Generalized metric spaces: completion, topology, and powerdomains
via the Yoneda embedding

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via the Yoneda Embedding

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Abstract

Generalized metric spaces are a common generalization of preorders and ordinary metric spaces (Lawvere 1973). Combining Lawvere's (1973) enriched-categorical and Smyth' (1988, 1991) topological view on generalized metric spaces, it is shown how to construct 1. completion, 2. topology, and 3. powerdomains for generalized metric spaces. Restricted to the special cases of preorders and ordinary metric spaces, these constructions yield, respectively: 1. chain completion and Cauchy completion; 2. the Alexandroff and the Scott topology, and the ϵ-ball topology; 3. lower, upper, and convex powerdomains, and the hyperspace of compact subsets. All constructions are formulated in terms of (a metric version of) the Yoneda (1954) embedding.

Contents

1 Overview

A generalized metric space consists of a set X together with a distance function $X(-,-) : X \times X \to [0, \infty]$, satisfying $X(x,x) = 0$ and $X(x,z) \leq X(x,y) + X(y,z)$, for all x, y, and z in X. The family of generalized metric spaces contains all ordinary metric spaces (for which the distance is moreover symmetric and different elements cannot have distance 0) as well as all preordered spaces (because a preorder relation can be viewed as a discrete distance function). Thus generalized metric spaces provide a common generalization of both preordered spaces and ordinary metric spaces, which is the main motivation for the present study.

Our sources of inspiration are the work of Lawvere on $\mathcal{V}$-categories and generalized metric spaces [Law73] and the work by Smyth on quasi metric spaces [Smy91], and we have been influenced by recent work of Flagg and Kopperman [FK95] and Wagner [Wag94]. The present paper continues earlier work [Rut95], in which part of the theory of generalized metric spaces has been developed.

The guiding principle throughout is Lawvere's view of metric spaces as $[0, \infty]$-*categories*, by which they are structures that are formally similar to (ordinary) categories. As a consequence, insights from category theory can be adapted to the world of metric spaces. In particular, we shall give the metric version of the famous *Yoneda Lemma*, which expresses, intuitively, that one may identify elements x of a generalized metric space X with a description of the distances between the elements of X and x (formally, the function that maps any y in X to $X(y,x)$). This elementary insight (with an easy proof) will be shown to be of fundamental importance for the theory of generalized metric spaces (and, a fortiori, both for order-theoretic and metric domain theory as well). Notably it will give rise to

1. a definition of *completion* of generalized metric spaces, generalizing both chain completion of preordered spaces and metric Cauchy completion;

2. a topology on generalized metric spaces generalizing both the Scott topology for *arbitrary* preorders, and the metric ϵ-ball topology;

3. the definition and characterization of three powerdomains generalizing on the one hand the familiar lower, upper, and convex powerdomains from order-theory; and on the other hand the metric powerdomain of compact subsets.

The present paper is a reworking of an earlier report [BBR95], in which generalized *ultra*metric spaces are considered, satisfying $X(x,z) \leq \max\{X(x,y), X(y,z)\}$, for all x, y, and z in X. There is but little difference between the two papers: as it turns out, none of the proofs about ultrametrics relies essentially on the strong triangle inequality. (See also [BBR96], which contains part of the present paper.)

As mentioned above, generalized metric spaces and the constructions that are given in the present paper both unify and generalize a substantial part of order-theoretic and metric domain theory. Both disciplines play a central role in (to a large extent even came into existence because of) the semantics of programming languages (cf. recent textbooks such as [Win93] and [BV96], respectively). The use of generalized metric spaces in semantics, or more precisely, in the study of transition systems, will be an important next step. The combination of results from [Rut95] (on domain equations) and the present paper will lead to the construction of domains for quantitative versions of simulation and bisimulation.

The paper is organized as follows. Sections 2 and 3 give the basic definitions and facts on generalized metric spaces. After the Yoneda Lemma in Section 4, completion, topology and powerdomains are discussed in Sections 5, 6, and 7. Finally Section 8 discusses related work, and the appendix recalls some basic facts from topology, and contains some proofs.

2 Generalized metric spaces as $[0, \infty]$-categories

Generalized metric spaces are introduced and the fact that they are $[0, \infty]$-categories is recalled from Lawvere's [Law73]. (For a brief recapitulation of Lawvere's enriched-categorical view of

(ultra)metric spaces see [Rut95].) The section concludes with a few basic definitions and properties to be used in the sequel.

A *generalized metric space* (gms for short) is a set X together with a mapping

$$X(-,-) : X \times X \to [0, \infty]$$

which satisfies, for all x, y, and z in X,

1. $X(x, x) = 0$, and

2. $X(x, z) \leq X(x, y) + X(y, z)$,

the so-called triangle inequality. The real number $X(x, y)$ will be called the distance from x to y. Examples of generalized metric spaces are:

1. The set A^∞ of finite and infinite words over some given set A with distance function, for v and w in A^∞,

$$A^\infty(v, w) = \begin{cases} 0 & \text{if } v \text{ is a prefix of } w \\ 2^{-n} & \text{otherwise,} \end{cases}$$

where n is the length of the longest common prefix of v and w.

2. Any *preorder* $\langle P, \leq \rangle$ (satisfying for all p, q, and r in P, $p \leq p$, and if $p \leq q$ and $q \leq r$ then $p \leq r$) can be viewed as a gms, by defining

$$P(p, q) = \begin{cases} 0 & \text{if } p \leq q \\ \infty & \text{if } p \not\leq q. \end{cases}$$

By a slight abuse of language, any gms stemming from a preorder in this way will itself be called a preorder.

3. The set $[0, \infty]$ with distance, for r and s in $[0, \infty]$,

$$[0, \infty](r, s) = \begin{cases} 0 & \text{if } r \geq s \\ s - r & \text{if } r < s. \end{cases}$$

Generalized metric spaces are $[0, \infty]$-enriched categories in the sense of [EK66, Law73, Kel82]. As shown in [Law73], $[0, \infty]$ is a complete and cocomplete symmetric monoidal closed category. It is a category because it is a preorder (objects are the non-negative real numbers including infinity; and for r and s in $[0, \infty]$ there is a morphism from r to s if and only if $r \geq s$). It is complete and cocomplete: equalizers and coequalizers are trivial (because there is at most one arrow between any two elements of $[0, \infty]$), the product $r \sqcap s$ of two elements r and s in $[0, \infty]$ is given by $\max\{r, s\}$, and their coproduct $r \sqcup s$ by $\min\{r, s\}$. More generally, products are given by sup, and coproducts are given by inf. Most important for what follows is the monoidal structure on $[0, \infty]$, which is given by

$$+ : [0, \infty] \times [0, \infty] \to [0, \infty],$$

assigning to two real numbers their sum. (As usual, $r + \infty = \infty + r = \infty$, for all $r \in [0, \infty]$.) Let $[0, \infty](-, -)$ be the ('internal hom-') functor that assigns to r and s in $[0, \infty]$ the distance $[0, \infty](r, s)$ as defined in the third example above. The following fundamental equivalence states that $[0, \infty](t, -)$ is right-adjoint to $t + -$, for any t in $[0, \infty]$:

Proposition 2.1 *For all r, s, and t in $[0, \infty]$,*

$t + s \geq r$ *if and only if* $s \geq [0, \infty](t, r)$.

$\square$

4

Many constructions and properties of generalized metric spaces are determined by the category $[0, \infty]$. Important examples are the definitions of limit and completeness, presented in Section 3. Also the category of all generalized metric spaces, which is introduced next, inherits much of the structure of $[0, \infty]$.

Let *Gms* be the category with generalized metric spaces as objects, and *non-expansive* maps as arrows: i.e., mappings $f : X \to Y$ such that for all x and x' in X,

$$Y(f(x), f(x')) \leq X(x, x').$$

A map f is *isometric* if for all x and x' in X,

$$Y(f(x), f(x')) = X(x, x').$$

Two spaces X and Y are called isometric (isomorphic) if there exists an isometric bijection between them. The *product* $X \times Y$ of two gms X and Y is defined as the Cartesian product of their underlying sets, together with distance, for $\langle x, y \rangle$ and $\langle x', y' \rangle$ in $X \times Y$,

$$X \times Y(\langle x, y \rangle, \ \langle x', y' \rangle) = \max\{X(x, x'), Y(y, y')\}.$$

The *exponent* of X and Y is defined by

$$Y^X = \{f : X \to Y \mid f \text{ is non-expansive }\},$$

with distance, for f and g in Y^X,

$$Y^X(f, g) = \sup\{Y(f(x), g(x)) \mid x \in X\}.$$

This section is concluded by a number of constructions and definitions for generalized metric spaces that will be used in the sequel.

A gms generally does not satisfy

3. if $X(x, y) = 0$ and $X(y, x) = 0$ then $x = y$,

4. $X(x, y) = X(y, x)$,

5. $X(x, y) < \infty$,

which are the additional conditions that hold for an *ordinary* metric space. Therefore it is sometimes called a *pseudo-quasi* metric space. A *quasi* metric space (qms for short) is a gms which satisfies axioms 1, 2, and 3. Note that $[0, \infty]$ is a quasi metric space. A gms satisfying 1, 2, and 4 is called a *pseudo* metric space. Finally, if a gms satisfies the so-called strong triangle inequality

2'. $X(x, z) \leq \max\{X(x, y), X(y, z)\}$,

then it is called a generalized *ultra*metric space (cf. [BBR95]).

The *opposite* X^{op} of a gms X is the set X with distance

$$X^{op}(x, x') = X(x', x).$$

With this definition, the distance function $X(-, -)$ can be described as a mapping

$$X(-, -) : X^{op} \times X \to [0, \infty].$$

Using Proposition 2.1 one can easily show that $X(-, -)$ is non-expansive.

We saw that any preorder P induces a gms. (Note that a partial order induces a quasi metric and that the non-expansive mappings between preorders are precisely the monotone maps.) Conversely, any gms X gives rise to a preorder $\langle X, \leq_X \rangle$, where $\leq_X$, called the *underlying* ordering of X, is given, for x and y in X, by

$$x \leq_X y \text{ if and only if } X(x, y) = 0.$$

Any (pseudo or quasi) metric space is a fortiori a gms. Conversely, any gms X induces a pseudo metric space X^s, the *symmetrization* of X, with distance

$$X^s(x, y) = \max\{X(x, y), X^{op}(x, y)\}.$$

For instance, the ordering that underlies A^∞ is the usual prefix ordering, and $(A^\infty)^s$ is a natural metric on words. The generalized metric on $[0, \infty]$ induces the reverse of the usual ordering: for r and s in $[0, \infty]$,

$$r \leq_{[0,\infty]} s \text{ if and only if } s \leq r;$$

and the symmetric version of $[0, \infty]$ is defined by

$$[0, \infty]^s(r, s) = \begin{cases} 0 & \text{if } r = s \\ |r - s| & \text{if } r \neq s. \end{cases}$$

Any gms X induces a quasi metric space $[X]$ defined as follows. Let $\approx$ be the equivalence relation on X defined, for x and y in X, by

$$x \approx y \text{ iff } (X(x, y) = 0 \text{ and } X(y, x) = 0).$$

Let $[x]$ denote the equivalence class of x with respect to $\approx$, and $[X]$ the collection of all equivalence classes. Defining $[X]([x], [y]) = X(x, y)$ turns $[X]$ into a quasi metric space. It has the following universal property: for any non-expansive mapping $f : X \to Y$ from X to a quasi metric space Y there exists a unique non-expansive mapping $f' : [X] \to Y$ with $f'([x]) = f(x)$, for $x \in X$.

3 Cauchy sequences, limits, and completeness

The notion of Cauchy sequence is introduced, followed by the definition of metric limit, first for Cauchy sequences in $[0, \infty]$ and then for Cauchy sequences in arbitrary generalized metric spaces. Furthermore the notions of completeness, finiteness, and algebraicity are introduced.

A sequence $(x_n)_n$ in a gms X is *forward-Cauchy* if

$$\forall \epsilon > 0 \; \exists N \; \forall n \geq m \geq N, \;\; X(x_m, x_n) \leq \epsilon.$$

Since our metrics need not be symmetric, the following variation exists: a sequence $(x_n)_n$ is *backward-Cauchy* if

$$\forall \epsilon > 0 \; \exists N \; \forall n \geq m \geq N, \;\; X(x_n, x_m) \leq \epsilon.$$

If X is an ordinary metric space then forward-Cauchy and backward-Cauchy both mean Cauchy in the usual sense. And if X is a preorder then Cauchy sequences are eventually increasing: there exists an N such that for all $n \geq N$, $x_n \leq x_{n+1}$. (Increasing sequences in a preorder are also called *chains*.) Similarly backward-Cauchy sequences are eventually decreasing.

