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# On the Symmetrical Shortest Queue and the Compensation Approach

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**Abstract**

The compensation approach has recently been introduced for the determination of the stationary distribution of two-dimensional nearest-neighbour random walks without one-step transitions to the North, the North-East and the East. The present study compares this approach for the symmetrical shortest queueing model with the analytical solution. It is shown that the final results are identical.

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## Introduction

The 'compensation approach' has recently been introduced for the determination of the stationary distribution of queueing processes which can be modelled by two-dimensional nearest-neighbour random walks for which one-step transition probabilities to the North, North-East and East are not possible, cf. [1],[2],[3],[4]. This approach leads to a series representation for the stationary state probabilities of which the terms are obtained by a recursive algorithm. In [1] and [2] it is shown that the compensation approach generates the exact solution for the stationary state probabilities of the symmetrical shortest queueing model, whenever it is positive recurrent. In order to obtain a deeper insight concerning the compensation approach we compare the results obtained by the compensation approach with those resulting from the analytic solution of the symmetrical shortest queue [5].

This comparison has been discussed in the report [7]. However, as Adan, Wessels and Zijm have pointed out (private communication) the conclusions drawn in that report are based on a serious mistake, and a misinterpretation of the algorithm of the compensation approach.

The present study is a corrected version of the report [7].

In Section 2 several results obtained in [5] are recapitulated. The two key functions $\Omega(r)$ and $\Phi(t)$ in the analysis of the symmetrical shortest queue are both meromorphic and have been represented in [5] as infinite product forms, see (2.8) below. For our present goal we need their partial fractions representation. From this representation the expressions for the state probabilities are derived in Section 3. With $\mathbf{x}_i$ the queue length of server $i$, $i = 1,2$, the $p_{km}$, $\Omega(r)$ and $\Phi(t)$ are defined by (note the symmetry): for $k, m \in \{0,1,2,\ldots\}$,

$$p_{km} := \Pr\{\mathbf{x}_1 = k, \mathbf{x}_2 = k+m\} = \Pr\{\mathbf{x}_2 = k, \mathbf{x}_1 = k+m\}, \tag{1.1}$$

$$\Omega(r) := \sum_{m=0}^{\infty} p_{0m} r^m, \qquad |r| \le 1, \qquad \Phi(t) := \sum_{k=0}^{\infty} p_{k0} t^k, \qquad |t| \le 1.$$

Section 4 starts with the description of the compensation approach, and the derivations of the expressions for $\Omega^{(a)}(r)$ and $\Phi^{(a)}(t)$, the analogous functions of those in (1.1) but based on the compensation approach. The expressions for $p_{km}^{(a)}$ are also derived in Section 4. Next we compare the expressions for

$\Omega(r)$ and $\Omega^{(a)}(r)$, and similarly those for $\Phi^{(a)}(t)$ and $\Phi(t)$. These comparisons lead to the conclusion that $\Omega(r)$ and $\Omega^{(a)}(r)$ are identical, similarly for $\Phi(t)$ and $\Phi^{(a)}(t)$.

From this it is concluded that the analytical analysis and the compensation approach lead to identical results. Section 4 concludes with a comparison of the steps in the iterative algorithm of the compensation approach and the successive determination of the poles and their residues in the construction of the meromorphic functions $\Phi(\cdot)$ and $\Omega(\cdot)$.

## 2. The partial fractions representation

The bivariate generating function of the stationary distribution of the queue lengths $(\mathbf{x}_1, \mathbf{x}_2)$ can be expressed as a linear combination of the generating functions

$$\Omega(r) := \mathrm{E}\{r^{\mathbf{x}_1}(\mathbf{x}_2 = 0)\}, \quad |r| \leq 1, \tag{2.1}$$

$$\Phi(t) := \mathrm{E}\{t^{\mathbf{x}_1}(\mathbf{x}_1 = \mathbf{x}_2)\}, \quad |t| \leq 1,$$

see [5]; here $\mathbf{x}_i$ is the queue length at server $i$, $i = 1, 2$. These functions are meromorphic functions, their infinite product form representations have been derived in [5]. For the purpose of the present study their partial fractions representation is needed. In the present section they will be derived.

Remark 2.1. The definitions of symbols used in the present text differs only in some minor points from those in [5]. □

Put, cf.(1.1),

$$F(r,t) := \mathrm{E}\{t^{\mathbf{x}_1} r^{\mathbf{x}_2 - \mathbf{x}_1}(\mathbf{x}_2 \geq \mathbf{x}_1)\} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} p_{km} t^k r^m, \qquad |t| \leq 1, \ |r| \leq 1. \tag{2.2}$$

From (2.3) of [5] it is seen that for $|r| \leq 1$, $|t| \leq 1$,

$$k_1(r,t)\mathrm{F}(r,t) = r[(r-t)\Omega(r) - \{r + \frac{1}{2}art - \frac{1}{2}(2+a)t\}\Phi(t)] + k_1(r,t)\Phi(t), \tag{2.3}$$

with

$$k_1(r,t) := at^2 + [1 - (2+a)r]t + r^2. \tag{2.4}$$

Denote by $t_1(r)$, $t_2(r)$ for fixed $r$ the two zeros of $k_1(r,t)$, they may be so defined that, cf.(3.1) of [5],

$$|t_1(r)| < |r| < |t_2(r)| \quad \text{for} \quad |r| \geq 1, \ r \neq 1, \tag{2.5}$$

$$t_1(1) = \min(1, \frac{1}{a}), \qquad t_2(1) = \max(1, \frac{1}{a});$$

note that

$$0 < a < 2, \tag{2.6}$$

is the necessary and sufficient condition for the existence of a stationary distribution.

In [5] the sequences $r_n$, $n = 0, 1, 2, \ldots$; $t_n$, $n = 0, 1, 2, \ldots$, are defined by

$$r_0 = \frac{2}{a}, \quad t_0 = \frac{4}{a^2}, \tag{2.7}$$

$$k_1(r_n, t_{n-1}) = 0, \qquad k_1(r_n, t_n) = 0,$$

$$r_0 < t_0 < r_1 < t_1 < r_2 \quad \dots \quad < t_{n-1} < r_n < t_n \dots,$$

$$t_{n-1} = t_1(r_n),\ n = 1, 2, \dots, \qquad t_n = t_2(r_n),\ n = 0, 1, 2, \dots.$$

From (4.7) of [5] we have for $|r| \leq 1$, $|t| \leq 1$,

$$\Omega(r) = \frac{1}{2}(2-a)\{\prod_{n=0}^{\infty} \frac{r_n^- - r}{r_n^- - 1}\}\{\prod_{n=1}^{\infty} \frac{r_n - 1}{r_n - r}\}, \tag{2.8}$$

$$\Phi(t) = \frac{1}{1+a}\{\prod_{n=1}^{\infty} \frac{t_n^- - t}{t_n^- - 1}\}\{\prod_{n=0}^{\infty} \frac{t_n - 1}{t_n - t}\}.$$

These functions can be continued meromorphically, and

| | | | |
|---|---|---|---|
| $r_n^-, n = 0, 1, 2, \dots,$ | are the zeros of | $\Omega(r),$ | $r_n^- < 0,$ |
| $t_n^-, n = 1, 2, \dots,$ | are the zeros of | $\Phi(t),$ | $t_n^- < 0,$ |
| $r_n, n = 1, 2, \dots$ | are the poles of | $\Omega(r),$ | $r_n > 0,$ |
| $t_n, n = 0, 1, \dots$ | are the poles of | $\Phi(t),$ | $t_n > 0,$ |

all poles and zeros have multiplicity one, see (3.8) and (3.10) of [5].

For the construction of the partial fractions representations of $\Omega(r)$ and $\Phi(t)$ we need their residues $\omega_n$ and $\phi_n$:

$$\begin{aligned} \phi_n &:= \lim_{t \to t_n} (t - t_n)\Phi(t), \qquad && n = 0, 1, 2, \dots, \\ \omega_n &:= \lim_{r \to r_n} (r - r_n)\Omega(r), \qquad && n = 1, 2, \dots. \end{aligned} \tag{2.9}$$

These residues can be obtained immediately from (2.8). However, since we need the recursive relations between these residues, it is more effective to start from the relations which have led in [5] to the expression (2.8), cf. (3.6) of [5]. These relations are: for $j = 1, 2$,

$$\begin{aligned} &\text{(i)} \quad \Omega(r) + k_2(r, t_j(r))\Phi(t_j(r)) = 0 \qquad && \text{for all} \quad r \neq r_n, \\ &\text{(ii)} \quad \lim_{r \to r_n} [\Omega(r) + k_2(r, t_j(r))\Phi(t_j(r))] = 0 \qquad && \text{for all} \quad n = 0, 1, 2, \dots, \end{aligned} \tag{2.10}$$

with

$$k_2(r, t) := -1 + \frac{1}{2}a^2 t - \frac{1}{2}ar. \tag{2.11}$$

The relations (2.10) follow from the condition that F$(r, t)$, cf.(2.2), should be finite for all $|r| \leq 1$, $|t| \leq 1$, see [5].

