index set J. Then*

$$H(q, \pi_J^q(p)) \leq H(q,p). \tag{6.38}$$

Proof By (6.16) the tangent space to V_J at a point $(q,p) \in V_J$ may be identified with the linear space

$$\ker \frac{\partial^T C_J}{\partial q}(q) \times \ker \frac{\partial^T C_J}{\partial q}(q) P(q) \subset T_q Q \times T_q^* Q \tag{6.39}$$

Clearly by (6.36), the second linear space on the left hand side, that is $\ker \frac{\partial^T C_J}{\partial q}(q)P(q)$, is orthogonal to $D_J(q,p)$ (interpreted as a linear subspace of T_q^*Q) with regard to the inner product determined by $P(q)$ on T_q^*Q. Therefore

$$\tfrac{1}{2}\left[\pi_J^q(p)\right]^T P(q)\pi_J^q(p) \leq \tfrac{1}{2} p^T P(q) p, \tag{6.40}$$

and (6.38) results. □

Remark 6.6 Note that since a jump point $(q,p,\lambda) \in \mathcal{W}_I$ already satisfies $(q,p) \in V_I$, the projection of (q,p) along $D_J(q,p)$ onto V_J is actually performed along the smaller subspace (cf. (6.33))

$$\text{span}\left\{ \frac{\partial C_i}{\partial q}(q) \mid i \in \Gamma_1(q,p,\lambda;I) \right\}. \tag{6.41}$$

Furthermore, by the definition of a jump point (q,p,λ)

$$\dot{y}_i(0) = \{H, C_i\}(q,p) \leq 0, \quad i \in J,$$

and so the momentum coordinates of a jump point (q,p,λ) necessarily satisfy

$$\frac{\partial^T C_J}{\partial q}(q) P(q) p \geq 0. \tag{6.42}$$

Example 6.3 (continued). Suppose the system is in mode $I = \emptyset$ (no active constraint and hence no constraint force, i.e., $\lambda = 0$). Let (q,p,λ) with $\lambda = 0$ be a jump point. Necessarily $C(q) = 0$. Project (q_1, q_2, p_1, p_2) along the forward integral curves of $-X_C(q) = \text{col}\,(0, 0, -2q_1, -2q_2)$ to V_K given by (6.22). This is done by computing τ satisfying

$$p_1(\tau) q_1 + p_2(\tau) q_2 = 0, \tag{6.43}$$

where

$$\begin{aligned} p_1(\tau) &= p_1 - \tau \cdot 2q_1 \\ p_2(\tau) &= p_2 - \tau \cdot 2q_2. \end{aligned} \tag{6.44}$$

Since $C(q) = 1 - q_1^2 - q_2^2 = 0$, this yields $\tau = \frac{1}{2}(p_1 q_1 + p_2 q_2)$, and hence

$$\pi_K^q(p_1, p_2) = (p_1 - q_1(p_1 q_1 + p_2 q_2), p_2 - q_2(p_1 q_1 + p_2 q_2)). \tag{6.45}$$

Suppose on the other hand that the system is in mode $I = K$ (that is, the constraint is active). Let $(q, p, \lambda_K^c(q,p))$ be a jump point, implying that (see (6.32) and (6.23))

$$\lambda_K^c(q(t), p(t)) = \tfrac{1}{2}(p_1^2(t) + p_2^2(t) - q_2(t)) < 0, \text{ for all } t \in (0, \varepsilon). \tag{6.46}$$

and $\lambda^c_K(q(0),p(0)) = 0$. Then the system continues in mode $I = \emptyset$. Physically, the state will leave the constrained state space V_K, and $C(q)$ will become positive—the rope will not be fully stretched anymore. □

REMARK 6.7 The projection rule given for Hamiltonian complementary-slackness systems agrees with the rule given in section 5 for *linear* complementary-slackness systems in cases where both are applicable, i.e. for linear Hamiltonian systems. Indeed for a such a system given by equations

$$\begin{aligned} \begin{bmatrix} \dot q \\ \dot p \end{bmatrix} &= \begin{bmatrix} 0 & P \\ -Q & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ C^T \end{bmatrix} u, \quad P = P^T > 0,\ Q = Q^T \\ y &= Cq, \qquad \operatorname{rank} C = \dim y \end{aligned} \tag{6.47}$$

the subspaces T_I and V_I corresponding to the constraints $C_I q = 0$ are easily seen to be equal to