The *forward-limit* of a forward-Cauchy sequence $(r_n)_n$ in $[0, \infty]$ is defined by

$$\lim_{\rightarrow} r_n = \sup_n \; \inf_{k \geq n} \; r_k.$$

Dually, the *backward-limit* of a backward-Cauchy sequence $(r_n)_n$ in $[0, \infty]$ is

$$\lim_{\leftarrow} r_n = \inf_n \; \sup_{k \geq n} \; r_k.$$

These numbers are what one intuitively would consider as metric limits of Cauchy sequences. If $[0, \infty]$ is taken with the standard symmetric Euclidian metric: $[0, \infty]^s(r, r') = |r - r'|$, for r and r' in $[0, \infty]$, then all bounded forward-Cauchy and backward-Cauchy sequences are Cauchy with respect to $[0, \infty]^s$, and the forward-limit and backward-limit defined above coincide with the usual notion of limit with respect to $[0, \infty]^s$ (cf. [Smy91]).

The following proposition shows how forward-limits and backward-limits in $[0, \infty]$ are related.

Proposition 3.1 *For a forward-Cauchy sequence $(r_n)_n$ in $[0,\infty]$, and all r in $[0,\infty]$,*

1. $[0,\infty](r,\lim_{\rightarrow} r_n) = \lim_{\rightarrow}[0,\infty](r,r_n);$

2. $[0,\infty](\lim_{\rightarrow} r_n,r) = \lim_{\leftarrow}[0,\infty](r_n,r).$

For a backward-Cauchy sequence $(r_n)_n$ in $[0,\infty]$, and all r in $[0,\infty]$,

3. $[0,\infty](r,\lim_{\leftarrow} r_n) = \lim_{\leftarrow}[0,\infty](r,r_n);$

4. $[0,\infty](\lim_{\leftarrow} r_n,r) = \lim_{\rightarrow}[0,\infty](r_n,r).$

$\square$

A proof follows easily using the following elementary facts:

Lemma 3.2 *For all non-empty subsets $V \subseteq [0,\infty]$ and r in $[0,\infty]$,*

1. $[0,\infty](r,\sup V) = \sup_{v\in V}[0,\infty](r,v);$

2. $[0,\infty](r,\inf V) = \inf_{v\in V}[0,\infty](r,v);$

3. $[0,\infty](\sup V,r) = \inf_{v\in V}[0,\infty](v,r);$

4. $[0,\infty](\inf V,r) = \sup_{v\in V}[0,\infty](v,r).$

$\square$

Forward-limits in an *arbitrary* gms X can now be defined in terms of backward-limits in $[0,\infty]$: an element x is a *forward-limit* of a forward-Cauchy sequence $(x_n)_n$ in X,

$$x = \lim_{\rightarrow} x_n \text{ iff } \forall y \in X, \quad X(x,y) = \lim_{\leftarrow} X(x_n,y).$$

This is well defined because of the following.

Proposition 3.3 *Let $(x_n)_n$ be a forward Cauchy sequence in X. Let $x \in X$.*

1. *The sequence $(X(x,x_n))_n$ is forward Cauchy in $[0,\infty]$.*

2. *The sequence $(X(x_n,x))_n$ is backward Cauchy in $[0,\infty]$.*

Note that our earlier definition of the forward-limit of forward-Cauchy sequences in $[0,\infty]$ is consistent with this definition for arbitrary generalized metric spaces: this follows from Proposition 3.1(2).

Further note that Cauchy sequences may have more than one limit. Therefore one has to be careful with an argument like:

$$\text{if } x = \lim_{\rightarrow} x_n \text{ and } y = \lim_{\rightarrow} x_n \text{ then } x = y,$$

which in general is not correct. All one can deduce from the assumptions is that $X(x,y) = 0$ and $X(y,x) = 0$. The conclusion $x = y$ *is* justified only in quasi metric spaces where as a consequence, limits are unique. For instance, limits in $[0,\infty]$ are unique.

In spite of the fact that in an arbitrary gms X limits are not uniquely determined, we shall nevertheless use expressions (for instance, in Proposition 3.4 below) such as

$$X(\lim_{\rightarrow} x_n,y)$$

(for a Cauchy sequence $(x_n)_n$ and an element y in X), because the value they denote does not depend on the particular choice of a limit. This is an immediate consequence of the fact that all limits have distance 0.

For ordinary metric spaces, the above defines the usual notion of limit:

$$x = \lim_{\to} x_n \text{ if and only if } \forall \epsilon > 0 \; \exists N \; \forall n \geq N, \quad X(x_n, x) < \epsilon.$$

If X is a partial order and $(x_n)_n$ is a chain in X then

$$x = \lim_{\to} x_n \text{ if and only if } \forall y \in X, \quad x \leq_X y \Leftrightarrow \forall n \geq 0, \; x_n \leq_X y,$$

i.e., $x = \bigsqcup x_n$, the least upperbound of the chain $(x_n)_n$.

One could also consider backward-limits for arbitrary gms. Since these will not play a role in the rest of this paper, this is omitted. For simplicity, we shall use *Cauchy* instead of forward-Cauchy. Similarly, we shall write

$$\lim x_n \text{ rather than } \lim_{\to} x_n,$$

and use *limit* instead of forward-limit.

Note that subsequences of a Cauchy sequence are Cauchy again. If a Cauchy sequence has a limit x, then all its subsequences have limit x as well.

The following fact will be useful in the future:

Proposition 3.4 *For a Cauchy sequence $(x_n)_n$ and an element x in a gms X,*

$$X(x, \lim_n x_n) \leq \lim_n X(x, x_n).$$

Proof: The inequality follows from

$$
\begin{aligned}
&[0, \infty](\lim_n X(x, x_n), \; X(x, \lim_n x_n)) \\
&= \quad \lim_{\leftarrow n} [0, \infty](X(x, x_n), \; X(x, \lim_n x_n)) \\
&\leq \quad \lim_{\leftarrow n} X(x_n, \lim_n x_n) \quad [\text{the mapping } X(x, -) : X \to [0, \infty] \text{ is non-expansive}] \\
&= \quad X(\lim_n x_n, \; \lim_n x_n) \\
&= \quad 0.
\end{aligned}
$$

$\square$

A gms X is *complete* if every Cauchy sequence in X has a limit. A subset $V \subseteq X$ is complete if every Cauchy sequence in V has a limit in V. For instance, $[0, \infty]$ is complete. If X is a partial order completeness means that X is a complete partial order, cpo for short: all ω-chains have a least upperbound. For ordinary metric spaces this definition of completeness is the usual one. There is the following fact (cf. Theorem 6.5 of [Rut95]).

Proposition 3.5 *Let X and Y be generalized metric spaces. If Y is complete then Y^X is complete. Moreover, limits are pointwise: let $(f_n)_n$ be a Cauchy sequence in Y^X and f an element in Y^X. Then $\lim f_n = f$ if and only if for all $x \in X$, $\lim f_n(x) = f(x)$. Furthermore, if Y is a quasi metric space then Y^X is a quasi metric space as well.* $\square$

A mapping $f : X \to Y$ between gms X and Y is *continuous* if it preserves Cauchy sequences and their limits: if $(x_n)_n$ is Cauchy and $x = \lim x_n$ in X, then $(f(x_n))_n$ is again Cauchy and $f(x) = \lim f(x_n)$ in Y. For ordinary metric spaces, this is the usual definition. For partial orders it amounts to preservation of least upperbounds of ω-chains.

An element b in a gms X is *finite* if the mapping

$$X(b, -) : X \to [0, \infty], \quad x \mapsto X(b, x)$$

is continuous. (So for finite elements, the inequality in Proposition 3.4 actually is an equality.) If X is a partial order this means that for any chain $(x_n)_n$ in X,

$$X(b, \bigsqcup x_n) = \lim_{\leftarrow} X(b, x_n),$$

or, equivalently,

$$b \leq_X \bigsqcup x_n \text{ iff } \exists n, \ b \leq_X x_n,$$

which is the usual definition of finiteness in ordered spaces. If X is an ordinary metric space then $X(b, -)$ is continuous for any b in X, hence all elements are finite.

A *basis* for a gms X is a subset $B \subseteq X$ consisting of finite elements such that every element x in X is the limit $x = \lim b_n$ of a Cauchy sequence $(b_n)_n$ of elements in B. A gms X is *algebraic* if there exists a basis for X. Note that such a basis is in general not unique. If X is algebraic then the collection B_X of all finite elements of X is the largest basis. Further note that algebraic does not imply complete. (Take any ordinary metric space which is not complete.) If there exists a countable basis then X is *ω-algebraic*.

For instance, the gms A^∞ from Section 2 is algebraic with basis A^*, the set of all finite words over A. If A is countable then A^∞ is ω-algebraic. Also the space $[0, \infty]$ is algebraic: by Proposition 3.1(1), all elements are finite. (It is even ω-algebraic, with the set of rational numbers as a basis.) This fact is somewhat surprising, since $[0, \infty]$ is not algebraic as a partial order.

4   The Yoneda Lemma

The following lemma turns out to be of great importance for the theory of generalized metric spaces. It is the $[0, \infty]$-categorical version of the famous *Yoneda Lemma* [Yon54] from category theory. We shall see in the subsequent sections that it gives rise to elegant definitions and characterizations of completion, topology, and powerdomains. A general proof of the Yoneda Lemma for arbitrary enriched categories can be found in [Kel82]. For generalized metric spaces, it is proved in [Law86].

The following notation will be used throughout the rest of this paper:

$$\hat{X} = [0, \infty]^{X^{op}},$$

i.e., the set of all non-expansive functions from X^{op} to $[0, \infty]$.

Lemma 4.1 (Yoneda Lemma) *Let X be a gms. For any $x \in X$ let*

$$X(-, x) : X^{op} \to [0, \infty], \ \ y \mapsto X(y, x).$$

This function is non-expansive and hence an element of $\hat{X}$. For any other element ϕ in $\hat{X}$, $\hat{X}(X(-, x), \phi) = \phi(x)$.

Proof: Because $X(-, -) : X^{op} \times X \to [0, \infty]$ is non-expansive, so is $X(-, x)$, for any x in X. Now let $\phi \in \hat{X}$. On the one hand,

$$
\begin{aligned}
\phi(x) &= [0, \infty](X(x, x), \ \phi(x)) \\
&\leq \sup_{y \in X} [0, \infty](X(y, x), \ \phi(y)) \\
&= \hat{X}(X(-, x), \ \phi).
\end{aligned}
$$

On the other hand, non-expansiveness of ϕ gives, for any y in X,

$$[0, \infty](\phi(x), \ \phi(y)) \leq X^{op}(x, y) = X(y, x),$$

which is equivalent by Proposition 2.1 to $[0, \infty](X(y, x), \ \phi(y)) \leq \phi(x)$. It follows that

$$\hat{X}(X(-, x), \ \phi) \leq \phi(x).$$

$\square$

The following corollary is immediate.

Corollary 4.2 *The **Yoneda embedding** $\mathbf{y} : X \to \hat{X}$, defined for x in X by $\mathbf{y}(x) = X(-, x)$ is isometric: for all x and x' in X,*

$$X(x, x') = \hat{X}(\mathbf{y}(x), \mathbf{y}(x')).$$

$\square$

The following fact will be of use when defining completion.

Lemma 4.3 *For any x in X, $\mathbf{y}(x)$ is finite in $\hat{X}$.*

Proof: We have to show that $\hat{X}(\mathbf{y}(x), -) : \hat{X} \to [0, \infty]$ is continuous: for any Cauchy sequence $(\phi_n)_n$ in $\hat{X}$,

$$
\begin{aligned}
\hat{X}(\mathbf{y}(x), \lim \phi_n) &= (\lim \phi_n)(x) \quad \text{[the Yoneda Lemma]} \\
&= \lim \phi_n(x) \quad \text{[Proposition 3.5]} \\
&= \lim \hat{X}(\mathbf{y}(x), \phi_n) \quad \text{[the Yoneda Lemma]}.
\end{aligned}
$$

$\square$

5 Completion via Yoneda

The completion of gms's is defined by means of the Yoneda embedding. It yields for ordinary metric spaces Hausdorff's standard Cauchy completion (as introduced in [Hau14]), for preorders the chain completion, and for qms's a completion given by Smyth (see [Smy91, page 214]).

Let X be a gms. Because $[0, \infty]$ is a complete qms (cf. Section 2 and 3), it follows from Proposition 3.5 that $\hat{X}$ is a complete qms as well. According to Corollary 4.2, the Yoneda embedding $\mathbf{y}$ isometrically embeds X in $\hat{X}$. The completion of X can now be defined as the smallest complete subspace of $\hat{X}$ which contains the $\mathbf{y}$-image of X.

Definition 5.1 The *completion* of a gms X is defined by

$$\bar{X} = \bigcap \{ V \subseteq \hat{X} \mid \mathbf{y}(X) \subseteq V \text{ and } V \text{ is a complete subspace of } \hat{X} \}.$$

The collection of which the intersection is taken is nonempty, since it contains $\hat{X}$. Because $\bar{X}$ is a complete subspace of the complete qms $\hat{X}$, also $\bar{X}$ is a complete qms, and, as a consequence, for any Cauchy sequence in $\bar{X}$, its limits in $\bar{X}$ and $\hat{X}$ coincide.

