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Abstract

We present solutions of the Quantum Yang-Baxter Equation that satisfy the condition

$$R^{ab}_{cd} \neq 0 \Rightarrow (\{a, b\} = \{c, d\}) \quad \text{or} \quad (b = \sigma(a) \quad \text{and} \quad d = \sigma(c)),$$

where σ denotes the involution on $\{1, \ldots, n\}$ given by $\sigma(i) = n + 1 - i$.

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1 Introduction

In this report we construct special solutions of the Quantum Yang-Baxter Equation (QYBE). The QYBE involves a regular $n^2 \times n^2$-matrix R over the field of complex numbers and can shortly be written as $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. In this equation, R_{12} denotes the $n^3 \times n^3$-matrix that arises by letting R act on the first and second factor of the tensor product $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$. The matrices R_{13} and R_{23} are defined similarly. Written out in components the QYBE takes the following form:

$$\sum_{i,j,k} R^{ab}_{ij} R^{ic}_{uk} R^{jk}_{vw} = \sum_{i,j,k} R^{bc}_{ij} R^{aj}_{kw} R^{ki}_{uv}. \tag{1.1}$$

The QYBE has a complex variety of solutions, it is therefore necessary to impose some additional restrictions on the structure of the matrix R. In [1] M. Hazewinkel studies R-matrices that are subject to the condition:

$$R^{ab}_{cd} \neq 0 \Rightarrow \{a, b\} = \{c, d\}. \tag{1.2}$$

For instance, the standard R-matrix for the quantum group of type A_n satisfies (1.2). The condition reduces the possible nonzero elements of R from n^4 to $n + 2n(n-1) = n(2n-1)$. The general solution of the QYBE satisfying this constraint is described in [1]. The method used to derive these solutions will be fundamental for our approach.

Although the known classical R-matrices of type B^1, C^1, D^1, A^2 do not arise in this way, they do satisfy a very similar condition. Let σ be the involution on $\{1, \ldots, n\}$ given by $\sigma(i) = n + 1 - i$, then the R-matrices of type B^1, C^1, D^1, A^2 have the property

$$R^{ab}_{cd} \neq 0 \Rightarrow (\{a, b\} = \{c, d\}) \quad \text{or} \quad (b = \sigma(a) \quad \text{and} \quad d = \sigma(c)). \tag{1.3}$$

In order to understand the consequences of this condition in terms of the structure of the matrix R, we consider the matrix to be built from $n \times n$ blocks in such a way that the element R^{ab}_{cd} is considered to be the (b, d)-th element of the (a, c)-th block. In this setting the condition states that the off diagonal blocks can contain at most two nonzero elements. This reduces the number of (possible) nonzero elements from n^4 to $n^2+2(n^2-2n)+n = 3n(n-1)$ if n is even and to $n^2+2(n^2-2n+1)+n-1 = 3n(n-1)+1$ if n is odd. For instance, for $n=3$ the matrix R will be of the following form

$$R = \begin{bmatrix} \begin{bmatrix} \star & & \\ & \star & \\ & & \star \end{bmatrix} & \begin{bmatrix} & & \\ \star & & \\ & * & \end{bmatrix} & \begin{bmatrix} & & \\ & & \\ \star & & \end{bmatrix} \\ \begin{bmatrix} & \star & \\ & & * \\ & & \end{bmatrix} & \begin{bmatrix} \star & & \\ & \star & \\ & & \star \end{bmatrix} & \begin{bmatrix} & & \\ * & & \\ & \star & \end{bmatrix} \\ \begin{bmatrix} & & \star \\ & & \\ & & \end{bmatrix} & \begin{bmatrix} & * & \\ & & \star \\ & & \end{bmatrix} & \begin{bmatrix} \star & & \\ & \star & \\ & & \star \end{bmatrix} \end{bmatrix}$$

where the symbol $\star$ denotes matrix entries that can be non-zero according to condition (1.2) and the symbol $*$ denotes matrix entries that can be non-zero due to its extension (1.3). Since the indices $\sigma(a)$ occur frequently, we will for the sake of convenience write a' instead of $\sigma(a)$ and we call this the *complementary index* of a or *complement* of a.

Lemma 1 *If there are nonzero terms in the LHS or RHS of (1.1) and R satisfies (1.3) then the multisets $\{a,b,c\}$ and $\{u,v,w\}$ have the following property:*

$$\{a,b,c\} = \{u,v,w\} \quad \text{or} \quad (\{a,b,c\} = \{p,q,q'\} \text{ and } \{u,v,w\} = \{p,r,r'\}).$$

Proof: From [1] we learn that, if R satisfies (1.2), the multisets $\{a,b,c\}$ and $\{u,v,w\}$ should coincide in order to have nonzero terms in LHS or RHS of (1.1). Therefore, we assume that $\{a,b,c\} \neq \{u,v,w\}$ and we show that the existence of nonzero terms yields the other form described above. Naturally there are a number of cases to be considered. We discuss only one of them which illustrates the kind of arguments involved in order to prove the Lemma. We suppose that a is not an element of $\{u,v,w\}$ and that there exists a nonzero term $R^{ab}_{ij}R^{ic}_{uk}R^{jk}_{vw}$ on the LHS of (1.1). Since $R^{ab}_{ij} \neq 0$ we can distuinguish the following three cases:

1 $i=a$, $j=b$ and $b \neq a'$.
If $c \neq a'$ then $R^{ac}_{uk} \neq 0$ implies $k=a$ and $u=c$ and thus $R^{jk}_{vw} = R^{ba}_{vw} \neq 0$ yields a contradiction. Hence, $c=a'$ and $k=u'$ and $R^{jk}_{vw} = R^{bu'}_{vw} \neq 0$ yields either $\{b,u'\} = \{v,w\}$ and $\{u,v,w\} = \{b,u,u'\}$, or $b=u$ and $w=v'$ which implies $\{u,v,w\} = \{b,v,v'\}$. In both cases we have $\{a,b,c\} = \{a,b,a'\}$.

2 $i=b$, $j=a$ and $b \neq a'$.
Due to $R^{ak}_{vw} \neq 0$ we have $k=a'$ and $w=v'$. Then $R^{bc}_{ua'} \neq 0$ implies $\{b,c\} = \{u,a'\}$ and therefore $\{a,b,c\} = \{a,a',u\}$ and $\{u,v,w\} = \{u,v,v'\}$.

3 $j=i'$ and $b=a'$.
Since $R^{i'k}_{vw} \neq 0$ there are two possibilities. The first is that $k=i$ and $w=v'$. On account of $R^{ic}_{ui} \neq 0$ we conclude $u=c$ yielding $\{a,b,c\} = \{a,a',c\}$ and $\{u,v,w\} = \{c,v,v'\}$. The second possibility is that $\{i',k\} = \{v,w\}$. The choice $i=v'$, $k=w$ and the fact that $R^{v'c}_{uw} \neq 0$ gives either $\{v',c\} = \{u,w\}$ or $v=c$ and $w=v'$. In the first case we obtain $\{u,v,w\} = \{c,v,v'\}$ and $\{a,b,c\} = \{a,a',c\}$ and in the second case $\{u,v,w\} = \{c,v,v'\}$ and $\{a,b,c\} = \{a,a',c\}$. The other choice ($i=w'$ and $k=v$) can be handled similarly.

□

This Lemma enables us to make a list of all multisets $\{a,b,c\}$ and $\{u,v,w\}$ for which nontrivial equations arise from (1.1). For that we need to distuinguish the cases of even and odd n. The reason for this is the possible existence of indices that are equal to their complement. In the case that n is even such indices simply do not exist and therefore we will from here on assume that n is even. The complete list of resulting equations is given in Appendix A.

The solution matrix R has to be invertible. It turns out that, for even n, the determinant of a matrix R satisfying condition (1.3) can be written as

$$det(R) = \prod_{i=1}^{n} R_{ii}^{ii} \prod_{\substack{i<j \\ i\neq j'}} (R_{ij}^{ij} R_{ji}^{ji} - R_{ji}^{ij} R_{ij}^{ji}) (\sum_{\tau \in S_n} (-1)^{\tau} \prod_{i=1}^{n} R_{\tau(i)\tau(i)'}^{ii'}) \tag{1.4}$$

where $(-1)^{\tau}$ denotes the sign of the permutation τ. The last term of this determinant is the essential part arising from the extension (1.3). It turns out to be related to the determinant of certain parameter matrices that describe solutions.

2 The classical solutions

As explained in the introduction, this study is an extension of [1] where solutions of the QYBE restricted to (1.2) are presented. We give a short résumé on the construction of those solutions which we will call *classical* from here on.

