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# Entropy Conditions for Heterogeneity Induced Shocks in Two-Phase Flow Problems

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**Abstract**

We study two-phase flow in a porous medium with piecewise smooth permeability. If capillary forces can not be neglected, the flow problem is parabolic. Without capillary forces this problem degenerates to a hyperbolic conservation law with a discontinuous flux function. The solution for this problem has stationary shocks at discontinuities in the permeability. In this paper we derive an entropy condition for these shocks by studying the limit case of vanishing capillarity.

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## 1 Introduction

We study two-phase flow in a 1D porous medium with piecewise constant permeability including gravitational effects. If capillary diffusion can not be neglected, the flow problem is parabolic. Assuming the classical Leverett model for the capillary pressure, the solution is discontinuous at heterogeneities in the permeability. However, except for some degenerate cases the capillary pressure is continuous.

Without capillary forces the flow problem degenerates to a first order hyperbolic conservation law with a discontinuous flux function. In general the solution for this problem is again discontinuous at heterogeneities in the permeability. These discontinuities can be considered as stationary shocks. Here we derive an entropy condition for these shocks by considering the limit case of vanishing capillarity.

An outline of this paper is as follows. In Section 2 we present the standard two-phase flow model, and in Section 3 we consider the derivation of the interface conditions at heterogeneities in the capillary dominated case. This is of interest because the procedure is the same as we use in Section 4 for deriving entropy conditions in the hyperbolic limit case of no capillarity. In Section 4 we first present the standard way for deriving the entropy condition, namely by a 'vanishing viscosity' argument. Next we derive the new entropy condition by a vanishing capillary diffusion argument. This new entropy conditions appears to be more restrictive than the standard one.

## 2 Equations

In this section we briefly state the two-phase flow model. For a more elaborate introduction the reader is referred to e.g. Bear [5] or Aziz and Settari [1]. The volumetric flow rate of two immiscible fluids in a porous medium is given by the generalized Darcy's law,

$$q_\alpha = -k \frac{k_{r\alpha}}{\mu_\alpha} \nabla (p_\alpha - \rho_\alpha \mathbf{g}), \quad \alpha = \mathbf{w}, \mathbf{n}, \tag{1}$$

with $k$ the absolute permeability, $\mathbf{g}$ acceleration due to gravity and $q_\alpha$, $k_{r\alpha}$, $\mu_\alpha$, $p_\alpha$ and $\rho_\alpha$ the Darcy velocity, relative permeability, viscosity, pressure and density of the wetting phase $w$ and the non-wetting phase $n$, respectively. The saturation of each phase $\alpha$ is denoted by $s_\alpha$, so

$$s_w + s_n = 1. \tag{2}$$

In the sequel we drop the subscript on $s_w$. In addition to these momentum equations we have mass conservation laws for both phases,

$$\phi \frac{\partial s_\alpha}{\partial t} + \operatorname{div} q_\alpha = 0, \quad \alpha = w, n, \tag{3}$$

with $\phi$ the (constant) porosity of the rock. The pressure of the non-wetting fluid differs from the wetting fluid because of interfacial tension on the microscopic pore level. This pressure difference, which is called the capillary pressure $p_c$, obeys the so called Leverett-relationship

$$p_n - p_w = p_c(x, s) = \sigma \sqrt{\frac{\phi}{k(x)}} J(s), \tag{4}$$

with $\sigma$ the interfacial tension.

We assume that the relative permeabilities $k_{rw}(s)$ and $k_{rn}(s)$ are continuously differentiable on $[0, 1]$, and that $J(s)$ is continuously differentiable on $(0, 1]$. Further we assume that they satisfy

- $k_{rw}$ is strictly increasing such that $k_{rw}(0) = 0$ and $k_{rw}(1) = 1$,

- $k_{rn}$ is strictly decreasing such that $k_{rn}(0) = 1$ and $k_{rn}(1) = 0$,
- $J' < 0$ on $(0, 1]$ and $J(1) \geq 0$.

