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Abstract

This paper studies a one-dimensional Markov chain $\{X_n, n = 0, 1, \ldots\}$ that satisfies the recurrence relation $X_n = max(0, X_{n-1} + \eta_n^{(m)})$ if $X_{n-1} = m \leq a$; for $X_{n-1} > a$ it satisfies the same relation with $\eta_n^{(m)}$ replaced by ξ_n. Here $\{\eta_n^{(m)}\}$ and $\{\xi_n\}$ are independent sequences of independent, integer-valued random variables. The limiting distribution of X_n is determined, using Wiener-Hopf factorization. It requires solving a set of $a + 1$ linear equations. The asymptotic behaviour of the limiting distribution is also described. Various applications to queueing models are discussed.

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1 Introduction

This paper is devoted to a one-dimensional Markov chain $\{X_n, n = 0, 1, \ldots\}$ of the following structure.

$$X_0 = 0, \tag{1}$$
$$X_n = \begin{cases} (X_{n-1} + \xi_n)^+ & \text{if } X_{n-1} > a, \\ (X_{n-1} + \eta_n^{(m)})^+ & \text{if } X_{n-1} = m,\ m = 0, \ldots, a. \end{cases}$$

Here a is a nonnegative integer; $(u)^+ := \max(0, u)$; and $\{\xi_n\}$, $\{\eta_n^{(m)}\}, m = 0, \ldots, a$ are independent sequences of independent integer-valued random variables, identically distributed within each sequence.
Random walks of the type $X_n = (X_{n-1} + \xi_n)^+$ have been studied extensively, in particular because of their natural occurrence in a queueing context (cf. [2], [4]). For example, this recursion is satisfied by the waiting time in the GI/G/1 queue (ξ_n then represents the difference between the service time of the $(n-1)$th customer and the interarrival time between the $(n-1)$th and nth customers; of course ξ_n is now not integer-valued). Several queueing models give rise to similar recursions, but with a *different* distribution for jumps from the origin, or even from states close to the origin. A classical example is the queue length process at service completion epochs in the M/G/1 queue: it satisfies (1) with $a = 0$ and $\xi_n \geq -1$, $\eta_n^{(0)} \geq 0$.
The purpose of this paper is to present a simple treatment of the class of random walks given by (1), based on Wiener-Hopf factorization. The generating function of the steady-state distribution $p_j := \lim_{n\to\infty} \mathbf{P}\{X_n = j\}$ is obtained; it is a linear combination of $a + 1$ terms, with weight factors $p_0, \ldots, p_a$. The latter probabilities are obtained by solving a set of linear equations. The results unify several known queueing results; some examples are gathered in Section 4. These examples include the M/G/1 queue with vacations, with set-up times, or with a different distribution for the first service time in a busy period; the case of an M/G/1 queue with additional negative customers [1] is also covered. But of course the applicability of the results far exceeds M/G/1-type queueing models - in particular by also allowing cases where $\xi_n \leq -1$.
The paper is organized as follows. In Section 2 we give an exact analysis of the steady-state distribution of the random walk (1), using a Wiener-Hopf approach. Section 3 is devoted to an asymptotic analysis of the steady-state

probabilities. In Section 4 several applications (mainly in the area of queues) are discussed.

2 The random walk

Consider the random walk specified by (1). The generating functions of the distributions of ξ_n and $\eta_n^{(m)}$ will be denoted by $f(z)$ and $g^{(m)}(z)$, respectively. We suppose throughout the paper that $\mathrm{E}\xi_1 < 0$. We also assume that the Markov chain is irreducible and aperiodic. Hence the limiting distribution $p_k := \lim_{n\to\infty} \mathbf{P}(X_n = k)$, $k = 0, 1, \ldots$ exists. The main goal of this section is to determine this limiting distribution.
It follows from (1) that

$$\begin{aligned}
\mathbf{P}(X_n = 0) &= \sum_{m=0}^{a} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\eta_n^{(m)} + m \leq 0) \\
&+ \sum_{m=a+1}^{\infty} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\xi_n + m \leq 0), \\
\mathbf{P}(X_n = k) &= \sum_{m=0}^{a} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\eta_n^{(m)} = k - m) \\
&+ \sum_{m=a+1}^{\infty} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\xi_n = k - m), \quad k = 1, 2, \ldots .
\end{aligned}$$

Introducing, for $|z| \leq 1$, $P_n(z) := \sum_{k=0}^{\infty} z^k \mathbf{P}(X_n = k)$ it follows that

$$\begin{aligned}
P_n(z) &= \sum_{m=0}^{a} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\eta_n^{(m)} + m < 0) \\
&+ \sum_{m=a+1}^{\infty} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\xi_n + m < 0) \\
&+ \sum_{k=0}^{\infty} z^k \sum_{m=0}^{a} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\eta_n^{(m)} = k - m) \\
&+ \sum_{k=0}^{\infty} z^k \sum_{m=a+1}^{\infty} \mathbf{P}(X_{n-1} = m)\mathbf{P}(\xi_n = k - m).
\end{aligned} \tag{2}$$