$$T_I = \operatorname{im} \begin{bmatrix} 0 & PC_I^T \\ C_I^T & 0 \end{bmatrix} \tag{6.48}$$

$$V_I = \ker \begin{bmatrix} C_I & 0 \\ 0 & C_I P \end{bmatrix} \tag{6.49}$$

On the other hand, given a point (q,p) satisfying the constraints $C_I q = 0$, projection along T_I onto V_I will really be only a projection along the subspace

$$\operatorname{span} \begin{bmatrix} 0 \\ C_I^T \end{bmatrix} \tag{6.50}$$

(since $C_I P C_I^T > 0$), which corresponds exactly to the distribution D_I (see (6.34)).

REMARK 6.8 Note that for a *nonlinear* Hamiltonian system

$$\begin{aligned} \dot x &= X_H(x) - \sum_{j=1}^m X_{C_j}(q) u_j, \quad x = (q,p) \\ y_j &= C_j(q), \quad j = 1, \cdots, m \end{aligned} \tag{6.51}$$

the analog of the *whole* subspace T_I would be the *distribution*

$$T_I(x) := \operatorname{span}\{X_{C_i}(x), [X_H, X_{C_i}](x) = X_{\{H,C_i\}}(x),\ i \in I\} \tag{6.52}$$

(which in fact is known as the *minimal conditioned invariant distribution* containing the input vector fields $X_{C_i}, i \in I$, cf. [15]). However, in general (without imposing extra conditions on the matrix $R_I(q)$) this distribution T_I will *not* be involutive, in which case it is *not* possible to integrate it to a foliation of the whole of T^*Q. As already noted above, the nonlinear analog of the subspace $\mathcal{T}_I$ for a nonlinear Hamiltonian system (6.51) is the following product distribution on (q,p,u)-space:

$$D_I \times \operatorname{span}\left\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_k}\right\} \tag{6.53}$$

which is complementary to $\mathcal{V}_J$ in $\ker \mathcal{C}_J$.

REMARK 6.9 The projection rule given for Hamiltonian complementary-slackness systems corresponds mechanically to the idealized situation in which the mechanical stops corresponding to the geometric inequality constraints are all assumed to be purely non-elastic; that is, there is no bouncing back. In principle, elasticity of (some of) the boundaries could be modelled by projecting $p \in T^*Q$ along $D_J(q,p)$ not *onto* the plane $\ker \frac{\partial^T C_J}{\partial q}(q)P(q)$ but instead to some point on the other side of this plane, for example to its mirror point if the boundaries are perfectly elastic (see e.g. [8]). Note that this implies that, although the mode J plays a role in determining the direction of the projection, the jump is not to mode J but instead to the originating mode I.

Recall that the set of feasible points for a complementary-slackness system with index set K is given as $\bigcup_{I \subset K} \mathcal{W}_I$. In section 5 it has been shown that *linear* complementary-slackness systems with a *scalar* constraint are well-posed, in the sense that there are *unique* solutions starting from each feasible point, provided a certain positivity condition is satisfied. For Hamiltonian complementary-slackness systems with a scalar constraint we derive the following nonlinear analog.

THEOREM 6.10 *Consider the Hamiltonian complementary-slackness system with scalar constraint*

$$\begin{aligned} \dot{x} &= X_H(x) - X_C(q)\lambda, \quad x = (q,p) \in T^*Q \\ y &= C(q), \\ u &= \lambda \\ y(t) &\geq 0, \quad u(t) \geq 0, \quad y(t)u(t) = 0, \quad u, y \in \mathbb{R} \end{aligned} \tag{6.54}$$

with H as in (6.4). Let Assumption 6.1 be satisfied, i. e. rank $\frac{\partial C}{\partial q}(q) = 1$ *for every q with $C(q) = 0$, and assume that H and C are real-analytic functions on T^*Q. Then the system is well-posed, in the sense that there are unique solutions starting from each feasible point.*

