Simulating TRSs by minimal TRSs: a simple, efficient, and correct compilation technique

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a Simple, Efficient, and Correct Compilation Technique

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Abstract

A simple, efficient, and correct compilation technique for left-linear Term Rewriting Systems (TRSs) is presented. TRSs are compiled into Minimal Term Rewriting Systems (MTRSs), a subclass of TRSs, presented in [WK95]. In MTRSs, the rules have such a simple form that they can be seen as instructions for an easily implementable abstract machine, the Abstract Rewriting Machine (ARM). In the correctness proof, it is shown that the MTRS resulting from compilation of a TRS simulates neither too much (soundness) nor too little (completeness), nor does it introduce unwarranted infinite sequences (termination conservation). The compiler and its correctness proof are largely independent of the reduction strategy.

1. Introduction

Term (graph) rewriting systems (TRSs) are becoming increasingly important for the implementation of theorem provers, verification tools, algebraic specifications, compiler generators, program analyzers and functional programming languages. Hence, a clear need arises for techniques enabling fast execution of TRSs.

A standard technique for speeding up the execution of a program in a formal (programming) language is compilation into the language of a concrete machine (e.g., a microprocessor). In compiler construction (c.f. [ASU86]), it is customary to use an abstract machine as abstraction of the concrete machine. On the one hand, this allows hiding details of the concrete machine in a small part of the compiler, and thus an easy reimplementation on other concrete machines. On the other hand, a good design of the abstract machine enables a simple mapping from source language into abstract machine language.

A compiler consists of zero or more transformations in the semantic domain of its source language, followed by a mapping to a lower-level language. This is repeated until the level

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of the concrete machine is reached. Because they take place in one domain, the source-to-
source transformations are easier to grasp semantically than the mappings to lower levels.
In this paper, we present a compilation technique for TRSs which stays entirely within the
well-known source language domain.

In [WK95], we have presented Minimal Term Rewriting Systems (MTRSs), a syntactic
restriction of TRSs, and shown that by a modest change of perspective, an MTRS can
be seen as a program for the Abstract Rewriting Machine (ARM), which is in turn easily
implemented on a concrete machine. In [WK95], we concentrate on the concretization of
MTRSs into abstract machine programs, we only show the plausibility of simulating arbitrary
pattern-matching by discussing an example, and we assume innermost rewriting with
syntactic specificity ordering throughout. In this paper, we concentrate on the mapping
from TRSs into simulating MTRSs, and the correctness proofs of these mappings, and we
have formulated our transformations in such a way as to minimize the assumptions regarding
strategy and rule ordering.

The idea to express pattern-matching of TRSs in the language of TRSs itself was inspired
by [Pet92], where pattern-matching of ML is expressed in ML itself. This paper does not
contain a correctness proof, and the algorithm is formulated in a less formal way than our
algorithm. The resulting pattern match code appears to have the same complexity as the
code produced by our algorithm.

The idea to include a correctness proof is taken from [HG94], in which steps towards a
provably correct compiler for OBJ3 are taken. Their compiler is less geared towards efficiency
than ours.

In the remainder of this paper, we proceed as follows. First, in Sections 2, 3 and 4, we
review TRSs, simulation and MTRSs, respectively. In Section 5 we discuss an example of
the application of our technique.

Then, in Section 6, we present a transformation that yields a simulating MTRS from a given
TRS, provided the latter is left-linear, and simply complete (in a simply complete TRS, every
defined function f has a most general rule, i.e., a rule with an LHS consisting of f applied
to a sufficient number of distinct variables). In Section 6.3, we drop the latter requirement
of simple completeness by a second transformation.

The first transformation has the remarkable property that the simulation holds for the
unrestricted rewrite relation, i.e. no assumptions regarding the rewrite strategy are made.

The second transformation is shown to be correct when we assume innermost rewriting
with priorities between rules (similar to the priorities defined in [BBKW89]). This is not as
bad as it seems, because given an implementation of innermost rewriting, other strategies
can be simulated by further transformations (for an example of this, see [KW95]).

We conclude our paper with a discussion of related work, conclusions and directions for
future work.

2. Term Rewriting
In this section, we mainly follow [Klo92], except for the notation of paths and contexts, which
is taken from [DJ90].
A signature $\Sigma$ consists of:

- A countably infinite set $V$ of variables: $x, y, \ldots$

- A non-empty set $F$ of function symbols $f, g, \ldots$, each with an arity ($\geq 0$), which is the number of arguments the function requires. We denote the arity of $f$ by $|f|$.

The set $T(\Sigma)$ of terms over $\Sigma$ is the smallest set satisfying

- $V \subseteq T(\Sigma)$,

- for all $f \in F$ with arity $n$, and $t_1, \ldots, t_n \in T(\Sigma)$, we have $f(t_1, \ldots, t_n) \in T(\Sigma)$.

We will write $\text{var}(t)$ for the set of variables occurring in $t$. Occasionally, we will abbreviate a sequence $t_1, \ldots, t_n$ to $\bar{t}$, and write $|\bar{t}|$ for $n$. We generalize this to empty sequences, which have $|\bar{t}| = 0$.