An extremely useful property of R-matrices satisfying (1.2) is that, for any subset J of the index set $I = \{1, 2, \ldots, n\}$, the $p^2 \times p^2$-matrix R_J ($p = |J|$, i.e. the number of elements of the index set J) which is defined by $(R_J)_{kl}^{ij} = R_{kl}^{ij}$, is also a solution of the QYBE satisfying the additional condition (1.2). This means that, for an arbitrary subset of indices, one can leave out the corresponding rows and columns and obtain what could naturally be called reduced solutions. This particular property enables one to construct solutions by pasting together certain entities of smaller dimensions. There are two special types of entities that play a prominent role in the construction of solutions. They are called components and blocks. The basic tool to define them is the partial ordering on the index set I given by

$$a \leq b \Longleftrightarrow R_{ba}^{ab} \neq 0.$$

Two indices a and b are called *connected* (notation: $a \sim b$) if $a \leq b$ or $b \leq a$ and the corresponding equivalence classes are called *blocks*. Similarly, two indices a and b are said to be *strongly connected* (notation: $a \simeq b$) if $a \leq b$ and $b \leq a$, the corresponding equivalence classes are called *components*.

In general a classical solution R is constructed by partitioning the index set I into blocks and each block into a number of components. For later use in this report it suffices to consider solutions consisting of one block. If this block consists of p components, then we have

$$I = \bigcup_{i=1}^{p} C_i \qquad C_1 < C_2 < \ldots < C_p$$

where $i < j$ denotes $i \leq j \wedge j \not\leq i$. Corresponding solutions are parametrized by $\mu, \delta \neq 0$ and $x_{ij} \neq 0$ $(1 \leq i < j \leq n)$ and λ_k $(1 \leq k \leq p)$ satisfying the quadratic equation $\lambda_k^2 = \lambda_k \mu + \delta$. The matrix R is described by

$$\begin{array}{lll} i, j \in C_k & R_{ii}^{ii} = R_{jj}^{jj} = R_{ji}^{ij} = R_{ij}^{ji} = \lambda_k & R_{ij}^{ij} = R_{ji}^{ji} = 0 \\ i \in C_k < j \in C_l & R_{ji}^{ij} = \mu, R_{ij}^{ij} = x_{ij}, R_{ji}^{ji} = \delta x_{ij}^{-1} & R_{ij}^{ji} = 0 \end{array} \tag{2.5}$$

Often one scales such a solution by putting δ equal to 1. In that case one can define $q = \lambda_1$ which implies $\mu = q - q^{-1}$ and all λ_k are equal to either q or $-q^{-1}$.

3 Special solutions: part I

We remark that R-matrices satisfying condition (1.3) in general do not allow reductions. Hence the choice of an arbitrary subset J of $\{1, 2, \ldots, n\}$ will not lead to a matrix R_J that satisfies the same conditions as R. In particular it will, in general, not be a solution of the QYBE. This is due to the fact that a number of equations involve summations over all indices, see e.g. equations 1(c)vii and 1(c)viii in the appendix. Nevertheless, if one chooses a subset J such that J and J' are disjoint, then the reduced matrix R_J satisfies condition (1.2) and will therefore be a classical solution. We call such a subset J a *reduction*. We will frequently use reductions in order to derive properties of solutions of the extended case. A reduction J is called *maximal* if $J \bigcup J' = \{1, 2, \ldots, n\}$. Two particular cases of maximal reductions are $J = L$ and $J = H$ where L denotes the set of lower indices $\{1, 2, \ldots, m = n/2\}$ and H the set of higher indices $\{m+1, m+2, \ldots, 2m = n\}$.

At first we consider solutions for which there exists a maximal reduction J with the property that both J and J' consist of one component. By a simple reordering we may assume that $J = L$. So, R_L and R_H consist of one component and according to the previous section, we have

$$i, j \in L(H) \quad R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda_L(\lambda_H) \quad R^{ij}_{ij} = R^{ji}_{ji} = 0 \tag{3.6}$$

For the sake of convenience we suppose that the components are large enough, i.e. $m \geq 3$. We investigate the possible connections between the lower and higher indices. It appears that the solutions can be divided into three categories.

Lemma 2 *If there exists a pair of indices (i, j) such that $i \in L$, $j \in H$ and $j \neq i'$ with the property $R^{ij}_{ji} \neq 0$ and $R^{ji}_{ij} \neq 0$, then $\lambda_L = \lambda_H = \lambda$ and for all similar pairs (k, l) holds*

$$R^{kl}_{lk} = R^{lk}_{kl} = \lambda \qquad R^{kl}_{kl} = R^{lk}_{lk} = 0.$$

Proof: The reduction $J = \{i, j\}$ yields one component which implies

$$\lambda_L = R^{ii}_{ii} = R^{ij}_{ji} = R^{ji}_{ij} = R^{jj}_{jj} = \lambda_H = \lambda \qquad R^{ij}_{ij} = R^{ji}_{ji} = 0.$$

By taking an index l from $H \setminus \{i', j\}$ the reductions $\{i, j, l\}$ and $\{j', i, l\}$ both yield one component and hence the pairs (i, l) and (j', l) have the stated property. Similarly one can cope with pairs (k, j) and (k, i') for $k \in L \setminus \{i, j'\}$. For pairs (k, l) with $k \in L \setminus \{i, j'\}$ and $l \in H \setminus \{i', j, k'\}$ one can use the reduction $\{i, j, k, l\}$. Finally, to deal with the pair (j', i') one takes an index k in $L \setminus \{i, j'\}$ and considers the reduction $\{k, j', i'\}$. □

Lemma 3 *If there exists a pair of indices (i, j) such that $i \in L$, $j \in H$ and $j \neq i'$ with the property $R^{ij}_{ji} = \mu \neq 0$ and $R^{ji}_{ij} = 0$, then for all similar pairs (k, l) holds*

$$R^{kl}_{lk} = \mu \qquad R^{lk}_{kl} = 0 \qquad R^{kl}_{kl} R^{lk}_{lk} = \delta$$

where δ is a nonzero complex number which is independent of k and l. Furthermore, the parameters λ_L and λ_H satisfy the quadratic equation $\lambda^2 = \lambda\mu + \delta$.

Proof: The proof is similar to the one of the preceding Lemma. The reduction $\{i, j\}$ yields one block with two components, so

$$\delta = R^{ij}_{ij} R^{ji}_{ji} \neq 0 \qquad \lambda_L^2 = \lambda_L \mu + \delta \qquad \lambda_H^2 = \lambda_H \mu + \delta.$$

By taking an index l from $H \setminus \{i', j\}$ the reductions $\{i, j, l\}$ and $\{j', i, l\}$ both yield one block with two components in such a way that $i < j \simeq l$ and $j' \simeq i < l$ hence the pairs (i, l) and (j', l) have the stated property. Similarly one can cope with pairs (k, j) and (k, i') for $k \in L \setminus \{i, j'\}$. For pairs (k, l) with $k \in L \setminus \{i, j'\}$ and $l \in H \setminus \{i', j, k'\}$ one can use the reduction $\{i, j, k, l\}$ which gives $k \simeq i < j \simeq l$. Finally, to deal with the pair (j', i') one takes a reduction $\{k, j', i'\}$ for an index k in $L \setminus \{i, j'\}$. □

Lemma 4 *If for all pairs of indices (i, j) with $i \in L$, $j \in H$ and $j \neq i'$ holds $R^{ij}_{ji} = R^{ji}_{ij} = 0$ then the product $R^{ij}_{ij} R^{ji}_{ji}$ is nonzero and does not depend on i and j.*

Proof: Consider similar reductions as in the preceding proofs and make use of the fact that in classical solutions consisting of several blocks the product $R^{ij}_{ij} R^{ji}_{ji}$ does not depend on the indices i and j explicitly. This product only depends on the blocks the indices belong to. □

The situations described in the Lemmas 2, 3 and 4 exhaust all possibilities and hence give rise to three types of solutions. For the sake of convenience we will speak of solutions of type III, II and I respectively. We restrict ourselves to the study of solutions of type I and II.

3.1 Solutions of Type I

One could say that solutions of type I are characterized by the fact that there is no classical coupling between L and H since all terms R^{ji}_{ij} and R^{ji}_{ij} are equal to zero for $i \in L$ and $j \in H \setminus \{i'\}$. The question is what kind of coupling there can appear between L and H using terms like $R^{ii'}_{jj'}$ with $i \in L$ and $j \in H$. We will show that all such terms are equal to zero.