As with preorders, completion is not idempotent, that is, the completion of the completion of X is in general not isomorphic to the completion of X. An interesting question is to characterize the family of gms's for which completion is idempotent (it contains at least all ordinary metric spaces). (Cf. [FS96].)

Completion for ordinary metric spaces is usually defined by means of (equivalence classes of) Cauchy sequences. The same applies to countable preorders: there the most common form of completion, ideal completion, is isomorphic to chain completion, and we have seen that chains are (special cases of) Cauchy sequences. It will be shown next that the completion introduced above can be expressed in terms of Cauchy sequences as well. This will at the same time enable us to prove its equivalence with the definition of the completion of qms's by Smyth.

Note that a sequence $(x_n)_n$ is Cauchy in a gms X if and only if $(\mathbf{y}(x_n))_n$ is Cauchy in $\hat{X}$, because the Yoneda embedding $\mathbf{y}$ is isometric. This is used in the following.

Proposition 5.2 *For any gms X,*

$$\bar{X} = \{ \lim_n \mathbf{y}(x_n) \mid (x_n)_n \text{ is a Cauchy sequence in } X \}.$$

Proof: The inclusion from right to left is immediate from the fact that the set on the right is contained in any complete subspace V of $\hat{X}$ which contains $\mathbf{y}(X)$. The reverse inclusion follows from the fact that the set on the right contains $\mathbf{y}(X)$, which is trivial, and the fact that it is a complete subspace of $\hat{X}$: this is a consequence of Lemma 4.3 and Proposition B.3 in the appendix.

$\square$

The elements of $\bar{X}$ can be seen to represent equivalence classes of Cauchy sequences. To this end, let $CS(X)$ denote the set of all Cauchy sequences in X, and let $\lambda : CS(X) \to \bar{X}$ map a Cauchy sequence $(v_n)_n$ in X to $\lim_n \mathbf{y}(v_n)$. This mapping induces a generalized metric structure on $CS(X)$ by putting, for Cauchy sequences $(v_m)_m$ and $(w_n)_n$,

$$CS(X)((v_m)_m, (w_n)_n) = \bar{X}(\lambda((v_m)_m), \lambda((w_n)_n)).$$

This metric can be characterized as follows.

$$
\begin{aligned}
& CS(X)((v_m)_m, (w_n)_n) \\
& = \bar{X}(\lambda((v_m)_m), \lambda((w_n)_n)) \\
& = \bar{X}(\lim_m \mathbf{y}(v_m), \lim_n \mathbf{y}(w_n)) \\
& = \widehat{X}(\lim_m \mathbf{y}(v_m), \lim_n \mathbf{y}(w_n)) \\
& = \varprojlim_m \widehat{X}(\mathbf{y}(v_m), \lim_n \mathbf{y}(w_n)) \\
& = \varprojlim_m \lim_n \widehat{X}(\mathbf{y}(v_m), \mathbf{y}(w_n)) \quad [\mathbf{y}(v_n) \text{ is finite in } \widehat{X}] \\
& = \varprojlim_m \lim_n X(v_m, w_n) \quad [\mathbf{y} \text{ is isometric}].
\end{aligned}
$$

The latter formula is what Smyth has used for a definition of the distance between Cauchy sequences of qms's. In his approach, the completion of a qms is defined as $[CS(X)]$, which is the qms obtained from $CS(X)$ by identifying all Cauchy sequences with distance 0 in both directions (cf. Section 2). Such sequences can be considered to represent the same limit. Both ways of defining completion are equivalent.

Proposition 5.3 *For any gms X, $\bar{X} \cong [CS(X)]$.*

Proof: Because $\bar{X}$ is a qms, the non-expansive mapping $\lambda : CS(X) \to \bar{X}$ induces a non-expansive mapping $\lambda' : [CS(X)] \to \bar{X}$ (cf. Section 2). Because λ is isometric by the definition of the metric on $CS(X)$, λ' is injective. Because λ is surjective by Proposition 5.2, λ' is also surjective. □

A corollary of this theorem is that the completion of gms's generalizes Cauchy completion of ordinary metric spaces and chain completion of preorders.

Recall that the category *Gms* has gms's as objects and non-expansive functions as arrows. Let *Acq* be the category with algebraic complete qms's as objects, and with non-expansive and continuous functions as arrows. We will show that completion can be extended to a functor from *Gms* to *Acq*, which is a left adjoint to the forgetful functor from *Acq* to *Gms*. First of all, the completion of a gms X is an object in *Acq*.

Theorem 5.4 *For any gms X, $\bar{X}$ is an algebraic complete qms.*

Proof: Since $\bar{X}$ is a complete subspace of the complete qms $\widehat{X}$, also $\bar{X}$ is a complete qms. Because all elements of $\mathbf{y}(X)$ are finite in $\widehat{X}$ according to Lemma 4.3, they are also finite in $\bar{X}$. From Proposition 5.2 we can conclude that every element of $\bar{X}$ is the limit of a Cauchy sequence in $\mathbf{y}(X)$. Consequently $\bar{X}$ is algebraic. □

The next theorem is the key to the extension of completion to a functor. It says that completion is a so-called free construction.

Theorem 5.5 *For any complete qms Y and non-expansive function $f : X \to Y$ there exists a unique non-expansive and continuous function $f^\# : \bar{X} \to Y$ such that $f^\# \circ \mathbf{y} = f$.*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mathbf{y}\ } & \bar{X} \\
& {\scriptstyle f}\searrow & \Big\downarrow {\scriptstyle f^\#} \\
& & Y
\end{array}
$$

Proof: For all Cauchy sequences $(v_n)_n$ and $(w_m)_m$ in X,

$$Y\left(\lim_n f\left(v_n\right), \lim_m f\left(w_m\right)\right)$$
$$= \quad \varprojlim_n Y\left(f\left(v_n\right), \lim_m f\left(w_m\right)\right)$$
$$\leq \quad \varprojlim_n \lim_m Y\left(f\left(v_n\right), f\left(w_m\right)\right) \quad \text{[Proposition 3.4]}$$
$$\leq \quad \varprojlim_n \lim_m X\left(v_n, w_m\right) \quad \text{[}f \text{ is non-expansive]}$$
$$= \quad \varprojlim_n \lim_m \widehat{X}\left(\mathbf{y}\left(v_n\right), \mathbf{y}\left(w_m\right)\right) \quad \text{[}\mathbf{y} \text{ is isometric]}$$
$$= \quad \varprojlim_n \widehat{X}\left(\mathbf{y}\left(v_n\right), \lim_m \mathbf{y}\left(w_m\right)\right) \quad \text{[}\mathbf{y}\left(v_n\right) \text{ is finite in } \widehat{X}\text{]}$$
$$= \quad \widehat{X}\left(\lim_n \mathbf{y}\left(v_n\right), \lim_m \mathbf{y}\left(w_m\right)\right).$$

Consequently,

$$\lim_n \mathbf{y}\left(v_n\right) = \lim_m \mathbf{y}\left(w_m\right)$$
$$\Rightarrow \quad \widehat{X}\left(\lim_n \mathbf{y}\left(v_n\right), \lim_m \mathbf{y}\left(w_m\right)\right) = 0 \wedge \widehat{X}\left(\lim_m \mathbf{y}\left(w_m\right), \lim_n \mathbf{y}\left(v_n\right)\right) = 0$$
$$\Rightarrow \quad Y\left(\lim_n f\left(v_n\right), \lim_m f\left(w_m\right)\right) = 0 \wedge Y\left(\lim_m f\left(w_m\right), \lim_n f\left(v_n\right)\right) = 0$$
$$\Rightarrow \quad \lim_n f\left(v_n\right) = \lim_m f\left(w_m\right).$$

According to Proposition 5.2, for all $\bar{x}$ in $\bar{X}$, there exists a Cauchy sequence $(x_n)_n$ in X, such that $\bar{x} = \lim_n \mathbf{y}\left(x_n\right)$. Since f is non-expansive, the sequence $(f\left(x_n\right))_n$ is also Cauchy. Because Y is a complete qms, $\lim_n f\left(x_n\right)$ exists. Hence, we can define $f^{\#} : \bar{X} \to Y$ by

$$f^{\#}\left(\lim_n \mathbf{y}\left(x_n\right)\right) = \lim_n f\left(x_n\right).$$

Since, for all Cauchy sequences $(v_n)_n$ and $(w_m)_m$ in X,

$$Y\left(f^{\#}\left(\lim_n \mathbf{y}\left(v_n\right)\right), f^{\#}\left(\lim_m \mathbf{y}\left(w_m\right)\right)\right)$$
$$= \quad Y\left(\lim_n f\left(v_n\right), \lim_n f\left(w_m\right)\right)$$
$$\leq \quad \widehat{X}\left(\lim_n \mathbf{y}\left(v_n\right), \lim_m \mathbf{y}\left(w_m\right)\right) \quad \text{[see above]}$$

the function $f^{\#}$ is non-expansive.

Next we prove that $f^{\#}$ is continuous. Let $(\bar{x}_n)_n$ be a Cauchy sequence in $\bar{X}$. Without loss of generality we can assume that

$$\forall n, \bar{X}\left(\bar{x}_n, \bar{x}_{n+1}\right) \leq \tfrac{1}{3n2^n}. \tag{1}$$

Because $\mathbf{y}$ is isometric, we can conclude from Proposition 5.2 that

$$\bar{X} = \left\{\, \lim_n \mathbf{y}\left(x_n\right) \mid (\mathbf{y}\left(x_n\right))_n \text{ is a Cauchy sequence in } \mathbf{y}\left(X\right) \,\right\}.$$

Because $\mathbf{y}\left(X\right)$ is a subspace of the complete qms $\widehat{X}$, and all elements of $\mathbf{y}\left(X\right)$ are finite in $\widehat{X}$ according to Lemma 4.3, we can conclude from Lemma B.1 and B.2 that there exist Cauchy sequences $(w_n^m)_m$ in $\mathbf{y}\left(X\right)$ satisfying

$$\forall m \forall n, \mathbf{y}\left(X\right)\left(w_n^m, w_{n+1}^m\right) \leq \tfrac{1}{n},$$
$$\forall n \forall m, \mathbf{y}\left(X\right)\left(w_n^m, w_n^{m+1}\right) \leq \tfrac{1}{m},$$
$$\forall n, \lim_m w_n^m = \bar{x}_n,$$
$$\lim_k w_k^k = \lim_n \bar{x}_n.$$

Since $\mathbf{y}$ is isometric, there exist Cauchy sequences $(x_n^m)_m$ in X satisfying

$$\forall m \forall n, X\left(x_n^m, x_{n+1}^m\right) \leq \tfrac{1}{n}, \tag{2}$$

$$\forall n \forall m, X\left(x_n^m, x_n^{m+1}\right) \le \tfrac{1}{m}, \tag{3}$$

$$\forall n, \lim_m \mathbf{y}\left(x_n^m\right) = \bar{x}_n, \tag{4}$$

$$\lim_k \mathbf{y}\left(x_k^k\right) = \lim_n \bar{x}_n. \tag{5}$$

As we have seen above, $f^\#$ is non-expansive. Consequently, $(f^\#\left(\bar{x}_n\right))_n$ is a Cauchy sequence in Y. Since f is non-expansive, we can derive from (2) and (3) that

$$\forall m \forall n, Y\left(f\left(x_n^m\right), f\left(x_{n+1}^m\right)\right) \le \tfrac{1}{n}, \tag{6}$$

$$\forall n \forall m, Y\left(f\left(x_n^m\right), f\left(x_n^{m+1}\right)\right) \le \tfrac{1}{m}. \tag{7}$$

From (4) we can deduce that

$$\forall n, \lim_m f\left(x_n^m\right) = f^\#\left(\bar{x}_n\right). \tag{8}$$

Since Y is a complete qms, it follows from (6), (7), (8), and Lemma B.2 that the sequence $(f\left(x_k^k\right))_k$ is Cauchy and

$$\lim_k f\left(x_k^k\right) = \lim_n f^\#\left(\bar{x}_n\right).$$

From (5) we can derive that

$$f^\#\left(\lim_n \bar{x}_n\right) = \lim_k f\left(x_k^k\right).$$

Hence $f^\#$ is continuous.

Let $g : \bar{X} \to Y$ be a non-expansive and continuous function such that $g \circ \mathbf{y} = f$. For all Cauchy sequences $(x_n)_n$ in X,

$$
\begin{aligned}
g\left(\lim_n \mathbf{y}\left(x_n\right)\right) \\
= \quad & \lim_n g\left(\mathbf{y}\left(x_n\right)\right) \quad [g \text{ is continuous}] \\
= \quad & \lim_n f\left(x_n\right) \quad [g \circ \mathbf{y} = f] \\
= \quad & f^\#\left(\lim_n \mathbf{y}\left(x_n\right)\right).
\end{aligned}
$$

This proves the unicity of $f^\#$. $\qquad\qquad\square$

Completion can be extended to a functor $(^-) : Gms \to Acq$, by defining its action on arrows in Gms in the following standard way: for gms's X and Y and a non-expansive mapping $f : X \to Y$, let $\bar{f} : \bar{X} \to \bar{Y}$ be defined by $\bar{f} = (\mathbf{y} \circ f)^\#$.