A path in a term is represented as a sequence of positive integers. By $t|_p$, we denote the sub-term of $t$ at path $p$. For example, if $t = f(g, h(f(y, z)))$, then $t|_{1, 2, 1}$ is the first sub-term of $t$'s second sub-term, which is $f(y, z)$. We write $p \subseteq s$ if $p$ is a valid path in $s$ (i.e., indicates a sub-term of $s$), and $p_1 \leq p_2$ if $p_1$ is a prefix of $p_2$ (i.e., $\exists p_3 : p_2 = p_1 + p_3$). We write $p[q$ iff neither $p \leq q$ nor $q \leq p$. The empty path (referring to root) is written as $\varepsilon$. We write $t|s|_p$ for the term resulting from the replacement at $p$ of $t|_p$ in $t$ by $s$. Following [HL91], we write $O(s)$ for the occurrences of $s$, that is $\{p | p \in s\}$.

We write $\text{of}(f(\bar{t})) = f$ for the outermost function symbol $f$ of a term $f($ $\bar{t})$, $\text{lhs}(l \rightarrow r) = l$ for the left hand side $l$ of a rule $l \rightarrow r$, and $\text{rhs}(l \rightarrow r) = r$ for the right hand side $r$ of a rule $l \rightarrow r$.

A context is a ‘term’ containing one occurrence of a special symbol $\Box$, denoting an empty place. A context is generally denoted by $C[\Box]$. If $t \in T(\Sigma)$ and $t$ is substituted for $\Box$, the result is $C[t] \in T(\Sigma)$ and $t$ is said to be a subterm of $C[t]$, notated as $C[t] \subseteq t$.

A substitution is a (total) map $\sigma : T(\Sigma) \rightarrow T(\Sigma)$ satisfying

$$\forall f \in F : \sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n)).$$

By convention, we often write $t^\sigma$ for $\sigma(t)$.

A rewrite rule is a pair of terms written as $s \rightarrow t$ with $s, t \in T(\Sigma)$. It is assumed that the left-hand side $s$ of a rule $s \rightarrow t$ is not a single variable, and that $\text{var}(t) \subseteq \text{var}(s)$.

A term rewriting system $R$ consists of a signature $\Sigma$ and a set of rewrite rules $R$ over $\Sigma$.

A term rewriting system defines a rewrite relation $\rightarrow_R$. Since the subscript $R$ is usually clear from the context, it is omitted. The overloading of $\rightarrow$ is by convention.

$$s \rightarrow t \iff \exists \sigma, p, u \rightarrow v \in R : s|_p = u^\sigma \land t = s|v^\sigma|_p.$$ 

The sub-term $u^\sigma$ is referred to as redex (for reducible expression); the sub-term $v^\sigma$, as reduct.
If we want to be specific about the rule and the redex position \( p \), we write \( s \xrightarrow{p \langle l \rightarrow r \rangle} t \).

We write \( \xrightarrow{*} \) for the transitive reflexive closure of \( \xrightarrow{-} \).

The rewrite relation is closed under contexts, i.e., if \( s \xrightarrow{-} t \), then for all \( C[] \), \( C[s] \xrightarrow{-} C[t] \).

A series of terms \( s = s_1, s_2, \ldots \) such that \( s_1 \xrightarrow{-} s_2 \xrightarrow{-} \ldots \) is called a rewrite sequence. A term \( s \) is said to be in normal form if there is no \( t \) such that \( s \xrightarrow{-} t \). A function-symbol \( f \) is called a defined function symbol if there is a rule \( f(t_1, \ldots, t_n) \rightarrow r \). A function-symbol \( c \) is called a constructor symbol if there is a normal form in which it occurs, and a free constructor if it is not a defined symbol.

A TRS is called left-linear if all left-hand sides are linear. A TRS is called confluent if, for all terms \( t_1, t_2, t_3 \), we have that \( t_1 \xrightarrow{*} t_2 \) and \( t_1 \xrightarrow{*} t_3 \) implies that there exists a term \( t_4 \) such that \( t_2 \xrightarrow{*} t_4 \) and \( t_3 \xrightarrow{*} t_4 \). A TRS is called terminating if there are no infinite rewrite sequences. Note that confluence and termination are generally undecidable.

Let \( r_1 : l \rightarrow r \) and \( r_2 : g \rightarrow d \) be rewrite rules. If there exists a context \( C[] \), a non-variable term \( s \), and a substitution \( \sigma \) such that \( l = C[s] \) and \( s^\sigma = g^\sigma \), then \( g \) overlaps with \( l \). We say there is an overlap between a rule \( r \) and a TRS \( T \) iff either \( r \) overlaps with a rule of \( T \), or there is a rule of \( T \) that overlaps with \( r \).

A TRS is called orthogonal if it is left-linear, and there is no overlap between the rules.

Following [HL91], we write \( R(s) \) for the set of paths to redexes in \( s \).

Given a rewrite step \( A : s \xrightarrow{p \langle l \rightarrow r \rangle} t \) and \( p \in R(s) \), where there is no overlap between \( l \) and the rule of \( p \), we define the set \( p \backslash A \) of residuals or descendants of \( p \) by \( A \) as a subset of \( O(s) \):

\[
\begin{align*}
p \backslash A = \begin{cases} \emptyset & \text{if } p = p_A; \\ \{ p \} & \text{if } p | p_A \text{ or } p \leq p_A; \\ \{ p | p_n r | r | p_m = x \} & \text{if } p = p_i p_m r \text{ and } l | p_m = x \in \mathcal{V}. \end{cases}
\end{align*}
\]

For rewrite sequences, we define \( p \backslash A \) by

\[
\begin{align*}
\{ p | \epsilon = \{ p \} \\
p \backslash A B = \{ p_n | p_n \in p \backslash A \}
\end{align*}
\]

For orthogonal systems (where there is no overlap at all) these definitions generalize to the ones given in [HL91].