In one space dimension the system of partial differential equations can be reduced to a single convection-diffusion equation using the incompressibility of the fluids. After appropriate scaling we obtain

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}\left(f_w(s)q_t - N_g k(x)\overline{\lambda}(s) + N_c k(x)\overline{\lambda}(s)\frac{\partial}{\partial x}\frac{J(s)}{\sqrt{k(x)}}\right) = 0, \quad (5)$$

with $q_t$ the (scaled) total flow which is determined by the boundary conditions, $\overline{\lambda}$ the total mobility that is defined by

$$\overline{\lambda}(s) = \frac{k_{rw}(s)k_{rn}(s)}{k_{rw}(s) + k_{rn}(s)\mu_w/\mu_n}, \quad (6)$$

and $N_g$ and $N_c$ the gravity and capillary numbers that are defined by

$$N_g = \frac{(\mathbf{g} \cdot \hat{\mathbf{e}}_{\mathbf{x}})(\rho_{\mathbf{n}} - \rho_{\mathbf{w}})\mathbf{k}^*}{\mathbf{q}^* \mu_{\mathbf{n}}}, \quad (7)$$

$$N_c = \frac{\sigma\sqrt{k^*\phi}}{q^* L^* \mu_n}, \quad (8)$$

with $k^*$, $q^*$ and $L^*$ characteristic values for the permeability, flow rate and length, respectively.

In the following two sections we study what happens if $k(x)$ is discontinuous. In Section 3 we consider the case that the capillary forces can not be neglected, i.e., if the capillary number $N_c$ is not small. In Section 4 we consider the case that $N_c$ vanishes, and (5) degenerates to a first-order hyperbolic conservation law.

# 3 Interface Conditions With Capillary Forces

At discontinuities in $k(x)$ the partial differential equation (5) ceases to hold. At those points we need interface conditions. To derive them we consider a medium with a single discontinuity at $x = 0$, i.e. $k$ satisfies

$$k(x) = \begin{cases} k^-, & x < 0, \\ k^+, & x > 0. \end{cases} \quad (9)$$

The first interface condition, the flux continuity condition, is straightforward. Assuming that the time derivative in (5) is bounded, we integrate this equation in a small neighborhood of $x = 0$. This leads to continuity of the flux,

$$\lim_{x \uparrow 0} q_w(x,t) = \lim_{x \downarrow 0} q_w(x,t), \quad (10)$$

where the wetting phase flux $q_w$ is given by

$$q_w = f_w(s)q_t - N_g k(x)\overline{\lambda}(s) + N_c k(x)\overline{\lambda}(s)\frac{\partial}{\partial x}\frac{J(s)}{\sqrt{k(x)}}. \tag{11}$$

The other interface condition, the *extended pressure condition*, is obtained by regularization of $k$. Following an earlier idea by Yortsos [7] we approximate the discontinuous function $k(x)$ by a smooth monotone function $k_\epsilon(x)$ that differs from $k$ only in a small interval $(-\epsilon, \epsilon)$ around the discontinuity. Then we determine the differential equation that should be satisfied by the limit function in the transition interval for vanishing $\epsilon$. From this differential equation a relation can be derived between the saturations at the end points of the interval. This relation is the other interface condition.

Let $s_\epsilon$ be the solution of the problem with the regularized permeability $k_\epsilon$ and $q_{w\epsilon}$ the corresponding wetting phase flux. By rescaling $y := x/\epsilon$ we blow up the interval $(-\epsilon, \epsilon)$. Inside this interval we have

$$q_{w\epsilon} = f_w(s_\epsilon)q_t - N_g k_\epsilon(y)\overline{\lambda}(s_\epsilon) + \frac{1}{\epsilon}N_c k_\epsilon(y)\overline{\lambda}(s_\epsilon)\frac{\partial}{\partial y}\frac{J(s_\epsilon)}{\sqrt{k_\epsilon(y)}}. \tag{12}$$

Boundedness of $q_{w\epsilon}$ for $\epsilon$ tends to zero implies

$$\overline{\lambda}(s_\epsilon)\frac{\partial}{\partial y}\left(\frac{J(s_\epsilon)}{\sqrt{k(y)}}\right) = 0. \tag{13}$$

A pair $(s^-, s^+)$ is said to satisfy the extended pressure condition if it satisfies the *first-order* ordinary differential equation (13), and the *two* boundary conditions

$$\lim_{y\to-\infty} s_\epsilon(y) = s^- \text{ and } \lim_{y\to+\infty} s_\epsilon(y) = s^+. \tag{14}$$