Introduce the following (useful) notation, for any $D \subset \{\ldots,-1,0,1,\ldots\}$ and for an arbitrary function of the form $\sum_{k=-\infty}^{\infty} a_k z^k$:

$$[\sum_{k=-\infty}^{\infty} a_k z^k]^D = \sum_{k \in D} a_k z^k.$$

Formula (2) can be rewritten as follows:

$$\begin{aligned} P_n(z) &= [P_{n-1}(z)f(z)]^{[0,\infty)} \qquad (3) \\ &+ \sum_{m=0}^{a} \mathbf{P}(X_{n-1}=m) \sum_{k=0}^{\infty} z^k \{\mathbf{P}(\eta_n^{(m)} = k-m) - \mathbf{P}(\xi_n = k-m)\} \\ &+ \sum_{m=0}^{a} \mathbf{P}(X_{n-1}=m)\{\mathbf{P}(\eta_n^{(m)} + m < 0) - \mathbf{P}(\xi_n + m < 0)\} \\ &+ \sum_{m=0}^{\infty} \mathbf{P}(X_{n-1}=m)\mathbf{P}(\xi_n + m < 0). \end{aligned}$$

Now let $n \to \infty$. Introducing

$$P(z) := \lim_{n\to\infty} P_n(z), \qquad |z| \le 1,$$

$$l(z) := [P(z)f(z)]^{(-\infty,-1]}, \qquad |z| \ge 1,$$

$$\begin{aligned} q_m(z) &:= \mathbf{P}(\eta_1^{(m)} + m < 0) - \mathbf{P}(\xi_1 + m < 0) \qquad (4) \\ &+ z^m [g_m(z) - f(z)]^{[-m,\infty)}, \quad m = 0,1,\ldots,a, \quad |z| \le 1, \end{aligned}$$

it follows from (3) that, for $|z| = 1$,

$$P(z) = [P(z)f(z)]^{[0,\infty)} + \sum_{m=0}^{a} p_m q_m(z) + l(1),$$

or, equivalently, for $|z| = 1$,

$$P(z)(1 - f(z)) = \sum_{m=0}^{a} p_m q_m(z) + l(1) - l(z). \qquad (5)$$

This basic identity is the starting point for the remaining analysis. Consider the standard Wiener-Hopf factorization (cf. [2])

$$1 - f(z) = R_+(z)R_-(z), \qquad (6)$$

where

$$R_+(z) := 1 - \mathrm{E}(z^{\chi_+}(\nu_+ < \infty)), \quad |z| \le 1, \tag{7}$$
$$R_-(z) := 1 - \mathrm{E}(z^{\chi_-}), \quad |z| \ge 1. \tag{8}$$

Here ν_+ is defined to be the first weak ascending ladder index:

$$\nu_+ := \min\{n \ge 1 : \xi_1 + \ldots + \xi_n \ge 0\};$$

similarly ν_- is defined to be the first descending ladder index,

$$\nu_- := \min\{n \ge 1 : \xi_1 + \ldots + \xi_n < 0\},$$

and $\chi_+ := \xi_1 + \ldots + \xi_{\nu_+}$, $\chi_- := \xi_1 + \ldots + \xi_{\nu_-}$.
Both sides of (5) vanish for $z = 1$, and also $R_-(1) = q_0(1) = \ldots = q_a(1) = 0$. Introducing, for $|z| \ge 1$, $\tilde{R}_-(z) := R_-(z)/(1-z)$ we obtain from (5) for $|z| = 1$:

$$P(z)R_+(z) = \sum_{m=0}^{a} p_m h_m(z) + \frac{l(1) - l(z)}{(1-z)\tilde{R}_-(z)}, \tag{9}$$

where

$$h_m(z) := \frac{q_m(z)}{(1-z)\tilde{R}_-(z)}. \tag{10}$$

For arbitrary integer s, t such that $s \le t$ denote by $S(s,t)$ the set of all convergent power series of the form $\sum_{k=s}^{t} b_k z^k$, $|z| = 1$. It is clear that $P(z) \in S(0,\infty)$, $R_+(z) \in S(0,\infty)$, therefore $P(z)R_+(z) \in S(0,\infty)$. The same must be true for the righthand side of (9). The functions $R_-(z)$, $\tilde{R}_-(z)$ are analytic for $|z| > 1$ and continuous up to $|z| = 1$. Furthermore, $\tilde{R}_-(z) \ne 0$ for $|z| \ge 1$, $\tilde{R}_-(z) \to 0$ as $z \to \infty$. Hence $\tilde{R}_-(z) \in S(-\infty,-1)$ and $\tilde{R}_-^{-1}(z) \in S(-\infty,1)$. For the latter statement, observe that $(1-z)/(zR_-(z)) \to -1$ for $z \to \infty$. Since $(l(1) - l(z))/(1-z) \in S(-\infty,-1)$ it now follows that