In general, a term may contain many redexes. A rewriting strategy determines which of these is chosen. Confluence guarantees unique normal forms, regardless of the strategy. A well-known strategy is rightmost innermost, which chooses the rightmost redex that does not contain another redex.

In priority rewrite systems (PRSs) [BBKW89], the rules are (partially) ordered, and a rule may be applied only if there are no applicable rules (i.e., even after reduction of subterms) with higher priority. We will also consider syntactic priority, in which the decision whether a rule is applicable is made without considering reductions of subterms.

The ordering we will use is syntactic specificity ordering, where a rule \( l \rightarrow r > s \rightarrow t \), when there exists a substitution \( \sigma \) such that \( s^\sigma = l \) (in [BBKW89], specificity ordering implies that
all ambiguities are between terms that are ordered according to specificity, which we do not demand for syntactic specificity ordering).

Under syntactic specificity ordering, any set of terms with the same outermost function symbol has a greatest lower bound (glb). We will call such a glb, a term of the form \( f(\overline{x}) \), a most general LHS.

A TRS is called sufficiently complete if defined functions do not appear in normal forms. In general, sufficient completeness is undecidable. We will call a TRS simply complete if every defined function has a most general rule. It is clear that simple completeness implies sufficient completeness.

3. Term Rewriting Simulations

In this section, we define the notion of simulation of a TRS by another TRS.

In principle, a TRS \( T = (\Sigma, R) \) is simulated by a TRS \( T' = (\Sigma', R') \) if every rewrite sequences w.r.t. \( R \) can be related to a rewrite sequence w.r.t. \( R' \). To this end, there must be a map from \( T(\Sigma') \) to \( T(\Sigma) \), which is called the simulation map.

This notion of simulation can be developed for arbitrary relations, but we will only use it in the more limited context of (minimal) term rewriting systems. In that context, as we will see, it is preferable to regard a simulating TRS of which the signature is an extension of that of the simulated TRS (i.e., \( \Sigma' \supseteq \Sigma \)), and for which the simulation map is identity on the common set of terms \( T(\Sigma) \).

3.1 Simulation maps between terms

**Definition 1** Let \( \Sigma = (F, V) \) and \( \Sigma' = (F', V') \) be signatures, such that \( \Sigma' \supseteq \Sigma \). A simulation map is a partial map \( S : F' \rightarrow F \) for which \( \forall f \in F : S(f) = f \). Let \( D_S \) be a predicate that holds precisely for all symbols in \( F' \) for which \( S \) is defined.

Note that the composition of two simulation maps is again a simulation map.

Under this definition, symbols in the original signature simulate themselves, and a simulating TRS may use intermediate symbols (terms) which are not a simulation of any symbol (term) in \( F \).

We extend \( S \) and \( D_S \) to \( T(\Sigma') \) by (partial) homomorphic extension.

As an example, consider \( F = \{ f, a \} \) and \( F' = \{ f, a, f_c, h \} \). In this example, \( f_c \) is a variant (a so-called constructor variant, discussed further in the sequel) of \( f \) with \( S(f_c) = f \), and \( h \) is an auxiliary function that has no counterpart in \( F \). Supposing that the arity of \( f \) is 1, and the arity of \( a \) is 0, we have (by partial homomorphic extension) that \( S(f(f_c(a))) = f(f(a)) \), and \( \neg D_S(f(h(a))) \), so \( S(f(h(a))) \) is undefined.

3.2 Simulating Relations

Let \( \mathcal{R} = (\Sigma, R) \) and \( \mathcal{R}' = (\Sigma', R') \) be TRSs, with the understanding that by \( R \) and \( R' \) we sometimes mean the rewrite relation, rather than the rewrite rules\(^1\), and let \( S : \Sigma' \rightarrow \Sigma \) be a simulation map. We will define simulation of \( \mathcal{R} \) by \( \mathcal{R}' \) under \( S \). First, we define three auxiliary

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\(^1\)This makes it easier to discuss restrictions of the rewrite relation, e.g. the relation with only innermost rewrites.
notions: soundness, completeness and termination conservation. In the figures illustrating the definitions below, dashed arrows are implied by solid arrows, closed points are universally quantified, and open points are existentially quantified.

Soundness of the triple \((R, S, R')\) means that sufficiently many sequences in \(R'\) are mapped (by \(S\)) to sequences in \(R\). If we have a sequence \(s \in R\) in the simulating system with \(S\) defined on \(s\) and \(t\), it is only reasonable to call \((R, S, R')\) sound when \(S(s) \in R\) (depicted in Fig. 1a). In case \(S\) is not defined on \(t\), we do not want the sequence to 'escape into undefinedness', so we demand that there is some \(u\) with \(t \in u\) and \(S\) defined on \(u\) (depicted in Fig. 1b). Formally, soundness is defined in Definition 2.

**Definition 2** A simulation \((S, R')\) of \(R\) is sound whenever
\[
\forall st \left( D_S(s) \land s \in R' \Rightarrow S(s) \in R \right) \land \left( \neg D_S(t) \lor \exists u \in u \land S(u) \right) t \in R' \land u
\]

The triple \((R, S, R')\) is complete, when every step \(S(t)\) in the simulated relation has a counterpart a simulating sequence \(s \in R\), with \(S(s) = t\), provided \(s\) is reachable, i.e. \(s R' s\), for some \(s_0 \in \text{Ter}(S)\), written \(\text{reachable}(S) = s_0\). This is defined formally in Definition 3, and depicted in Fig. 2.