Lemma 5 *Let R be a type I solution, then for all $i \in L$ and $j \in H$ holds $R^{ii'}_{jj'} = R^{jj'}_{ii'} = 0$.*

Proof: We take $a, c \in L$ and $b \in H$, then equation (2(a)i) yields

$$R^{aa'}_{bb'} R^{bc}_{bc} R^{ca'}_{ca'} = 0 \Longrightarrow R^{aa'}_{bb'} = 0.$$

The choice $a \in L$ and $c \in H$ in (1(c)xiv) gives

$$R^{aa'}_{a'a} R^{ca}_{ca} R^{ca'}_{a'c} = 0 \Longrightarrow R^{aa'}_{a'a} = 0.$$

This takes care of the terms $R^{ii'}_{jj'}$, the terms $R^{jj'}_{ii'}$ can be handled similarly. □

In order to determine the general structure of solutions of type I, we introduce the following $m \times m$-matrices:

$$[L] = (L^i_j) \quad L^i_j = R^{ii'}_{jj'} \quad \text{and} \quad [H] = (H^i_j) \quad H^i_j = R^{i'i}_{j'j} \quad (i, j \in L). \tag{3.7}$$

That these matrices play a significant role in the description of solutions can be illustrated by writing out the determinant of R subject to the conditions we have imposed so far.

$$det(R) = Q \prod_{i=1}^{n} R^{ii}_{ii} \prod_{\substack{i<j \\ i \neq j'}} (R^{ij}_{ij} R^{ji}_{ji} - R^{ij}_{ji} R^{ji}_{ij}) = Q \lambda_L^m \lambda_H^m \times$$

$$\prod_{\substack{i,j \in L \\ i<j}} (R^{ij}_{ij} R^{ji}_{ji} - R^{ij}_{ji} R^{ji}_{ij}) \prod_{\substack{i,j \in H \\ i<j}} (R^{ij}_{ij} R^{ji}_{ji} - R^{ij}_{ji} R^{ji}_{ij}) \prod_{\substack{i \in L, j \in H \\ i \neq j'}} (R^{ij}_{ij} R^{ji}_{ji} - R^{ij}_{ji} R^{ji}_{ij}) =$$

$$Q \lambda_L^{m^2} \lambda_H^{m^2} \delta^{m(m-1)}$$

where

$$Q = \sum_{\tau \in S_n} (-1)^\tau \prod_{i=1}^{n} R^{ii'}_{\tau(i)\tau(i)'} = \sum_{\tau \in S_n} (-1)^\tau \prod_{i=1}^{m} R^{ii'}_{\tau(i)\tau(i)'} \prod_{i=1}^{m} R^{i'i}_{\tau(i')\tau(i')'}.$$

According to Lemma 5 a term $R^{ii'}_{\tau(i)\tau(i)'}$ can only be nonzero if i and $\tau(i)$ belong to the same component. Hence, the sum over all permutations can be replaced by a sum over all elements $\tau \in S_n$ that leave L and H invariant. The restriction of such a τ to L can be considered to be an element of S_m. By use of the involution σ, one can transform its restriction to H to become an element of S_m given by $\hat{\tau} = \sigma \circ \tau \circ \sigma$. This has the following consequence

$$Q = (\sum_{\tau \in S_m} (-1)^\tau \prod_{i=1}^{m} R^{ii'}_{\tau(i)\tau(i)'})(\sum_{\hat{\tau} \in S_m} (-1)^{\hat{\tau}} \prod_{i=1}^{m} R^{i'i}_{\hat{\tau}(i)'\hat{\tau}(i)}) =$$

$$(\sum_{\tau \in S_m} (-1)^\tau \prod_{i=1}^{m} L^i_{\tau(i)})(\sum_{\hat{\tau} \in S_m} (-1)^{\hat{\tau}} \prod_{i=1}^{m} H^i_{\hat{\tau}(i)}) = det([L]) det([H]).$$

So, due to the invertibility of the matrix R we conclude that the matrices $[L]$ and $[H]$ need to be invertible.

Lemma 6 *Let R be a solution of type I and let $[L]$ and $[H]$ be defined as given in (3.7), then $[L][H] = \delta I_m$ where I_m denotes the identity matrix of size m.*

Proof: In equation (1(c)ix) we take $a \in L$ and $c \in H$, this gives

$$R^{ac}_{ac}R^{ca}_{ca}R^{ca'}_{a'c} = \delta\lambda_H = \sum_{p=1}^{n} R^{aa'}_{p'p}R^{pp'}_{a'a}R^{cp}_{pc} = \lambda_H \sum_{p\in H} R^{aa'}_{p'p}R^{pp'}_{a'a} = \lambda_H \sum_{p=1}^{m} L^a_p H^p_a.$$

In equation (2(a)ii) we take $a, c \in L$ and $b \in H$:

$$R^{aa'}_{bb'}R^{a'c}_{ca'}R^{bc}_{cb} = 0 = \sum_{p=1}^{n} R^{aa'}_{pp'}R^{p'p}_{bb'}R^{pc}_{cp} = \lambda_L \sum_{p\in L} R^{aa'}_{pp'}R^{p'p}_{bb'} = \lambda_L \sum_{p=1}^{m} L^a_p H^p_{b'}.$$

□

It turns out that for any choice of the matrices $[L]$ and $[H]$ satisfying $[L][H] = \delta I_m$, one obtains a solution R of the QYBE. The determinant of such a solution R is equal to $\lambda_L^{m^2}\lambda_H^{m^2}\delta^{m^2}$.

Theorem 1 *A matrix R, with all entries equal to zero with the exception of the following*

$$R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda_L(\lambda_H) \quad i,j \in L(H)$$

$$R^{ij}_{ij} = x_{ij} \qquad R^{ji}_{ji} = x_{ij}^{-1}\delta \qquad i \in L, j \in H \setminus \{i'\}$$

$$R^{ii'}_{jj'} = \alpha^i_j \qquad R^{i'i}_{j'j} = \beta^i_j \qquad i,j \in L$$

is a solution of the QYBE if the parameters satisfy the additional restrictions:

- $\lambda_L, \lambda_H, x_{ij}$ *and* δ *are non-zero.*
-

$$\begin{bmatrix} \alpha^1_1 & \alpha^1_2 & \dots & \alpha^1_m \\ \alpha^2_1 & \alpha^2_2 & \dots & \alpha^2_m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^m_1 & \alpha^m_2 & \dots & \alpha^m_m \end{bmatrix} \begin{bmatrix} \beta^1_1 & \beta^1_2 & \dots & \beta^1_m \\ \beta^2_1 & \beta^2_2 & \dots & \beta^2_m \\ \vdots & \vdots & \ddots & \vdots \\ \beta^m_1 & \beta^m_2 & \dots & \beta^m_m \end{bmatrix} = \delta I_m$$

Proof: This is just a tedious computation involving the verification of the equations presented in the appendix. It turns out that all equations give rise to one of the special conditions presented above. □

To illustrate these type of solutions, we consider the non-classical example of smallest dimension, i.e. $n = 4$ so $L = \{1, 2\}$ and $H = \{3, 4\}$. The matrix R described in the theorem above is:

$$\begin{bmatrix}
\lambda_L &&&&&&&&&&&&&&& \\
& 0 &&& \lambda_L &&&&&&&&&&& \\
&& x_{13} &&&&&& 0 &&&&&&& \\
&&& \alpha^1_1 &&& \alpha^1_2 &&& 0 &&& 0 &&& \\
& \lambda_L &&& 0 &&&&&&&&&&& \\
&&&&& \lambda_L &&&&&&&&&& \\
&&& \alpha^2_1 &&& \alpha^2_2 &&& 0 &&& 0 &&& \\
&&&&&&& x_{24} &&&&&& 0 && \\
&& 0 &&&&&& \delta x_{13}^{-1} &&&&&&& \\
&&& 0 &&& 0 &&& \beta^2_2 &&& \beta^2_1 &&& \\
&&&&&&&&&& \lambda_H &&&&& \\
&&&&&&&&&&& 0 &&& \lambda_H & \\
&&& 0 &&& 0 &&& \beta^1_2 &&& \beta^1_1 &&& \\
&&&&&&& 0 &&&&&& \delta x_{24}^{-1} && \\
&&&&&&&&&&& \lambda_H &&& 0 & \\
&&&&&&&&&&&&&&& \lambda_H
\end{bmatrix}$$

All the matrix entries in positions that are left blank are equal to zero due to condition (1.3). The zeros indicate matrix entries that are equal to zero due to the special structure of Type I.

3.2 Solutions of Type II

From Lemma 3 we know that a solution of type II is determined by the parameters λ_L, λ_H, μ and δ which satisfy the equations $\lambda_L^2 = \lambda_L\mu + \delta$ and $\lambda_H^2 = \lambda_H\mu + \delta$. We will first prove that the elements $R^{ii'}_{i'i}$ behave like classical elements R^{ij}_{ji}.