$$\Psi(z) := \frac{l(1) - l(z)}{(1-z)\tilde{R}_-(z)} \in S(-\infty,0).$$

Hence, writing $c := \Psi(\infty)$ and applying Liouville's theorem to (9), it follows that

$$P(z)R_+(z) = \sum_{m=0}^{a} p_m [h_m(z)]^{[0,\infty)} + c, \quad |z| \le 1, \tag{11}$$

$$\sum_{m=0}^{a} p_m [h_m(z)]^{(-\infty,1]} \; + \; \frac{l(1)-l(z)}{(1-z)\tilde{R}_-(z)} \; - \; c \; = \; 0, \qquad |z| \geq 1. \tag{12}$$

Remark 2.1 In fact (12) can be derived from (11), and hence (11) and (9) are equivalent. This equivalence, which is used at the end of this section, is proven as follows (we first take $|z| = 1$, and later use analytic continuation). Using the relations $[P(z)]^{(-\infty,-1]} = 0$ and (11) we get

$$\begin{aligned} l(z) &= -[P(z)(1-f(z)) - P(z)]^{(-\infty,-1]} \\ &= -[P(z)R_+(z)R_-(z)]^{(-\infty,-1]} \\ &= -\sum_{m=0}^{a} p_m \left[\left(h_m(z) - [h_m(z)]^{(-\infty,-1]} \right) R_-(z) \right]^{(-\infty,-1]} - c[R_-(z)]^{(-\infty,-1]}. \end{aligned}$$

Since $h_m(z)R_-(z) = q_m(z) \in S(0,\infty)$ (see (4) and (10)), it gives

$$\begin{aligned} l(z) &= \sum_{m=0}^{a} p_m \left[[h_m(z)]^{(-\infty,-1]} R_-(z) \right]^{(-\infty,-1]} - c(R_-(z) - R_-(\infty)) \\ &= \sum_{m=0}^{a} p_m [h_m(z)]^{(-\infty,-1]} R_-(z) - c(R_-(z) - R_-(\infty)). \end{aligned}$$

Taking into account that $|h_m(1)| < \infty$ and $R_-(1) = 0$, we have rederived (12):

$$\begin{aligned} \frac{l(1)-l(z)}{R_-(z)} &= \Bigg(\sum_{m=0}^{a} p_m [h_m(z)]^{(-\infty,-1]} R_-(z)|_{z=1} \\ &\quad - \sum_{m=0}^{a} p_m [h_m(z)]^{(-\infty,-1]} R_-(z) \; + \; cR_-(z) \Bigg) R_-^{-1}(z) \\ &= -\sum_{m=0}^{a} p_m [h_m(z)]^{(-\infty,-1]} \; + \; c. \end{aligned}$$

Summation of the equations (11) and (12) gives (9), i.e. (9) and (11) are equivalent.

Formula (11) now gives the main result of the paper: for $|z| \leq 1$,

$$P(z) \; = \; \sum_{m=0}^{a} p_m R_+^{-1}(z) [h_m(z)]^{[0,\infty)} \; + \; cR_+^{-1}(z). \tag{13}$$

The numbers $p_0, \dots, p_a$ and c are yet unknown. They are determined as follows. Let the sequences $\{\gamma_k^{(m)}\}$ and r_k be defined by the following expansions, for $|z| \le 1$:

$$R_+^{-1}(z)[h_m(z)]^{[0,\infty)} = \sum_{k=0}^{\infty} z^k \gamma_k^{(m)}, \tag{14}$$

$$R_+^{-1}(z) = \sum_{k=0}^{\infty} z^k r_k. \tag{15}$$

Then we obtain from (13) the equations:

$$p_k = \sum_{m=0}^{a} p_m \gamma_k^{(m)} + c r_k, \qquad k = 0, 1, \dots . \tag{16}$$

The first $a+1$ equations together with the normalizing condition $P(1) = 1$ determine the $a+2$ constants $p_0, \dots, p_a, c$. Suppose that these equations can give another solution $p'_0, \dots, p'_a, c'$. It means that we can obtain another function $P'(z)$ satisfying (11) and also (9) and (5) (here we use the equivalence of (9) and (11)). But this will contradict the uniqueness property of the steady-state distribution. Therefore the solution of (16) is unique.