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & Y \\
{\scriptstyle \mathbf{y}}\downarrow & & \downarrow{\scriptstyle \mathbf{y}} \\
\bar{X} & \dashrightarrow[{(\mathbf{y} \circ f)^\#}]{} & \bar{Y}
\end{array}
$$

According to Theorem 5.5, the function $\bar{f}$ is non-expansive and continuous, and hence an arrow in Acq. One can easily verify that we have extended completion to a functor. It is an immediate consequence of Theorem 5.5 that it is left adjoint to the forgetful functor from Acq to Gms (cf. [ML71, Chapter 4]). The Yoneda embedding $\mathbf{y}$ is the unit of the adjunction.

For every complete qms X with basis B, $X \cong \bar{B}$. More generally:

Theorem 5.6 *Let X be a complete qms. Let $B \subseteq X$. Then the following three conditions are equivalent.*

1. *B is a basis for X.*

2. The function $\mathbf{y}_B : X \to \widehat{B}$ defined, for $x \in X$, by

$$\mathbf{y}_B(x) = \lambda b \in B \,.\, X(b, x),$$

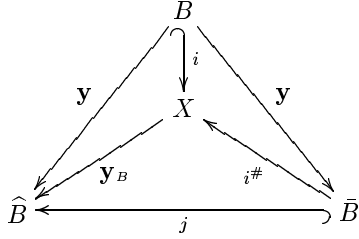   i.e., the restriction of $\mathbf{y}(x) \in \widehat{X}$ to B, is isometric and continuous.

3. The inclusion function $i : B \to X$ induces an isomorphism $i^\# : \bar{B} \to X$.

Proof:

1. $\Rightarrow$ 2. According to Corollary 4.2, $\mathbf{y}$ is isometric. Consequently, $\mathbf{y}_B$ is non-expansive. Because, for all Cauchy sequences $(x_n)_n$ in X,

$$\lim_n \mathbf{y}_B(x_n)$$
$$= \lim_n \lambda b \in B \,.\, X(b, x_n)$$
$$= \lambda b \in B \,.\, \lim_n X(b, x_n) \quad [\text{Proposition 3.5}]$$
$$= \lambda b \in B \,.\, X(b, \lim_n x_n \quad [b \text{ is finite in } X]$$
$$= \mathbf{y}_B(\lim_n x_n),$$

$\mathbf{y}_B$ is continuous. Consider the following diagram:

where j is the inclusion of $\bar{B}$ in $\widehat{B}$. One can easily verify that $\mathbf{y}_B \circ i^\# \circ \mathbf{y} = \mathbf{y}$ and $j \circ \mathbf{y} = \mathbf{y}$. Therefore by Theorem 5.5,

$$\mathbf{y}_B \circ i^\# = j. \tag{9}$$

Since B is a basis for X, $i^\#$ is surjective. Because $i^\#$ is furthermore non-expansive and j is isometric, $\mathbf{y}_B$ is isometric.

2. $\Rightarrow$ 3. For all Cauchy sequences $(b_n)_n$ in B,

$$(\mathbf{y}_B \circ i^\#)(\lim_n \mathbf{y}(b_n))$$
$$= \mathbf{y}_B(\lim_n i^\# \circ \mathbf{y}(b_n)) \quad [i^\# \text{ is continuous}]$$
$$= \mathbf{y}_B(\lim_n i(b_n))$$
$$= \lim_n \mathbf{y}_B \circ i(b_n) \quad [\mathbf{y}_B \text{ is continuous}]$$
$$= \lim_n \mathbf{y}(b_n),$$

from which (9) follows. Thus $\mathbf{y}_B$ actually maps into $\bar{B}$. Because $\mathbf{y}_B$ is isometric it is injective. As a consequence, $i^\# \circ \mathbf{y}_B = 1_X$ follows from

$$\mathbf{y}_B \circ (i^\# \circ \mathbf{y}_B) = (\mathbf{y}_B \circ i^\#) \circ \mathbf{y}_B = \mathbf{y}_B = \mathbf{y}_B \circ 1_X,$$

where 1_X is the identity on X. Thus $i^\#$ is an isomorphism with $\mathbf{y}_B$ as inverse.

3. $\Rightarrow$ 1. As we have already seen in the proof of Theorem 5.4, all elements of $\mathbf{y}(B)$ are finite in $\bar{B}$. Since $i^\#$ is isometric and surjective, all elements in $(i^\# \circ \mathbf{y})(B)$ are finite in X. Because $i = i^\# \circ \mathbf{y}$, all elements of B are finite in X. Since $i^\#$ is surjective, every element of X is the limit of a Cauchy sequence in B. Hence, B is a basis for X. $\qquad \square$

A subset B of a gms X for which the function $\mathbf{y}_B$ of the second clause above is isometric, is called *adequate* in [Law73, page 154].

This section is concluded by the introduction of the notion of adjoint pairs of mappings between gms's, and a characterization of completeness in terms thereof. This will not be used in the rest of the paper.

Let X and Y be gms's. A pair of non-expansive mappings $f : X \to Y$ and $g : Y \to X$ form an *adjunction*, with f left adjoint to g denoted by $f \dashv g$, if

$$\forall x \in X \forall y \in Y, Y(f(x), y) = X(x, g(y)).$$

An equivalent condition is that $X^X(1_X, g \circ f) = 0$ and $Y^Y(f \circ g, 1_Y) = 0$. Expressed in terms of the underlying orderings, this can be read as $1_X \leq g \circ f$ and $f \circ g \leq 1_Y$, saying that f and g form an adjunction as monotone maps between the underlying preorders $\langle X, \leq_X \rangle$ and $\langle Y, \leq_Y \rangle$.

The following lemma was suggested to us by Bart Jacobs.

Lemma 5.7 *Let X be a qms. Consider the (corestriction of the) Yoneda embedding $\mathbf{y} : X \to \bar{X}$. The space X is complete if and only if there exists a non-expansive and continuous mapping $f : \bar{X} \to X$ with $f \dashv \mathbf{y}$.*

Proof: Suppose X is complete. By Theorem 5.5, there exists a unique non-expansive and continuous extension $1_X^\# : \bar{X} \to X$ of the identity mapping on X, defined, for $\phi = \lim_n \mathbf{y}(x_n)$ in $\bar{X}$ with $(x_n)_n$ a Cauchy sequence in X, by

$$1_X^\#(\phi) = \lim_n x_n.$$

For any $x \in X$,

$$
\begin{aligned}
X(1_X^\#(\phi), x) &= X(\lim_n x_n, x) \\
&= \varprojlim_n X(x_n, x) \\
&= \varprojlim_n \bar{X}(\mathbf{y}(x_n), \mathbf{y}(x)) \quad \text{[the Yoneda embedding is isometric]} \\
&= \bar{X}(\lim_n \mathbf{y}(x_n), \mathbf{y}(x)) \\
&= \bar{X}(\phi, \mathbf{y}(x)),
\end{aligned}
$$

showing that $1_X^\# \dashv \mathbf{y}$. For the converse suppose we are given a non-expansive and continuous mapping $f : \bar{X} \to X$ with $f \dashv \mathbf{y}$. For any Cauchy sequence $(x_n)_n$ in X and $x \in X$,

$$
\begin{aligned}
X(f(\lim_n \mathbf{y}(x_n)), x) &= \bar{X}(\lim_n \mathbf{y}(x_n), \mathbf{y}(x)) \\
&= \varprojlim_n \bar{X}(\mathbf{y}(x_n), \mathbf{y}(x)) \\
&= \varprojlim_n X(x_n, x) \quad \text{[the Yoneda embedding is isometric]},
\end{aligned}
$$

proving that $\lim_n x_n = f(\lim_n \mathbf{y}(x_n))$. $\qquad\square$

6 Topology via Yoneda

Let X be a gms. Recall that $\hat{X}$ is a gms with the supremum distance, and that it contains as a subset an isometric copy of X via the Yoneda embedding. The Yoneda embedding of a gms X into $\hat{X}$ gives rise to two topological closure operators. Their corresponding topologies are shown to generalize both the ϵ-ball topology of ordinary metric spaces and the Alexandroff and Scott topologies of preordered spaces.

The main idea (stemming from [Law86]) is to interpret an element ϕ of $\hat{X}$ as a 'fuzzy' predicate (or 'fuzzy' subset) on X: the value that ϕ assigns to an element x in X is thought of as a measure for 'the extent to which x is an element of ϕ'. The smaller this number is, the more x should be

viewed as an element of the fuzzy subset ϕ. In fact, the only real elements are the ones where ϕ is 0. By taking only its real elements we obtain its *extension*,

$$\int_A \phi = \{x \in X \mid \phi(x) = 0\},$$

where the subscript A stands for Alexandroff. For instance, for x in X, $\int_A \mathbf{y}(x) = \int_A X(-, x) = \{z \in X \mid X(z, x) = 0\} = x{\downarrow}$. More generally, for any ϕ in $\hat{X}$,

$$
\begin{aligned}
\int_A \phi \;&=\; \{x \in X \mid \phi(x) = 0\} \\
&=\; \{x \in X \mid \hat{X}(X(-, x),\ \phi) = 0\} \quad \text{[the Yoneda Lemma 4.1]} \\
&=\; \{x \in X \mid \hat{X}(\mathbf{y}(x),\ \phi) = 0\} \quad \text{[definition of the Yoneda embedding]} \\
&=\; \{x \in X \mid \mathbf{y}(x) \leq_{\hat{X}} \phi\}.
\end{aligned}
$$

Any subset $V \subseteq X$ defines, conversely, a predicate $\rho_A(V) : X^{op} \to [0, \infty]$ which is referred to as the *character* of the subset V. It is defined, for $x \in X$, by

$$\rho_A(V)(x) = \inf\{X(x, v) \mid v \in V\},$$

i.e., the distance from x to the set V. Note that, by definition of the Yoneda embedding, this is equivalent to

$$\rho_A(V) = \lambda x \in X. \inf\{\mathbf{y}(v)(x) \mid v \in V\}.$$

The mappings $\int_A : \hat{X} \to \mathcal{P}(X)$ and $\rho_A : \mathcal{P}(X) \to \hat{X}$ can be nicely related by considering $\hat{X}$ with the underlying preorder $\leq_{\hat{X}}$, and $\mathcal{P}(X)$ ordered by subset inclusion (cf. [Law86]):

Proposition 6.1 *Let X be a gms. The maps $\int_A : \langle \hat{X}, \leq_{\hat{X}} \rangle \to \langle \mathcal{P}(X), \subseteq \rangle$ and $\rho_A : \langle \mathcal{P}(X), \subseteq \rangle \to \langle \hat{X}, \leq_{\hat{X}} \rangle$ are monotone. Moreover ρ_A is left adjoint to $\int_A$.*

Proof: Monotonicity of $\int_A$ and ρ_A follows directly from their definitions. We will hence concentrate on the second part of the proposition by proving for all $V \in \mathcal{P}(X)$ and $\phi \in \hat{X}$,

$$V \subseteq \int_A \rho_A(V) \quad \text{and} \quad \rho_A(\textstyle\int_A \phi) \leq_{\hat{X}} \phi,$$

which is equivalent to ρ_A being left adjoint to $\int_A$, (cf. Theorem 0.3.6 of [GHK$^+$80]). Because, for all $V \in \mathcal{P}(X)$ and v in V, $\mathbf{y}(v) \leq_{\hat{X}} \rho_A(V)$, we have that

$$V \subseteq \{x \in X \mid \mathbf{y}(x) \leq_{\hat{X}} \rho_A(V)\} = \textstyle\int_A \rho_A(V).$$

Furthermore, for $\phi \in \hat{X}$ and $x \in X$,

$$
\begin{aligned}
\rho_A(\textstyle\int_A \phi)(x) \;&=\; \inf\{X(x, y) \mid y \in X\ \&\ \mathbf{y}(y) \leq_{\hat{X}} \phi\} \\
&=\; \inf\{\mathbf{y}(y)(x) \mid y \in X\ \&\ \forall z \in X,\ \mathbf{y}(y)(z) \geq \phi(z)\} \\
&\geq\; \inf\{\mathbf{y}(y)(x) \mid y \in X\ \&\ \mathbf{y}(y)(x) \geq \phi(x)\} \\
&\geq\; \phi(x).
\end{aligned}
$$

Consequently, $\rho_A(\int_A \phi) \leq_{\hat{X}} \phi$ (note that the ordering underlying $[0, \infty]$ is the reverse of the usual one). $\qquad\square$

The above fundamental adjunction relates character of subsets and extension of predicates and is often referred to as the *comprehension schema* (cf. [Law73, Ken88]). As with any adjoint

pair between preorders, the composition $\int_A \circ \rho_A$ is a closure operator on X (cf. Theorem 0.3.6 of [GHK$^+$80]). It satisfies, for $V \subseteq X$,

$$
\begin{aligned}
(\textstyle\int_A \circ \rho_A)(V) \quad &= \quad \{x \in X \mid \rho_A(V)(x) = 0\} \\
&= \quad \{x \in X \mid \hat{X}(\mathbf{y}(x), \rho_A(V)) = 0\} \quad [\text{the Yoneda Lemma 4.1}] \\
&= \quad \{x \in X \mid \forall y \in X, \ [0, \infty](\mathbf{y}(x)(y), \rho_A(V)(y)) = 0\} \\
&= \quad \{x \in X \mid \forall y \in X, \ \mathbf{y}(x)(y) \geq \rho_A(V)(z)\} \\
&= \quad \{x \in X \mid \forall \epsilon > 0 \ \forall y \in X, \ \mathbf{y}(x)(y) < \epsilon \Rightarrow (\exists v \in V, \ X(y, v) < \epsilon)\} \\
&= \quad \{x \in X \mid \forall \epsilon > 0 \ \forall y \in X, \ X(y, x) < \epsilon \Rightarrow (\exists v \in V, \ X(y, v) < \epsilon)\} \quad (10) \\
&\quad\quad [\text{the Yoneda Lemma 4.1}].
\end{aligned}
$$

By using the above characterization (10) we can prove the following lemma.