To derive the recursive relations for the residues note that $k_1(r, t_j(r)) = 0$, $j = 1, 2$, implies that

$$\frac{\mathrm{d}t_j(r)}{\mathrm{d}r} = -\frac{2r - (2+a)t_j(r)}{2at_j(r) + 1 - (2+a)r}. \tag{2.12}$$

Because it follows from the properties of the roots of the quadratic equation $k_1(r, t) = 0$, cf. (b.4) of [5], that

$$r_n + r_{n+1} = (2+a)t_n, \quad t_n + t_{n+1} = \frac{1}{a}[(2+a)r_{n+1} - 1], \tag{2.13}$$

we obtain

$$\frac{\mathrm{d}t_2(r)}{\mathrm{d}r}|_{r=r_n} = -\frac{1}{a}\frac{r_n - r_{n+1}}{t_n - t_{n-1}}, \qquad \frac{\mathrm{d}t_1(r)}{\mathrm{d}r}|_{r=r_n} = \frac{r_n - r_{n-1}}{t_{n-1} - t_n}. \tag{2.14}$$

From (2.7) and (2.9) we obtain: for $j = 1, 2$,

$$\lim_{r\to r_n} (r - r_n)\Phi(t_2(r)) = \phi_n \left[\frac{\mathrm{d}t_2(r)}{\mathrm{d}r}\right]^{-1}_{r=r_n}, \quad n = 0, 1, 2, \ldots, \tag{2.15}$$

$$\lim_{r\to r_n} (r - r_n)\Phi(t_1(r)) = \phi_{n-1} \left[\frac{\mathrm{d}t_1(r)}{\mathrm{d}r}\right]^{-1}_{r=r_n}, \quad n = 1, 2, \ldots.$$

Hence from (2.9), (2.10)ii, (2.14) and (2.15), for $n = 1, 2, \ldots$,

$$\begin{aligned} \omega_n &= ak_2(r_n, t_n)\frac{t_n - t_{n-1}}{r_n - r_{n+1}}\phi_n \\ &= ak_2(r_n, t_{n-1})\frac{t_{n-1} - t_n}{r_n - r_{n-1}}\phi_{n-1}, \end{aligned} \tag{2.16}$$

and so: for $n = 1, 2, \ldots$

$$\frac{\phi_n}{\phi_{n-1}} = \frac{-1 + \frac{1}{2}a^2 t_{n-1} - \frac{1}{2}ar_n}{-1 + \frac{1}{2}a^2 t_n - \frac{1}{2}ar_n}\ \frac{r_{n+1} - r_n}{r_n - r_{n-1}}, \tag{2.17}$$

$$\frac{\omega_{n+1}}{\omega_n} = \frac{-1 + \frac{1}{2}a^2 t_n - \frac{1}{2}ar_{n+1}}{-1 + \frac{1}{2}a^2 t_n - \frac{1}{2}ar_n}\ \frac{t_{n+1} - t_n}{t_n - t_{n-1}}.$$

From (2.16) and (2.17) it is seen that the $\omega_n$ and $\phi_n$ can be recursively calculated once $\phi_0$ is known. For the determination of $\phi_0$ see (2.29) below.

We consider the relations (2.17) for $n \to \infty$. It is readily seen from (2.4) and (2.7) that $r_n \to \infty$, $t_n \to \infty$ for $n \to \infty$ and that, cf. (4.2), (4.3), (b.7) of [5]: for $n \to \infty$,

$$\frac{t_n}{r_n} \to \delta, \qquad \frac{r_{n+1}}{t_n} \to a\delta, \qquad \frac{t_{n+1}}{t_n} \to a\delta^2, \qquad \frac{r_{n+1}}{r_n} \to a\delta^2, \tag{2.18}$$

$$\delta := \frac{1}{2a}[2 + a + \sqrt{a^2 + 4}] > \frac{2}{a} > 1.$$

From (2.17) and (2.18) it is readily seen that $\phi_n/\phi_{n-1}$, and similarly $\omega_{n+1}/\omega_n$, has a limit for $n \to \infty$ and

$$\begin{aligned} &\lim_{n\to\infty} \frac{\phi_n}{\phi_{n-1}} = -\mu_\phi, &&\lim_{n\to\infty} \frac{\omega_n}{\omega_{n-1}} = -\mu_\omega, \\ &\mu_\phi := a\delta\frac{\delta - 1}{a\delta - 1}, &&\mu_\omega := a^2\delta^3\frac{\delta - 1}{a\delta - 1}. \end{aligned} \tag{2.19}$$

Because $\delta$, cf.(2.18), satisfies

$$a\delta^2 - (2+a)\delta + 1 = 0, \tag{2.20}$$

it follows readily from (2.18) that

$$\frac{1}{\delta}\frac{\delta-1}{a\delta-1}<1, \qquad \mu_\phi>1, \qquad \mu_\omega>1. \tag{2.21}$$

Put

$$\begin{aligned}
&\text{(i)} \quad \tilde{\Phi}(t):=\sum_{n=0}^{\infty}\frac{\phi_n}{t-t_n},\\
&\text{(ii)} \quad \tilde{\Omega}(r):=\sum_{n=1}^{\infty}\frac{\omega_n}{r-r_n}\frac{r}{r_n}.
\end{aligned} \tag{2.22}$$

To justify the definitions note that for $n\to\infty$,

$$\begin{aligned}
&\frac{\phi_{n+1}}{t-t_{n+1}}/\frac{\phi_n}{t-t_n}=\frac{\phi_{n+1}}{\phi_n}\frac{t_n}{t_{n+1}}\frac{1-\frac{t}{t_n}}{1-\frac{t}{t_{n+1}}}\to-\frac{1}{\delta}\frac{\delta-1}{a\delta-1},\\
&\frac{\omega_{n+1}}{(r-r_{n+1})r_{n+1}}/\frac{\omega_n}{(r-r_n)r_n}=\frac{\omega_{n+1}}{\omega_n}\left(\frac{r_n}{r_{n+1}}\right)^2\frac{1-\frac{r}{r_n}}{1-\frac{r}{r_{n+1}}}\to-\frac{1}{\delta}\frac{\delta-1}{a\delta-1},
\end{aligned} \tag{2.23}$$

for every finite $t\neq t_n$, $t_{n+1}$ and $r\neq r_n$, $r_{n+1}$, respectively. Hence, the sum in the righthand side of (2.22)i converges absolutely for every finite $t\neq t_n$, $n=0,1,2,\ldots$, because of (2.21); similarly does the sum in (2.22)ii for every finite $r\neq r_n$, $n=1,2,\ldots$. Consequently the functions $\tilde{\Phi}(t)$ and $\tilde{\Omega}(r)$ are well-defined meromorphic functions.

Obviously $\Phi(t)$, cf.(2.8), and $\tilde{\Phi}(t)$ have the same pole set and their poles have the same multiplicity, similarly for $\Omega(r)$ and $\tilde{\Omega}(r)$. Consequently we may write

$$\begin{aligned}
\Phi(t)&=\hat{\Phi}(t)+\tilde{\Phi}(t),\\
\Omega(r)&=\hat{\Omega}(r)+\tilde{\Omega}(r),
\end{aligned} \tag{2.24}$$

with $\hat{\Phi}(t)$ and $\hat{\Omega}(r)$ entire functions. In appendix A it is shown that $\hat{\Phi}(t)$ is identically zero and that $\hat{\Omega}(r)$ is constant. Hence from (2.22) and (2.24) we have

$$\begin{aligned}
\Phi(t)&=\sum_{n=0}^{\infty}\frac{\phi_n}{t-t_n},\\
\Omega(r)&=\Omega(0)+\sum_{n=1}^{\infty}\frac{\omega_n}{r-r_n}\frac{r}{r_n}.
\end{aligned} \tag{2.25}$$

From (4.8) and (4.9) of [5] we have

$$\Phi(1)=\frac{1}{1+a}, \qquad \Omega(1)=\Phi(\frac{1}{a})=\frac{1}{2}(2-a), \tag{2.26}$$

and from (2.1) and (2.25),

$$\Omega(0)=\Phi(0)=-\sum_{n=0}^{\infty}\frac{\phi_n}{t_n}. \tag{2.27}$$

In Section 3 it is also shown, see (3.15), that

$$\Omega^{(1)}(0) = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{(1+at_n)t_n}. \tag{2.28}$$