Lemma 7 *Let R be a type II solution, then*

$$R^{ii'}_{i'i} = \begin{cases} \mu & i \in L \\ 0 & i \in H \end{cases}$$

Proof: Consider equation (1(c)xi). If we take $a, c \in L$ then

$$R^{a'c}_{a'c}\lambda_L(\mu - R^{aa'}_{a'a}) = 0 \Longrightarrow R^{aa'}_{a'a} = \mu$$

and for $a, c \in H$ we have

$$R^{a'c}_{a'c}R^{aa'}_{a'a}(\mu - \lambda_H) = 0 \Longrightarrow R^{aa'}_{a'a} = 0.$$

□

Further, as in the case of type I solutions, there appears to be no essential non-classical coupling between the components L and H.

Lemma 8 *Let R be a type II solution, then for all indices $i \in L$ and $j \in H$ with $j \neq i'$ holds*

$$R^{ii'}_{jj'} = R^{jj'}_{ii'} = 0.$$

Proof: We take $a \in L$ and $b \in H$ in equation (2(b)ix):

$$0 = R^{aa'}_{bb'}R^{ba}_{ba}(0 - \lambda_L) \Longrightarrow R^{aa'}_{bb'} = 0.$$

This takes care of the terms $R^{ii'}_{jj'}$, the terms $R^{jj'}_{ii'}$ can be handled similarly. □
As in the case of Type I we introduce the matrices $[L]$ and $[H]$. The regularity argument of the preceding section is also valid here. It turns out that the structure is completely similar: $[H] = \delta[L]^{-1}$ is a necessary and sufficient condition for R to satisfy the QYBE.

Lemma 9 *Let R be a solution of type II and let $[L]$ and $[H]$ be defined as given in (3.7), then $[L][H] = \delta I_m$.*

Proof: In equation (1(c)ix) we take $a \in L$ and $c \in H$, this gives

$$\delta\lambda_H = \lambda_H \sum_{p \in H} R^{aa'}_{p'p}R^{pp'}_{a'a} \Longrightarrow \sum_{p=1}^{m} L^a_p H^p_a = \delta.$$

In equation (2(a)ii) we take $a \in L$ and $b, c \in H$:

$$R^{aa'}_{bb'}R^{a'c}_{ca'}R^{bc}_{cb} = 0 = \sum_{p=1}^{n} R^{aa'}_{pp'}R^{p'p}_{bb'}R^{pc}_{cp} = \mu \sum_{p \in L} R^{aa'}_{pp'}R^{p'p}_{bb'} =$$

$$\mu \sum_{p=1}^{m} L^a_p H^p_{b'} \Longrightarrow \sum_{p=1}^{m} L^a_p H^p_{b'} = 0.$$

□

Theorem 2 *A matrix R, with all entries equal to zero with the exception of the following*

$$R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda_L(\lambda_H) \quad i, j \in L(H)$$

$$R^{ij}_{ji} = \mu \qquad R^{ji}_{ij} = 0 \qquad i \in L, j \in H$$

$$R^{ij}_{ij} = x_{ij} \qquad R^{ji}_{ji} = x^{-1}_{ij}\delta \qquad i \in L, j \in H \setminus \{i'\}$$

$$R^{ii'}_{jj'} = \alpha^i_j \qquad R^{i'i}_{j'j} = \beta^i_j \qquad i, j \in L$$

is a solution of the QYBE if the parameters satisfy the additional restrictions:

- *μ, δ and x_{ij} are non-zero.*
- *λ_L and λ_H satisfy the quadratic equation $X^2 = X\mu + \delta$.*
- *$[L][H] = \delta I_m$ where $[L] = (\alpha^i_j)$ and $[H] = (\beta^i_j)$.*

Proof: As mentioned in the proof of Theorem 1, this is just a tedious computation involving the verification of the equations presented in the appendix. □

Again to illustrate these type of solutions, we consider the non-classical example of smallest dimension, i.e. the matrix R for $n = 4$:

$$\left[\begin{array}{cccccccccccccccc}
\lambda_L &&&&&&&&&&&&&&& \\
& 0 &&& \lambda_L &&&&&&&&&&& \\
&& x_{13} &&&&&& \mu &&&&&&& \\
&&& \alpha^1_1 &&& \alpha^1_2 &&& 0 &&& \mu &&& \\
& \lambda_L &&& 0 &&&&&&&&&&& \\
&&&&& \lambda_L &&&&&&&&&& \\
&&& \alpha^2_1 &&& \alpha^2_2 &&& \mu &&& 0 &&& \\
&&&&&&& x_{24} &&&&&& \mu && \\
&& 0 &&&&&& \delta x_{13}^{-1} &&&&&&& \\
&&& 0 &&& 0 &&& \beta^2_2 &&& \beta^2_1 &&& \\
&&&&&&&&&& \lambda_H &&&&& \\
&&&&&&&&&&& 0 &&& \lambda_H & \\
&&& 0 &&& 0 &&& \beta^1_2 &&& \beta^1_1 &&& \\
&&&&&&& 0 &&&&&& \delta x_{24}^{-1} && \\
&&&&&&&&&&& \lambda_H &&& 0 & \\
&&&&&&&&&&&&&&& \lambda_H
\end{array}\right]$$

4 Special solutions: part II

In the previous section we discussed solutions which have a maximal reduction J with the property that both J and J' consist of one component. In this section we allow J' to be a block. By means of reordering we may assume that $J = L$. Hence $J' = H = \bigcup H_j$ which are assumed to be ordered such that $H_1 < H_2 < \ldots < H_p$ where p denotes the number of components of H. From the knowledge of the classical solutions we conclude

$$(4.8)\qquad \begin{array}{rll}
i, j \in L & R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda_L & R^{ij}_{ij} = R^{ji}_{ji} = 0 \\
i, j \in H_l & R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda^l_H & R^{ij}_{ij} = R^{ji}_{ji} = 0 \\
i \in H_k, j \in H_l, k < l & R^{ij}_{ji} = \mu, R^{ji}_{ij} = 0 & R^{ij}_{ij} R^{ji}_{ji} = \delta
\end{array}$$

and the parameters λ^l_H satisfy the quadratic equation $(\lambda^l_H)^2 = \lambda^l_H \mu + \delta$. We will show that analogous to the previous section there are three types of solutions.

Lemma 10 *If there exists a pair of indices (i, j) such that $i \in L$, $j \in H$ and $j \neq i'$ with the property*

$$R^{ij}_{ji} \neq 0 \quad \textit{and} \quad R^{ji}_{ij} \neq 0,$$

then $\lambda_L = \lambda^s_H$ where s is the index of the component that contains the index j, and furthermore

$$\begin{array}{lll}
R^{kl}_{lk} = R^{lk}_{kl} = \lambda^s_H & R^{kl}_{kl} = R^{lk}_{lk} = 0 & k \in L, l \in H_s, k \neq l' \\
R^{kl}_{lk} = \mu, R^{lk}_{kl} = 0 & R^{kl}_{kl} R^{lk}_{lk} = \delta & k \in L, l \in H_t, t > s, k \neq l' \\
R^{kl}_{lk} = 0, R^{lk}_{kl} = \mu & R^{kl}_{kl} R^{lk}_{lk} = \delta & k \in L, l \in H_t, t < s, k \neq l'
\end{array}$$

Proof: The results for pairs (k, l) with $k \in L$ and $l \in H_s$ are a direct consequence of the reasoning given in the proof of Lemma 2. Suppose we have indices $k \in L$ and $l \in H_t$ with $t > s$. Take an arbitrary index $j \in H_s$ with $j \neq k'$. The reduction $\{k, j, l\}$ yields a two component solution with $k \simeq j < l$ implying the above mentioned result for the indices k and l. □

The solutions that correspond to the situation described in the Lemma above can be characterized by the following ordering:

$$(4.9)\qquad H_1 < H_2 < \ldots H_{s-1} < L \simeq H_s < H_{s+1} < \ldots < H_{p-1} < H_p.$$

The interpretation of this ordering is the following. For any reduction J the solution consists of one block with several components which are ordered by (4.9).

Next we consider solutions for which there does not exist a pair (i,j) as described in Lemma 10.