Lemma 6.2 *For a gms X, the closure operator $\int_A \circ \rho_A$ on X is topological.*

Proof: It is an immediate consequence of (10) that $\int_A \circ \rho_A(\emptyset) = \emptyset$. Moreover, for $V, W \subseteq X$,

$$
\textstyle\int_A \circ \rho_A(V \cup W) \quad \supseteq \quad \int_A \circ \rho_A(V) \cup \int_A \circ \rho_A(W),
$$

because $\int_A \circ \rho_A$ is a closure operator. For the reverse inclusion, let $x \in \int_A \circ \rho_A(V \cup W)$. Suppose $x \notin \int_A \circ \rho_A(V)$. We will show $x \in \int_A \circ \rho_A(W)$. Let y_W in X and $\epsilon_W > 0$ with $X(y_W, x) < \epsilon_W$. We should find a w in W with $X(y_W, w) < \epsilon_W$. Because x not in $\int_A \circ \rho_A(V)$ there exist a y_V in X and an $\epsilon_V > 0$ such that

$$
X(y_V, x) < \epsilon_V \quad \& \quad (\forall v \in V, \ X(y_V, v) \geq \epsilon_V). \tag{11}
$$

Let $\epsilon = \min\{\epsilon_V - X(y_V, x), \epsilon_W - X(y_W, x)\}$. Because x in $\int_A \circ \rho_A((V \cup W)$ and $X(x, x) < \epsilon$, there exists a w in $V \cup W$ with $X(x, w) < \epsilon$. The assumption that w in V contradicts (11) because

$$
X(y_V, w) \leq X(y_V, x) + X(x, w) < \epsilon_V.
$$

Thus $y \in W$. Furthermore,

$$
X(y_W, w) \leq X(y_W, x) + X(x, w) < \epsilon_W.
$$

$\square$

The above lemma implies that the closure operator $\int_A \circ \rho_A$ induces a topology on X, which in Proposition 6.3 below is proved equivalent to the following generalized ϵ-ball topology: For $x \in X$ and $\epsilon > 0$ define the *ϵ-ball centered in x* by

$$
B_\epsilon(x) = \{z \in X \mid X(x, z) < \epsilon\}.
$$

A subset $o \subseteq X$ of a gms X is *generalized Alexandroff* open (gA-open, for short) if, for all $x \in X$,

$$
x \in o \quad \Rightarrow \exists \epsilon > 0, \ B_\epsilon(x) \subseteq o.
$$

The set of all gA-open subsets of X is denoted by $\mathcal{O}_{gA}(X)$. For instance, for every $x \in X$ the ϵ-ball $B_\epsilon(x)$ is a gA-open set. The pair $\langle X, \mathcal{O}_{gA}(X) \rangle$ can be shown to be topological space with $B_\epsilon(x)$, for every $\epsilon > 0$ and $x \in X$, as basic open sets (cf. [FK95]). For a subset V of X we write $cl_A(V)$ for the closure of V in the generalized Alexandroff topology.

Proposition 6.3 *For every subset V of a gms X, $cl_A(V) = \int_A \circ \rho_A(V)$.*

Proof: It follows from the characterization (10) of $\int_A \circ \rho_A$ that it is sufficient to prove

$$cl_A(V) = \{x \in X \mid \forall \epsilon > 0 \,\forall z \in X, \ X(z,x) < \epsilon \ \Rightarrow \ (\exists v \in V, \ X(z,v) < \epsilon)\}.$$

Because $cl_A(V) = V \cup V^d$, where V^d is the so-called derived set of V (cf. Section A of the appendix), it follows from the definition of derived set and the fact that the set of all ϵ-balls is a basis for the generalized Alexandroff topology, that for every $x \in X$,

$$
\begin{aligned}
x \in V^d \quad &\Longleftrightarrow \quad \forall o \in \mathcal{O}_{gA}(X), \ x \in o \ \Rightarrow \ o \cap (V \setminus \{x\}) \neq \emptyset \\
&\Longleftrightarrow \quad \forall \epsilon > 0 \,\forall z \in X, \ x \in B_\epsilon(z) \ \Rightarrow \ B_\epsilon(z) \cap (V \setminus \{x\}) \neq \emptyset \\
&\Longleftrightarrow \quad \forall \epsilon > 0 \,\forall z \in X, \ X(z,x) < \epsilon \ \Rightarrow \ \exists v \in (V \setminus \{x\}), \ X(z,v) < \epsilon.
\end{aligned}
$$

Therefore,

$$cl_A(V) = V \cup V^d = \{x \in X \mid \forall \epsilon > 0 \,\forall z \in X, \ X(z,x) < \epsilon \ \Rightarrow \ (\exists v \in V, \ X(z,v) < \epsilon)\}. \qquad \square$$

For ordinary metric spaces, gA-open sets are just the usual open sets. For preorders, a set is gA-open precisely when it is Alexandroff open (upper closed) because if X is a preorder then for every $\epsilon > 0$,

$$
\begin{aligned}
B_\epsilon(x) \ &= \ \{y \in X \mid X(x,y) < \epsilon\} \\
&= \ \{y \in X \mid X(x,y) = 0\} \\
&= \ \{y \in X \mid x \leq_X y\}
\end{aligned}
$$

The specialization preorder on a gms X induced by its generalized Alexandroff topology coincides with the preorder underlying X.

Proposition 6.4 *Let X be a gms. For all x and y in X, $x \leq_{\mathcal{O}_{gA}} y$ if and only if $x \leq_X y$.*

Proof: For any gA-open set V, if x in V and $X(x,y) = 0$ then y in V. From this observation the implication from right to left is clear. For the converse, suppose $x \leq_{\mathcal{O}_{gA}} y$. Then, for every $\epsilon > 0$, $x \in B_\epsilon(x)$ implies $y \in B_\epsilon(x)$, because generalized ϵ-balls are gA-open sets. Hence $X(x,y) < \epsilon$. Since ϵ was arbitrary, $X(x,y) = 0$, that is $x \leq_X y$. $\qquad \square$

The above proposition tells us that the underlying preorder of a gms can be reconstructed from its generalized Alexandroff topology. Note that the specialization preorder $\leq_{\mathcal{O}_{gA}}$ is a partial order—this is equivalent to the generalized Alexandroff topology being T_0—if and only if X is a qms.

For computational reasons we are interested in complete spaces, in which one can model infinite behaviors by means of limits. A topology for a complete space X can then be considered satisfactory if limits in X are topological limits. This is not the case for the generalized Alexandroff topology: for instance, for complete partial orders $\mathcal{O}_{gA}(X)$ coincides with the standard Alexandroff topology, for which the coincidence of the least upperbounds of chains and their topological limits does not hold. Therefore the *Scott* topology is usually considered to be preferable: it is the coarsest topology refining the Alexandroff topology, in which least upper bounds of chains are topological limits (cf. Section II-1 of [GHK+80]. See also [Mel89, Smy92]). Also for gms's, a suitable refinement of the generalized Alexandroff topology exists.

This topology will be introduced, first, by defining which sets are open, and next—for algebraic gms's—by means of the Yoneda embedding.

A subset $o \subseteq X$ of a gms X is *generalized Scott* open (gS-open, for short) if for all Cauchy sequences $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$,

$$x \in o \Rightarrow \exists N \,\exists \epsilon > 0 \,\forall n \geq N, \ B_\epsilon(x_n) \subseteq o.$$

The set of all gS-open subsets of X is denoted by $\mathcal{O}_{gS}(X)$. Below it will be shown that this defines a topology indeed. Note that every gS-open set $o \subseteq X$ is gA-open because every point $x \in X$ is the limit of the constant Cauchy sequence $(x)_n$ in X. Therefore this topology refines the generalized Alexandroff topology. Furthermore it will be shown to

1. coincide with the ϵ-ball topology in case X is a metric space; and to

2. coincide with the Scott topology in case X is a complete partial order.

The following proposition gives an example of gS-open sets:

Proposition 6.5 *For every gms X, an element $b \in X$ is finite if and only if for every $\epsilon > 0$, the set $B_\epsilon(b)$ is gS-open.*

Proof: Let b be finite in X and $\epsilon > 0$. We have to show that the generalized ϵ-ball $B_\epsilon(b)$ is gS-open. Let $(x_n)_n$ be a Cauchy sequence in X and assume $x \in B_\epsilon(b)$ with $x = \lim x_n$. It suffices to prove that

$$\exists \delta > 0 \; \exists N \; \forall n \geq N, \quad X(b, x_n) < \epsilon - \delta. \tag{12}$$

Because x in $B_\epsilon(b)$, we have that there exists $\delta > 0$ such that $X(b, x) < \epsilon - \delta$. Since

$$\begin{aligned}
\epsilon - \delta \quad &> \quad X(b, x) \\
&= \quad \lim X(b, x_n) \quad [b \text{ is finite in } X]
\end{aligned}$$

and the sequence $(X(b, x_n))_n$ is Cauchy, we can conclude (12).

Conversely, assume that, for all $\epsilon > 0$, the set $B_\epsilon(b)$ is gS-open. We need to prove, for every Cauchy sequence $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$, that

$$\lim X(b, x_n) \leq X(b, x) \tag{13}$$

(the reverse inequality is given by Proposition 3.4). We have

$$\forall \epsilon > 0, \quad x \in B_{X(b,x)+\epsilon}(b).$$

Because the set $B_{X(b,x)+\epsilon}(b)$ is gS-open,

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \exists N \; \forall n \geq N, \quad B_\delta(x_n) \subseteq B_{X(b,x)+\epsilon}(b).$$

Hence, $\lim X(b, x_n) \leq X(b, x)$. $\qquad\qquad\square$

Next we prove that the collection of all gS-open sets forms indeed a topology.

Proposition 6.6 *For every gms X the pair $\langle X, \mathcal{O}_{gS}(X) \rangle$ is a topological space. If X is also algebraic with basis B, then the set $\{B_\epsilon(b) \mid b \in B \ \& \ \epsilon > 0\}$ forms a basis for the generalized Scott topology $\mathcal{O}_{gS}(X)$.*

Proof: We first prove that $\mathcal{O}_{gS}(X)$ is closed under finite intersections and arbitrary unions. Let I be a finite index set (possibly empty) and let $o = \bigcap_I o_i$ with $o_i \in \mathcal{O}_{gS}(X)$ for all $i \in I$. If $x \in o$ for a Cauchy sequence $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$, then for every $i \in I$ there exist $N_i \geq 0$ and $\epsilon_i > 0$ such that $B_{\epsilon_i}(x_n) \subseteq o_i$ for all $n \geq N_i$. Take $N = \max_I N_i$ and $\epsilon = \min_I \epsilon_i$ (here $\max_\emptyset = 0$ and $\min_\emptyset = \infty$). Then $B_\epsilon(x_n) \subseteq o$ for all $n \geq N$, that is, o is gS-open.

Next let I be an arbitrary index set and let $o = \bigcup_I o_i$ with $o_i \in \mathcal{O}_{gS}(X)$ for all $i \in I$. If $x \in o$ for a Cauchy sequence $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$, then there exists $i \in I$ such that $x \in o_i$. Therefore there exists $N \geq 0$ and $\epsilon > 0$ such that $B_\epsilon(x_n) \subseteq o_i \subseteq o$ for all $n \geq N$, that is, o is gS-open.

Finally assume that X is an algebraic gms with basis B. We have already seen that for every $\epsilon > 0$ and finite element $b \in B$ the set $B_\epsilon(b)$ is gS-open. We claim that every gS-open set $o \subseteq X$ is the union of ϵ-balls of finite elements. Let $x \in o$. Since X is algebraic there is a Cauchy sequence $(b_n)_n$ in B with $x = \lim b_n$. Because o is gS-open, there exists $\epsilon_x > 0$ and $N_x \geq 0$ such that $B_{\epsilon_x}(b_n) \subseteq o$ for all $n \geq N_x$ and with $x \in B_{\epsilon_x}(b_n)$ for N_x big enough. Therefore $o \subseteq \bigcup_{x \in o} B_{\epsilon_x}(b_{N_x})$. Since the other inclusion trivially holds we have that the collection of all ϵ-balls of finite elements forms a basis for the generalized Scott topology. $\qquad\square$

Any ordinary metric space X is an algebraic gms where all elements are finite. Therefore, by the previous proposition, the basic open sets of the generalized Scott topology are all the ϵ-balls $B_\epsilon(x)$, with $x \in X$. Hence for ordinary metric spaces the generalized Scott topology coincides with the standard ϵ-ball topology.