3 Asymptotics

In this section we determine the asymptotic behaviour of the steady-state probabilities p_k for $k \to \infty$. Denote $R := \sup\{z \ge 1 : f(z) < \infty\}$. The asymptotic behavior of the probabilities $p_k, k \to \infty$ is different for the cases $R > 1, f(R) > 1$; $R > 1, f(R) \le 1$; and $R = 1$. It depends also on the relation between the tail probabilities $\mathbf{P}(\xi_1 \ge x)$ and $\mathbf{P}(\eta_1^{(m)} \ge x)$, $m = 0, \dots, a$. All these problems were studied in detail in [3] for the case $a = 0$. It was shown there also that for the purpose of asymptotic analysis the case $a > 0$ can be reduced to the study of a *modified* random walk with $a = 0$.
We demonstrate here the calculation of the asymptotics only for the first case mentioned above. Suppose that there exists a number $\zeta > 1$ such that

$$|f(z)| < \infty, \quad 1 \le |z| \le \zeta, \qquad f(\zeta) > 1, \tag{17}$$

and also

$$|g_m(z)| < \infty, \qquad 1 \le |z| \le \zeta, \qquad m = 0, \dots, a. \tag{18}$$

Then clearly

$$\begin{aligned}|q_m(z)| &= |\mathbf{P}(\eta_1^{(m)} + m < 0) - \mathbf{P}(\xi_1 + m < 0) \\ &+ z^m[g_m(z) - f(z)]^{[-m,\infty)}| < \infty\end{aligned}$$

for $|z| \leq \zeta$. Since $q_m(1) = 0$, we have $|q_m(z)(1-z)^{-1}| < \infty$ for $|z| \leq \zeta$, and also $|\tilde{R}_-^{-1}(z)| < \infty$ for $|z| \geq 1$. This means that $|h_m(z)| < \infty$ in the ring $1 \leq |z| \leq \zeta$. Therefore, the function $[h_m(z)]^{[0,\infty)}$ is analytic for $|z| < \zeta$ and continuous up to the boundary $|z| = \zeta$. It follows from (17) and the fact that $f(1) = 1$ with $f'(1) = \mathrm{E}\xi_1 < 0$ that there exists a number q such that $1 < q < \zeta$ and $f(q) = 1$ (note that $f(z)$ is real for real z, and that q is unique). We conclude that $R_+(q) = 0$ since $R_-(q) \neq 0$. The function $(1 - f(z))R_-^{-1}(z)$ can be analytically continued in the set $|z| < \zeta$. This function has no zeros except $z = q$ in the circle $|z| \leq q + \delta$ for some $\delta > 0$ (observe that $|f(z)| < f(q) = 1$ for each z with $1 < |z| < q$). This means that the function $R_+^{-1}(z)$ has a unique prime pole $z = q$ in the set $|z| \leq q+\delta$. The same is true for the function $R_+^{-1}(z)[h_m(z)]^{[0,\infty)}$. Let numbers ρ_k, $\tilde{\rho}_k$ $(k \geq 0)$, b_m $(m = 0, \ldots, a)$ and b be defined by the expansions ($|z| < q$)

$$\begin{aligned}R_+^{-1}(z)[h_m(z)]^{[0,\infty)} &= \frac{b_m}{q-z} + \sum_{k=0}^{\infty} z^k \rho_k, \\ R_+^{-1}(z) &= \frac{b}{q-z} + \sum_{k=0}^{\infty} z^k \tilde{\rho}_k.\end{aligned}$$

The functions

$$\sum_{k=0}^{\infty} z^k \rho_k, \qquad \sum_{k=0}^{\infty} z^k \tilde{\rho}_k$$

can be continued analytically in the circle $|z| < q + \delta$ and, therefore,

$$\rho_k = O((q+\delta)^{-k}); \qquad \tilde{\rho}_k = O((q+\delta)^{-k}), \qquad k \to \infty.$$

So we get

$$\begin{aligned}\gamma_k^{(m)} &= b_m q^{-k-1} + O((q+\delta)^{-k}), \\ r_k &= b q^{-k-1} + O((q+\delta)^{-k}).\end{aligned}$$

Substituting these relations in (16) we get

$$p_k = (\sum_{m=0}^{a} p_m b_m + cb) q^{-k-1} + O((q+\delta)^{-k}), \qquad k \to \infty. \tag{19}$$

Remark 3.1 Note that the coefficient of q^{-k-1} is completely specified. The number δ can also be specified. It depends on the radius of a circle containing no zeros of the function $R_+(z)$ except $z = q$.

Remark 3.2 If condition (18) is violated for some m, the asymptotic behavior of p_k will be determined by $\gamma_k^{(m_0)}$, where m_0 corresponds to the function $[g_{m_0}(z)]^{[0,\infty)}$ having minimal radius of convergence among other functions $[g_m(z)]^{[0,\infty)}$.

4 Applications

In Section 2 we have derived a formal expression for the steady-state distribution of the random walk (1). In the present section we consider various special cases for which this formal expression can be evaluated. For example, we shall take $a = 0$, and we shall take one or both of the Wiener-Hopf factors $R_+(z)$, $R_-(z)$ rational, etc. Several of these applications have an interpretation in queueing theory.