In appendix B it is derived that

$$\phi_0 = -\frac{4}{a^2}\frac{(2-a)(4-a)}{4+a}, \tag{2.29}$$

$$\omega_1 = -\frac{2}{a^3}\frac{(2-a)(4-a)}{4+a}(a^2+4a+16).$$

## 3. The expression for $\Pr\{\mathbf{x}_1 = k,\ \mathbf{x}_2 = k+m\}$

In this section we derive the explicit expressions for the stationary state probabilities, cf.(1.1),

$$p_{km} := \Pr\{\mathbf{x}_1 = k,\ \mathbf{x}_2 = k+m\}, \quad k = 0,1,2,\ldots;\ m = 0,1,2,\ldots. \tag{3.1}$$

From (2.3) we have for $|t| = 1$, $m = 1,2,\ldots$,

$$\sum_{k=0}^{\infty} p_{km}t^k = \frac{1}{2\pi\mathrm{i}}\int_{|r|=1}\frac{\mathrm{d}r}{r^m}\frac{(r-t)\Omega(r) - [r+\frac{1}{2}art - \frac{1}{2}(2+a)t]\Phi(t)}{k_1(r,t)} + \frac{1}{2\pi\mathrm{i}}\int_{|r|=1}\frac{\mathrm{d}r}{r^{m+1}}\Phi(t). \tag{3.2}$$

Note that

$$\frac{1}{2\pi\mathrm{i}}\int_{|r|=1}\frac{\mathrm{d}r}{r^{m+1}} \begin{aligned} &= 0 \quad \text{for} \quad m = 1,2,\ldots, \\ &= 1 \quad \text{for} \quad m = 0. \end{aligned} \tag{3.3}$$

To evaluate the integral in (3.2) it is noted that (2.10) implies that the zeros of $k_1(r,t)$ with $|t| = 1$ are no poles of the integrand so that its only poles are those of $\Omega(r)$. Put

$$R_N := \frac{1}{2}(r_{N+1} + r_N), \tag{3.4}$$

and note that, cf.(2.5),

$$k_1(r,t) = a[t - t_1(r)][t - t_2(r)]. \tag{3.5}$$

By applying Cauchy's theorem we obtain from (3.2) and (3.3) for $|t| = 1$, $m = 1,2,\ldots$,

$$\begin{aligned} \sum_{k=0}^{\infty} p_{km}t^k = & -\sum_{n=1}^{N}\frac{1}{r_n^m}\lim_{r\to r_n}(r-r_n)\frac{(r-t)\Omega(r) - [r+\frac{1}{2}art - \frac{1}{2}(2+a)t]\Phi(t)}{a(t-t_1(r))(t-t_2(r))} \\ & + \frac{1}{2\pi\mathrm{i}}\int_{|r|=R_N}\frac{\mathrm{d}r}{r^m}\frac{(r-t)\Omega(r) - [r+\frac{1}{2}art - \frac{1}{2}(2+a)t]\Phi(t)}{a(t-t_1(r))(t-t_2(r))}. \end{aligned} \tag{3.6}$$

In appendix C it is shown that for $N \to \infty$,

$$\left[\frac{r-t}{r^2}\,\Omega(r)\right]_{r=R_N} \to 0 \qquad \text{for } N \to \infty. \tag{3.7}$$

Because $t_j(r)/r$, $j = 1,2$, has a finite limit for $r \to \infty$, it is seen that the integrand of the integral in (3.6) behaves as $R_N^{-m}$ for $N \to \infty$. Consequently the limit of this integral for $N \to \infty$ is zero for every $m = 1, 2, \ldots$, cf. also (3.7).

Hence it follows from (3.6) that: for $|t| = 1$, $m \geq 1$,

$$\sum_{k=0}^{\infty} p_{km} t^k = \sum_{n=1}^{\infty} \frac{\omega_n}{r_n^m} \frac{t - r_n}{a[t - t_1(r_n)][t - t_2(r_n)]} = \frac{1}{a} \sum_{n=1}^{\infty} \frac{-\omega_n}{t_n - t_{n-1}} \left[ \frac{t_n - r_n}{t_n - t} - \frac{t_{n-1} - r_n}{t_{n-1} - t} \right]. \quad (3.8)$$

From (3.8) it follows by expanding $[t_n - t]^{-1}$ and $[t_{n-1} - t]^{-1}$ in power series of $t$, note that $t_n > 1$, $n = 0, 1, 2, \ldots$, and by equating the coefficients of $t^h$, $h = 0, 1, 2, \ldots$, that: for $m = 1, 2, \ldots$; $k = 0, 1, 2, \ldots$,

$$p_{km} = \frac{1}{a} \sum_{n=1}^{\infty} \frac{-\omega_n}{r_n^m} \left[ \frac{t_n - r_n}{t_n - t_{n-1}} t_n^{-(k+1)} + \frac{r_n - t_{n-1}}{t_n - t_{n-1}} t_{n-1}^{-(k+1)} \right]. \quad (3.9)$$

For $m = 1$ we note that the first integral in (3.2) has a simple pole in $r = 0$, and so we obtain for $|t| = 1$,

$$\sum_{k=0}^{\infty} p_{k1} t^k = \frac{-2\Omega(0) + (2 + a)\Phi(t)}{2(at + 1)}. \quad (3.10)$$

Because the lefthand side in (3.10) cannot have a pole in $t = -\frac{1}{a}$ if $2 > a \geq 1$, it follows that

$$\Phi(-\frac{1}{a}) = \frac{2}{2 + a} \Omega(0) \text{ for } 2 > a \geq 1. \quad (3.11)$$

We next show that (3.11) also holds for $0 < a < 1$. From (3.8) we have for $m = 1$, $|t| = 1$,

$$\sum_{k=0}^{\infty} p_{k1} t^k = \frac{1}{a} \sum_{n=1}^{\infty} \frac{\omega_n}{r_n} \frac{1}{t_n - t_{n-1}} \left[ \frac{t_{n-1} - r_n}{t_{n-1} - t} - \frac{t_n - r_n}{t_n - t} \right]. \quad (3.12)$$

It is readily verified that the righthand side is a well-defined meromorphic function and that its only poles are $t_n$, $n = 0, 1, 2, \ldots$. Consequently $t = -\frac{1}{a}$ is not a pole of the lefthand side, and so (3.11) holds for $0 < a < 2$.

From (2.25) and (3.10) we have for $|t| = 1$,

$$\begin{aligned} \sum_{k=0}^{\infty} p_{k1} t^k &= -\frac{\Omega(0)}{1 + at} + \frac{1}{2}(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{t - t_n} \frac{1}{1 + at} \quad (3.13) \\ &= \frac{-\Omega(0)}{1 + at} + \frac{1}{2}(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{1 + at_n} \{ \frac{1}{t - t_n} - \frac{a}{1 + at} \} = \\ &- \frac{1}{1 + at} \left[ \Omega(0) + \frac{1}{2} a(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{1 + at_n} \right] + \frac{1}{2}(2 + a) \sum_{n=0}^{\infty} \frac{\phi_n}{1 + at_n} \frac{1}{t - t_n}. \end{aligned}$$

The relation (3.13) for $|t| = 1$ can be continued meromorphically, note that the last sum is a well-defined meromorphic function. Above it has been shown that $t = t_n$, $n = 0, 1, 2, \ldots$, are the only poles of the lefthand side of (3.13). So $t = -\frac{1}{a}$ is not a pole and consequently

$$\begin{aligned}
&\text{(i)} \quad \Omega(0) = p_{00} = \frac{1}{2}a(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{1+at_n},\\
&\text{(ii)} \quad \sum_{k=0}^{\infty} p_{k1}t^k = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{\phi_n}{1+at_n}\frac{1}{t-t_n}.
\end{aligned} \tag{3.14}$$

Because for $t=0$ the function $k_1(r,0)$ has $r=0$ as a zero of multiplicity two it follows from (2.10)i that

$$\left[\Omega^{(1)}(r) + [\frac{1}{2}a^2\frac{\mathrm{d}t_1(r)}{\mathrm{d}r} - \frac{1}{2}a]\Phi(r) - \Phi^{(1)}(t_1(r))\frac{\mathrm{d}t_1(r)}{\mathrm{d}r}\right]_{r=0} = 0,$$

so from (2.12),

$$p_{01} = \Omega^{(1)}(0) = \frac{1}{2}a\Phi(0) = \frac{1}{2}ap_{00} = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{(1+at_n)t_n}, \tag{3.15}$$

here the last relation follows from (3.14)ii. From (3.14)i and (3.15) we obtain again, cf. (2.27),

$$p_{00} = \Omega(0) = \Phi(0) = \sum_{n=0}^{\infty}\frac{-\phi_n}{t_n}. \tag{3.16}$$