**Definition 3** A simulation \((S, R')\) of a relation \(R\) is complete whenever
\[
\forall st \left( \text{reachable}(S) \land D_S(s) \land S(t) = t \right) \Rightarrow \exists u \in u \land S(u) \land t \in R' \land u
\]

Termination conservation of the triple \((R, S, R')\) means that only terms taking part in infinite sequences in \(R\), have origins (under \(S\)) occurring in infinite sequences in \(R'\).

**Definition 4** A simulation \((S, R')\) is termination preserving whenever
\[
\forall s \in \text{inf}(R) : D_S(s) \Rightarrow \exists t \in \text{inf}(R) \land S(t) = t
\]

where \(\text{inf}(R)\) is the set of infinite sequences in \(R\), and we denote the \(i\)th term in a rewrite sequence \(s\) by \(s_i\).

**Fig. 1a.**

**Fig. 1b.**

**Fig. 2.** Completeness

**Fig. 3.** Conservation of termination
Definition 5 (Simulation) Let $\mathcal{R} = \langle \Sigma, R \rangle$ and $\mathcal{R}' = \langle \Sigma', R' \rangle$ be TRSs with $\Sigma \subseteq \Sigma'$ and let $S : \Sigma' \rightarrow \Sigma$ be a simulation map. We say that $\mathcal{R}$ is simulated by $\mathcal{R}'$ under $S$, written as $\mathcal{R} \parallel_S \mathcal{R}'$, iff the triple $(R, S, R')$ is sound, complete and termination conserving.

When $S$ undefined on $\Sigma' \setminus \Sigma$, or if $S$ is clear from the context, we will write $\mathcal{R} \parallel \mathcal{R}'$.

Normal forms, confluence, and strong and weak normalization are preserved under simulation. It is easy to verify that normal forms are preserved under simulation, that is, if we have $S(m) = n$ with $n$ a normal form then for all $mR^*m'$, we have that $S(m') = n$, and from termination preservation it follows that there are no infinite sequences starting with $m$. Confluence follows directly from completeness. Conservation of strong normalization follows directly from termination preservation. With regard to weak normalization we remark that from completeness it follows that the sequence leading to a normal form $n$ can be simulated.

Note that our notion of simulation is transitive: given that $\mathcal{R} \parallel_S \mathcal{R}'$ and $\mathcal{R}' \parallel_S \mathcal{R}''$, we have that $\mathcal{R} \parallel_S S \circ \mathcal{R}''$.

In a simple simulation, the effect of a single rule is simulated by a pair of complementary rules.

Lemma 1 (Simple Simulation) Let $\mathcal{R} = \langle \Sigma, R \rangle$ and $\mathcal{R}' = \langle \Sigma', R' \rangle$ such that:

1. $\Sigma' = \Sigma \cup \{f\}$ ($f \notin \Sigma$);
2. $R = R_0 \cup \{r_0 : l \rightarrow r\}$ ($r_0 \notin R_0$);
3. $R' = R_0 \cup \{r_1 : l \rightarrow f(t), r_2 : f(x) \rightarrow r'\}$;
4. $s \rightarrow_{r_0} t \land s \rightarrow_{r_1} t' \Leftrightarrow s \rightarrow_{r_1} t', s' \rightarrow_{r_2} t$;
5. All (sub)terms occurring in $t$ also occur in $l$ or in $r$.

Then $\mathcal{R} \parallel_{2} \mathcal{R}'$.

Proof We have to prove completeness, soundness, and termination conservation of the triple $(R, I_{\Sigma}, R')$.

Completeness is trivial, it follows directly from requirement 4.

For soundness, we first observe that given a sequence $sR^*t$ with $D_S(s) \land D_S(t)$ (i.e., both $s, t \in \text{Ter}(\Sigma)$) we have $sR^*t$. This follows from the fact that applications of rule $r_2$ are only possible on terms created by applications of rule $r_1$. Because $r_2$ has no overlap with other rules, and no redexes of $r_2$ remain in $t$ (this follows from $D_S(t)$), we can replace the applications of $r_1$ in $sR^*t$ by applications of $r_0$, delete the applications of $r_2$, and thus obtain $sR^*t$. Second, we observe that when we have $sR^*t$ with $\neg D_S(t)$, this must be because there are some $r_2$-redexes left in $t$. We can rewrite these and obtain $sR^*tR^*t'$, with $D_S(t')$. Now, by the first observation, $sR^*t$, which completes the proof of soundness.

We prove termination conservation by considering the number of $r_1$-contractions. If there are no $r_1$-contractions in an infinite $R'$-sequence starting in a term $t$ with $D_S(t)$, there are no $r_2$-contractions either, so the infinite sequence is itself an $R$-sequence. If there is only a finite number of $r_1$-contractions in an infinite sequence, there can only be an infinite number of $r_T$-contractions if there is some context $C[\cdot]$ in which (descendants of) an $r_2$ redex can be duplicated infinitely many times. But because $r_2$ has no overlap with other rules, this means that (descendants of) the $r_1$-redex can already be duplicated infinitely many times in $C[\cdot]$,
which is a contradiction, so all \( r_1 \) and \( r_2 \)-contractions occur in a finite prefix of the infinite sequence, and the infinite suffix corresponds to an infinite \( R \)-sequence. Finally, if there is an infinite number of \( r_1 \) contractions, then there is also an infinite number of \( r_0 \)-contractions possible, because all subterms in an instantiated RHS of \( r_1 \) are also in an instantiated RHS of \( r_0 \), and an instance of the RHS of \( r_1 \) itself can only be contracted by \( r_2 \), with the same result as a direct contraction by \( r_0 \). 