Case 1: $a = 0$ and $\mathbf{P}(\xi_1 \geq -1) = 1$
Observe that the definition of $R_-(z)$ (see (8)) immediately implies that, when $\mathbf{P}(\xi_1 \geq -1) = 1$ and $\mathrm{E}\xi_1 < 0$,

$$R_-(z) = 1 - 1/z, \tag{20}$$

so that

$$R_+(z) = \frac{1 - f(z)}{1 - 1/z}. \tag{21}$$

From (4) and (10) it now follows that

$$\begin{aligned} h_0(z)^{[0,\infty)} &= \left[\frac{\mathbf{P}\{\eta_1^{(0)} < 0\} + g_0(z)^{[0,\infty)} - f(z) - (1 - 1/z)\mathbf{P}\{\xi_1 = -1\}}{1 - 1/z} \right]^{[0,\infty)} \\ &= \frac{g_0(z)^{[0,\infty)} - \mathbf{P}\{\eta_1^{(0)} \geq 0\}}{1 - 1/z} + \frac{1 - f(z)}{1 - 1/z} - \mathbf{P}\{\xi_1 = -1\} \\ &= z \sum_{l=0}^{\infty} \mathbf{P}\{\eta_1^{(0)} = l\} \frac{z^l - 1}{z - 1} + R_+(z) - \mathbf{P}\{\xi_1 = -1\}. \end{aligned} \tag{22}$$

Since $R_+(0) = \lim_{z\to\infty} -z(1-f(z))/(1-z) = \mathbf{P}\{\xi_1 = -1\}$, it follows in particular that $\gamma_0^{(0)} = 0$ (cf. (14)); and, cf. (15), $r_0 = 1/R_+(0)$. Hence it follows from (16) that $p_0 = cr_0 = c/R_+(0)$. The constant c can be determined from the normalizing condition $P(1) = 1$. Equation (13) can now be written in a more explicit form:

$$P(z) = \frac{R_+(1)}{R_+(z)} \left[\frac{r_0 h_0(z)^{[0,\infty)} + 1}{r_0 h_0(z)^{[0,\infty)}|_{z=1} + 1} \right], \tag{23}$$

with $R_+(z)$ given by (21) and $h_0(z)^{[0,\infty)}$ by (22).

Case 2: $a = 0$ and $f(z) = B^*(\lambda(1-z))/z$

Here $B^*(\lambda(1-z))$ is the generating function of the number of Poisson events (rate λ) that take place during a length of time with mean $\mathrm{E}B$, distribution $B(\cdot)$ and Laplace-Stieltjes Transform (LST) $B^*(\cdot)$. This is a special case of Case 1, which naturally arises in M/G/1-type queues. $P(z)$ represents the GF (Generating Function) of the equilibrium queue length distribution at departure epochs of an M/G/1 queue with arrival rate λ and service time distribution $B(\cdot)$. The freedom to still choose $g_0(\cdot)$ leads to variants of the M/G/1 queue like the M/G/1 queue with server vacations, with set-up times and with a different distribution for the first service time in a busy period; below we discuss several choices for $g_0(\cdot)$.

Substitution of $f(z) = B^*(\lambda(1-z))/z$ in (23) yields, with $\rho := \lambda \mathrm{E}B$:

$$P(z) = \frac{1-\rho}{B^*(\lambda(1-z)) - z} \frac{B^*(\lambda(1-z)) - z\sum_{l=0}^{\infty} \mathbf{P}\{\eta_1^{(0)} = l\} z^l}{1 - \rho + \sum_{l=0}^{\infty} l\mathbf{P}\{\eta_1^{(0)} = l\}}. \tag{24}$$

If $\mathbf{P}\{\eta_1^{(0)} \geq 0\} = 1$ then the two sums in the numerator and the denominator of (24) reduce to $g_0(z)$ and $\mathrm{E}\eta_1^{(0)}$, respectively. We mention some special cases where this holds.

Case 2(i): $g_0(z) = B^*(\lambda(1-z))$

This is the ordinary M/G/1 queue. Indeed (24) reduces to the well-known Pollaczek-Khintchine formula:

$$P(z) = \frac{(1-\rho)(1-z)B^*(\lambda(1-z))}{B^*(\lambda(1-z)) - z}. \tag{25}$$

Case 2(ii): $g_0(z) = B^*(\lambda(1-z))S^*(\lambda(1-z))$
Here $S^*(\cdot)$ is the LST of a probability distribution of a non-negative random variable with mean ES, that e.g. represents a set-up time before the first service in a busy period of the M/G/1 queue. Formula (24) reduces to

$$P(z) = \frac{(1-\rho)B^*(\lambda(1-z))}{B^*(\lambda(1-z)) - z} \frac{1 - zS^*(\lambda(1-z))}{1 + \lambda \mathrm{E}S}. \tag{26}$$

The result agrees with Formula (2.49a) on p. 131 of [5].