From the results derived above we obtain

$$\begin{aligned}
&\text{(i)} \quad p_{km} = -\frac{1}{a}\sum_{n=1}^{\infty}\frac{\omega_n}{r_n^m}\left[\frac{t_n-r_n}{t_n-t_{n-1}}t_n^{-(k+1)} + \frac{r_n-t_{n-1}}{t_n-t_{n-1}}t_{n-1}^{-(k+1)}\right], k\geq 0,\ m\geq 1,\\
&\text{(ii)} \quad p_{k1} = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{1+at_n}t_n^{-(k+1)}, \quad k\geq 0,\\
&\text{(iii)} \quad p_{k0} = \textstyle\sum_{n=0}^{\infty}(-\phi_n)t_n^{-(k+1)}, \quad k\geq 0,\\
&\text{(iv)} \quad p_{0m} = 2\sum_{n=1}^{\infty}(-\omega_n)r_n^{-(m+1)}, \quad m\geq 2,\\
&\text{(v)} \quad p_{00} = \Omega(0) = \Phi(0) = \sum_{n=0}^{\infty}\frac{-\phi_n}{t_n},\\
&\text{(vi)} \quad p_{01} = \frac{1}{2}(2+a)\sum_{n=0}^{\infty}\frac{-\phi_n}{(1+at_n)t_n}.
\end{aligned} \tag{3.17}$$

From (3.17) it is simple to obtain the asymptotic relations

$$p_{k0} = -\phi_0 t_0^{-(k+1)}\{1+\mathrm{O}(\left(\frac{t_0}{t_1}\right)^{k+1})\} \qquad \text{for} \qquad k\to\infty, \tag{3.18}$$

$$p_{0m} = -\omega_1 r_1^{-(m+1)}\{1 + \mathrm{O}(\left(\frac{r_1}{r_2}\right)^{m+1})\} \qquad \text{for} \qquad m \to \infty,$$

$$p_{k1} = \frac{1}{2}(2+a)\frac{-\phi_0}{1+at_0}t_0^{-(k+1)}\{1 + \mathrm{O}(\left(\frac{t_0}{t_1}\right)^{k+1})\} \quad \text{for} \qquad k \to \infty.$$

## 4. The compensation approach

The 'compensation approach' applied to the symmetrical shortest queueing problem is exposed in the studies [1] and [2]. By $p_{km}^{(w)}$ and $p_{km}^{(a)}$ we shall indicate the stationary state probabilities when calculated by the compensation approach as applied in [1] and [2], respectively, analogously for $\Omega^{(a)}(r)$, $\Phi^{(a)}(t)$, $\Phi^{(w)}(t)$. Note that the notations in [1] and [2] differ slightly.

The solution obtained by the compensation approach as exposed in [1] and [2] reads: for $k = 0, 1, 2, \dots$,

$$p_{km}^{(a)} = p_{km}^{(w)} = C^{-1}\sum_{i=0}^{\infty} d_i[c_i\alpha_i^k + c_{i+1}\alpha_{i+1}^k]\beta_i^m, \quad m = 1, 2, \dots, \tag{4.1}$$

$$p_{k0}^{(a)} = C^{-1}\sum_{i=0}^{\infty} c_i f_i \alpha_i^k, \tag{4.2}$$

for $p_{k0}^{(w)}$ see (4.8)iv, below.

The constants in (4.1) and (4.2) are defined by, cf.(3.20),…,(3.25) of [2]:

$$\alpha_n^{-1} = t_n, \quad \beta_n^{-1} = r_{n+1}, \quad n = 0, 1, 2, \dots, \text{ cf.(2.8)}, \tag{4.3}$$

and for $i = 0, 1, 2, \dots$,

$$c_{i+1} = \frac{\alpha_{i+1} - \beta_i}{\beta_i - \alpha_i}c_i, \qquad c_0 = 1, \tag{4.4}$$

$$d_{i+1} = -\frac{(\alpha_{i+1} + \frac{1}{2}a)/\beta_{i+1} - (1 + \frac{1}{2}a)}{(\alpha_{i+1} + \frac{1}{2}a)/\beta_i - (1 + \frac{1}{2}a)}d_i, \qquad d_0 = 1,$$

$$f_{i+1} = \frac{\alpha_{i+1}}{\alpha_{i+1} + \frac{1}{2}a}(d_i + d_{i+1}), \qquad f_0 = \frac{\alpha_0}{\alpha_0 + \frac{1}{2}a};$$

$C$ is determined by the norming condition

$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} p_{km}^{(a)} + \sum_{k=0} p_{k0}^{(a)} = 1. \tag{4.5}$$

Remark 4.1. An explicit expression for $p_{k0}^{(w)}$ is not given in [1], its determination is only indicated; below we give the expression for $p_{k0}^{(w)}$ according to [1], see (4.8)iv. □

From the Kolmogorov equations for the stationary state probabilities the following equivalent set of equations is derived in [1]:

$$
\begin{aligned}
&\text{(i)} && (a+2)p_{km} = ap_{k-1,m+1} + p_{k,m+1} + p_{k+1,m-1}, && k \geq 1,\ m \geq 2;\\
&\text{(ii)} && (a+2)p_{k1} = ap_{k-1,2} + p_{k2} + \frac{2}{a+2}(ap_{k1} + p_{k+1,1}) + \frac{a}{a+2}(ap_{k-1,1} + p_{k1}), && k \geq 1;\\
&\text{(iii)} && (a+1)p_{0m} = p_{0,m+1} + p_{1,m-1}, && m \geq 2;\\
&\text{(iv)} && -(a+1)p_{01} + p_{02} + \frac{2}{a+2}(ap_{01} + p_{11}) + p_{01} = 0, &&\\
&\text{(v)} && \frac{1}{2}(a+2)p_{k0} = ap_{k-1,1} + p_{k1}, && k \geq 1,\\
&\text{(vi)} && ap_{00} = p_{01}. &&
\end{aligned}
\tag{4.6}
$$

For our analysis of the compensation approach we need some asymptotic relations for the coefficients $c_i$, $d_i$ and $f_i$, see [2]. They are easily derived from (4.4) by using (2.18) and (4.3). It results: for $i \to \infty$,

$$
\begin{aligned}
&\text{(i)} && 0 < \frac{c_{i+1}}{c_i} \to \frac{1}{\delta}\frac{\delta - 1}{a\delta - 1} < 1,\\
&\text{(ii)} && -\frac{d_{i+1}}{d_i} \to a\delta^2 > 1,\\
&\text{(iii)} && -\frac{d_{i+1}(c_{i+1} + c_{i+2})}{d_i(c_i + c_{i+1})} \to a\delta\frac{\delta - 1}{a\delta - 1} > 1,\\
&\text{(iv)} && -\frac{d_{i+1}(c_{i+1} + c_{i+2})\beta_{i+1}}{d_i(c_i + c_{i+1})\beta_i} \to \frac{1}{\delta}\frac{\delta - 1}{a\delta - 1} < 1,\\
&\text{(v)} && \frac{c_{i+1}f_{i+1}}{c_i f_i} \to \frac{1}{\delta^2}\left[\frac{\delta - 1}{a\delta - 1}\right]^2 < 1.
\end{aligned}
\tag{4.7}
$$

From the relations above the expressions for the following generating functions are readily derived. For $|r| \leq 1$, $|t| \leq 1$,

$$
\begin{aligned}
&\text{(i)} && \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} p_{km}^{(a)} t^k r^m = C^{-1}\sum_{i=0}^{\infty} d_i\left[\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1}t}\right]\frac{\beta_i r}{1-\beta_i r},\\
&\text{(ii)} && \Omega^{(a)}(r) := \sum_{m=0}^{\infty} p_{0m}^{(a)} r^m = p_{00}^{(a)} + C^{-1}\sum_{i=0}^{\infty} d_i(c_i + c_{i+1})\frac{\beta_i r}{1-\beta_i r},\\
&\text{(iii)} && \Phi^{(a)}(t) := \sum_{k=0}^{\infty} p_{k0}^{(a)} t^k = C^{-1}\sum_{i=0}^{\infty}\frac{c_i f_i}{1-\alpha_i t},\\
&\text{(iv)} && \Phi^{(w)}(t) := \sum_{k=0}^{\infty} p_{k0}^{(w)} t^k = \frac{2}{a+2}\left[p_{00}^{(w)} + (1+at)\sum_{k=0}^{\infty} p_{k1}^{(a)} t^k\right],\\
& && = \frac{2}{a+2}\left[p_{00}^{(w)} + (1+at)C^{-1}\sum_{i=0}^{\infty} d_i\beta_i\left[\frac{c_i}{1-\alpha_i t} + \frac{c_{i+1}}{1-\alpha_{i+1}t}\right]\right],
\end{aligned}
\tag{4.8}
$$

(v) $p^{(w)}_{00} = \frac{2}{a} C^{-1} \sum_{i=0}^{\infty} d_i \beta_i (c_i + c_{i+1}).$

The relation (4.8)i follows directly from (4.1), also (4.8)ii is obtained from (4.1); (4.8)iii is derived from (4.2)i. The relation for $\Phi^{(w)}(t)$ is obtained from (4.6)v and iv, and the second expression in (4.18)iv is obtained by using (4.8)i. By taking $t = 0$ in (4.8)iv and noting that $\Phi^{(w)}(0) = p^{(w)}_{00}$, the relation (4.8)v results. By using the asymptotic expressions (4.7) it is readily verified that the righthand sides in (4.8)i,…,iv, are for $|r| > 1$ and/or $|t| > 1$ well-defined meromorphic functions and so the lefthand sides in (4.8)i,…,iv, can be continued meromorphically into $|r| > 1$, and/or $|t| > 1$.