For a complete partial order X, a set $o \subseteq X$ is gS-open precisely when it is Scott open: if $o \in \mathcal{O}_{gS}(X)$ then it is upper closed because the gS-topology refines the gA-topology. Moreover, if $\bigsqcup x_n \in o$ for an ω-chain $(x_n)_n$ in X then—because o is gS-open—there exists $\epsilon > 0$ and $N \geq 0$ such that $B_\epsilon(x_n) \subseteq o$ for all $n \geq N$. But $x_n \in B_\epsilon(x_n)$ for all ϵ, therefore o is an ordinary Scott open set. Conversely, assume o is Scott open and let $x \in o$, for a Cauchy sequence $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$. Because o is Scott open (and limits are least upper bounds) there exists $N \geq 0$ such that $x_n \in o$ for all $n \geq N$. This is enough to prove that o is also gS-open because for every $x \in X$ and $\epsilon > 0$, $B_\epsilon(x) = x\!\uparrow$.

As usual, a subset c of a gms X is *gS-closed* if its complement $X \backslash c$ is gS-open. This is equivalent to the following condition: for all Cauchy sequences $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$,

$$(\forall N \ \forall \epsilon > 0 \ \exists n \geq N \ \exists y \in c, \ \ X(x_n, y) < \epsilon) \ \Rightarrow \ x \in c. \tag{14}$$

For a subset V of X we write $cl_S(V)$ for the closure of V in the generalized Scott topology, that is, $cl_S(V)$ is the smallest generalized Scott closed set containing V. From the definition of limits we have that for any Cauchy sequence $(x_n)_n$ in V and $x \in X$ with $x = \lim x_n$, $x \in cl_S(V)$. The latter implies that if X is an algebraic gms with basis B then B is *dense* in X, that is $cl_S(B) = X$. Indeed, $B \subseteq X$ implies $cl_S(B) \subseteq cl_S(X) = X$. For the converse we use the fact that every element of X is the limit of a Cauchy sequence in B. Since (the image under $\mathbf{y}$ of) every gms X is a basis for its completion $\bar{X}$ it follows that every gms is dense in its completion.

The following lemma, suggested to us by Flagg and Sünderhauf, gives an example of gS-closed sets.

Lemma 6.7 *Let X be a gms. For all x in X and $\delta \geq 0$, the set $\bar{B}_\delta^{op}(x) = \{y \in X \mid X(y, x) \leq \delta\}$ is gS-closed.*

Proof Let $(z_n)_n$ be a Cauchy sequence in X and let $z \in X$, with $z = \lim z_n$, be such that

$$\forall N \ \forall \epsilon > 0 \ \exists n \geq N \ \exists y \in \bar{B}_\delta^{op}(x), \ \ X(z_n, y) < \epsilon.$$

Then

$$\forall N \ \forall \epsilon > 0 \ \exists n \geq N, \ \ X(z_n, x) < \epsilon + \delta.$$

Because the sequence $(z_n)_n$ is Cauchy,

$$\forall \epsilon > 0 \ \exists N \ \forall n \geq N, \ \ X(z_n, x) < \epsilon + \delta.$$

Consequently, $\varprojlim X(z_n, x) \leq \delta$, and hence $X(z, x) \leq \delta$. $\qquad\qquad\square$

Like for the generalized Alexandroff topology, the specialization preorder on a gms X induced by its gS-topology coincides with the preorder underlying X.

Proposition 6.8 *Let X be a gms. For all x and y in X, $x \leq_{\mathcal{O}_{gS}} y$ if and only if $x \leq_X y$.*

Proof: For any gS-open set o, if $x \in o$ and $X(x, y) = 0$, then also $y \in o$. From this observation, the implication from right to left is clear. For the converse, suppose $X(x, y) \neq 0$. Then $x \notin \bar{B}_0^{op}(y)$ but $y \in \bar{B}_0^{op}(y)$. Since, by Lemma 6.7, the set $X \setminus \bar{B}_0^{op}(y)$ is gS-open it follows that $x \not\leq_{\mathcal{O}_{gS}} y$. $\quad\square$

As promised above, next we show that the generalized Scott topology also encodes all information about convergence.

Proposition 6.9 *Let X be a gms, $(x_n)_n$ a Cauchy sequence in X, and let $x \in X$ be such that $x = \lim x_n$. For all $y \in X$, $\mathcal{N}((x_n)_n) \to y$ if and only if $y \leq_{\mathcal{O}_{gS}} x$, that is, limits are maximal topological limits.*

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Proof: By definition of gS-open sets, $\mathcal{N}((x_n)_n) \to x$. Hence $y \leq_{\mathcal{O}_{gS}} x$ implies $\mathcal{N}((x_n)_n) \to y$. For the converse, let $\mathcal{N}((x_n)_n) \to y$ and assume $y \not\leq_{\mathcal{O}_{gS}} x$. According to Proposition 6.8 there is a $\delta > 0$ such that $X(y,x) \not\leq \delta$. Hence, $y \in X \setminus \bar{B}_\delta^{op}(x)$, which is a gS-open set by Lemma 6.5. Since $\mathcal{N}((x_n)_n) \to y$,

$$\exists N \ \forall n \geq N, \ \ x_n \in X \setminus \bar{B}_\delta^{op}(x).$$

But

$$0 \ \ = X(x,x) = \ \ \varprojlim X(x_n, x)$$

so there exists M such that for all $m \geq M$, $X(x_m, x) \leq \delta$. This gives a contradiction. Therefore, $y \leq_{\mathcal{O}_{gS}} x$. $\square$

From the above proposition we can conclude that in a complete gms every Cauchy sequence topologically converges to its metric limits. However, not every topologically convergent sequence is Cauchy. For example, provide the set $X = \{1, 2, \ldots, \omega\}$ with the distance function

$$X(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x = \omega \text{ and } y = n \\ 1 & \text{otherwise} \end{cases}$$

Then X is an algebraic complete qms with X itself as basis, since there are no non-trivial Cauchy sequences. The sequence $(n)_n$ topologically converges to ω but is not Cauchy.

For an algebraic gms X with basis B, topological convergence (with respect to the gS-topology on X) is easily characterized: a sequence $(x_n)_n$ in X converges to $x \in X$ if and only if

$$\forall \epsilon > 0 \ \forall b \in B, \ \ X(b, x) < \epsilon \ \Rightarrow \ (\exists N \ \forall n \geq N, \ X(b, x_n) < \epsilon).$$

Continuity is also encoded by the generalized Scott topology.

Proposition 6.10 *Let X and Y be two complete gms's. A non-expansive function $f : X \to Y$ is metrically continuous if and only if it is topologically continuous.*

Proof: Let $f : X \to Y$ be a non-expansive and metrically continuous function and let $o \subseteq Y$ be gS-open. We need to prove $f^{-1}(o) \subseteq X$ in order to conclude that f is topologically continuous. Indeed, for any Cauchy sequence $(x_n)_n$ in X and $x \in X$ with $x = \lim x_n$ we have

$$
\begin{aligned}
x \in f^{-1}(o) \quad &\Longleftrightarrow \quad f(x) \in o \\
&\Longleftrightarrow \quad \lim f(x_n) \in o \quad [f \text{ is metrically continuous}] \\
&\Rightarrow \quad \exists N \ \exists \epsilon > 0 \ \forall n \geq N, \ B_\epsilon(f(x_n)) \subseteq o \\
&\qquad [f \text{ is non-expansive, } (f(x_n))_n \text{ is a Cauchy sequence, } o \text{ is gS-open}] \\
&\Rightarrow \quad \exists N \ \exists \epsilon > 0 \ \forall n \geq N, \ B_\epsilon(x_n) \subseteq f^{-1}(o) \quad [f \text{ is non-expansive}].
\end{aligned}
$$

For the converse assume $f : X \to Y$ to be non-expansive and topologically continuous. Let $(x_n)_n$ be a Cauchy sequence in X and $x \in X$ with $x = \lim x_n$. Since f is non-expansive, $(f(x_n))_n$ is a Cauchy sequence in Y. Let $y = \lim f(x_n)$. According the definition of metric limit, it suffices to prove, that $Y(y, f(x)) = 0$ and $Y(f(x), y) = 0$. We have that

$$
\begin{aligned}
Y(y, f(x)) \ &= \ \varprojlim Y(f(x_n), f(x)) \\
&\leq \ \varprojlim X(x_n, x) \quad [f \text{ is non-expansive}] \\
&= \ X(x,x) \quad [x = \lim x_n] \\
&= \ 0.
\end{aligned}
$$

Since f is topologically continuous and, by Proposition 6.9, $(x_n)_n$ converges to x, also $(f(x_n))_n$ converges to $f(x)$. By Proposition 6.9 again, $f(x) \leq_{\mathcal{O}_{gS}} y$. Therefore, by Proposition 6.8, $Y(f(x), y) = 0$. $\qquad\square$

This section is concluded with a characterization of the generalized Scott topology for algebraic complete metric spaces in term of the Yoneda embedding. A key step towards the definition of a topological closure operator for the generalized Scott topology is to compare the fuzzy subsets of a basis B of an algebraic complete gms X, rather than the fuzzy subsets of X as we have done for the generalized Alexandroff topology, with the ordinary subsets of X. To this end, the previously defined extension and character functions are extended as follows:

$$\int : \hat{B} \to \mathcal{P}(X) \quad \text{and} \quad \rho : \mathcal{P}(X) \to \hat{B}, \tag{15}$$
$$\phi \mapsto \{x \in X \mid \mathbf{y}_B(x) \leq_{\hat{B}} \phi\} \qquad V \mapsto \lambda b \in B. \inf\{\mathbf{y}_B(v)(b) \mid v \in V\},$$

where $\mathbf{y}_B : X \to \hat{B}$ is the restriction of the Yoneda embedding as defined in Theorem 5.6.

Similar to Proposition 6.1 we have that the mappings $\int : \langle \hat{B}, \leq_{\hat{B}} \rangle \to \langle \mathcal{P}(X), \subseteq \rangle$ and $\rho : \langle \mathcal{P}(X), \subseteq \rangle \to \langle \hat{B}, \leq_{\hat{X}} \rangle$ are monotone, and ρ is left adjoint to $\int$. Thus, $\int \circ \rho$ is a closure operator on X. Since a basis is generally not unique, one might think that its definition depends on the choice of the basis. In Theorem 6.12 below we will demonstrate that this is not the case.

In a way similar to (10), the closure operator $\int \circ \rho$ can be characterized, for an algebraic complete gms X with basis B and $V \subseteq X$, by

$$(\textstyle\int \circ \rho)(V) \quad = \quad \{x \in X \mid \forall b \in B \; \forall \epsilon > 0, \; X(b, x) < \epsilon \Rightarrow (\exists v \in V, \; X(b, v) < \epsilon)\}. \tag{16}$$

An alternative characterization of $\int \circ \rho$, which will be useful in the next section, is the following. For an algebraic complete gms X with basis B and $V \subseteq X$,

$$
\begin{aligned}
(\textstyle\int \circ \rho)(V) \quad &= \quad \{x \in X \mid \mathbf{y}_B(x) \leq_{\hat{B}} \rho(V)\} \\
&= \quad \{x \in X \mid \hat{B}(\mathbf{y}_B(x), \rho(V)) = 0\} \\
&= \quad \{x \in X \mid \forall b \in B, \; [0, \infty](\mathbf{y}_B(x)(b), \rho(V)(b)) = 0\} \\
&= \quad \{x \in X \mid \forall b \in B, \; \inf_{v \in V} \mathbf{y}_B(v)(b) \leq \mathbf{y}_B(x)(b)\} \\
&= \quad \{x \in X \mid \forall b \in B, \; \inf_{v \in V} X(b, v) \leq X(b, x)\} \\
&= \quad \{x \in X \mid \forall b \in B \; \forall \epsilon > 0 \; \exists v \in V, \; X(b, v) \leq X(b, x) + \epsilon\}. \tag{17}
\end{aligned}
$$

The closure operator $\int \circ \rho$ is topological.