Case 2(iii): $g_0(z) = B^*(\lambda(1-z))[V^*(\lambda(1-z)) - V^*(\lambda)]/[z(1 - V^*(\lambda))]$
This corresponds to an M/G/1 queue 'with multiple server vacations', with vacation time LST $V^*(\cdot)$. Formula (24) reduces to

$$P(z) = \frac{(1-\rho)(1-z)B^*(\lambda(1-z))}{B^*(\lambda(1-z)) - z} \frac{1 - V^*(\lambda(1-z))}{\lambda \mathrm{E}V(1-z)}, \tag{27}$$

with EV the mean of the vacation time. Formula (27) exhibits the well-known decomposition property: $P(z)$ is the product of the GF of the queue length distribution in the ordinary M/G/1 queue and the GF of the number of arrivals during the past (or residual) part of a vacation period (cf. [5], p. 112).

Case 2(iv): $g_0(z) = G^*(\lambda(1-z))$
Here $G^*(\cdot)$ is the LST of a service time distribution; the situation corresponds to that of an M/G/1 queue with service time LST $B^*(\cdot)$ except for the first service time in a busy period, which has LST $G^*(\cdot)$. Formula (24) now immediately yields Formula (2.38) on p. 129 of [5].

Remark 4.1 It follows from the previous section that the behaviour of p_k for $k \to \infty$ is determined by the smallest (real) zero of $B^*(\lambda(1-z)) - z$ outside the unit circle.

Case 3: $a = 0$ and $\mathbf{P}(\xi_1 \leq 1) = 1$
The definition of $R_+(z)$ (see (7)) immediately implies that, when $\mathbf{P}(\xi_1 \leq 1) = 1$,

$$R_+(z) = 1 - dz, \tag{28}$$

so that

$$R_-(z) = \frac{1 - f(z)}{1 - dz}. \tag{29}$$

Here $d = \mathbf{P}(\nu_+ < \infty)$, cf. (7). Note that (see below (10)) $R_-(z) \neq 0$ in $|z| > 1$; hence $1/d$ is determined as the unique zero of $1 - f(z)$ in $|z| > 1$ ($1/d$ cannot be a pole of $R_-(z)$ in $|z| > 1$, because the latter function is analytic in that area). It clearly follows from Section 3 that $p_k \sim d^k$, $k \to \infty$, but we can be much more specific here. From (4) and (10) it follows that

$$\begin{aligned} h_0(z) = \frac{q_0(z)}{R_-(z)} &= \frac{\mathbf{P}(\eta_1^{(0)} < 0) + g_0(z)^{[0,\infty)} - 1 + (1-z)\mathbf{P}(\xi_1 = 1)}{R_-(z)} \\ &= \frac{\mathbf{P}(\xi_1 = 1) - \sum_{l=0}^{\infty} \mathbf{P}(\eta_1^{(0)} = l)\frac{1-z^l}{1-z}}{\tilde{R}_-(z)}. \end{aligned} \tag{30}$$

Now let us make the additional assumption that $\mathbf{P}(\eta_1^{(0)} \leq 1) = 1$. Hence

$$h_0(z) = \frac{\mathbf{P}(\xi_1 = 1) - \mathbf{P}(\eta_1^{(0)} = 1)}{\tilde{R}_-(z)}, \tag{31}$$

and using the observation (see below (10)) that $\tilde{R}_-^{-1}(z) \in S(-\infty, 1)$,

$$h_0(z)^{[0,\infty)} = U + Vz. \tag{32}$$

Observing that $r_0 = 1$ and $\gamma_0^{(0)} = U$, so that (cf. (16)) $p_0 = Up_0 + c$, it follows from (13) that

$$P(z) = p_0 \frac{1 + Vz}{1 - dz} = \frac{1 - d}{1 - dz} \frac{1 + Vz}{1 + V}. \tag{33}$$

In the last equality we have used the normalizing condition. The constant V equals

$$V = \lim_{z \to \infty} \frac{\mathbf{P}(\xi_1 = 1) - \mathbf{P}(\eta_1^{(0)} = 1)}{z\tilde{R}_-(z)}. \tag{34}$$

Let us consider two special cases.