We next compare the results according to the compensation approach with those obtained in Sections 2 and 3. First we compare some asymptotic results.

From (4.8) the following asymptotic results are readily derived.

$$\begin{aligned}
&\text{(i)} \quad p^{(a)}_{k0} = C^{-1} c_0 f_0 \alpha_0^k [1 + \mathrm{O}(\left(\frac{\alpha_1}{\alpha_0}\right)^k)] && \text{for} \quad k \to \infty,\\
&\text{(ii)} \quad p^{(w)}_{k0} = \frac{2}{a+2} C^{-1} d_0 \beta_0 c_0 \alpha_0^k (1 + \frac{a}{\alpha_0})[1 + \mathrm{O}(\left(\frac{\alpha_1}{\alpha_0}\right)^k)] && \text{for} \quad k \to \infty,\\
&\text{(iii)} \quad p^{(a)}_{k1} = C^{-1} d_0 \beta_0 \alpha_0^k [1 + \mathrm{O}(\left(\frac{\alpha_1}{\alpha_0}\right)^k)] && \text{for} \quad k \to \infty,\\
&\text{(iv)} \quad p^{(a)}_{0m} = C^{-1} d_0 (c_0 + c_1) \beta_0^m [1 + \mathrm{O}(\left(\frac{\beta_1}{\beta_0}\right)^m)] && \text{for} \quad m \to \infty.
\end{aligned} \tag{4.9}$$

From (2.7), (4.3) and (4.4) it is seen that

$$\begin{aligned}
&\alpha_0^{-1} = t_0 = \frac{4}{a^2}, && \beta_0^{-1} = r_1 = \frac{2}{a^2}(a+4),\\
&\alpha_1^{-1} = t_1 = \frac{(a+4)^2}{a^3}, && \beta_1^{-1} = r_2 = \frac{a+4}{a^3}(a^2 + 4a + 8),\\
&c_1 + c_0 = \frac{a^2 + 4a + 16}{(2+a)(4+a)}, && f_0 = \frac{a}{a+2}.
\end{aligned} \tag{4.10}$$

It follows from (3.18), (2.19) and from (4.9) and (4.10),

$$\begin{aligned}
&\text{(i)} \quad p_{k0} \sim \tfrac{(2-a)(4-a)}{4+a} \left(\tfrac{a^2}{4}\right)^k, && p^{(a)}_{k0} \sim p^{(w)}_{k0} \sim C^{-1} \tfrac{a}{2+a} \left(\tfrac{a^2}{4}\right)^k, && k \to \infty,\\
&\text{(ii)} \quad p_{k1} \sim a \tfrac{(4-a^2)(4-a)}{2(4+a)^2} \left(\tfrac{a^2}{4}\right)^k, && p^{(a)}_{k1} \sim C^{-1} \tfrac{a^2}{2(4+a)} a \left(\tfrac{a^2}{4}\right)^k, && k \to \infty,\\
&\text{(iii)} \quad p_{0m} \sim \tfrac{(2-a)(4-a)}{a(4+a)^2} (a^2 + 4a + 16) \left[\tfrac{a^2}{2(a+4)}\right]^m, && p^{(a)}_{0m} \sim C^{-1} \tfrac{a^2+4a+16}{(2+a)(4+a)} \left[\tfrac{a^2}{2(a+4)}\right]^m, && m \to \infty.
\end{aligned} \tag{4.11}$$

In [1] and [2] the constant $C$ is shown to be given by

$$C = \frac{a(4+a)}{(4-a^2)(4-a)}; \tag{4.12}$$

its derivation is based on $\Omega^{(a)}(1) = 1 - \frac{1}{2}a$, a result which follows from the norming condition, and the relation (2.3), cf.(3.34) of [2] or (32) of [1], and by considering (2.3) for $t = t_0 = 4/a^2$.

When inserting the expression (4.12) for $C$ into (4.11) it is seen that the corresponding relations in (4.11) are identical.

Comparison of the expressions for $\Omega^{(a)}(r)$, cf. (4.8)ii, and $\Omega(r)$, cf. (2.25), (3.16), shows that these functions have the same pole set and the same structure, it is readily seen that the similar conclusion holds for $\Phi^{(a)}(t)$, cf. (4.8)iii, and $\Phi(t)$, cf. (2.25).

For a further comparison we consider the residues of the functions just mentioned.

Put, cf. (4.3) and (4.8), for $i = 0, 1, 2, \ldots$,

$$\begin{aligned} \omega_{i+1}^{(a)} &:= \lim_{r \to r_{i+1}} (r - r_{i+1})\Omega^{(a)}(r) = -C^{-1} d_i (c_i + c_{i+1}) r_{i+1}, \\ \phi_i^{(a)} &:= \lim_{t \to t_i} (t - t_i)\Phi^{(a)}(t) = -C^{-1} c_i f_i t_i. \end{aligned} \tag{4.13}$$

In appendix D it is shown that for all $i = 1, 2, \ldots$,

$$\begin{aligned} &\text{(i)} \quad \frac{\omega_{i+1}^{(a)}}{\omega_i^{(a)}} = \frac{\omega_{i+1}}{\omega_i}, \\ &\text{(ii)} \quad \frac{\phi_i^{(a)}}{\phi_{i-1}^{(a)}} = \frac{\phi_i}{\phi_{i-1}}. \end{aligned} \tag{4.14}$$

From the comparison of the relations in (4.11) discussed above it follows that

$$\frac{\omega_1^{(a)}}{\omega_1} = \frac{\phi_0^{(a)}}{\phi_0}. \tag{4.15}$$

From (2.29), (4.10), (4.12) and

$$\phi_0^{(a)} = -C^{-1} c_0 f_0 t_0, \tag{4.16}$$

it is readily seen that

$$\phi_0^{(a)} = \phi_0. \tag{4.17}$$

Consequently, from (4.14), (4.15) and (4.16) it follows that: for $i = 1, 2, \ldots$,

$$\omega_i^{(a)} = \omega_i, \qquad \phi_{i-1}^{(a)} = \phi_{i-1}. \tag{4.18}$$

Because $\Omega^{(a)}(r)$ and $\Omega(r)$ have the same structure and the same pole set, and because their residues at corresponding poles are equal, cf. (4.18), it follows that,

$$\begin{aligned} \Omega^{(a)}(r) &= \Omega(r) \quad \text{for all } r, \\ \Phi^{(a)}(t) &= \Phi(t) \quad \text{for all } t, \end{aligned} \tag{4.19}$$

the second relation in (4.19) is similarly shown. From (2.2), (2.3) and (4.19) it is readily seen that, cf. (3.17), (4.1) and (4.2),

$$p_{km}^{(a)} = p_{km}, \tag{4.20}$$

for all $m = 0, 1, 2, \ldots$, and all $k = 0, 1, 2, \ldots$.

Next we compare the construction of the solution according to the compensation approach with the construction of the solution described in sections 2 and 3. The compensation approach is based on the assumption that the $p^{(a)}_{km}$ possesses a series representation of the following form, cf. (4.1) and (4.2), for $k = 0, 1, 2, \ldots$,

$$p^{(a)}_{km} = C^{-1} \sum_{i=0}^{\infty} \tilde{d}_i [\tilde{c}_i \tilde{\alpha}_i^k + \tilde{c}_{i+1} \tilde{\alpha}_{i+1}^k] \tilde{\beta}_i^m, \qquad m = 1, 2, \ldots, \tag{4.21}$$

$$p^{(a)}_{k0} = C^{-1} \sum_{i=0}^{\infty} \tilde{c}_i \tilde{f}_i \tilde{\alpha}_i^k; \tag{4.22}$$

here $C$ is a constant, to be determined by the norming condition, and

$$(\tilde{\alpha}_n^{-1}, \tilde{\beta}_{n+1}^{-1}) = (\tilde{t}_n, \tilde{r}_n), \quad (\tilde{\alpha}_n^{-1}, \tilde{\beta}_{n+2}^{-1}) = (\tilde{t}_n, \tilde{r}_{n+1}), \tag{4.23}$$

are zero tuples of $k_1(r, t)$, cf. (2.4) and (2.7).

Whenever the series in (4.21) converges it follows from (4.23) that $p^{(a)}_{km}$ satisfies the equations (4.6)i for all $k \geq 1$, $m \geq 2$, i.e. the Kolmogorov equations, because each of the terms in (4.21) satisfies these equations.