PROOF Denote for simplicity $\mathcal{V}_{\{1\}} = \mathcal{V}_1, \mathcal{W}_{\{1\}} = \mathcal{W}_1, V_{\{1\}} = V_1, R_{\{1\}} = R, \pi_{\{1\}} = \pi$, and $\lambda^c_{\{1\}} = \lambda^c$. We shall refer to the mode corresponding to $I = \emptyset$, that is

$$\dot{x} = X_H(x), \quad x = (q,p) \in T^*Q, \tag{6.55}$$

as *mode 0*. The mode corresponding to $I = \{1\}$, that is

$$\dot{x} = X_H(x) - X_C(x)\lambda^c(x), \quad x \in V_1, \tag{6.56}$$

with

$$\lambda^c(q,p) = -R^{-1}(q)\{H,\{H,C\}\}(x), \tag{6.57}$$

and $R(q) = \frac{\partial^T C}{\partial q}(q)P(q)\frac{\partial C}{\partial q}(q)$, $V_1 = \{(q,p) \in T^*Q \mid C(q) = 0, \{H,C\}(q,p) = 0\}$, will be called *mode 1.*

Now consider an arbitrary point $(q_0,p_0) \in T^*Q$. Denote solutions $(q(t),p(t))$ starting from (q_0,p_0) at $t = 0$ corresponding to mode 0 as $(q(t;0),p(t;0))$, and solutions corresponding to mode 1 (when existing) as $(q(t;1),p(t;1))$. Recall the last equation of (6.14), that is

$$\ddot{y} = \{H,\{H,C\}\}(q,p) + R(q)\lambda. \tag{6.58}$$

Obviously, if $C(q_0) < 0$, then (q_0,p_0) is not feasible. On the other hand, if $C(q_0) > 0$, then there is unique smooth continuation in mode 0.

Thus, let $C(q_0) = 0$. Then there are two possibilities (compare the proof of Theorem 5.8):

(1) $(q_0,p_0) \notin V_1$

(2) $(q_0,p_0) \in V_1$.

For possibility (1) there are two sub-cases.

(1.1) The solution $(q(t;0),p(t;0))$ starting from (q_0,p_0) satisfies $C(q(t;0)) \geq 0$ for t small. In this case, there is unique smooth continuation in mode 0.

(1.2) $\exists \varepsilon > 0$ such that $C(q(t;0),0) < 0$, for all $t \in (0,\varepsilon)$.

In the latter case (1.2) a jump to mode 1 must occur and (q_0,p_0) is projected to $(q_0,\pi_1^{q_0}(p_0)) \in V_1$, and we end up in possibility (2).

So, let $(q_0,p_0) \in V_1$. Then there are three possibilities:

(2.1) $\lambda^c(q_0, p_0) > 0$. Then there is smooth continuation in mode 1, while there is no smooth continuation in mode 0.

(2.2) $\lambda^c(q_0, p_0) < 0$. Then there is no smooth continuation in mode 1. In this case we must jump to mode 0 by setting $\lambda = 0$. Because of (6.57) and positivity of $R(q)$ we have $\{H, \{H, C\}\}(q_0, p_0) > 0$, and so by (6.58) with $\lambda = 0$ we conclude $\ddot{y}(0) > 0$, and hence $C(q(t; 0)) > 0$ for t small, implying smooth continuation in mode 0.

(2.3) $\lambda^c(q_0, p_0) = 0$. This is the hard case. By definition of $\lambda^c(q, p)$:

$$\{H, \{H, C\}\}(q(t; 1), p(t; 1)) + R(q(t; 1))\lambda^c(q(t; 1), p(t; 1)) = 0 \tag{6.59}$$

for all t. Differentiation of (6.59) to t yields by (6.24)

$$\{H - \lambda^c C, \{H, \{H, C\}\}\} + \{H - \lambda^c C, R\}\lambda^c + R\{H - \lambda^c C, \lambda^c\} = 0 \tag{6.60}$$

where everything is evaluated in $(q(t; 1), p(t; 1))$. Substituting $t = 0$, that is $q(0; 1) = q_0$, $p(0; 1) = p_0$, writing out the Poisson brackets, and using $C(q_0) = 0$, $\lambda^c(q_0, p_0) = 0$, one obtains

$$\{H, \{H, \{H, C\}\}\}(q_0, p_0) + R(q_0)\{H - \lambda^c C, \lambda^c\}(q_0, p_0) = 0. \tag{6.61}$$

Now suppose $\{H, \lambda^c\}(q_0, p_0) = \{H - \lambda^c C, \lambda^c\}(q_0, p_0) > 0$. Then $\lambda(q(t; 1), p(t; 1)) > 0$ for t small, and there is continuation in mode 1. Furthermore, since $\{H, \{H, \{H, C\}\}\} = y^{(3)}$ in mode 0, it follows by (6.61) that in this case $y^{(3)}(0) < 0$ and thus $y(t) < 0$, for t small in mode 0. So the only continuation is in mode 1.