Case 3(i): $f(z) = g_0(z) = zA^*(\mu(1 - 1/z))$
Here $A^*(\mu(1 - z))$ is the generating function of the number of Poisson events (rate μ) that take place during a length of time with distribution $A(\cdot)$ and

LST $A^*(\cdot)$. This leads to the G/M/1 queue with interarrival time distribution $A(\cdot)$ and service rate μ: $P(\cdot)$ is the GF of the queue length distribution as seen by arriving customers.
The fact that $\mathbf{P}(\xi_1 = 1) = \mathbf{P}(\eta_1^{(0)} = 1)$ implies that $h_0(z) = 0$, so that $V = 0$. Hence

$$P(z) = \frac{1-d}{1-dz}. \tag{35}$$

As observed above, $1/d$ is the unique zero of $1 - f(z)$ in $|z| > 1$. Hence d is the unique zero of $y - A^*(\mu(1-y))$ in $|y| < 1$ - a fact that is well-known in G/M/1 theory, cf. [4].

Case 3(ii): $f(z) = zA^*(\mu(1-1/z))$, $g_0(z) = zA^*(\mu_0(1-1/z))$
This corresponds to a G/M/1 queue in which the service rate is μ_0 at the beginning of the busy period, until the next arrival. It follows from (34) that $V = d[A^*(\mu_0) - A^*(\mu)]/A^*(\mu)$.

Case 4: $f(z) = B_a^*(\lambda(1-z))/z$, $g_0(z) = B_b^*(\lambda(1-z))$, $g_m(z) = B_b^*(\lambda(1-z))/z$, $m = 1, \dots, a$
This case corresponds to an M/G/1 queue with service time distribution $B_a(\cdot)$ with LST $B_a^*(\cdot)$ when the last departing customer leaves more than a customers behind, and service time distribution $B_b(\cdot)$ with LST $B_b^*(\cdot)$ otherwise. This model seems to us of interest, as it arises rather naturally in a control setting. $P(z)$ is the GF of the queue length distribution in this M/G/1 queue immediately after departure epochs. As in the ordinary M/G/1 queue, that distribution coincides with the queue length distribution at arrival epochs, and with the steady-state queue length distribution. Also note that, as in the ordinary M/G/1 queue, $P(z) = S^*(\lambda(1-z))$, with $S^*(\cdot)$ the LST of the sojourn time distribution, so that we also have the latter distribution once we have obtained $P(z)$.
Formulas (20) and (21) hold again; (4) and (10) yield after some simple calculations:

$$h_0(z)^{[0,\infty)} = -B_a^*(\lambda) + \frac{B_a^*(\lambda(1-z)) - zB_b^*(\lambda(1-z))}{1-z}, \tag{36}$$

$$h_m(z)^{[0,\infty)} = z^m + \frac{B_a^*(\lambda(1-z)) - B_b^*(\lambda(1-z))}{1-z}, \quad m = 1, \dots, a. \tag{37}$$

Substitution in (13) gives:

$$
\begin{aligned}
P(z) &= \frac{1-z}{B_a^*(\lambda(1-z))-z}[p_0(-B_a^*(\lambda)+\frac{B_a^*(\lambda(1-z))-zB_b^*(\lambda(1-z))}{1-z}) \\
&+ \sum_{m=1}^{a} p_m z^m \frac{B_a^*(\lambda(1-z))-B_b^*(\lambda(1-z))}{1-z}+c]. \qquad (38)
\end{aligned}
$$

The $a+2$ constants $p_0,\dots,p_a$ and c are determined from (16) and the normalizing condition. We find from (36), (37) and (14), cf. (38):

$$
\sum_{k=0}^{\infty} z^k \gamma_k^{(0)} = -B_a^*(\lambda) + \frac{B_a^*(\lambda(1-z))-zB_b^*(\lambda(1-z))}{B_a^*(\lambda(1-z))-z}, \qquad (39)
$$

$$
\sum_{k=0}^{\infty} z^k \gamma_k^{(m)} = z^m \frac{B_a^*(\lambda(1-z))-B_b^*(\lambda(1-z))}{B_a^*(\lambda(1-z))-z}, \quad m=1,\dots,a. \qquad (40)
$$

Observe that $h_m(z)^{[0,\infty)} = h_m(z)^{[m,\infty)}$, so that $\gamma_k^{(m)}=0$ for $k=0,\dots,m-1$, $m=0,\dots,a$.
Let us now restrict ourselves to the case of exponential service times: $B_a^*(s) = \alpha/(\alpha+s)$, $B_b^*(s)=\beta/(\beta+s)$. Rather lengthy but straightforward calculations reveal that

$$
r_0 = 1+\frac{\lambda}{\alpha}, \quad r_k = (\frac{\lambda}{\alpha})^{k+1}, \quad k=1,2,\dots; \qquad (41)
$$

$$
\gamma_k^{(m)} = \frac{\alpha\lambda-\beta\lambda}{\lambda+\beta-\alpha}[\frac{1}{\alpha}(\frac{\lambda}{\alpha})^{k-m} - \frac{1}{\beta+\lambda}(\frac{\lambda}{\beta+\lambda})^{k-m}], \quad k=m,\dots,a, \;\; m=1,\dots,a. \qquad (42)
$$

For $\alpha=\beta+\lambda$, the righthand side has to be replaced by the limit for $\alpha\to\beta+\lambda$. It is now easy to solve (16) and to determine the $a+2$ constants.