The expressions (4.21), (4.22) should also satisfy the other equations of (4.6) and moreover the conditions

$$\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} |p^{(a)}_{km}| < \infty, \ \sum_{k=0}^{\infty} |p^{(a)}_{k0}| < \infty. \tag{4.24}$$

Whenever the conditions mentioned above can be satisfied then the representation (4.21), (4.22) is justified since the set of Kolmogorov equations possesses only one absolute convergent solution apart from a constant factor.

The conditions (4.24) obviously require that: for all $n = 0, 1, 2, \ldots$,

$$|\tilde{\alpha}_n| < 1, \ |\tilde{\beta}_n| < 1, \quad k_1(\tilde{\beta}_n^{-1}, \tilde{\alpha}_n^{-1}) = 0, \ k_1(\tilde{\beta}_n^{-1}, \tilde{\alpha}_{n+1}^{-1}) = 0. \tag{4.25}$$

The $d_i, c_i, f_i$ in (4.21) and (4.22) should obviously be determined by the conditions that (4.21) and (4.22) satisfy all the equations (4.6), note that (4.21) satisfies (4.6)i. Consider the following partial sums $x^{(j)}_{km}$ of the right-hand side of (4.21):

$$
\begin{aligned}
&\text{(i)} \quad x^{(0)}_{km} := \tilde{d}_0 \tilde{c}_0 \tilde{\alpha}_0^k \tilde{\beta}_0^m, \\
&\text{(ii)} \quad x^{(1)}_{km} := \tilde{d}_0 \tilde{c}_0 \tilde{\alpha}_0^k \tilde{\beta}_0^m + \tilde{d}_0 \tilde{c}_1 \tilde{\alpha}_1^k \tilde{\beta}_0^m, \\
&\text{(iii)} \quad x^{(2)}_{km} := \tilde{d}_0 \tilde{c}_0 \tilde{\alpha}_0^k \tilde{\beta}_0^m + \tilde{d}_0 \tilde{c}_1 \tilde{\alpha}_1^k \tilde{\beta}_0^m + \tilde{d}_1 \tilde{c}_1 \tilde{\alpha}_1^k \tilde{\beta}_1^m, \\
&\text{(iv)} \quad x^{(3)}_{km} := \tilde{d}_0 \tilde{c}_0 \tilde{\alpha}_0^k \tilde{\beta}_0^m + \tilde{d}_0 \tilde{c}_1 \tilde{\alpha}_1^k \tilde{\beta}_0^m + \tilde{d}_1 \tilde{c}_1 \tilde{\alpha}_1^k \tilde{\beta}_1^m + \tilde{d}_1 \tilde{c}_2 \tilde{\alpha}_2^k \tilde{\beta}_1^m, \\
&\quad \ldots \qquad\qquad \ldots \qquad\qquad \ldots
\end{aligned}
\tag{4.26}
$$

with $\tilde{d}_0 = \tilde{c}_0 = 1$.

Suppose there exist an $\tilde{\alpha}_0$ and $\tilde{\beta}_0$ satisfying (4.25) and such that $x^{(a)}_{km}$ satisfies the condition (4.6)ii (or (4.6)iii). Then determine $\tilde{c}_1$ and $\tilde{\alpha}_1$ such that $x^{(1)}_{km}$ satisfies the condition (4.6)iii ((4.6)ii). Next determine $d_1$ and $\tilde{\beta}_1$ such that $x^{(2)}_{km}$ satisfies (4.6)ii ((4.6)iii). Hence the iterative procedure for the determination of $\tilde{c}_i$ and $\tilde{d}_i$ amounts to alternatingly satisfying the boundary conditions (4.6)ii, (4.6)iii.

In [1] it is shown that

$$\tilde{\alpha}_0 = \frac{a^2}{4}, \tag{4.27}$$

is an appropriate starting value and that $\tilde{\alpha}_0^k \tilde{\beta}_0^m$ satisfies the equation (4.6)ii for all $k \geq 1$. Actually it is shown in [1] that with the starting value (4.27) the iteration procedure produces a sequence of $\tilde{c}_i, \tilde{d}_i$, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ for which the $x_{km}^{(N)}$ converges for $N \to \infty$, and for which the series in (4.21) and (4.22) are absolute convergent and satisfy (4.24) and the equations (4.6)i, ii and iii. With the solution so obtained the coefficients in (4.22) for $k \geq 1$ follow from (4.6)v and for $k = 0$ from (4.6)iv. The factor $C$ is then determined by the norming condition.

The determination of $\tilde{\alpha}_0$ is obviously crucial. It amounts finding an $\tilde{\alpha}_0$ and $\tilde{\beta}_0$ satisfying (4.25) and such that $x_{km}^{(0)}$ satisfies (4.6)ii or (4.6)iii. In both cases this leads to an equation of the fourth degree for $\alpha_0$ or $\beta_0$. This equation for $\alpha_0$ whenever $x_{km}^{(0)}$ should satisfy (4.6)ii has four zeros, viz. $\frac{a^2}{4}$, 1 and the double zero -1. The condition (4.25) shows that $\tilde{\alpha}_0$ as given in (4.27) should be the starting value.

From the iteration procedures described above it is seen that by the alternating adaptation of the partial sums of (4.21) to the boundary equations (4.6)ii and (4.6)iii a new pole is added to the bi-generating functions of these partial sums; the pole as well as its residue is uniquely determined. This iteration corresponds obviously to the analytic continuation of $\Phi(t)$ and $\Omega(r)$ as described in [5], and used in Section 2, cf.(2.10) and (2.16), and it clearly indicates that the starting value $\tilde{\alpha}_0 = t_0^{-1}$ should be the common zero in $t > 1$ of (2.4) and (2.11).

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# Appendix A

In this appendix it will be shown that, cf.(2.24), $\hat{\Phi}(t)$ is a constant and $\hat{\Omega}(r)$ a first degree polynomial in $r$.

Consider the sequence

$$\rho_0 < \tau_0 < \rho_1 < \tau_1 \ldots < \tau_{n-1} < \rho_n < \tau_n < \ldots, \tag{a.1}$$

with: for $n = 1, 2, \ldots$,

$$1 \le \rho_0 < \frac{2}{a}, \tag{a.2}$$

$$\tau_{n-1} := t_1(\rho_n), \qquad \tau_n := t_2(\rho_n).$$

Because $\rho_0 \neq 2/a$ it is readily seen that the $\tau_n$ are neither poles nor zeros of $\Phi(t)$, similarly the $\rho_n$ are neither zeros nor poles of $\Omega(r)$, so

$$\rho_m \neq r_n, \qquad \tau_m \neq t_n \text{ for all } m, n \in \{0, 1, 2, \ldots\}. \tag{a.3}$$

Next note that

$$k_1(r,t) = 0 \text{ and } k_2(r,t) = 0 \Longleftrightarrow r = \frac{2}{a},\ t = \frac{4}{a^2} \text{ or } r = -1 - \frac{2}{a},\ t = -\frac{1}{a}.$$

Hence from (2.10), (2.11) and (a.2) we have for $n = 1, 2, \ldots$,

$$\frac{\Phi(\tau_n)}{\Phi(\tau_{n-1})} = \frac{k_2(\rho_n, \tau_{n-1})}{k_2(\rho_n, \tau_n)} = \frac{-1 + \frac{1}{2}a^2\tau_{n-1} - \frac{1}{2}a\rho_n}{-1 + \frac{1}{2}a^2\tau_n - \frac{1}{2}a\rho_n}. \tag{a.4}$$

From (a.4) it is seen that $\Phi(\tau_n)$ is bounded for every finite $n$. Obviously $\rho_n \to \infty$, $\tau_n \to \infty$, for $n \to \infty$ and it is readily verified that: for $n \to \infty$,

$$\frac{\tau_{n-1}}{\rho_n} \to \frac{1}{a\delta} < 1, \qquad \frac{\tau_n}{\rho_n} \to \delta > 1, \qquad \frac{\rho_{n+1}}{\rho_n} = a\delta^2 > 1. \tag{a.5}$$

From (a.4) we have for $m = 1, 2, \ldots$,

$$\frac{\Phi(\tau_{N+m})}{\Phi(\tau_N)} = \prod_{i=1}^{m} \frac{-1 + \frac{1}{2}a^2\tau_{N-1+i} - \frac{1}{2}a\rho_{N+i}}{-1 + \frac{1}{2}a^2\tau_{N+i} - \frac{1}{2}a\rho_{N+i}}. \tag{a.6}$$

For $n$ large we have from (a.5), by using (2.20),

$$\frac{-1 + \frac{1}{2}a^2\tau_{n-1} - \frac{1}{2}a\rho_n}{-1 + \frac{1}{2}a^2\tau_n - \frac{1}{2}a\rho_n} = -\frac{\delta - 1}{\delta(a\delta - 1)}\left[1 - \frac{1}{\rho_n}\,\frac{2}{a}\,\frac{-2 + (1+a)\delta}{a\delta - 1}\right] + \mathrm{o}(\frac{1}{\rho_n}). \tag{a.7}$$