Lemma 11 *If there exists a pair of indices (i,j) such that $i \in L$, $j \in H$ and $j \neq i'$ with the property*

$$R^{ij}_{ji} \neq 0 \quad \textit{and} \quad R^{ji}_{ij} = 0,$$

then there exists an index s such that

$$\begin{array}{lll} R^{kl}_{lk} = \mu, R^{lk}_{kl} = 0 & R^{kl}_{kl} R^{lk}_{lk} = \delta & k \in L, l \in H_t, t > s, k \neq l' \\ R^{kl}_{lk} = 0, R^{lk}_{kl} = \mu & R^{kl}_{kl} R^{lk}_{lk} = \delta & k \in L, l \in H_t, t \leq s, k \neq l' \end{array}$$

and the parameter λ_L satisfies $\lambda_L^2 = \lambda_L \mu + \delta$.

Proof: Suppose that $j \in H_m$, then on account of the reasoning given in the proof of Lemma 3 and the fact that $H_s < H_t$ for $s < t$, the results of the first line obviously hold for $k \in L$ and $l \in H_t$ with $t \geq m$. Next we consider indices $k \in L$ and $l \in H_t$ with $t < m$. The reduction $\{k, j, l\}$ satisfies $k < j$ and $l < j$. From the classical solutions we know that this implies that k and l are connected, so we have $k \leq l$ or $l \leq k$. Since we have excluded the existence of a pair (k,l) with the property $k \simeq l$ this reasoning proves the existence of an index s as described in the Lemma. □

Solutions that correspond to the situation described in Lemma 11 can be characterized by the ordering:

$$H_1 < H_2 < \ldots H_{s-1} < H_s < L < H_{s+1} < \ldots < H_{p-1} < H_p. \tag{4.10}$$

Finally there are solutions that have no classical coupling between L and H at all. This is completely analogous to the case described in Lemma 4.

Lemma 12 *If for all pairs of indices (i,j) such that $i \in L$, $j \in H$ and $j \neq i'$*

$$R^{ij}_{ji} = R^{ji}_{ij} = 0$$

then the product $R^{ij}_{ij} R^{ji}_{ji}$ is nonzero and does not depend on i and j.

The situations described in the Lemmas 10, 11 and 12 exhaust all possibilities and hence give rise to three types of solutions which will again be denoted by solutions of type III, II and I respectively. Again we restrict ourselves to the study of solutions of type I and II.

4.1 Solutions of type I

The solutions of type I are characterized by the component L with parameter λ_L, the block H with components $H_1 < H_2 < \ldots < H_p$ and defining parameters μ, δ and λ^i_H. Furthermore, for all $i \in L$ and $j \in H$ with $i \neq j'$ the classical terms R^{ij}_{ji} and R^{ji}_{ij} are equal to zero and the product $R^{ij}_{ij} R^{ji}_{ji}$ does not depend on i and j and defines the parameter δ'. By the same reasoning as given in the proof of Lemma 5 one finds that the coupling terms $R^{ii'}_{jj'}$ with $i \in L$ and $j \in H$ are equal to zero. In order to describe the solutions we make use of the matrices $[L]$ and $[H]$ as defined in (3.7). We write L_l to denote H'_l and call this the l-th component of L.

We consider equation (2(a)iii) and take $a \in L, b \in H$ and $c \in H_k$ which yields

$$0 = \sum_{q \in H} L^a_{q'} H^{q'}_{b'} R^{cq}_{qc} = \lambda^k_H \sum_{q \in L_k} L^a_q H^q_{b'} + \mu \sum_{\substack{q \in L_s \\ s > k}} L^a_q H^q_{b'}. \tag{4.11}$$

In order to rewrite this in a more compact form we look at the matrix $[L]$ as builded from blocks in the following way:

$$[L] = \begin{bmatrix} [L]^1_1 & [L]^1_2 & \ldots & [L]^1_p \\ [L]^2_1 & [L]^2_2 & \ldots & [L]^2_p \\ \vdots & \vdots & \ddots & \vdots \\ [L]^p_1 & [L]^p_2 & \ldots & [L]^p_p \end{bmatrix} \tag{4.12}$$

The matrix $[L]^i_j$ is the matrix of size $|L_i| \times |L_j|$ with the entries L^k_l where $k \in L_i$ and $l \in L_j$. Naturally the matrix $[H]$ has a similar block structure for which we will use the same notations. If we choose $a \in L_i$ and $b \in H_j$ in (4.11), then the equation can be rewritten in terms of the blocks as follows

$$[L]^i_k[H]^k_j + \frac{\mu}{\lambda^k_H}\sum_{q>k}[L]^i_q[H]^q_j = 0 \qquad i \neq j. \tag{4.13}$$

The term on the righthandside is the matrix of size $|L_i| \times |L_j|$ with all entries equal to zero.

Next we consider the case $i = j$. Since it is not allowed to choose $b = a'$ in (4.11) we consider equation (1(c)ix) and take $a \in L_i$ and $c \in H_k$. For $i < k$ we have

$$0 = \sum_{q\in H} L^a_{q'} H^{q'}_a R^{cq}_{qc} = \lambda^k_H \sum_{q\in L_k} L^a_q H^q_a + \mu \sum_{\substack{q\in L_s\\ s>k}} L^a_q H^q_a \tag{4.14}$$

and for $i = k$ we obtain

$$\delta' \lambda^k_H = \lambda^k_H \sum_{q\in L_k} L^a_q H^q_a + \mu \sum_{\substack{q\in L_s\\ s>k}} L^a_q H^q_a \tag{4.15}$$

From (4.11) and (4.14) we find that

$$[L]^i_k[H]^k_i + \frac{\mu}{\lambda^k_H}\sum_{q>k}[L]^i_q[H]^q_i = 0 \qquad (i < k) \tag{4.16}$$

and from (4.11) and (4.15) follows

$$[L]^k_k[H]^k_k + \frac{\mu}{\lambda^k_H}\sum_{q>k}[L]^k_q[H]^q_k = \delta' I_{\hat{k}} \tag{4.17}$$

with $I_{\hat{k}}$ the identity matrix of size $|L_k|$. From these equations we can easily derive the structure of the blocks of $[L]$ and $[H]$.

Lemma 13 *For all $1 \leq q \leq p$ holds*

$$[L]^q_q[H]^q_q = \delta' I_{\hat{q}} \quad [L]^i_q = 0 \quad [H]^q_i = 0 \quad (i \neq q).$$

Proof: We prove the Lemma by downward induction. First we take $k = p$ in (4.17) and obtain $[L]^p_p[H]^p_p = \delta' I_{\hat{p}}$. Then $j = k = p$ and $i < p$ in (4.13) yields $[L]^i_p[H]^p_p = 0$ and hence $[L]^i_p = 0$. Similarly we can take $i = k = p$ and $j < p$ which gives $[H]^p_j = 0$. This proves the statement (13) for $q = p$. Suppose the statement holds for all $q \geq l$, then put $k = l - 1$ in (4.17) which gives $[L]^{l-1}_{l-1}[H]^{l-1}_{l-1} = \delta' I_{\hat{l-1}}$ and $j = k = l - 1$ and $i \neq l - 1$ in (4.13) yielding $[L]^i_{l-1}[H]^{l-1}_{l-1} = 0$. This completes the proof. □

Theorem 3 *Given any partitioning of the upper index set H of the form*

$$H_1 < H_2 < \ldots < H_p,$$

a matrix R with all entries equal to zero with the exception of the following:

$$\begin{array}{rl}
i, j \in L & R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda_L \\
i, j \in H_l & R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda^l_H \\
i \in H_k, j \in H_l, k < l & R^{ij}_{ji} = \mu, R^{ij}_{ij} = x_{ij}, R^{ji}_{ji} = x^{-1}_{ij}\delta \\
i \in L, j \in H, j \neq i' & R^{ij}_{ij} = x_{ij}, R^{ji}_{ji} = x^{-1}_{ij}\delta' \\
i, j \in L_l = H'_l & R^{ii'}_{jj'} = \alpha(l)^i_j, R^{i'i}_{j'j} = \beta(l)^i_j
\end{array}$$

is a solution of the QYBE if the parameters satisfy the additional constraints:

- *$\lambda_L, x_{ij}, \delta, \delta'$ and μ are non-zero.*
- *λ^l_H satisfies the quadratic equation $X^2 = X\mu + \delta$.*
- *$\alpha(l)\beta(l) = \delta' I_{\hat{l}}$.*

Proof: The same argument as given in Theorems 1 and 2. □

To illustrate these type of solutions, for $n = 4$ the matrix R is given below:

$$
\left[\begin{array}{cccccccccccccccc}
\lambda_L & & & & & & & & & & & & & & & \\
 & 0 & & & \lambda_L & & & & & & & & & & & \\
 & & x_{13} & & & & & & 0 & & & & & & & \\
 & & & \alpha(1) & & & 0 & & & 0 & & & 0 & & & \\
 & \lambda_L & & & 0 & & & & & & & & & & & \\
 & & & & & \lambda_L & & & & & & & & & & \\
 & & & 0 & & & \alpha(2) & & & 0 & & & 0 & & & \\
 & & & & & & & x_{24} & & & & & & 0 & & \\
 & & 0 & & & & & & \frac{\delta'}{x_{13}} & & & & & & & \\
 & & & 0 & & & 0 & & & \beta(2) & & & 0 & & & \\
 & & & & & & & & & & \lambda_H^1 & & & & & \\
 & & & & & & & & & & & x_{34} & & & \mu & \\
 & & & 0 & & & 0 & & & 0 & & & \beta(1) & & & \\
 & & & & & & & 0 & & & & & & \frac{\delta'}{x_{24}} & & \\
 & & & & & & & & & & & 0 & & & \frac{\delta}{x_{34}} & \\
 & & & & & & & & & & & & & & & \lambda_H^2
\end{array}\right]
$$

The zeros indicate entries that could have been non-zero according to (1.3). We note that, due to the fact that the components of H consists of only one element, this solution is classical. The first non-classical case is already to large to be presented in this report.