A careful analysis of this problem (see Van Duijn and De Neef [3] and Van Duijn e.a. [2]) yields the following condition:

$$\begin{cases} \frac{J(s^-)}{\sqrt{k^-}} = \frac{J(s^+)}{\sqrt{k^+}}, & \text{if } s^+ \le s^*, \\ s^- = 1, & \text{if } s^* < s^+ \le 1, \end{cases} \tag{15}$$

where $s^-$ and $s^+$ correspond to the limit value of $s$ at the side of the lowest permeability and the highest permeability, respectively. The first part of the condition (15) is a jump condition for the saturation: it states the continuity of capillary pressure. The second part indicates that the capillary pressure need not always be continuous if $J(1) > 0$. In this case there exists a critical saturation $s^*$,

$$\frac{J(s^*)}{\sqrt{k^+}} = \frac{J(1)}{\sqrt{k^-}}, \tag{16}$$

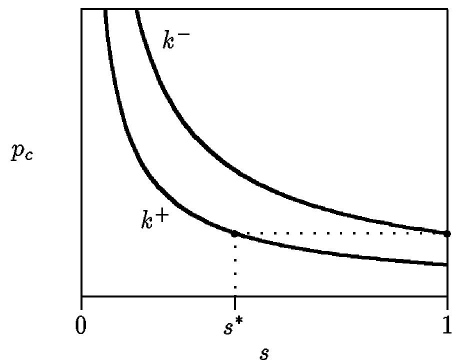


Figure 1: Capillary pressure $p_c$ as a function of $s$ with $k^+ > k^-$. If $J(1) > 0$ there exists a critical saturation $s^*$ such that $p_c$ can not be continuous if $s^+ > s^*$.

such that the capillary pressure can not be continuous if $s^+ > s^*$ (see Figure 1). Notice that $s^* = 1$ if $J(1) = 0$, which implies that the capillary pressure is always continuous if $J(1) = 0$. Considering the capillary pressure curves used in practice, we observe that $J(1) > 0$ for the Brooks-Corey curve and that $J(1) = 0$ for the Van Genuchten curve.

## 4 Entropy Condition for Hyperbolic Limit

If capillary forces can be neglected completely, i.e. if $N_c = 0$, the convection-diffusion equation (5) reduces to the first-order hyperbolic conservation law

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x} F(s, k) = 0, \tag{17}$$

with

$$F(s, k) = f_w(s) q_t - N_g k(x) \overline{\lambda}(s). \tag{18}$$

Again we consider the case that $k$ has a single heterogeneity at the origin as in (9). Continuity of flux at the heterogeneity implies

$$F(s^-, k^-) = F(s^+, k^+) = \overline{F}, \tag{19}$$

which is the Rankine-Hugoniot condition for a shock that moves with zero speed. It is well known that the Rankine-Hugoniot condition is in general insufficient to single out unique solutions of initial value problems, and an extra condition, an entropy condition, is needed.

The standard way to derive such an entropy condition for (17) is to regularize the problem by adding a small diffusion term (see e.g. Langtangen and Tveito [6] or Gimse and Risebro [4]),

$$\frac{\partial s_\epsilon}{\partial t} + \frac{\partial}{\partial x} F(s_\epsilon, k) = \epsilon \frac{\partial^2 s_\epsilon}{\partial x^2}, \tag{20}$$

which enforces continuity of $s_\epsilon$. As in the previous section we consider the behavior of $s_\epsilon$ in the neighborhood of $x = 0$ if $\epsilon$ tends to zero. We assume that the time derivative of $s_\epsilon$ remains bounded, and blow up the transition region around the origin by rescaling $y := x/\epsilon$. A shock $(s^-, s^+)$ satisfies the entropy condition if it has a viscous profile, that is, if there exist a solution $s_\epsilon$ that satisfies the *first-order* ordinary differential equation (cf. Equation (13))

$$\frac{\mathrm{d}s_\epsilon}{\mathrm{d}y} = F(s_\epsilon, k) - \overline{F}, \tag{21}$$

with the *two* boundary conditions

$$\lim_{y \to -\infty} s_\epsilon(y) = s^- \text{ and } \lim_{y \to +\infty} s_\epsilon(y) = s^+. \tag{22}$$

The shock $(s^-, s^+)$ with $s^- < s^+$ satisfies this entropy condition if (see e.g. Langtangen and Tveito [6]): $\exists \tilde{s} \in [s^-, s^+]$ such that

$$\begin{aligned} F(s, k^-) - \overline{F} \geq 0, \quad \forall s \in (s^-, \tilde{s}), \\ F(s, k^+) - \overline{F} \geq 0, \quad \forall s \in (\tilde{s}, s^+). \end{aligned} \tag{23}$$

As similar condition is found for the case $s^+ < s^-$. This condition implies that the solution of the regularized problem $s_\epsilon$ is monotone increasing for all values of $y$. Figure 2 shows the condition (23) in the $(k, s)$-phase plane. Here it is important to note that according to Equation (9) $k(y)$ only assumes the values $k^-$ and $k^+$.