Lemma 6.11 *Let X be an algebraic complete gms. The closure operator $\int \circ \rho$ on X is topological.*

Proof: This lemma is proved using the characterization (16) along the same lines as Lemma 6.2, but one needs the following additional observation. If B is a basis for X then, for any b_V and b_W in B, $\epsilon_V, \epsilon_W > 0$, and $x \in X$, such that $X(b_V, x) < \epsilon_V$ and $X(b_W, x) < \epsilon_W$, there exists a b in B such that $X(b_V, b) < \epsilon_V$, $X(b_W, b) < \epsilon_W$, and $X(b, x) < \epsilon$, where $\epsilon = \min\{\epsilon_V - X(b_V, b), \epsilon_W - X(b_W, b)\}$. This fact can be proved as follows. Because X is an algebraic complete gms with B as basis, there exists a Cauchy sequence $(b_n)_n$ in B with $x = \lim b_n$. Because

$$
\begin{aligned}
\epsilon_V \quad &> \quad X(b_V, x) \\
&= \quad \lim X(b_V, b_n) \quad [x = \lim b_n, \; b_V \text{ is finite in } X]
\end{aligned}
$$

there exists an N_V such that, for all $n \geq N_V$, $X(b_V, b_n) < \epsilon_V$. Similarly, there exists an N_W such that, for all $n \geq N_W$, $X(b_W, b_n) < \epsilon_W$. Since

$$
\begin{aligned}
0 \quad &= \quad X(x, x) \\
&= \quad \lim_{\leftarrow} X(b_n, x) \quad [x = \lim b_n]
\end{aligned}
$$

there exists an N such that, for all $n \geq N$, $X(b_n, x) < \epsilon$. The element $b_{\max\{N_V, N_W, N\}}$ in B is the one we were looking for. $\qquad\square$

Thus, the closure operator $\int \circ \rho$ induces a topology on algebraic complete gms's. According to (17), a subset V of an algebraic complete gms X with basis B is closed in this topology if and only if

$$V \;=\; (\textstyle\int \circ \rho)(V) = \;\; \{x \in X \mid \forall b \in B \; \forall \epsilon > 0 \; \exists v \in V, \;\; X(b,v) \leq X(b,x) + \epsilon\}. \qquad (18)$$

In the case that X is an algebraic complete partial order with basis B it follows from characterization (16) that for every $V \subseteq X$,

$$(\textstyle\int \circ \rho)(V) = \{x \in X \mid \forall b \in B, \;\; b \leq_X x \Rightarrow \exists v \in V, \;\; b \leq_X v\},$$

which we recognize as the closure operator induced by the ordinary Scott topology.

Next we show that the topology induced by $\int \circ \rho$ on an algebraic complete gms coincides with the generalized Scott topology. Recall that, for $V \subseteq X$, we write $cl_S(V)$ for the closure of V in the generalized Scott topology.

Theorem 6.12 *Let X be an algebraic complete gms with basis B. For all $V \subseteq X$, $cl_S(V) = (\int \circ \rho)(V)$.*

Proof: This theorem can be proved along the same lines as Theorem 6.3. It follows from characterization (16) of $\int \circ \rho$ and the fact that the generalized ϵ-balls of finite elements form a basis for the generalized Scott topology. $\qquad\square$

Since the definition of the closure operator cl_S does not use the basis, the above theorem implies that the choice of the basis is irrelevant for the definition of the closure operator $\int \circ \rho$.

7 Powerdomains via Yoneda

A generalized lower (or *Hoare*) powerdomain for algebraic complete generalized metric spaces is defined, again by means of the Yoneda embedding. Next this powerdomain is characterized in terms of completion and topology. Also the definition of generalized upper and convex powerdomains will be given. Their characterizations will be discussed elsewhere.

For the rest of this section let X be an algebraic complete gms and let B be a basis for X. Recall (Theorem 5.6) that $\mathbf{y}_B : X \to \widehat{B}$, defined for $x \in X$ by

$$\mathbf{y}_B(x) = \lambda b \in B \,.\, X(b, x),$$

is continuous and isometric. This fact justifies the following

convention: $\mathbf{y}_B(x)$ will often be denoted by x.

We shall define a powerdomain on X as a subspace of $\hat{B}$, using the Yoneda embedding $\mathbf{y}_B$. Let $\amalg : [0, \infty] \times [0, \infty] \to [0, \infty]$ map elements r and s in $[0, \infty]$ to (their coproduct) $\min\{r, s\}$. This makes $\langle [0, \infty], \amalg \rangle$ a *semi-lattice*: for all r, s, and t in $[0, \infty]$,

$$(i)\; r \amalg r = r, \quad (ii)\; r \amalg s = s \amalg r, \quad (iii)\; (r \amalg s) \amalg t = r \amalg (s \amalg t).$$

Furthermore, the following inequality holds for all r and s in $[0, \infty]$:

$$(iv)\; r \leq_{[0,\infty]} r \amalg s.$$

It is immediate that $\langle \hat{B}, \amalg \rangle$ is a semi-lattice as well, with $\amalg$ taken pointwise: for ϕ and ψ in $\hat{B}$ and b in B,

$$(\phi \amalg \psi)(b) = \phi(b) \amalg \psi(b).$$

Recalling the idea that elements in $\hat{B}$ are fuzzy subsets of B, the semi-lattice operation II may be viewed as fuzzy subset union. A *generalized lower powerdomain* on X is now defined as the smallest subset of $\hat{B}$ which contains the image of X under the Yoneda embedding $\mathbf{y}_B$; is metrically complete (i.e., contains limits of Cauchy sequences); and is closed under the operation II. Formally,

$$\mathcal{P}_{gl}(X) = \bigcap \{V \subseteq \hat{B} \mid \mathbf{y}_B(X) \subseteq V, \ V \text{ is a complete subspace of } \hat{B}, \text{ and } V \text{ is closed under } \mathrm{II}\}.$$

This definition is very similar to the definition of completion in Section 5. It will be a consequence of Theorem 7.14 below that this definition is independent of the choice of the basis B.

A generalized Hausdorff distance

The powerdomain $\mathcal{P}_{gl}(X)$ can be described in a number of ways. The main tool will be the adjunction (15) of Section 6:

$$\int : \hat{B} \to \mathcal{P}(X), \ \ \phi \mapsto \{x \in X \mid \mathbf{y}_B(x) \leq_{\hat{B}} \phi\};$$

$$\rho : \mathcal{P}(X) \to \hat{B}, \ \ V \mapsto \lambda b \in B. \inf\{\mathbf{y}_B(v)(b) \mid v \in V\}.$$

Before turning to the characterizations of $\mathcal{P}_{gl}(X)$, let us first show how this adjunction induces a distance on $\mathcal{P}(X)$: for subsets V and W of X, define

$$\mathcal{P}(X)(V, W) = \hat{B}(\rho(V), \rho(W)).$$

Identifying $\mathbf{y}_B(v)$ with v, and observing that the infimum of a set of functions is taken pointwise, the function ρ can also be described as

$$\rho(V) = \inf V,$$

by which the distance $\mathcal{P}(X)(V, W)$ can be written as

$$\mathcal{P}(X)(V, W) = \hat{B}(\inf V, \inf W).$$

It satisfies the following equation.

Theorem 7.1 *For all V and W in $\mathcal{P}(X)$,*

$$\mathcal{P}(X)(V, W) = \inf\{\epsilon > 0 \mid \forall b \in B \ \forall v \in V \ \exists w \in W, \ X(b, w) \leq \epsilon + X(b, v)\}.$$

For ordinary metric spaces, where all elements are finite, the above equality is equivalent with

$$\mathcal{P}(X)(V, W) = \inf\{\epsilon > 0 \mid \forall v \in V \ \exists w \in W, \ X(v, w) \leq \epsilon\}.$$

Therefore the distance above is called the *generalized Hausdorff distance*.

Proof: First note that it follows from Theorem 5.6 that

$$
\begin{aligned}
X(b, x) \ &= \ \hat{B}(\mathbf{y}_B(b), \mathbf{y}_B(x)) \\
&= \ \hat{B}(b, x) \quad [\text{our convention}],
\end{aligned}
$$

for every $b \in B$ and $x \in X$. Thus we have to prove:

$$\mathcal{P}(X)(V, W) = \inf\{\epsilon > 0 \mid \forall b \in B \ \forall v \in V \ \exists w \in W, \ \hat{B}(b, w) \leq \epsilon + \hat{B}(b, v)\}.$$

Let I denote the set on the right of the equality. In order to show that $\mathcal{P}(X)(V, W) \leq \inf I$ consider $\epsilon \in I$. (If $I = \emptyset$ then $\inf I = \infty$, and we are done.) If $V = \emptyset$ then $\mathcal{P}(X)(V, W) = 0 \leq \inf I$. Next let $v = \lim b_n$ be an element of V, with b_n in B, for all n. Because $\epsilon \in I$ there exists for every n an element $w \in W$ such that

$$\hat{B}(b_n, w) \leq \epsilon + \hat{B}(b_n, v).$$

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Therefore

$$\begin{aligned}
\hat{B}(b_n, \inf W) &= (\inf W)(b_n) \quad \text{[Yoneda lemma]}\\
&\leq w(b_n)\\
&= \hat{B}(b_n, w) \quad \text{[Yoneda lemma]}\\
&\leq \epsilon + \hat{B}(b_n, v),
\end{aligned}$$

whence

$$\begin{aligned}
&\hat{B}(v, \inf W)\\
&= \hat{B}(\mathbf{y}_B(v), \inf W) \quad \text{[our convention]}\\
&= \hat{B}(\lim \mathbf{y}_B(b_n), \inf W) \quad \text{[Theorem 5.6]}\\
&= \varprojlim \hat{B}(\mathbf{y}_B(b_n), \inf W)\\
&= \varprojlim \hat{B}(b_n, \inf W) \quad \text{[our convention]}\\
&\leq \varprojlim \epsilon + \hat{B}(b_n, v)\\
&= \epsilon + \varprojlim \hat{B}(b_n, v) \quad \text{[one easily shows that } + \text{ preserves backward-limits]}\\
&= \epsilon + \hat{B}(v, v)\\
&= \epsilon.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{P}(X)(V, W) &= \hat{B}(\inf V, \inf W)\\
&= \sup_{v \in V} \hat{B}(v, \inf W) \quad \text{[see Lemma 7.2 below]}\\
&\leq \epsilon.
\end{aligned}$$

Hence $\mathcal{P}(X)(V, W) \leq \inf I$.

For the reverse let $\delta > 0$ be arbitrary and define

$$\epsilon = \mathcal{P}(X)(V, W) + \delta.$$

We shall show that $\epsilon \in I$, which implies that $\inf I \leq \mathcal{P}(X)(V, W)$. Consider $b \in B$ and $v \in V$. The existence of $w \in W$ such that $\hat{B}(b, w) < \epsilon + \hat{B}(b, v)$ follows from

$$\begin{aligned}
&\inf_{w \in W} \hat{B}(b, w)\\
&= \hat{B}(b, \inf W) \quad \text{[Yoneda lemma]}\\
&\leq \hat{B}(b, v) + \hat{B}(v, \inf W)\\
&\leq \hat{B}(b, v) + \sup_{u \in V} \hat{B}(u, \inf W)\\
&= \hat{B}(b, v) + \hat{B}(\inf V, \inf W) \quad \text{[see Lemma 7.2 below]}\\
&= \hat{B}(b, v) + \mathcal{P}(X)(V, W)\\
&< \hat{B}(b, v) + \epsilon.
\end{aligned}$$

$\square$

The following lemma, used above, is an immediate consequence of Lemma 3.2.

Lemma 7.2 *For any $V \subseteq X$ and $\phi \in \hat{B}$, $\hat{B}(\inf V, \phi) = \sup_{v \in V} \hat{B}(v, \phi)$.*

Proof: For $V \subseteq X$ and $\phi \in \hat{B}$,

$$\hat{B}(\inf V, \phi)$$
$$= \sup_{x \in B} [0, \infty]((\inf V)(x), \phi(x))$$
$$= \sup_{x \in B} [0, \infty](\inf_{v \in V} v(x), \phi(x))$$
$$= \sup_{x \in B} \sup_{v \in V} [0, \infty](v(x), \phi(x)) \quad [\text{Lemma 3.2}]$$
$$= \sup_{v \in V} \sup_{x \in B} [0, \infty](v(x), \phi(x))$$
$$= \sup_{v \in V} \hat{B}(v, \phi).$$

$\square$

The restriction of the distance on $\mathcal{P}(X)$ to subsets of B gives the familiar (non-symmetric) Hausdorff distance (cf. [Law86]). More precisely:

Theorem 7.3 *For all $V \subseteq X$ and $W \subseteq X$ such that either $V \subseteq B$ or W is finite,*

$$\mathcal{P}(X)(V, W) = \sup_{v \in V} \inf_{w \in W} X(v, w).$$

Proof: Applying the Yoneda lemma twice gives, for all v in B, $\inf_{w \in W} \hat{B}(v, w) = \hat{B}(v, \inf W)$. If W is finite the same equality holds for arbitrary $v \in X$ (by an extension of Lemma 7.2 similar to Lemma 3.2). Therefore, if either $V \subseteq B$ or W is finite,

$$\sup_{v \in V} \inf_{w \in W} \hat{B}(v, w) = \sup_{v \in V} \hat{B}(v, \inf W)$$
$$= \hat{B}(\inf V, \inf W) \quad [\text{Lemma 7.2}]$$
$$= \mathcal{P}(X)(V, W).$$

$\square$

For a complete partial order X with basis B, the above amounts to

$$V \leq_{\mathcal{P}(X)} W \text{ iff } \forall v \in V \; \exists w \in W, \; v \leq_X w,$$

which is the usual Hoare ordering. More generally, for a gms X, there is the following characterization of the order induced by $\mathcal{P}(X)$.