4. Minimal Term Rewriting Systems

Here, we repeat the definition of minimal term rewriting systems (MTRSs), a syntactic restriction of TRSs that can be interpreted as the language of an abstract machine (see [WK95]).

In MTRSs, all rules have an extremely simple form. The most conspicuous aspect is that any rule has at most three function symbols, of which at most two are found on either side. Even the SKI calculus ([Klo92]), which is minimal in the number of rules (3), and in the total number of function symbols (4: S, K, I, and \( \cdot \)), needs 7 function symbols in its most complicated rule \( (S \cdot x \cdot y \cdot z \rightarrow (x \cdot y) \cdot (y \cdot z)) \). Somewhat less conspicuous, but equally important for the interpretation as a machine language, is the fact that the ‘action’ (adding, changing or deleting function symbols or variables) performed by application of a rule is ‘local’, i.e. restricted to a number of consecutive arguments and the outermost function symbol.

**Definition 6 (MTRS)** Let \( \mathcal{R} = (\Sigma, R) \) be a TRS, and \( r : s \rightarrow t \) a rule in \( R \). The rule \( r \) is called minimal if it is left-linear and it is in one of the following six forms:

\[
\begin{align*}
C : & \quad f(x, y, z) \rightarrow h(x, g(y), z) \\
R : & \quad f(y) \rightarrow y \\
M : & \quad f(x, g(y), z) \rightarrow h(x, y, z) \\
A : & \quad f(x, z) \rightarrow h(x, y, z) \quad \text{(} y \text{ is } x; \text{ or } z; \text{)} \\
D : & \quad f(x, y, z) \rightarrow h(x, z) \quad \text{(} |y| \neq 0 \text{)} \\
I : & \quad f(x) \rightarrow h(x)
\end{align*}
\]

A TRS \( \mathcal{R} \) is called a Minimal Term Rewriting System (MTRS) if all its rules are minimal.

We have labeled the forms with mnemonics reminding of their basic purpose (in the context of innermost rewriting). The mnemonic \( C \) stands for continuation, in the sense that \( h \) is the continuation after the evaluation of \( g \). Conversely, \( R \) stands for return, in the sense that control is passed to a continuation if that was issued earlier, or rewriting is finished if there is no such continuation. Rules of the form \( M \) take apart a term, when there is a match of the symbol \( g \). The forms \( A, D \) and \( I \) are for addition, deletion and identity on the set of variables.

5. An illustrative example compilation

Before we present our compilation technique in its general form, we would like to give an intuitive impression by showing how a concrete TRS is transformed into a simulating MTRS. Consider the following example of a simply complete TRS:

\[
\begin{align*}
f(g(X), h(Y)) & \rightarrow a(g(Y), X) \quad (5.1) \\
f(X, f(Y, Z)) & \rightarrow b \quad (5.2) \\
f(X, Y) & \rightarrow c \quad (5.3)
\end{align*}
\]
In Section 6.1, we will show how pattern-matching is simulated by a transformation. On the TRS above, this transformation would yield:

\[
\begin{align*}
    f(g(X), Y) & \rightarrow f(g(X), Y) \\
    f(X, Y) & \rightarrow f(S(X, Y)) \\
    f(g(X, h(Y))) & \rightarrow f(g(X, Y)) \\
    f(g(X, Y)) & \rightarrow f(S(g(X), Y)) \\
    f(S(X, f(Y, Z))) & \rightarrow f(S(X, Y, Z)) \\
    f(S(X, Y)) & \rightarrow f(S(S(X, Y))) \\
    f(g(X, Y)) & \rightarrow a(g(Y), X) \\
    f_{gh}(X, Y) & \rightarrow b \\
    f_{S}(X, Y) & \rightarrow c \\
\end{align*}
\]

The newly introduced functions \( f_{g}, f^{S}, f^{S}_{g}, \) and \( f^{SS} \) can be understood as representants of the states of a matching automaton for LHS patterns (inspired by [HO82, Wal91, Pet92]). In this TRS, all rules are minimal, except for (5.10), in which a non-trivial RHS appears (the variables are in the wrong order). In Section 6.2, we will show how the construction of RHSs is simulated by a transformation. This transformation replaces (5.10) by the rules

\[
\begin{align*}
    f_{gh}(X, Y) & \rightarrow a^{R}(X, Y, X) \\
    a^{R}(X, Y, X') & \rightarrow a^{RR}(Y, X') \\
    a^{RR}(Y', X') & \rightarrow a(g(Y'), X') \\
\end{align*}
\]

The result is an MTRS. The reader is invited to verify that this MTRS simulates the original TRS, using as simulation map \( I_{\Sigma} \), the identity on \( \Sigma \).

Now suppose that rule (5.3) were not present in the original system, then the system would not be simply complete. But, as is shown in general in Section 6.3, the TRS

\[
\begin{align*}
    f(g(X), h(Y)) & \rightarrow a(g(Y), X) \\
    f(X, f_{c}(Y, Z)) & \rightarrow b \\
    f(X, Y) & \rightarrow f_{c}(X, Y) \\
\end{align*}
\]

simulates this system, when we assume innermost rewriting with specificity, and \( S(f_{c}) = f \).