From (a.5) it is seen that

$$0 < \sum_{n=1}^{\infty} \frac{1}{\rho_n} < \infty, \tag{a.8}$$

so by using (2.21) it is seen from (a.6) that for sufficiently large $N$ and $m = 1, 2, \ldots$,

$$|\frac{\Phi(\tau_{N+m})}{\Phi(\tau_N)}| \sim \left[\frac{\delta-1}{\delta(a\delta-1)}\right]^m \left[1-\frac{2}{a}\frac{(1+a)\delta-2}{a\delta-1}\sum_{k=1}^{m}\frac{1}{\rho_k}\right], \tag{a.9}$$

note that the righthand side of (a.9) tends to zero for $m \to \infty$.
From (2.10) we have for $m = 1, 2, \ldots$,

$$\frac{1}{\rho_{N+m}}\Omega(\rho_{N+m}) = -\frac{k_2(\rho_{N+m}, \tau_{N+m})}{\rho_{N+m}}\frac{\Phi(\tau_{N+m})}{\Phi(\tau_N)}\Phi(\tau_N). \tag{a.10}$$

For sufficiently large $N$ it is readily seen that $k_2(\rho_{N+m}, \tau_{N+m})/\rho_{N+m}$ is bounded in $m = 1, 2, \ldots$, and so we obtain from (a.9) and (a.10), since $\Phi(\tau_N)$ is finite for finite $N$ that

$$\begin{aligned} &|\Phi(\tau_{N+m})| \leq \epsilon^{(\phi)}_{N+m} && \text{with} \quad \epsilon^{(\phi)}_{N+m} \to 0 \quad \text{for} \quad m \to \infty, \\ &|\Omega(\rho_{N+m})|/\rho_{N+m} \leq \epsilon^{(\omega)}_{N+m} && \text{with} \quad \epsilon^{(\omega)}_{N+m} \to 0 \quad \text{for} \quad m \to \infty, \end{aligned} \tag{a.11}$$

for every finite $N$ and every $\rho_0$ satisfying (a.2).
From (2.10)i we have for every $r$ and $j = 1, 2$,

$$\frac{1}{r}\Omega(r) + \frac{1}{r}k_2(r, t_j(r))\Phi(t_j(r)) = 0. \tag{a.12}$$

For $r = r_n + \epsilon$ with $|\epsilon| << 1$ it is seen from (2.16), since $\hat{\Phi}(t)$ and $\hat{\Omega}(r)$ are entire functions and $t_j(r_n)$ and $r_n$ are simple poles of $\tilde{\Phi}(t)$ and $\tilde{\Omega}(r)$, respectively, that

$$\frac{1}{r}\hat{\Omega}(r) + \frac{1}{r}k_2(r, t_j(r))\tilde{\Phi}(t_j(r)) + \frac{1}{r}\mathrm{O}(r - r_n) = 0 \text{ for } r \to r_n. \tag{a.13}$$

Consequently we obtain from (2.24), (a.11), (a.12) and (a.13) that for $j = 1, 2$, and $n \to \infty$,

$$\frac{1}{\rho_n}\hat{\Omega}(\rho_n) + \frac{1}{\rho_n}k_2(\rho_n, t_j(\rho_n))\hat{\Phi}(t_j(\rho_n)) \to 0. \tag{a.14}$$

Because

$$\lim_{n\to\infty} |\frac{1}{\rho_n}k_2(\rho_n, t_j(\rho_n))| \neq 0,$$

and, cf. (a.11),

$$\hat{\Phi}(t_j(\rho_n)) \to 0 \text{ for } n \to \infty,$$

and so $\hat{\Phi}(\rho_n) \to 0$ for $n \to \infty$, (a.14) implies that $\hat{\Omega}(\rho_n)/\rho_n \to 0$ for $n \to \infty$. By noting that $\hat{\Phi}(t)$ and $\hat{\Omega}(r)$ are both entire functions, Liouville's theorem implies that

$$\begin{aligned} &\hat{\Phi}(t) = 0, \\ &\hat{\Omega}(r) = \hat{\Omega}(0), \end{aligned} \tag{a.15}$$

because for varying $\rho_0$ with $1 \leq \rho_0 < \frac{2}{a}$ the set of $\rho_n = \rho_n(\rho_0)$ will be dense in $(r, \infty)$ for some $r > 0$.
A slightly different approach for the derivation of the results in this appendix from the product form representation (2.8) is Cauchy's method of decomposing meromorphic functions, see [6] Section VIII.4.

# Appendix B

In this appendix we derive explicit expressions for $\phi_0$ and $\omega_1$.

Consider first the case

$$\frac{1}{a} \geq 1. \tag{b.1}$$

It follows, cf.(2.5),

$$t_2(1) = \frac{1}{a}, \qquad t_1(1) = 1, \qquad t_1(\frac{2}{a}) = \frac{1}{a}, \qquad t_2(\frac{2}{a}) = \frac{4}{a^2} = t_0. \tag{b.2}$$

Hence from, cf.(2.10),

$$\Omega(\frac{2}{a}) + \left[-1 + \frac{1}{2}a^2 t_1(\frac{2}{a}) - \frac{1}{2}a\frac{2}{a}\right]\Phi(t_1(\frac{2}{a})) = 0,$$

and (2.26) we obtain

$$\Omega(\frac{2}{a}) = \frac{1}{4}(2-a)(4-a). \tag{b.3}$$

So from, cf.(2.10),

$$\Omega(\frac{2}{a}) + \lim_{r \to \frac{2}{a}} \frac{-1 + \frac{1}{2}a^2 t_2(\frac{2}{a}) - \frac{1}{2}ar}{t_2(r) - 4/a^2}(t_2(r) - \frac{4}{a^2})\Phi(t_2(r)) = 0,$$

the relation (2.29) for $\phi_0$ follows.

Next consider the case

$$\frac{1}{a} < 1.$$

Then

$$t_1(\frac{1}{a}) = \frac{1}{a}, \qquad t_2(1) = 1, \qquad k_1(\frac{2}{a}, \frac{1}{a}) = 0, \qquad t_0 = t_2(\frac{2}{a}) = \frac{4}{a^2},$$

and (b.3) follows again by using (2.10). As before we obtain again the relation (2.29) for $\phi_0$.

To determine $\omega_1$ note that $k_1(\frac{2}{a}, \frac{4}{a^2}) = 0$, hence

$$r_1 = \frac{at^2 + t}{r_0}\Big|_{t_0 = \frac{4}{a^2}} = \frac{2}{a^2}(a+4). \tag{b.4}$$

From (2.9) and (2.10) we have

$$\omega_1 + \lim_{r \to r_1}\left[-1 + \frac{1}{2}a^2 t_0 - \frac{1}{2}ar_1\right]\frac{r - r_1}{t_1(r) - 4/a^2}(t_1(r) - 4/a^2)\Phi(t_1(r)) = 0.$$

So

$$\omega_1 + \phi_0\left[-1 + 2 - \frac{a+4}{a}\right]\left[\frac{\mathrm{d}t_1(r)}{\mathrm{d}r}\right]^{-1}_{r=r_1} = 0,$$

and by using (2.12) the expression (2.29) for $\omega_1$ results.

# Appendix C

To evaluate the integral in (3.6) with $m = 1, 2, \ldots$, for $N \to \infty$ note that (2.10) implies that: for $|t| = 1$,

$$\frac{1}{r^2}(r-t)\Omega(r) = \frac{r-t}{r^2}\{1 - \frac{1}{2}a^2 t_2(r) + \frac{1}{2}ar\}\Phi(t_2(r)). \tag{c.1}$$

From (a.2) and (a.4) we have

$$\Phi(t_2(\rho_n)) \to 0 \qquad \text{for} \quad n \to \infty. \tag{c.2}$$

The integral in (3.6) does not change its value if the contour $|r| = R_N$ is replaced by the contour $|r| = \rho_{N+1}$, note that $r_N < \rho_{N+1} < r_{N+1}$ and apply Cauchy's theorem. Hence by using (2.18) it follows that: for $|t| = 1$,

$$\frac{1}{\rho_{N+1}^2}(\rho_{N+1} - t)\Omega(\rho_{N+1}) \to 0 \quad \text{for} \quad N \to \infty. \tag{c.3}$$

# Appendix D

In this appendix it will be shown that for all $n = 1, 2, \ldots$,

$$\frac{\omega_{n+1}^{(a)}}{\omega_n^{(a)}} = \frac{\omega_{n+1}}{\omega_n}, \tag{d.1}$$

$$\frac{\phi_n^{(a)}}{\phi_{n-1}^{(a)}} = \frac{\phi_n}{\phi_{n-1}}. \tag{d.2}$$

We start with the proof of (d.1). From (4.13) we have for $i = 1, 2, \ldots$,

$$\frac{\omega_{i+1}^{(a)}}{\omega_i^{(a)}} = \frac{d_i}{d_{i-1}} \frac{c_i + c_{i+1}}{c_{i-1} + c_i} \frac{r_{i+1}}{r_i}. \tag{d.3}$$