Although this way of regularizing the hyperbolic problem in order to obtain the entropy condition is well established from the mathematical point of view, it is less satisfying from the physical point of view. Instead of just adding a small *linear* diffusion term, it seems to be more appropriate in the context of two-phase flow to use the capillary diffusion term for regularizing the problem. In the previous section we have outlined the procedure to deal with the regularization of problems in cases that capillary diffusion can not be neglected. For any small nonzero value of the capillary number $N_c$ we obtain the interface condition given by Equation (15). This interface condition involves the Leverett

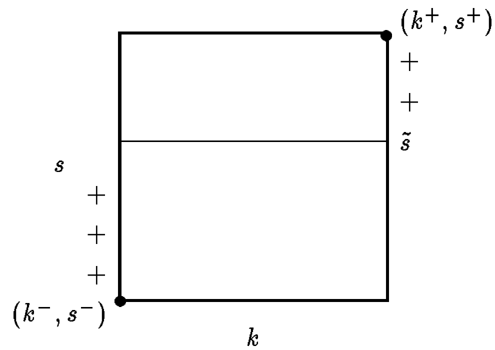


Figure 2: Standard entropy condition in the $(k, s)$-phase plane for the case $s^- < s^+$. Indicated is the sign of $F(s, k) - \overline{F}$

$J$-function. However we want the entropy condition to be independent of the particular choice for the $J$-function that is being used.

To avoid this dependence on the $J$-function we consider a different procedure. Instead of letting the interval in which the regularized permeability $k_\epsilon$ differs from $k$ vanish and keeping the capillary number $N_c$ fixed, we not let both terms tend to zero. Suppose that $k_\epsilon$ varies in an interval of length $L_\epsilon$, and that the capillary number is of the form $\epsilon N_c$ with $N_c$ fixed. As before we rescale by $y := x/\epsilon$ and obtain the following expression for the flux in the regularized problem:

$$\overline{F} = F(s_\epsilon, k_\epsilon) + \epsilon N_c \overline{\lambda}(s_\epsilon) k_\epsilon \frac{1}{\epsilon} \left( \frac{J'(s_\epsilon)}{\sqrt{k_\epsilon}} \frac{\mathrm{d}s_\epsilon}{\mathrm{d}y} - \frac{J(s_\epsilon)}{2k_\epsilon^{3/2}} k'_\epsilon(y) \right). \tag{24}$$

In order to get rid of the explicit dependence on the shape of the smooth, monotone function $k_\epsilon$ that is used to regularize $k$, we estimate

$$|k'_\epsilon(y)| = \mathcal{O}(\frac{\epsilon}{L_\epsilon}) \text{ for } \epsilon \to 0, \tag{25}$$

and assume that

$$\lim_{\epsilon \to 0} \frac{\epsilon}{L_\epsilon} = 0. \tag{26}$$

A shock $(s^-, s^+)$ satisfies the new entropy condition if there exists a $s_\epsilon$ such that for any admissible regularization $k_\epsilon$ (cf. (13,21))

$$\left( -N_c \overline{\lambda}(s_\epsilon) \sqrt{k_\epsilon} J'(s_\epsilon) \right) \frac{\mathrm{d}s_\epsilon}{\mathrm{d}y} = F(s_\epsilon, k_\epsilon) - \overline{F}, \tag{27}$$

with

$$\lim_{y \to -\infty} s_\epsilon(y) = s^-, \qquad \lim_{y \to +\infty} s_\epsilon(y) = s^+, \tag{28}$$

$$\lim_{y \to -\infty} k_\epsilon(y) = k^-, \qquad \lim_{y \to +\infty} k_\epsilon(y) = k^+. \tag{29}$$

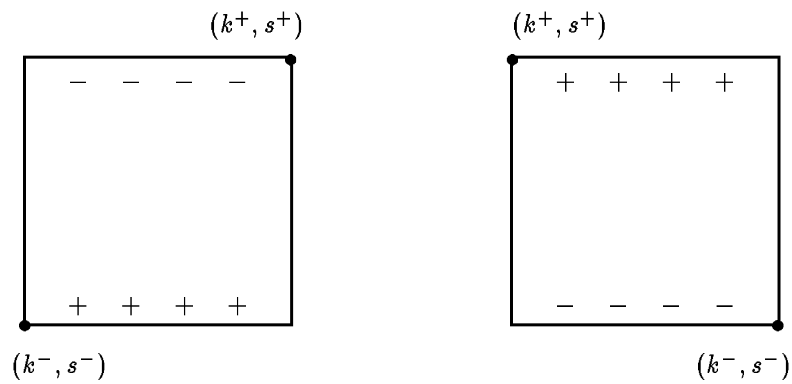


Figure 3: Sign of $\frac{ds_\epsilon}{dy}$ in the $(k, s)$-phase plane with $k^+ > k^-$ (left), and $k^+ < k^-$ (right).