The idea is that rule (5.18) only applies when none of the other rules apply, so that in normal forms, \( f_{c} \) occurs exactly where in the original system \( f \) would have occurred. Rule (5.17) is adapted to match such normal forms (in innermost rewriting, the proper subterms of a redex are in normal form by definition).

6. **Every simply complete TRS can be simulated by an MTRS**

We will now show that every simply complete left-linear TRS can be simulated by an MTRS. We will give a constructive proof by providing a terminating transformation that transforms any simply complete left-linear TRS into a simulating MTRS. Because simulation is transitive, it suffices to prove simulation for every individual step of the transformation.
6.1 Transforming complicated LHSs
We will now present a specification of the function \( \text{sim} \), and prove that applying \( \text{sim} \) to a simply complete TRS \( \langle \Sigma, R \rangle \) yields a TRS \( \langle \Sigma', R' \rangle \) such that \( R \upharpoonright_{\text{tr}} R' \). The specification of \( \text{sim} \) is itself nonambigous and terminating, so it can be used as a pattern matching compiler.

In the specification, we will extensively use union of TRSs:

\[
\langle \Sigma, R \rangle \cup \langle \Sigma', R' \rangle = \langle \Sigma \cup \Sigma', R \cup R' \rangle
\]

Given some index set \( I = \{ i_1, \ldots, i_n \} \), we will use the notation \( \bigcup_{i \in I} T_i \) for the (finite) union \( T_{i_1} + \ldots + T_{i_n} \).

If all rules in \( R \) are most general, then:

\[
[\text{sim-base}] \text{sim}(\Sigma, R) = \langle \Sigma, R \rangle.
\]

Otherwise, let \( i \) be the least index such that for some rule \( f(t) \to s \in R \), we have \( t_i \notin V \), and let \( G = \{ g | f(t) \to r \in R \land t_i = g(u) \} \), the set of function symbols found at position \( i \) of LHSs defining \( f \) in \( R \). Then, taking \( |x| = |t| = i - 1 \), and \( f^S, f_g \notin \Sigma \) fresh symbols, we have:

\[
[\text{sim-rec}] \text{sim}(\Sigma, R) = (\bigcup_{g \in G} \text{Match}_g \cup \text{Matched}_g) \cup \text{Skip} \cup \text{Other},
\]

where

\[
\text{Match}_g = \langle \{ f, g, f_g \}, \{(m1 : f(x,y,z) \to f_g(x,y,z)) \} \rangle,
\]

\[
\text{Matched}_g = \text{sim}(\Sigma \cup \{ f_g, f^S \}, \{(m2 : f_g(t,u,v) \to r \mid (m3 : f(t,g(u), v) \to r \in R) \\
\cup \{(m4 : f_g(x,y,z) \to f^S(x,y,z)) \} \} \rangle;
\]

\[
\text{Skip} = \langle \Sigma \cup \{ f^S \}, \{(s1 : f(x) \to f^S(x)) \} \rangle;
\]

\[
\text{Other} = \text{sim}(\Sigma \cup f^S, \{(o1 : f_o^S(i) \to s \mid (o2 : f(i) \to s \in R \land i \in V) \\
\cup \{(o3 : r \in R \land \text{of}(\text{lhs}(r)) \neq f) \} \rangle.
\]

An intuitive explanation of [sim-rec] is, that \( \text{Match}_g \) has a rule \( m1 \) that matches a symbol \( g \) at position \( i \), \( \text{Matched}_g \) deals with a successful match of \( g \) at position \( i \), by either completing a match of \( m3 \) by applying \( m2 \), or restoring the LHS of \( m1 \) (up to \( f^S \)) by applying \( m4 \), \( \text{Skip} \) just replaces \( f \) by \( f^S \) (with the effect of sharing an 'automaton' state between reconstructed terms and terms for which matching fails right away), and \( \text{Other} \) simulates the rules \( o2 \) that have a variable at position \( i \) with rules \( o1 \), and rules \( o3 \) for other function symbols than \( f \).

Let \( NVP(R) \) be the number of paths to nonvariable proper subterms of LHSs of the rewrite rules \( R \). It is clear that \( NVP(R) \) is a well-founded measure on TRSs. It is easily established that, when read from left to right, the recursive rule [sim-rec] is decreasing in this measure. Furthermore, the conditions are decidable, so the specification of \( \text{sim} \) is an executible specification that can be used as a pattern-match compiler.

**Theorem 1** Let \( \langle \Sigma', R' \rangle = \text{sim}(\langle \Sigma, R \rangle) \). Then the triple \( (R, \text{tr}, R') \) is complete.

**Proof** By induction on \( NVP(R) \). If \( NVP(R) = 0 \), we have case [sim-base], which is trivially complete. Otherwise, [sim-rec] must be applied. Because the simulation map is \( \text{tr} \), we only have to consider terms in \( \text{Term}(\Sigma) \). There are three cases: either a rule of type \( m3 \) is applied,
or another rule defining \( f \), or a rule defining another function symbol than \( f \). A rule of type \( m3 \) is simulated by applying \( m1 \) and then \( m2 \); other rules defining \( f \) are simulated by applying \( s1 \) and then \( m2 \); rules defining other symbols than \( f \) are available as rules of type \( c3 \).

**Theorem 2** Let \( \langle \Sigma', R' \rangle = \text{sim}(\langle \Sigma, R \rangle) \). Then the triple \((R, I_\Sigma, R')\) is sound.