4.2 Solutions of type II

We study solutions that are determined by the ordering (4.10). The corresponding parameters are μ, δ, λ_L and λ_H^i which satisfy $\lambda_L^2 = \lambda_L\mu + \delta$ and similarly for all the λ_H^i, in particular each parameter λ_H^i equals λ_L or $\mu - \lambda_L$. It turns out that the ordering also holds for the non-classical terms.

Lemma 14 *Let R be a type II solution satisfying the classical ordering (4.10), then*

$$R_{i'i}^{ii'} = \begin{cases} \mu & i < i' \\ 0 & i > i' \end{cases}$$

where the ordering is the one given in (4.10).

Proof: We give the proof for $i \in L$, the case $i \in H$ can de handled similarly. Let a be an element of L. If $a < a'$ then we take $c \in L$ in equation (1(c)xi) yielding

$$R_{a'c}^{a'c}\lambda_L(\mu - R_{a'a}^{aa'}) = 0 \Longrightarrow R_{a'a}^{aa'} = \mu.$$

For $a > a'$ we choose an element c in the same component H_i as a' and consider equation (1(c)xii)

$$R_{ac}^{ac}R_{a'a}^{aa'}(\mu - \lambda_H^i) = 0 \Longrightarrow R_{a'a}^{aa'} = 0.$$

□

The further description of solutions turns out to be completely similar to the one given for type I.

Lemma 15 *Let R be a type II solution, then for all indices $i \in L$ and $j \in H$ with $j \neq i'$ holds*

$$R_{jj'}^{ii'} = R_{ii'}^{jj'} = 0.$$

Proof: We take $a \in L$ and $b \in H$ in equation (2(b)ix), in case $a < a'$ we obtain

$$0 = R_{bb'}^{aa'}R_{ba}^{ba}(0 - \lambda_L) \Longrightarrow R_{bb'}^{aa'} = 0.$$

and in case $a' < a$ we find

$$0 = R_{bb'}^{aa'}R_{ba}^{ba}(\mu - \lambda_L) \Longrightarrow R_{bb'}^{aa'} = 0.$$

This takes care of the terms $R^{ii'}_{jj'}$, the terms $R^{jj'}_{ii'}$ can be handled similarly. □

We consider equation (2(a)iii) and take $a \in L_l$ and $b \in H_k$ such that $b \neq a'$. It is irrelevant whether k is equal to l or not. For this particular choice the equation yields:

$$0 = \sum_{p=1}^{n} R^{aa'}_{p'p} R^{pp'}_{bb'} R^{cp}_{pc}$$

We take an element c of H_m and distinguish the cases $H_m < L$ and $L < H_m$. In both cases the equation above gives

$$\lambda_H^m \sum_{p \in L_m} L^a_p H^p_{b'} + \mu \sum_{p \in L_s, s > m} L^a_p H^p_{b'} = 0$$

By introducing block structures for the matrices $[L]$ and $[H]$ as described in the previous subsection, we can write this result in a more compact form as follows:

$$\left([L]^l_m [H]^m_k + \frac{\mu}{\lambda_H^m} \sum_{s>m} [L]^l_s [H]^s_k \right)^a_b = 0 \qquad a \in L_l, b \in L_k, a \neq b \tag{4.18}$$

The question is what the diagonal matrix entries are in the particular case that $k = l$. In order to answer this question we will use equation (1(c)ix) and take a to be an element of L. Again we have to distinguish between the cases $a < a'$ and $a' < a$. We will only give the details in the first case. Suppose that $a \in L_l$ and $c \in H_k$. We first consider the case that $c < a$, so $H_k < L$. Equation (1(c)ix) yields

$$\mu^3 + \mu\delta = \mu^3 + \lambda_H^k \sum_{p \in H_k} L^a_{p'} H^{p'}_a + \mu \sum_{p \in H_s, s>k} L^a_{p'} H^{p'}_a$$

or

$$\delta = \left(\frac{\lambda_H^k}{\mu} [L]^l_k [H]^k_l + \sum_{s>k} [L]^l_s [H]^s_l \right)^a_a \qquad a \in L_l, H_k < L \tag{4.19}$$

Analogously we find for $a < c < a'$

$$\delta = \left(\frac{\lambda_H^k}{\mu} [L]^l_k [H]^k_l + \sum_{s>k} [L]^l_s [H]^s_l \right)^a_a \qquad a \in L_l, L < H_k < H_l \tag{4.20}$$

and for $a < a' \simeq c$

$$\delta = \left([L]^l_l [H]^l_l + \frac{\mu}{\lambda_H^l} \sum_{s>l} [L]^l_s [H]^s_l \right)^a_a \qquad a \in L_l, L < H_l \tag{4.21}$$

and finally for $a < a' < c$

$$0 = \left([L]^l_k [H]^k_l + \frac{\mu}{\lambda_H^k} \sum_{s>k} [L]^l_s [H]^s_l \right)^a_a \qquad a \in L_l, L < H_l < H_k \tag{4.22}$$

Lemma 16 *For all* $1 \leq q \leq p$ *holds*

$$[L]^q_q [H]^q_q = \delta I_{\hat{q}} \quad [L]^i_q = 0 \quad [H]^q_i = 0 \quad (i \neq q).$$

Proof: We prove the Lemma by downward induction on q. First we take $l = p$ in (4.21) and $k = l = m = p$ in (4.18) and obtain $[L]^p_p [H]^p_p = \delta I_{\hat{p}}$. Then $l = m = p$ and $k < p$ in (4.18) yields $[L]^p_p [H]^p_k = 0$ and hence $[H]^p_k = 0$. Similarly we can take $k = m = p$ and $l < p$ which gives $[L]^l_p = 0$. This proves the statement for $q = p$. Suppose the statement holds for all $q \geq n$, then put $k = l = m = n-1$ in (4.18) and $l = n-1$ in (4.21) which gives $[L]^{n-1}_{n-1} [H]^{n-1}_{n-1} = \delta I_{\widehat{n-1}}$. Finally $k = m = n-1$ and $l \neq n-1$ in (4.18) yields $[L]^l_{n-1} [H]^{n-1}_{n-1} = 0$. This completes the proof. □

Theorem 4 *Given any partitioning of the upper index set H of the form*

$$H_1 < H_2 < \ldots < H_p,$$

a matrix R with all entries equal to zero with the exception of the following:

$$\begin{array}{ll}
i,j \in L & R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda_L \\
i,j \in H_l & R^{ii}_{ii} = R^{jj}_{jj} = R^{ij}_{ji} = R^{ji}_{ij} = \lambda^l_H \\
i \in H_k, j \in H_l, k < l & R^{ij}_{ji} = \mu, R^{ij}_{ij} = x_{ij}, R^{ji}_{ji} = x^{-1}_{ij}\delta \\
i \in H_k, j \in L, j \neq i', k < m & R^{ij}_{ji} = \mu, R^{ij}_{ij} = x_{ij}, R^{ji}_{ji} = x^{-1}_{ij}\delta \\
i \in L, j \in H_k, j \neq i', k \geq m & R^{ij}_{ji} = \mu, R^{ij}_{ij} = x_{ij}, R^{ji}_{ji} = x^{-1}_{ij}\delta \\
i \in L_k, k \geq m & R^{ii'}_{i'i} = \mu \\
i \in L_k, k < m & R^{i'i}_{ii'} = \mu \\
i,j \in L_l = H'_l & R^{ii'}_{jj'} = \alpha(l)^i_j, R^{i'i}_{j'j} = \beta(l)^i_j
\end{array}$$

is a solution of the QYBE if the parameters satisfy the additional constraints:

- x_{ij}, δ *and* μ *are non-zero.*
- λ^l_H *and* λ_L *satisfy the quadratic equation* $X^2 = X\mu + \delta$.
- $\alpha(l)\beta(l) = \delta I_{\hat{l}}$.