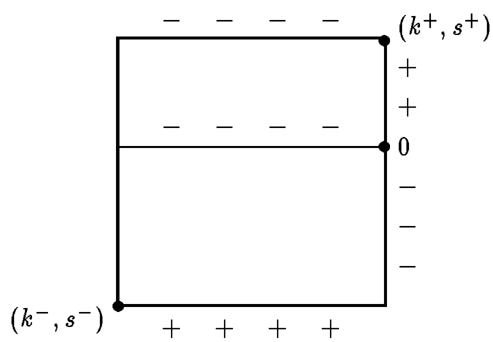


Figure 4: Sign of $F(s, k) - \overline{F}$ in $(k, s)$-phase plane for the case that $F(s, k^+) - \overline{F}$ changes sign.

The fundamental difference between this approach and the standard approach (21) is that Equation (27) now involves values for $k_\epsilon$ in between $k^-$ and $k^+$. For the flux function $F(s, k)$ it follows from (18) that

$$\text{sign}\left(\frac{\partial F}{\partial k}\right) = \text{sign}\,(-N_g). \tag{30}$$

Let us consider the case $s^- < s^+$ and $N_g < 0$. In Figure 3 the sign of $\frac{ds_\epsilon}{dy}$ is shown in the $(k, s)$-phase plane for the two different values of $\text{sign}\,(k^+ - k^-)$. Here we have used (19) and (30). If we want $s_\epsilon$ to be bounded by $s^-$ and $s^+$ for any $k_\epsilon$ it is clearly necessary to require

$$\text{sign}\,(k^+ - k^-) = \text{sign}\,(s^+ - s^-) * \text{sign}\,(-N_g). \tag{31}$$

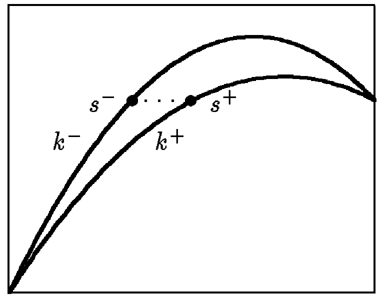


Figure 5: Example of a shock that satisfies the standard entropy condition, but violates the new entropy condition.

However this is condition alone is insufficient to guarantee the existence of a solution to (27) that satisfies the boundary conditions (28). Suppose that $s^- < s^+$, $N_g < 0$ and that condition (31) holds, so $k^+ > k^-$. If $F(s,k^+) - \overline{F}$ changes sign for some $\tilde{s} \in [s^-, s^+]$ this implies (cf. (30))

$$F(\tilde{s}, k) - \overline{F} < 0, \quad \forall k < k^+. \tag{32}$$

This situation is shown in Figure 4. Clearly the solution $s_\epsilon$ can not cross the line $s = \tilde{s}$ in the phase plane. Conversely, if $F(s,k^+) - \overline{F}$ does not change sign there is always a solution $s_\epsilon$ possible. This consideration leads to the following entropy condition for the case $s^- < s^+$:

$$\operatorname{sign}(k^+ - k^-) = \operatorname{sign}(-N_g), \tag{33}$$

$$F(s,k^+) - \overline{F} \geq 0, \quad \forall s \in (s^-, s^+). \tag{34}$$

A similar condition holds for the case $s^- > s^+$.

A comparison of this new entropy condition with the standard entropy condition (23) is interesting. If condition (33) holds, then condition (34) and the standard entropy condition (23) are equivalent (see Figure 4). However there exist pairs $(s^-, s^+)$ that violate condition (33) but do satisfy (23). An example of this situation is shown in Figure 5. Here we have taken $N_g < 0$, $k^+ < k^-$ and $s^+ > s^-$, which clearly violates condition (33). However by taking $\tilde{s} = s^+$ we see that the standard entropy condition (23) is satisfied.

## Conclusion

We have studied two-phase flow through heterogeneous porous media including gravity effects. In the case that there is no capillarity, the solution has stationary shocks at discontinuities in the permeability. An entropy condition for these shocks has been derived by considering the case of vanishing capillarity. This new entropy condition does not depend on the particular shape of the capillary pressure curve or permeability function that is used for the regularization. The new entropy condition is more restrictive than the standard one that is derived by a 'vanishing viscosity' argument. This difference is caused by the fact that in our approach also the discontinuity in the permeability is regularized, so that we have to consider intermediate values for the permeability.

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