**Proof** With induction on \( NVP(R) \). When \( NVP(R) = 0 \), it is clear we have case [\text{sim-base}], and soundness is trivial. Otherwise, let \( i \) be the least index such that there is a rule \( r \in R \) with \( \text{lhs}(r) \mid i \not\in V \). Without loss of generality, we can assume \( r \) to be \( f(t, g(u), v) \to s \), with \( |t| = i - 1 \). We have to prove that

\[
\forall t \ (D_S(s) \land sR^*t) \implies S(s)R^*S(t) \lor (\neg D_S(t) \land \exists uD_S(u) \land tR^*u)
\]

By the induction hypothesis, we may assume \( \text{Matched}_g \) to simulate

\[
\langle \Sigma \cup \{f_g, f^S\}, \{\langle m2 : f_g(t, u, v) \to r \rangle \mid m3 : f(t, g(u), v) \to r \in R\} \\
\cup \{\langle m4 : f_g(x, y, z) \to f^S(x, g(y), z) \rangle \}
\]

and \( \text{Other} \) to simulate

\[
\langle \Sigma \cup \{^S\}, \{\langle o1 : r^S \rangle \mid r \in R \land \text{ofo}(\text{lhs}(r)) = f \land \text{lhs}(r) \mid i \in V\} \\
\cup \{\langle c3 \rangle \mid r \in R \land \text{ofo}(\text{lhs}(r)) \neq f\}
\]

It is clear that when the sequence \( sR^*t \) does not contain a (sub)term with a function symbol \( f_g \) or \( f^S \), \( sR^*t \) holds trivially. A (sub)term with function symbol \( f_g \) is necessarily a descendant of a term introduced by rule \( m1 \). Such a term is a redex for \( m4 \), and potentially also for \( m2 \). Five things can happen to the descendants: they may be contracted according to \( m2 \) or \( m4 \), they may persist in \( t \), they may be duplicated and they may be deleted by contraction of a higher redex. Duplication and deletion are trivially copied in \( R \). If the redex persists in \( t \), we can construct \( t' \) by applying \( m4 \), which brings us to the case where we have a (sub)term with function symbol \( f^S \) (see below). The result of a contraction according to \( m2 \) can be obtained in \( R \) by contracting the original term according to \( m3 \). The result of a contraction according to \( m4 \) brings us again to the case where we have a (sub)term with function symbol \( f^S \). Apart from the two cases above, a (sub)term with function symbol \( f^S \) can be introduced by \( s1 \). Because of simple completeness, such a (sub)term is a redex of at least one of the rules in \( \text{Other} \), but there can only be root-overlap with other rules, because \( f^S \) does not occur in \( \Sigma \). Therefore, the descendants of this redex may be duplicated, deleted, or contracted according to one of the rules in \( \text{Other} \). Duplication and deletion are again trivially copied in \( R \), contraction according to \( o1 \) or \( o3 \) corresponds to a contraction according to \( o2 \) or \( o3 \), respectively. Finally, if a (sub)term with \( f^S \) is left in \( t \), we have \( \neg D_S(t) \), but we have \( tR^*t' \) by a most general rule in \( \text{Other} \). By the argument above, we now have that \( sRt' \).

**Theorem 3** Let \( \langle \Sigma', R' \rangle = \text{sim}(\langle \Sigma, R \rangle) \). Then the simulation \((I_\Sigma, \langle \Sigma', R' \rangle)\) of \( \langle \Sigma, R \rangle \) is termination preserving under innermost rewriting.

**Proof** Obvious by induction on \( NVP(R) \).
6.2 Transforming Complicated RHSs

Here we present a transformation that will transform a TRS \( N \), which may have RHSs that do not conform to the restrictions imposed on MTRSs, into a simulating TRS \( M \), whose RHSs are minimal. Any rule with a minimal LHS and a non-minimal RHS has the form \( l(x, y, z) \rightarrow h(x, t, u, z) \), where \( u \) is either a variable (not equal to the last variable of \( y \)) or a term \( g(u) \), and \( x \) and \( z \) contain only variables, and are taken of maximal length. The goal is to reduce the non-compliant segment \( t, u \).

In case \( u \) is a variable, we replace the rule by the following rules:

\[
\begin{align*}
l(x, y, z) & \rightarrow h(x, t, u, z) \\
& (6.19) \\
\end{align*}
\]

\[
\begin{align*}
h(x, y, u, z) & \rightarrow h(x, t, u, z), \\
& (6.20)
\end{align*}
\]

Where \( u' \) is a fresh variable. Rule (6.19) is an instance of \( A \), and rule (6.20) has a shorter non-compliant segment \( t \).

In case \( u = g(u) \), we replace the rule by the following rules:

\[
\begin{align*}
l(x, y, z) & \rightarrow h(x, t, u, z) \\
& (6.21) \\
\end{align*}
\]

\[
\begin{align*}
h(x', y', z) & \rightarrow h(x', g(y'), z), \\
& (6.22)
\end{align*}
\]

where \( |x'| = |x| + |t| \), \( |y'| = |u| \); \( h \) is a fresh function symbol which did not already occur in the TRS; and \( x' \) and \( y' \) consist entirely of fresh variables. Rule (6.21) contains one function symbol less than the original rule, and rule (6.22) is an instance of \( C \).

We take the simulation map \( S \) to be undefined for \( h \). Repeated application of the transformation above to a TRS with minimal LHSs leads to an MTRS.