On the other hand, suppose $\{H, \lambda^c\}(q_0, p_0) < 0$, then $\lambda(q(t; 1), p(t; 1)) < 0$ for t small, and so no continuation in mode 1 is possible. Moreover in this case by (6.61) $y^{(3)}(0) > 0$, and therefore $y(t) > 0$, for t small in mode 0. As a consequence the *only* continuation is in mode 0.

Therefore, suppose

$$\{H - \lambda^c C, \lambda^c\}(q_0, p_0) = 0 = \{H, \{H, \{H, C\}\}\}(q_0, p_0). \tag{6.62}$$

The idea is now to compute the *second* time-derivative of (6.59), or, what is the same, the first time-derivative of (6.60):

$$\begin{aligned} &\{H - \lambda^c C, \{H - \lambda^c C, \{H, C\}\}\}\} + \\ &+ \{H - \lambda^c C, \{H - \lambda^c C, R\}\}\lambda^c + 2\{H - \lambda^c C, R\}\{H - \lambda^c C, \lambda^c\} + \\ &+ R\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\} \quad = \quad 0 \end{aligned} \tag{6.63}$$

and to evaluate this expression in $t = 0$, yielding, since $C(q_0) = 0$, $\lambda^c(q_0, p_0) = 0$ and because of (6.62)

$$\{H, \{H, \{H, \{H, C\}\}\}\}(q_0, p_0) + R(q_0)\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) = 0. \tag{6.64}$$

The reasoning now follows the same lines as above. Suppose $\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) > 0$. Then $\lambda(q(t; 1), p(t; 1)) > 0$ for t small. Furthermore, since

$$y^{(4)}(0) = \{H, \{H, \{H, \{H, C\}\}\}\}(q_0, p_0) < 0,$$

we have $y(t) < 0$ for t small in mode 0. Therefore the only continuation is in mode 1.

On the other hand, suppose $\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) < 0$, then $\lambda(q(t; 1), p(t; 1)) < 0$ for t small, while $y(t) > 0$ for t small in mode 0. So the only continuation is in mode 0.

Therefore we are left with the case that apart from $C(q_0) = 0, \lambda^c(q_0, p_0) = 0$, not only (6.62) holds, but also

$$\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) = 0 = \{H, \{H, \{H, \{H, C\}\}\}\}(q_0, p_0). \tag{6.65}$$

In this case we differentiate (6.63) once more in $t=0$, and apply the same reasoning. Define inductively for $k \in \mathbb{N}$

$$\begin{aligned} &\mathrm{ad}_H^k C = \{H, \mathrm{ad}_H^{k-1} C\}, \quad \mathrm{ad}_H^0 C = C \\ &\mathrm{ad}_{H-\lambda^c C}^k \lambda^c = \{H - \lambda^c C, \mathrm{ad}_{H-\lambda^c C}^{k-1} \lambda^c\}, \quad \mathrm{ad}_{H-\lambda^c C}^0 \lambda^c = \lambda^c. \end{aligned} \tag{6.66}$$

It follows that either for some $\ell < \infty$, $\mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) \neq 0$, or $\mathrm{ad}_{H-\lambda^c C}^{k} \lambda^c(q_0, p_0) = 0$ for all $k \in \mathbb{N}$.

In the first case, by taking the smallest ℓ such that $\mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) \neq 0$, we obtain by induction

$$\left\{\mathrm{ad}_H^{\ell+2}, C\right\}(q_0, p_0) + R(q_0)\, \mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) = 0 \tag{6.67}$$

while $\{\mathrm{ad}_H^{k+2}, C\}(q_0, p_0) = \mathrm{ad}_{H-\lambda^c C}^{k} \lambda^c(q_0, p_0) = 0$ for $k < \ell$. If $\mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) > 0$, then as above it follows that there is unique continuation in mode 1, while if $\mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) < 0$, then there is unique continuation in mode 0.