From (4.4),

$$c_{i+1} + c_i = \frac{r_{i+1}}{t_{i+1}} \frac{t_i - t_{i+1}}{t_i - r_{i+1}} c_i,$$

$$\frac{c_{i+1}}{c_i} = \frac{t_i}{t_{i+1}} \frac{r_{i+1} - t_{i+1}}{t_i - r_{i+1}},$$

so that

$$\frac{c_{i+1} + c_i}{c_i + c_{i-1}} = \frac{r_{i+1}}{r_i} \frac{t_{i-1}}{t_{i+1}} \frac{r_i - t_i}{t_i - r_{i+1}} \frac{t_{i+1} - t_i}{t_i - t_{i-1}},$$

$$\frac{d_{i+1}}{d_i} = -\frac{(2 + at_{i+1})r_{i+2} - (2 + a)t_{i+1}}{(2 + at_{i+1})r_{i+1} - (2 + a)t_{i+1}}.$$

By using the relation $r_{i+1}^2 = at_{i+1}t_i$, which follows from (2.7) and the properties of the zeros of $k_1(r, t)$ cf. (2.4), we obtain by inserting the relations above into (4.3) that for $i = 1, 2, \ldots$,

$$\frac{\omega_{i+1}^{(a)}}{\omega_i^{(a)}} \frac{t_i - t_{i-1}}{t_{i+1} - t_i} = \frac{r_i - t_i}{r_{i+1} - t_i} \frac{(2 + at_i)r_{i+1} - (2 + a)t_i}{(2 + at_i)r_i - (2 + a)t_i}. \tag{d.4}$$

From (2.17) we have

$$\frac{\omega_{i+1}}{\omega_i} \frac{t_i - t_{i-1}}{t_{i+1} - t_i} = \frac{1 - \frac{1}{2}a^2 t_i + \frac{1}{2}ar_{i+1}}{1 - \frac{1}{2}a^2 t_i + \frac{1}{2}ar_i}. \tag{d.5}$$

So in order to prove (d.1) it remains to show that

$$\frac{1 - \frac{1}{2}a^2 t_i + \frac{1}{2}ar_{i+1}}{1 - \frac{1}{2}a^2 t_i + \frac{1}{2}ar_i} = \frac{r_i - t_i}{r_{i+1} - t_i} \frac{(2 + at_i)r_{i+1} - (2 + a)t_i}{(2 + at_i)r_i - (2 + a)t_i}. \tag{d.6}$$

For the lefthand side $L_6$ of (d.6) we have

$$L_6 = 1 + \frac{\frac{1}{2}a(r_{i+1} - r_i)}{1 - \frac{1}{2}a^2 t_i + \frac{1}{2}ar_i}. \tag{d.7}$$

For the righthand side $R_6$ of (d.6) we have by using $r_i r_{i+1} = (at_i + 1)t_i$, cf. (2.4) and (2.7),

$$R_6 = 1 + \frac{(2+a)t_i(r_{i+1}-r_i) - (2+at_i)(r_{i+1}-r_i)t_i}{(2+at_i)t_i(at_i+1) - (2+a)t_i r_{i+1} - (2+at_i)r_i t_i + (2+a)t_i^2}. \quad \text{(d.8)}$$

In order that

$$L_6 = R_6, \quad \text{(d.9)}$$

we should have cf. (d.7) and (d.8) after division by $t_i \neq 0$ and $r_{i+1} - r_i \neq 0$,

$$\frac{\frac{1}{2}a}{1 - \frac{1}{2}a^2 t_i + \frac{1}{2}ar_i} = \frac{a(1-t_i)}{(2+at_i)(at_i+1-r_i) - (2+a)(r_{i+1}-t_i)}. \quad \text{(d.10)}$$

From which it follows that (d.10) is equivalent to

$$t_i\{a^2 + 4a + 4\} - (2+a)(r_i + r_{i+1}) = 0. \quad \text{(d.11)}$$

Because, cf. (2.4) and (2.7), we have $r_i + r_{i+1} = (2+a)t_i$, it is seen that (d.11) is an identity for all $i = 1, 2, \ldots$. Hence (d.6) holds for $i = 1, 2, \ldots$, i.e. the validity of (d.1) has been proved for all $n = 1, 2, \ldots$.

Next we prove (d.2). From (4.13) we have for $i = 0, 1, 2, \ldots$,

$$\frac{\phi_{i+1}^{(a)}}{\phi_i^{(a)}} = \frac{c_{i+1}}{c_i}\frac{f_{i+1}}{f_i}\frac{t_{i+1}}{t_i}. \quad \text{(d.12)}$$

We have, see below (d.3),

$$\frac{c_{i+1}}{c_i}\frac{t_{i+1}}{t_i} = \frac{r_{i+1}-t_{i+1}}{t_i - r_{i+1}}. \quad \text{(d.13)}$$

From (4.4),

$$\frac{f_{i+1}}{f_i} = \frac{1}{1+\frac{1}{2}at_{i+1}}(d_i + d_{i+1}), \quad \text{(d.14)}$$

$$d_{i+1} + d_i = d_i \frac{(2+at_{i+1})(r_{i+1}-r_{i+2})}{(2+at_{i+1})r_{i+1} - (2+a)t_{i+1}}.$$

From (d.14) we obtain

$$\frac{d_{i+1}+d_i}{d_i+d_{i-1}} = -\frac{2+at_{i+1}}{2+at_i}\frac{(2+at_i)r_{i+1}-(2+a)t_i}{(2+at_{i+1})r_{i+1}-(2+a)t_{i+1}}\frac{r_{i+1}-r_{i+2}}{r_i - r_{i+1}}, \quad \text{(d.15)}$$

$$\frac{f_{i+1}}{f_i} = -\frac{(2+at_i)r_{i+1}-(2+a)t_i}{(2+at_{i+1})r_{i+1}-(2+a)t_{i+1}}\frac{r_{i+1}-r_{i+2}}{r_i-r_{i+1}}.$$

Hence from (d.12), ..., (d.15),

$$\frac{\phi_{i+1}^{(a)}}{\phi_i^{(a)}} = \frac{r_{i+1}-t_{i+1}}{r_{i+1}-t_i}\frac{(2+at_i)r_{i+1}-(2+a)t_i}{(2+at_{i+1})r_{i+1}-(2+a)t_{i+1}}\frac{r_{i+1}-r_{i+2}}{r_i-r_{i+1}}. \quad \text{(d.16)}$$

From (2.17) we have

$$\frac{\phi_{i+1}}{\phi_i} = \frac{1-\frac{1}{2}a^2t_i+\frac{1}{2}ar_{i+1}}{1-\frac{1}{2}a^2t_{i+1}+\frac{1}{2}ar_{i+1}}\frac{r_{i+2}-r_{i+1}}{r_{i+2}-r_i}. \tag{d.17}$$

So in order to prove (d.2) we have to show that for $i = 0, 1, 2, \ldots$,

$$\frac{1-\frac{1}{2}a^2t_i+\frac{1}{2}ar_{i+1}}{1-\frac{1}{2}a^2t_{i+1}+\frac{1}{2}ar_i} = \frac{r_{i+1}-t_{i+1}}{r_{i+1}-t_i}\frac{(2+at_i)r_{i+1}-(2+a)t_i}{(2+at_{i+1})r_{i+1}-(2+a)t_{i+1}}. \tag{d.18}$$

For the lefthand side of (d.18) we write

$$L_{18} = 1 - \frac{\frac{1}{2}a^2(t_i - t_{i+1})}{1-\frac{1}{2}a^2t_{i+1}+\frac{1}{2}ar_{i+1}}. \tag{d.19}$$

The righthand side of (d.18) may be written as:

$$R_{18} = 1-(t_{i+1}-t_i)\frac{a(r_{i+1}-1)}{(\frac{2+3a}{a}+at_{i+1})r_{i+1}-2(t_{i+1}+t_i)-at_{i+1}(t_i+1)}. \tag{d.20}$$

Note that (2.4) and (2.7) imply that

$$t_{i+1}+t_i = \frac{1}{a}[(2+a)r_{i+1}-1],\ at_{i+1}t_i = r_{i+1}^2.$$

Hence in order that

$$L_{18} = R_{18},$$

we should have

$$\frac{a}{2-a^2t_{i+1}+ar_{i+1}} = \frac{1-r_{i+1}}{(\frac{2+3a}{a}+at_{i+1})r_{i+1}-\frac{2}{a}[(2+a)r_{i+1}-1]-at_{i+1}-r_{i+1}^2}. \tag{d.21}$$

It is readily verified that (d.21) is indeed an identity for all $i = 0, 1, 2, \ldots$, and so (d.2) has been proved.

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