Case 5: $a=0$ and $g_0(z)=zf(z)$, with $f(\cdot)$ a rational function
We include this case for two reasons: (i) it gives an illustration of the important case that $f(\cdot)$ is a rational function. Rationality assumptions allow a much more explicit evaluation of $P(z)$. (ii) This particular case contains a recently studied M/G/1 model with negative customers [1]. In that model, negative customers arrive according to a Poisson process and require no service. At the end of a service of an ordinary customer, each negative customer removes exactly one ordinary customer; if there are more negative customers than ordinary customers after a service completion, then the system becomes

empty.
Without loss of generality, under the rationality assumption regarding $f(z)$ we can write:

$$1 - f(z) = D(1-z)\frac{\prod_i (z-\alpha_i)}{\prod_j (z-\beta_j)}, \tag{43}$$

with D a constant. We have taken out the factor $(z-1)$ as $f(1) = 1$; we assume that $|\alpha_i| \neq 1$ for all i, thus excluding the possibility that the distribution of ξ_1 is periodic. The factors $R_+(z)$ and $R_-(z)$ are given by (cf. [1]):

$$R_-(z) = (z-1)\frac{\prod_{|\alpha_i|<1}(z-\alpha_i)}{\prod_{|\beta_j|<1}(z-\beta_j)}, \tag{44}$$

$$R_+(z) = D\frac{\prod_{|\alpha_i|>1}(z-\alpha_i)}{\prod_{|\beta_j|>1}(z-\beta_j)}. \tag{45}$$

Indeed $R_-(z)$ is analytic in $|z|>1$, and $\lim_{z\to\infty}\frac{zR_-(z)}{1-z} = -1$; the degree of the product in the denominator exceeds that of the product in the numerator in (44) by one. As indicated below (10), $R_+(z) \in S(0,\infty)$ and $R_-(z) \in S(-\infty, 0)$.
According to (4) and (10), with $g_0(z) = zf(z)$:

$$\begin{aligned} h_0(z)^{[0,\infty)} &= \left\{\frac{\mathbf{P}(\eta_1^{(0)}<0) - \mathbf{P}(\xi_1<0) + \{(z-1)f(z)\}^{[0,\infty)}}{R_-(z)}\right\}^{[0,\infty)} \\ &= \frac{\mathbf{P}(\eta_1^{(0)}<0) - \mathbf{P}(\xi_1<0)}{R_-(0)} + \left\{\frac{(z-1)f(z)}{R_-(z)}\right\}^{[0,\infty)} \\ &= \frac{\mathbf{P}(\eta_1^{(0)}<0) - \mathbf{P}(\xi_1<0)}{R_-(0)} + \left\{\frac{(z-1)}{R_-(z)}\right\}^{[0,\infty)} + R_+(z)(1-z). \end{aligned} \tag{46}$$

The second equality uses the fact that the $(-\infty,-1]$ part of $(z-1)f(z)$ will not contribute to the $[0,\infty)$ part of $h_0(z)$. In the last equality, we replaced $f(z)$ by $1-(1-f(z))$, and we used that $R_+(z) \in S(0,\infty)$.
From (46) and (13):

$$\begin{aligned} P(z) &= p_0\frac{h_0(z)^{[0,\infty)}}{R_+(z)} + \frac{c}{R_+(z)} \\ &= \frac{X}{R_+(z)} + \frac{p_0 Y z}{R_+(z)} + p_0(1-z). \end{aligned} \tag{47}$$

In fact Y is the coefficient of z^1 in $(z-1)/R_-(z) = \prod_{|\beta_j|<1}(z-\beta_j)/\prod_{|\alpha_i|<1}(z-\alpha_i)$. $Y = 1$ since the degree of the product in the numerator exceeds the degree of the product in the denominator by one. Substitution of $z = 0$ (which amounts to applying (16)) yields $X = 0$. Substitution of $z = 1$ yields $p_0 = R_+(1)/Y = R_+(1)$. Finally,

$$P(z) = R_+(1)[\frac{z}{R_+(z)} + 1 - z], \tag{48}$$

in agreement with Formula (11) of [1]; $R_+(z)$ is explicitly given in (45). In the case of an M/G/1 queue with additional negative customers with Poisson arrival rate ν, $f(z)$ equals $B^*(\lambda(1-z) + \nu(1-1/z))/z$. For rational $B^*(\cdot)$, $P(z)$ is now immediately given by (48).

Remark 4.2 The results of Section 3, combined with Formula (45), imply that the asymptotic behaviour of p_k for $k \to \infty$ is determined by the in absolute value smallest β_i outside the unit circle.

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