In the second case, if $\mathrm{ad}_{H-\lambda^c C}^{k} \lambda^c(q_0, p_0) = 0$, for all k, then also $\mathrm{ad}_H^k C(q_0, p_0) = 0$, for all k, while already $C(q_0) = \{H, C\}(q_0, p_0) = 0$. From analyticity it follows that

$$\begin{aligned} &\lambda^c(q(t;1), p(t;1)) = 0, \qquad \text{for } t \text{ small} \\ &C(q(t;0)) = 0, \ \text{for } t \text{ small}\ . \end{aligned} \tag{6.68}$$

This implies that there is continuation in mode 0 as well as in mode 1. However, because of (6.67) it follows that in mode 1 not only $C(q(t;1))$ but also $\lambda^c(q(t;1), p(t;1))$ is zero for t small. Since in all points (q,p) for which both $C(q) = 0$ *and* $\lambda^c(q,p) = 0$

$$X_H(q,p) = X_{H-\lambda^c C}(q,p) \tag{6.69}$$

it thus follows that the continuation in mode 1 is the *same* as the continuation in mode 0. □

EXAMPLE 6.3 (continued). Consider an initial point (q_0, p_0) with $q_{01} = 0$, $q_{20} = 1$, $p_{01} = 0$, $p_{02} > 0$. (The mass has been thrown upwards to meet the constraint $q_1^2 + q_2^2 = 1$.) This jump point $(0,1,0,p_{02})$ will be projected to $(0,1,0,0) \in V_1$. However, since $\lambda^c(0,1,0,0) = -\frac{1}{2} < 0$ (see (6.46)) no continuation is possible in mode 1, but instead there is a jump in the constraint force to $\lambda = 0$, and the continuation will be in mode 0; the mass will fall down again. Notice that if instead of $q_{20} = 1$ we have e.g. $q_{20} = -1$, while $p_{02} < 0$, then the continuation will be in mode 1. □

It has been already remarked that for every $I \subset K$ the resulting mode is Hamiltonian with regard to the Dirac bracket $\{\,,\}_I$ on the constrained state space V_I, and with regard to the restriction H_I of H to V_I. In fact, we can be more explicit about this Hamiltonian dynamics, due to the fact that the phase space is T^*Q, and the Hamiltonian has the special form (6.4). First of all, we define for every $I \subset K$ the *constrained configuration space*

$$Q_I := \{q \in Q \mid C_I(q) = 0\} \tag{6.70}$$

which is (because of Assumption 6.1) a submanifold of Q, of dimension $n_I := n - |I|$. It follows that we may choose local coordinates $q = (q_1, \cdots, q_n)$ for Q, such that locally $C_I = (q_{n_I+1}, \cdots, q_n)$, and $q_I = (q_1, \cdots, q_{n_I})$ are local coordinates for Q_I.

Associated with $q = (q_1, \cdots, q_n)$ natural momentum coordinates $p = (p_1, \cdots, p_n)$ are defined. Now it has been shown in [18, Example 12.43] that

$$(q_I, p_I) := (q_1, \cdots, q_{n_I}, p_1, \cdots, p_{n_I}) \tag{6.71}$$

are local coordinates for V_I, and moreover that they are *canonical* with regard to the Dirac bracket $\{\,,\,\}_I$, that is

$$\{p_i, q_j\}_I = \delta_{ij}, \{p_i, p_j\}_I = \{q_i, q_j\}_I = 0, \quad i, j = 1, \cdots, n_I. \tag{6.72}$$

So V_I together with the Dirac bracket $\{\,,\,\}_I$ can be identified with T^*Q_I with its natural Poisson bracket. Hence the Hamiltonian dynamics of every mode I given by (6.25) can be identified with the standard Hamiltonian dynamics

$$\begin{array}{rcl} \dot{q}_I &=& \frac{\partial H_I}{\partial p_I}(q_I, p_I) \\ \dot{p}_I &=& -\frac{\partial H_I}{\partial q_I}(q_I, p_I) \end{array} \qquad (q_I, p_I) \in T^*Q_I \tag{6.73}$$

where $H_I(q_I, p_I)$ is the Hamiltonian H *restricted* to V_I, and subsequently expressed in the local coordinates q_I, p_I for V_I. The following relation exists between the momentum variables p_I for different modes I, including the mode $I = \emptyset$. Note that

$$T^*_{q_I}Q_I \simeq T^*_{q_I}Q/_{(T_{q_I}Q_I)^\perp}, \quad q_I \in Q_I \tag{6.74}$$