Proof: The same argument as given in Theorems 1 and 2. □
To illustrate these type of solutions, for $n = 4$ the matrix R is given below. In this example $H_1 = \{3\}$ and $H_2 = \{4\}$ and we have chosen the ordering such that $H_1 < L < H_2$.

$$\begin{bmatrix}
\lambda_L &&&&&&&&&&&&&&& \\
& 0 &&& \lambda_L &&&&&&&&&&& \\
&& x_{13} &&&&&& 0 &&&&&&& \\
&&& \alpha(2) &&& 0 &&& 0 &&& \mu &&& \\
& \lambda_L &&& 0 &&&&&&&&&&& \\
&&&&& \lambda_L &&&&&&&&&& \\
&&& 0 &&& \alpha(1) &&& 0 &&& 0 &&& \\
&&&&&&& x_{24} &&&&&& \mu && \\
&& \mu &&&&&& \frac{\delta}{x_{13}} &&&&&&& \\
&&& 0 &&& \mu &&& \beta(1) &&& 0 &&& \\
&&&&&&&&&& \lambda^1_H &&&&& \\
&&&&&&&&&&& x_{34} &&& \mu & \\
&&& 0 &&& 0 &&& 0 &&& \beta(2) &&& \\
&&&&&&& 0 &&&&&& \frac{\delta}{x_{24}} && \\
&&&&&&&&&&& 0 &&& \frac{\delta}{x_{34}} & \\
&&&&&&&&&&&&&&& \lambda^2_H
\end{bmatrix}$$

We note that, due to the fact that the components of H consists of only one element, this solution is classical. The first non-classical case is already to large to be presented in this report.

A The complete set of equations

We present the list of equations one obtains when writing out the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in the special case that the matrix R has property (1.3). In components the equation takes the following form:

$$\sum_{i,j,k} R^{ab}_{ij} R^{ic}_{uk} R^{jk}_{vw} = \sum_{i,j,k} R^{bc}_{ij} R^{aj}_{kw} R^{ki}_{uv}$$

We will denote the first index set $\{a, b, c\}$ by I and the second $\{u, v, w\}$ by J. With the result of Lemma 1 one can systematically make a list of all the distinct cases in which nonzero terms on the left- or righthandside occur. We take n to be an even integer. In some equations there is an index p present, this is just a formal way of denoting a summation over all indices. So, for instance equation (1(c)vii) is equal to

$$R^{aa'}_{aa'} R^{a'c}_{ca'} R^{ac}_{ca} = \sum_{p=1}^{n} R^{aa'}_{pp'} R^{p'p}_{aa'} R^{pc}_{cp}$$

1. The index sets I and J are equal.
   There are four subcases to be considered:
   (a) $I = J = \{a, b, c\}$ with $a \neq b \neq c \neq a$ and $a \neq b'$, $b \neq c'$ and $a \neq c'$
      i. $R^{bc}_{bc}(R^{ab}_{ba} R^{ac}_{ca} - R^{ab}_{ba} R^{bc}_{cb} - R^{ac}_{ca} R^{cb}_{bc}) = 0$
      ii. $R^{ab}_{ab}(R^{ac}_{ca} R^{bc}_{cb} - R^{ab}_{ba} R^{bc}_{cb} - R^{ac}_{ca} R^{ba}_{ab}) = 0$
      iii. $R^{ab}_{ba} R^{bc}_{cb}(R^{ab}_{ba} - R^{bc}_{cb}) = R^{ac}_{ca}(R^{bc}_{bc} R^{cb}_{cb} - R^{ab}_{ab} R^{ba}_{ba})$
   (b) $I = J = \{a, a, b\}$ with $b \neq a, a'$
      i. $R^{ab}_{ab} R^{ab}_{ba} R^{ba}_{ab} = 0$
      ii. $R^{ab}_{ba} R^{ba}_{ab}(R^{ab}_{ba} - R^{ba}_{ab}) = 0$
      iii. $R^{ab}_{ba}(R^{aa}_{aa} R^{aa}_{aa} - R^{aa}_{aa} R^{ab}_{ba} - R^{ab}_{ab} R^{ba}_{ba}) = 0$
      iv. $R^{ba}_{ab}(R^{aa}_{aa} R^{aa}_{aa} - R^{aa}_{aa} R^{ba}_{ab} - R^{ab}_{ab} R^{ba}_{ba}) = 0$
   (c) $I = J = \{a, a', c\}$ with $c \neq a, a'$
      i. $R^{aa'}_{cc'} R^{cc'}_{aa'} R^{c'a'}_{c'a'} = 0$
      ii. $R^{aa'}_{cc'} R^{cc'}_{aa'} R^{ac}_{ac} = 0$
      iii. $R^{aa'}_{cc'} R^{cc'}_{aa'}(R^{a'c'}_{c'a'} - R^{c'a'}_{a'c'}) = 0$
      iv. $R^{aa'}_{cc'} R^{cc'}_{aa'}(R^{ac}_{ca} - R^{ca}_{ac}) = 0$
      v. $R^{aa'}_{cc'} R^{cc'}_{a'a}(R^{a'c'}_{c'a'} - R^{c'a}_{ac'}) = 0$
      vi. $R^{aa'}_{cc'} R^{cc'}_{a'a}(R^{a'c}_{ca'} - R^{ca}_{ac}) = 0$
      vii. $R^{aa'}_{aa'} R^{a'c}_{ca'} R^{ac}_{ca} = R^{aa'}_{pp'} R^{p'p}_{aa'} R^{pc}_{cp}$
      viii. $R^{aa'}_{aa'} R^{ca'}_{a'c} R^{ca}_{ac} = R^{aa'}_{p'p} R^{pp'}_{aa'} R^{cp}_{pc}$
      ix. $R^{aa'}_{a'a} R^{ca}_{ac} R^{ca}_{ac} + R^{ac}_{ac} R^{ca}_{ca} R^{ca'}_{a'c} = R^{aa'}_{p'p} R^{pp'}_{a'a} R^{cp}_{pc}$
      x. $R^{aa'}_{a'a} R^{a'c}_{ca'} R^{a'c}_{ca'} + R^{a'c}_{a'c} R^{ca'}_{ca'} R^{ac}_{ca} = R^{aa'}_{pp'} R^{p'p}_{a'a} R^{pc}_{cp}$
      xi. $R^{aa'}_{a'a} R^{a'c}_{a'c} R^{a'c}_{ca'} + R^{a'c}_{a'c} R^{ac}_{ca} R^{ca'}_{a'c} = R^{aa'}_{a'a} R^{a'c}_{a'c} R^{ac}_{ca} + R^{aa'}_{c'c} R^{c'c}_{a'a} R^{ca}_{ca}$
      xii. $R^{aa'}_{a'a} R^{ac}_{ac} R^{ca}_{ac} + R^{ac}_{ac} R^{ac}_{ca} R^{ca'}_{a'c} = R^{aa'}_{a'a} R^{ac}_{ac} R^{ca'}_{a'c} + R^{aa'}_{c'c} R^{c'c}_{a'a} R^{ca'}_{ca'}$
      xiii. $R^{aa'}_{a'a} R^{a'c}_{ca'} R^{ca'}_{ca'} + R^{ac}_{ca} R^{ca'}_{a'c} R^{ca'}_{ca'} = R^{aa'}_{a'a} R^{ac}_{ca} R^{ca'}_{ca'} + R^{aa'}_{cc'} R^{cc'}_{a'a} R^{ac}_{ac}$
      xiv. $R^{aa'}_{a'a} R^{ca}_{ca} R^{ca}_{ac} + R^{ac}_{ca} R^{ca}_{ca} R^{ca'}_{a'c} = R^{aa'}_{a'a} R^{ca}_{ca} R^{ca'}_{a'c} + R^{aa'}_{cc'} R^{cc'}_{a'a} R^{a'c}_{a'c}$
      xv. $R^{aa'}_{a'a} R^{a'c}_{a'c} R^{ca'}_{ca'} + R^{ac}_{ca} R^{ca'}_{a'c} R^{ca'}_{a'c} = R^{aa'}_{a'a} R^{ac}_{ac} R^{ca}_{ca} + R^{ac}_{ca} R^{ac}_{ca} R^{ca'}_{a'c}$
   (d) $I = J = \{a, a, a'\}$
      i. $R^{aa}_{aa} R^{aa'}_{a'a} R^{aa'}_{aa'} = R^{aa'}_{p'p} R^{pp'}_{aa'} R^{ap}_{pa}$
      ii. $R^{aa}_{aa} R^{aa}_{aa} R^{aa'}_{a'a} = R^{aa'}_{p'p} R^{pp'}_{a'a} R^{ap}_{pa}$
      iii. $R^{aa}_{aa} R^{a'a}_{aa'} R^{aa'}_{aa'} = R^{aa'}_{pp'} R^{p'p}_{aa'} R^{pa}_{ap}$
      iv. $R^{aa}_{aa} R^{a'a}_{a'a} R^{aa'}_{a'a} = R^{a'a}_{p'p} R^{pp'}_{a'a} R^{ap}_{pa}$
      v. $R^{aa}_{aa} R^{aa}_{aa} R^{a'a}_{aa'} = R^{a'a}_{pp'} R^{p'p}_{aa'} R^{pa}_{ap}$
      vi. $R^{aa}_{aa} R^{a'a}_{a'a} R^{a'a}_{aa'} = R^{a'a}_{pp'} R^{p'p}_{a'a} R^{pa}_{ap}$
      vii. $R^{a'a}_{p'p} R^{pp'}_{aa'} R^{ap}_{pa} = R^{aa'}_{pp'} R^{p'p}_{a'a} R^{pa}_{ap}$
2. $I = \{a, p, p'\}$ and $J = \{a, q, q'\}$ and $I \neq J$
   There are three subcases to be considered:
   (a) $I = \{a, a', c\}$, $J = \{b, b', c\}$ with $a \neq b, b'$ and $c \neq a, b, a', b'$
      i. $R^{aa'}_{bb'}(R^{ac}_{ac} R^{cb'}_{cb'} - R^{bc}_{bc} R^{ca'}_{ca'}) = 0$
      ii. $R^{aa'}_{bb'} R^{a'c}_{ca'} R^{bc}_{cb} = R^{aa'}_{pp'} R^{p'p}_{bb'} R^{pc}_{cp}$
      iii. $R^{aa'}_{bb'} R^{ca}_{ac} R^{cb'}_{b'c} = R^{aa'}_{p'p} R^{pp'}_{bb'} R^{cp}_{pc}$