We note that both the variable case and the nonvariable case fulfill the requirements for a simple simulation (see Lemma 1), so the RHS transformation yields a simulating TRS.

6.3 Simulating General Left-Linear TRSs by Simply Complete left-linear TRSs

Until now, we have only dealt with simply complete TRSs. Unfortunately, simple completeness is a rare property. Here we will show that, under the restriction to innermost rewriting with syntactic specificity ordering, every TRS can be simulated by a simply complete TRS.

Let the TRS \( \langle \Sigma, R \rangle \) be given, and let \( \Sigma_p \subseteq \Sigma \) be the set of function symbols for which \( R \) has no general rule. Let \( \Sigma_c \) contain a so-called constructor variant \( f_c \) for every \( f \in \Sigma_p \), and let \( S(f_c) = f \). Given a term \( t \) or a sequence \( t \), define \( t_c \) or \( t_c \) to be the term or sequence obtained by replacing all \( f \in \Sigma_u \) by their constructor variants \( f_c \). Taking \( R' = \{ c_1: f(t_c) \rightarrow s(f(t)) \in R \} \cup \{ c_1: f(x) \rightarrow f_c(x) | f \in \Sigma_u \} \), we have obtained a simply complete TRS \( \langle \Sigma \cup \Sigma_c, R' \rangle \).

It is easy to see that the triple \( \langle \Sigma, R \rangle, S, \langle \Sigma', R' \rangle \) is sound, complete and termination conserving, so \( R \models_S R' \). For soundness, we observe that, given a rewrite sequence \( t_1 \rightarrow t_2 \rightarrow \ldots \rightarrow t_n \) in \( R' \), it follows that either \( S(t_i) = S(t_{i+1}) \) (in case \( c_1 \) is applied), or \( S(t_i) R S(t_{i+1}) \) (in case \( c_1 \) is applied), so always \( t_i R' \star t_n \).

For completeness, we observe that a step \( S(t_i) R S(t_{i+1}) \) may not be possible in \( R' \) because some subterms of \( t_i \) have an original function symbol where a constructor variant is needed. We may first rewrite exactly these subterms with rules of type \( c_1 \), however, \( t_i R' \star t'_i \), with
$S(t_i') = S(t_i)$, and then we have $t_i' R t_{i+1}$. Because a step $t_i R t_{i+1}$ is only taken when all subterms of $t_i$ are already in normal form, rewriting $t_i$ to $t_i'$ does not invalidate future $R$-rewrites and because of specificity ordering, rules of type $c1$ can only rewrite terms $t$ for which $S(t)$ is a normal form.

Finally, conservation of termination follows from the fact that only a finite number of applications of rules of type $c1$ is possible on any term, so if there is an infinite reduction on $t$ according to $R'$, there is necessarily an infinite reduction on $S(t)$ according to $R$.

Without proof, we mention that non-linear TRSs can be simulated in a similar vein, under the same restrictions and for given $\Sigma$.

6.4 Efficiency Considerations

The efficiency of compilers can be expressed by several measures:

- The size (in number of rules) of the target program;
- The time and space taken for compilation from source to target language;
- The time and space taken by an execution of the target program, compared to the time and space taken by execution of the source program.

It is clear that the size of the target program depends in a linear fashion on the total number of occurrences of function symbols in the source program, and rules in the target program are at least as simple as the rules in the source program.

With regard to the space taken by the compilation, we observe that the number of new rules constructed depends in a linear fashion on the total number of occurrences of function symbols in the source program. Thus, a naive implementation needs at most an amount of space linear in the size of the source program.

With respect to the time taken, even a naive implementation of $\text{sim}$ that scans all rules to find a rule with nonvariable arguments, will only be quadratic in the number of rules and linear in the number of symbol occurrences in LHSs.

Considering the time taken by the execution of the target program, we remark that indeed the number of rewriting steps is linearly increased by the compilation. The complexity of executing a single step, however, is decreased. In practice, this leads to comparable performance. The big gain, however, is in the fact that the MTRS can easily be seen as a program to be executed by a concrete machine (see [WK95]). The resulting machine code and its performance is similar to that of existing compilers for functional languages [HF+96].

7. Conclusions and Future Work

We have presented transformations from arbitrary left-linear TRSs into simulating MTRSs. The transformations can be expressed in a concise way, and their correctness proofs are short and easy to grasp. Furthermore, the transformations are described as executable specifications, which can be used as an efficient compiler. The resulting code is similar to the code generated by an earlier version of our TRS compiler, with which favorable results have been reached [HF+96].

In [KW95], we presented a transformation to simulate lazy rewriting by eager (innermost) rewriting. It appears that this transformation can be simplified greatly by first applying the
transformations in this paper, and then the laziness transformation, which is much simpler when only MTRSSs have to be considered.

Similarly, we expect that the transformations given in this paper could simplify other research on TRSs (e.g., Hans Zantema suggested that termination proofs might be simpler after our transformations, but we have not yet investigated this issue in any depth).

In the future, we hope to find a bigger class of TRSs for which a strategy-independent simulation by MTRSSs can be given. For our current implementation requirements, however, the current class is sufficient.

An interesting class of TRSs (which unfortunately has no inclusion relation with simply complete TRSs) is the class that admits specificity ordering as defined in [BBKW89]. It appears that applying the transformation in this paper to a member of this class yields a simulating MTRSS, if we consider the priority rewrite relation, without any further assumption about the strategy. We would like to establish this rigorously.

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References


References