while furthermore $T_{q_I}Q_I \subset T_{q_I}Q$ is given as $\ker \frac{\partial^T C_I}{\partial q}(q_I)$. Hence

$$T^*_{q_I}Q_I \simeq T^*_{q_I}Q/_{\mathrm{im}\,\frac{\partial C_I}{\partial q}(q_I)}, \quad q_I \in Q_I \tag{6.75}$$

and so a momentum vector $p_I \in T_{q_I}Q_I$ can be identified with the equivalence class $[p]$, where

$$[p^1] = [p^2] \Leftrightarrow p^1 - p^2 \in \mathrm{im}\,\frac{\partial C_I}{\partial q}(q_I), \quad p^1, p^2 \in T^*_{q_I}Q. \tag{6.76}$$

Comparing this to the *projection rule* for Hamiltonian complementary-slackness systems as given above (see e.g. Proposition 6.4) we conclude that at the occasion of a *jump* in the state space (projection along $\mathrm{im}\,\frac{\partial C_I}{\partial q}(q_I)$) there is *conservation of momentum* in the sense that the equivalence class given in (6.76) remains the same.

EXAMPLE 6.3 (continued). Define new coordinates adapted to $C(q) = 1 - q_1^2 - q_2^2$

$$\tilde{q}_1 = \arctan(q_2/q_1), \quad \tilde{q}_2 = 1 - q_1^2 - q_2^2 \tag{6.77}$$

for $Q = \mathbb{R}^2 \backslash \{(q_1, q_2) \mid q_1 = 0)\}$. Carrying out the canonical transformation (see [16]), one finds the corresponding new momentum variables:

$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} = \begin{bmatrix} -q_2 & q_1 \\ -\frac{q_1}{2(q_1^2+q_2^2)} & -\frac{q_2}{2(q_1^2+q_2^2)} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}. \tag{6.78}$$

Therefore $\tilde{q}_1 = \arctan q_2/q_1$, $\tilde{p}_1 = q_1 p_2 - q_2 p_1$ are canonical coordinates for T^*Q_K, with $Q_K = \{(q_1, q_2) \mid q_1^2 + q_2^2 = 1\}$; this is in accordance with (6.28). Furthermore, as in (6.30), $H_K(\tilde{q}_1, \tilde{p}_1) = \frac{1}{2}\tilde{p}_1^2 + \sin\tilde{q}_1$. □

7 CONCLUSIONS

We have studied a class of dynamical systems which we have called the complementary-slackness class. One interesting feature of these dynamical systems is that they combine continuous and discrete characteristics so that they can be considered as a subclass of the class of hybrid systems. Considered in this way the complementary-slackness systems form a rather small class but nevertheless there are already interesting conclusions to be drawn which may also be of relevance to the much larger class of hybrid systems. In particular we have seen that a simple transition rule does not suffice, and instead we have used a rule allowing for multiple jumps. We have also begun a study of existence and uniqueness of solutions, which has led us to some nontrivial problems. The description of the transitions between modes has been in terms of a complementary foliation associated to the consistent manifold of each mode. We have seen that such foliations can be given in a natural way for linear and for Hamiltonian systems, and one may ask under what conditions complementary foliations can also be obtained for other types of constrained dynamics.

Although we have obtained some first results, clearly there are many questions still to be answered. Both in the linear and in the Hamiltonian case, the issue of well-posedness has only been resolved for bimodal systems. In particular for Hamiltonian complementary-slackness systems and linear passive networks with diodes this is unsatisfactory since physical intuition suggests strongly (at least if our modeling is correct) that such systems should be well-posed also when there are multiple constraints.

Several extensions of the class of systems studied in this paper could be considered. Take for instance electrical or hydraulic networks, or mechanical systems with friction, which will lead to complementary-slackness systems that are neither linear nor Hamiltonian. In many situations it will be of interest to add external inputs, and in particular one may formulate control problems for complementary-slackness systems. But there are also other connections with control theory. An obvious relation is the one via optimal control problems with state inequality constraints, but one may also think of using the Hamiltonian complementary-slackness structure, and in particular the energy inequality that we proved, as a means of finding nonsmooth stabilizing controllers for certain nonlinear systems. The inequality constraints that we used could be replaced by more complicated constraints, for instance two-sided constraints which would be needed to describe Coulomb friction, or ones involving hysteresis such as would occur in the modeling of stiction. Clearly some of these extensions would call for a considerable expansion of the framework that we have used here. Nevertheless, the limited setting that we used in this paper already has a significant range of applications and moreover has allowed us to concentrate on a few basic issues that might be snowed under in a more extensive environment.