iv. $R^{aa'}_{cc'}R^{cc'}_{bb'}R^{c'a'}_{c'a'} = R^{aa'}_{bb'}R^{ac'}_{ac'}(R^{c'b'}_{b'c'} - R^{c'a'}_{a'c'})$

v. $R^{aa'}_{cc'}R^{cc'}_{bb'}R^{ac}_{ac} = R^{aa'}_{bb'}R^{ca'}_{ca'}(R^{bc}_{cb} - R^{ac}_{ca})$

vi. $R^{aa'}_{cc'}R^{cc'}_{bb'}R^{bc}_{bc} = R^{aa'}_{bb'}R^{cb'}_{cb'}(R^{ca}_{ac} - R^{cb}_{bc})$

vii. $R^{aa'}_{bb'}(R^{ca'}_{ca'}R^{ca}_{ca} - R^{cb'}_{cb'}R^{cb}_{cb}) = R^{aa'}_{cc'}R^{cc'}_{bb'}(R^{bc}_{cb} - R^{ca}_{ac})$

viii. $R^{aa'}_{bb'}(R^{a'c}_{a'c}R^{ac}_{ac} - R^{b'c}_{b'c}R^{bc}_{bc}) = R^{aa'}_{c'c}R^{c'c}_{bb'}(R^{cb'}_{b'c} - R^{a'c}_{ca'})$

(b) $I = \{a, a, a'\}$ and $J = \{b, b', a\}$ such that $a \neq b, b'$

i. $R^{a'a'}_{a'a'}R^{aa'}_{bb'}R^{ba'}_{a'b} = R^{aa'}_{pp'}R^{p'p}_{bb'}R^{pa'}_{a'p}$

ii. $R^{aa}_{aa}R^{aa'}_{bb'}R^{ab'}_{b'a} = R^{aa'}_{p'p}R^{pp'}_{bb'}R^{ap}_{pa}$

iii. $R^{aa'}_{bb'}(R^{aa}_{aa}R^{ab'}_{ab'} - R^{aa'}_{aa'}R^{ba}_{ba} - R^{ab'}_{ab'}R^{ab}_{ba}) = 0$

iv. $R^{aa'}_{bb'}(R^{aa}_{aa}R^{aa'}_{aa'} - R^{aa'}_{aa'}R^{ba}_{ab} - R^{ab'}_{ab'}R^{ab}_{ab}) = 0$

v. $R^{aa'}_{bb'}(R^{a'a'}_{a'a'}R^{aa'}_{aa'} - R^{aa'}_{aa'}R^{a'b'}_{b'a'} - R^{b'a'}_{b'a'}R^{ba'}_{ba'}) = 0$

vi. $R^{aa'}_{bb'}(R^{a'a'}_{a'a'}R^{ba'}_{ba'} - R^{aa'}_{aa'}R^{a'b'}_{a'b'} - R^{b'a'}_{a'b'}R^{ba'}_{ba'}) = 0$

vii. $R^{a'a}_{aa'}R^{aa'}_{bb'}R^{a'b'}_{b'a'} + R^{a'a}_{bb'}R^{b'a'}_{b'a'}R^{ba'}_{ba'} = R^{aa'}_{p'p}R^{pp'}_{bb'}R^{a'p}_{pa'}$

viii. $R^{a'a}_{aa'}R^{aa'}_{bb'}R^{ba}_{ab} + R^{a'a}_{bb'}R^{ab'}_{ab'}R^{ab}_{ab} = R^{aa'}_{pp'}R^{p'p}_{bb'}R^{pa}_{ap}$

ix. $R^{a'a}_{bb'}R^{ab'}_{ab'}(R^{aa'}_{a'a} - R^{ab}_{ba}) = R^{aa'}_{bb'}R^{ba}_{ba}(R^{a'a}_{aa'} - R^{b'a}_{ab'})$

(c) $I = \{a, a', b\}$ and $J = \{b, b, b'\}$ with $a \neq b, b'$

i. $R^{bb}_{bb}R^{ba}_{ab}R^{aa'}_{b'b} = R^{aa'}_{p'p}R^{pp'}_{b'b}R^{bp}_{pb}$

ii. $R^{bb}_{bb}R^{a'b}_{ba'}R^{aa'}_{bb'} = R^{aa'}_{pp'}R^{p'p}_{bb'}R^{pb}_{bp}$

iii. $R^{aa'}_{bb'}(R^{b'b'}_{b'b'}R^{bb'}_{bb'} - R^{a'b'}_{a'b'}R^{ab'}_{ab'} - R^{a'b'}_{b'a'}R^{bb'}_{bb'}) = 0$

iv. $R^{aa'}_{bb'}(R^{bb}_{bb}R^{ba'}_{ba'} - R^{ab}_{ab}R^{bb'}_{bb'} - R^{ab}_{ba}R^{ba'}_{ba'}) = 0$

v. $R^{aa'}_{bb'}(R^{b'b'}_{b'b'}R^{ab'}_{ab'} - R^{bb'}_{bb'}R^{b'a'}_{b'a'} - R^{ab'}_{ab'}R^{b'a'}_{a'b'}) = 0$

vi. $R^{aa'}_{bb'}(R^{bb}_{bb}R^{bb'}_{bb'} - R^{ba'}_{ba'}R^{ba}_{ba} - R^{bb'}_{bb'}R^{ba}_{ab}) = 0$

vii. $R^{aa'}_{b'b}R^{ba'}_{ba'}R^{ba}_{ba} + R^{aa'}_{bb'}R^{bb'}_{b'b}R^{ba}_{ab} = R^{aa'}_{p'p}R^{pp'}_{bb'}R^{bp}_{pb}$

viii. $R^{aa'}_{bb'}R^{a'b}_{a'b}R^{ab}_{ab} + R^{aa'}_{b'b}R^{b'b}_{bb'}R^{a'b}_{ba'} = R^{aa'}_{pp'}R^{p'p}_{b'b}R^{pb}_{bp}$

ix. $R^{aa'}_{b'b}R^{ba'}_{ba'}(R^{b'b}_{bb'} - R^{ab}_{ba}) = R^{aa'}_{bb'}R^{ab}_{ab}(R^{bb'}_{b'b} - R^{ba'}_{a'b})$

References

[1] Hazewinkel, M (1993). *Multiparameter Quantum Groups and Multiparameter R-matrices.* CWI-report R-9307, CWI, Amsterdam.