References

[1] V.A. Armentano (1986). The pencil $(sE - A)$ and controllability-observability for generalized linear systems: a geometric approach. *SIAM J. Contr. Optimiz.* 24, 616–638.

[2] A. Back, J. Guckenheimer and M. Myers (1992). A dynamical simulation facility for hybrid systems. Tech. Rep. 92-6, Math. Sci. Inst., Cornell Univ.

[3] D. Bosscher, I. Polak and F. Vaandrager (1994). Verification of an audio control protocol. In *Formal Techniques in Real-Time and Fault-Tolerant Systems* (Proc. 3rd Intern. Symp., Lübeck, Germany, 1994), H. Langmaack, W.-P. de Roever and J. Vytopil (eds.), Lect. Notes Comp. Sci. 863. Springer, Berlin.

[4] M. S. Branicky, V. S. Borkar and S. K. Mitter (1994). *A unified framework for hybrid control: background, model, and theory.* Tech. Rep. LIDS-P-2239 (revised), Lab. for Information and Decision Systems, MIT, June 1994.

[5] K. E. Brenan, S. L. Campbell and L. R. Petzold (1989). *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations.* North-Holland, Amsterdam.

[6] R. W. Brockett (1984). Smooth multimode control systems, in *Proc. 1983 Berkeley-Ames Conference on Nonlinear Problems in Control and Fluid Mechanics*, 103–110, L. Hunt and C. Martin (eds.). Math. Sci. Press.

[7] R. W. Brockett (1993). Hybrid models for motion control systems. In *Essays on Control. Perspectives in the Theory and its Applications* (lectures and the mini-courses of the European control conference (ECC'93), Groningen, the Netherlands, June/July 1993), 29–53, H. L. Trentelman and J.C. Willems (eds.). Birkhäuser, Boston.

[8] B. Brogliato, S.-I. Niculescu and P. Orhant (1995). *On the control of finite-dimensional mechanical systems with unilateral constraints.* Preprint, Lab. d'Automatique de Grenoble, Sept. 1995.

[9] F. R. Gantmacher (1959). *The Theory of Matrices (Vol. I).* Chelsea, New York.

[10] F. R. Gantmacher (1959). *The Theory of Matrices (Vol. II).* Chelsea, New York.

[11] C. W. Gear (1988). Differential-algebraic equation index transformations. *SIAM J. Sci. Stat. Comput.* 9, 39–47.

[12] R.L. Grossman, A. Nerode, A.P. Ravn and H. Rischel (eds.) (1993). *Hybrid Systems.* Lect. Notes Comp. Sci. 736. Springer, New York.

[13] J. Guckenheimer (1995). A robust hybrid stabilization strategy for equilibria. *IEEE Trans. Automat. Contr.* AC-40, 321–326.

[14] M. L. J. Hautus and L. M. Silverman (1983). System structure and singular control. *Lin. Alg. Appl.* 50, 369–402.

[15] A. Isidori, A.J. Krener, C. Gori-Giorgi and S. Monaco (1981). Nonlinear decoupling via feedback: a differential-geometric approach. *IEEE Trans. Automat. Contr.* AC-26, 331–345.

[16] C. Lanczos (1986). *The Variational Principles of Mechanics.* Dover, New York, First edition 1949.

[17] Z. Manna and A. Pnueli (1992). *The Temporal Logic of Reactive and Concurrent Systems.* Springer, New York.

[18] H. Nijmeijer and A. J. van der Schaft (1990). *Nonlinear Dynamical Control Systems.* Springer-Verlag, Berlin.

[19] J.-E. Strömberg (1994). *A Mode Switching Modelling Philosophy.* Ph.D. thesis, Linköping University, Dept. of Electrical Engineering.

[20] J. C. Willems (1972). Dissipative dynamical systems. Part II: Linear systems with quadratic supply rates. *Arch. Rational Mech. Anal.* 45, 352–393.

[21] J. C. Willems (1991). Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Automat. Control* AC-36, No. 3, 259–294.

[22] W. M. Wonham (1979). *Linear Multivariable Control: A Geometric Approach*, 2nd ed. Springer Verlag, Heidelberg.