Lemma 7.4 *For subsets V and W of X, if W is gS-closed then*

$$V \leq_{\mathcal{P}(X)} W \text{ if and only if } V \subseteq W.$$

Proof: If $V \subseteq W$ then $\mathcal{P}(X)(V, W) = 0$ by Theorem 7.1. Conversely, assume $\mathcal{P}(X)(V, W) = 0$ and let $v \in V$. We shall prove that $v \in W$. Recall from Section 6 (17) that W is closed if and only if

$$W = \{x \in X \mid \forall b \in B \; \forall \epsilon > 0 \; \exists w \in W, \; X(b, w) \leq \epsilon + X(b, x)\}.$$

Therefore it is sufficient to show that v satisfies

$$\forall \epsilon > 0 \; \forall b \in B \; \exists w \in W, \; X(b, w) \leq \epsilon + X(b, v).$$

This follows from $\mathcal{P}(X)(V, W) = 0$ by Theorem 7.1. $\square$

Because $V \subseteq cl_S(V)$, for every $V \subseteq X$, the above lemma implies $\mathcal{P}(X)(V, cl_S(V)) = 0$. Also $\mathcal{P}(X)(cl_S(V), V) = 0$: this follows from Theorem 7.1 and the characterization of the generalized Scott closure operator (17). This leads to the following.

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Lemma 7.5 *For subsets V and W of X,*

$$\mathcal{P}(X)(V, W) = \mathcal{P}(X)(cl_S(V), W) \ \ and \ \ \mathcal{P}(X)(V, W) = \mathcal{P}(X)(V, cl_S(W)).$$

Proof: Immediate from the fact that $\mathcal{P}(X)(V, cl_S(V)) = 0 = \mathcal{P}(X)(cl_S(V), V)$, and the triangle inequality. $\square$

Characterizing $\mathcal{P}_{gl}(X)$ as a completion

Let $\mathcal{P}_{nf}(B)$ be the gms consisting of all non-empty and finite subsets of B with the non-symmetric Hausdorff distance defined above: for V and W in $\mathcal{P}_{nf}(B)$,

$$
\begin{aligned}
\mathcal{P}_{nf}(B)(V, W) &= \hat{B}(\rho(V), \rho(W)) \\
&= \max_{v \in V} \min_{w \in W} X(v, w) \quad [\text{by Theorem 7.3}].
\end{aligned}
$$

Its completion $\overline{\mathcal{P}_{nf}(B)}$ will be shown to be isomorphic to $\mathcal{P}_{gl}(X)$. We shall need two lemmas and a theorem.

The following lemma generalizes Lemma 4.3.

Lemma 7.6 *For any V in $\mathcal{P}_{nf}(B)$, $\rho(V)$ is finite in $\hat{B}$.*

Proof: We only treat the case that $V = \{v_1, v_2\}$ (the general case follows by induction on the number of elements of V). For any Cauchy sequence $(\phi_n)_n$ in $\hat{B}$,

$$
\begin{aligned}
&\widehat{B}(\rho(V), \lim \phi_n) \\
&= \ \widehat{B}(\min\{v_1, v_2\}, \lim \phi_n) \\
&= \ \max\{\widehat{B}(v_1, \lim \phi_n), \ \widehat{B}(v_2, \lim \phi_n)\} \quad [\text{Lemma 7.2}] \\
&= \ \max\{\lim \widehat{B}(v_1, \phi_n), \ \lim \widehat{B}(v_2, \phi_n)\} \quad [\text{Lemma 4.3}] \\
&= \ \lim \max\{\widehat{B}(v_1, \phi_n), \ \widehat{B}(v_2, \phi_n)\} \quad [\text{max is continuous}] \\
&= \ \lim \widehat{B}(\min\{v_1, v_2\}, \phi_n) \quad [\text{Lemma 7.2}] \\
&= \ \lim \widehat{B}(\rho(V), \phi_n).
\end{aligned}
$$

$\square$

The lemma above is used in the proof of the following.

Lemma 7.7 $\qquad \overline{\mathcal{P}_{nf}(B)} \cong \{\lim \rho(V_n) \mid V_n \in \mathcal{P}_{nf}(B), \ for \ all \ n, \ and \ (\rho(V_n))_n \ is \ Cauchy \ in \ \hat{B}\}.$

Proof: Let us denote the set on the right by R. Because the quasi metric space $\widehat{B}$ is complete, the isometric, and hence non-expansive, function $\rho : \mathcal{P}_{nf}(B) \to \widehat{B}$ induces a non-expansive and continuous function $\rho^{\#} : \overline{\mathcal{P}_{nf}(B)} \to \widehat{B}$ according to Theorem 5.5, making the following diagram commute:

It follows from Proposition 5.2 that the image of $\rho^{\#}$ is precisely R. Furthermore $\rho^{\#}$ is isometric: for all Cauchy sequences $(V_n)_n$ and $(W_m)_m$ in $\mathcal{P}_{nf}(B)$,

$$\widehat{B}\left(\rho^{\#}(\lim_n \mathbf{y}(V_n)), \rho^{\#}(\lim_m \mathbf{y}(W_m))\right)$$

$$= \widehat{B}(\lim_n \rho(V_n), \lim_m \rho(W_m))$$

$$= \varprojlim_n \lim_m \widehat{B}(\rho(V_n), \rho(W_m)) \quad [\rho(V_n) \text{ is finite in } \widehat{B} \text{ by Lemma 7.6}]$$

$$= \varprojlim_n \lim_m \mathcal{P}_{nf}(B)(V_n, W_m) \quad [\rho \text{ is isometric}]$$

$$= \varprojlim_n \lim_m \widehat{\mathcal{P}_{nf}(B)}(\mathbf{y}(V_n), \mathbf{y}(W_m)) \quad [\mathbf{y} \text{ is isometric}]$$

$$= \widehat{\mathcal{P}_{nf}(B)}(\lim_n \mathbf{y}(V_n), \lim_m \mathbf{y}(W_m)) \quad [\mathbf{y}(V_n) \text{ is finite in } \widehat{\mathcal{P}_{nf}(B)}]$$

$$= \overline{\mathcal{P}_{nf}(B)}(\lim_n \mathbf{y}(V_n), \lim_m \mathbf{y}(W_m)).$$

Thus $\rho^{\#}$ is injective and hence an isomorphism from $\overline{\mathcal{P}_{nf}(B)}$ to R. $\qquad\square$

The following theorem will be often used in the sequel.

Theorem 7.8 $\qquad \mathcal{P}_{gl}(X) = \{\lim \rho(V_n) \mid V_n \in \mathcal{P}_{nf}(B), \text{ for all } n, \text{ and } (\rho(V_n))_n \text{ is Cauchy in } \hat{B}\}.$

Proof: Let R again denote the righthand side. The set R contains $\mathbf{y}_B(X)$, because $\mathbf{y}_B$ is continuous. Moreover, R is complete (by Lemma 7.7), and is closed under $\amalg$:

$$\lim \rho(V_n) \amalg \lim \rho(W_n) = \lim(\rho(V_n) \amalg \rho(W_n)) \quad [\amalg \text{ is continuous on } \hat{B}]$$

$$= \lim \rho(V_n \cup W_n),$$

for Cauchy sequences $(\rho(V_n))_n$ and $(\rho(W_n))_n$. It follows that $\mathcal{P}_{gl}(X) \subseteq R$.

For the converse note that any subset V of $\hat{B}$ which is closed under $\amalg$ and contains $\mathbf{y}_B(X)$, also contains $\rho(V)$ for any $V \in \mathcal{P}_{nf}(B)$. If V is moreover complete than $\lim \rho(V_n)$ is in V, for any Cauchy sequence $(\rho(V_n))_n$ in $\hat{B}$ with $V_n \in \mathcal{P}_{nf}(B)$, for all n. Consequently, R is contained in any V having all three properties. Thus $R \subseteq \mathcal{P}_{gl}(X)$. $\qquad\square$

Combining Lemma 7.7 and Theorem 7.8 yields the following.

Corollary 7.9 $\mathcal{P}_{gl}(X) \cong \overline{\mathcal{P}_{nf}(B)}.$ $\qquad\square$

The above description of the generalized lower powerdomain can be used to give the following categorical characterization. Let a *metric lower semi-lattice* be an algebraic complete quasi metric space S together with a non-expansive and continuous operation $\uplus : S \times S \to S$ such that, for all x, y, and z in S,

$$(i)\ x \uplus x = x, \quad (ii)\ x \uplus y = y \uplus x, \quad (iii)\ (x \uplus y) \uplus z = x \uplus (y \uplus z), \quad (iv)\ x \leq_X x \uplus y.$$

For example, $\langle \mathcal{P}_{gl}(X), \amalg \rangle$ is a metric lower semi-lattice because $\mathcal{P}_{gl}(X)$ is an algebraic complete quasi metric space by the above corollary, and $\amalg$ is continuous and non-expansive.

As a consequence of Theorem 7.8, the lower powerdomain construction can be seen to be free. First note that every x in X is mapped by $\mathbf{y}_B : X \to \hat{B}$ to an element of $\mathcal{P}_{gl}(X)$. Thus we may consider $\mathbf{y}_B$ as a non-expansive and continuous map $\mathbf{y}_B : X \to \mathcal{P}_{gl}(X)$.

Theorem 7.10 *For every metric lower semi-lattice $\langle S, \uplus \rangle$, and non-expansive and continuous function $f : X \to S$ there exists a unique non-expansive, continuous and additive mapping $f^{\star} : \langle \mathcal{P}_{gl}(X), \amalg \rangle \to \langle S, \uplus \rangle$ such that $f^{\star} \circ \mathbf{y}_B = f$:*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \mathbf{y}_B\ } & \mathcal{P}_{gl}(X) \\
 & \searrow\scriptstyle{f} & \big\downarrow\scriptstyle{f^{\star}} \\
 & & S.
\end{array}
$$

$\qquad\square$

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(This theorem can be proved similarly to Theorem 5.5.)

Now let $Lsl(Acq)$ denote the category of metric lower semi-lattices with continuous, non-expansive and additive functions as morphisms. There is a forgetful functor $\mathcal{U} : Lsl(Acq) \to Acq$ which maps every metric lower semi-lattices $\langle S, \uplus \rangle$ to S. As a consequence of Theorem 7.10, the lower powerdomain construction can be extended to a functor $\mathcal{P}_{gl}(-) : Acq \to Lsl(Acq)$ which is left adjoint to $\mathcal{U}$. As usual, this implies that the functor $\mathcal{U} \circ \mathcal{P}_{gl}(-) : Acq \to Acq$ is locally non-expansive and locally continuous (cf. [Plo83, Rut95]), by which it can be used in the construction of recursive domain equations.

Characterizing $\mathcal{P}_{gl}(X)$ topologically

In the rest of this section (in Theorem 7.12, to be precise), we shall make the following

assumption: the basis B of our gms X is countable.

(In other words, X is an ω-algebraic complete gms.) The main result of this subsection is:

$$\mathcal{P}_{gl}(X) \cong \mathcal{P}_{gS}^{+}(X),$$

where

$$\mathcal{P}_{gS}^{+}(X) = \{V \subseteq X \mid V \text{ is gS-closed and non-empty}\}.$$

The proof makes use of the adjunction $\rho \vdash \int$ as follows. As with any adjunction between preorders, the co-restrictions of ρ and $\int$ give an isomorphism

$$\rho : Im(\textstyle\int) \to Im(\rho), \quad \textstyle\int : Im(\rho) \to Im(\textstyle\int).$$

Recall that the gS-closed subsets of X are precisely the fixed points of $\int \circ \rho$ (Theorem 6.12). Because $\int \circ \rho \circ \int = \int$ (as with any adjunction between preorders), all elements of $Im(\int)$ are gS-closed. Thus

$$
\begin{aligned}
\mathcal{P}_{gS}(X) &= \{V \subseteq X \mid V \text{ is gS-closed}\} \\
&= \{V \subseteq X \mid V = \textstyle\int \circ \rho(V)\} \\
&= Im(\textstyle\int).
\end{aligned}
$$

In order to conclude that $\mathcal{P}_{gl}(X) \cong \mathcal{P}_{gS}^{+}(X)$, it is now sufficient to prove $\mathcal{P}_{gl}(X) = Im^{+}(\rho)$, where

$$Im^{+}(\rho) = \{\rho(V) \in \hat{B} \mid V \subseteq X, \; V \text{ non-empty}\}.$$

The inclusion $\mathcal{P}_{gl}(X) \subseteq Im^{+}(\rho)$ is an immediate consequence of Theorem 7.8 and the following.

Lemma 7.11 *For all Cauchy sequences $(\rho(V_n))_n$ in $\hat{B}$ such that V_n is a finite and non-empty subset of B for all n, $\lim \rho(V_n) \in Im^{+}(\rho)$.*

Proof: Let $(V_n)_n$ be a sequence of finite and non-empty subsets of B such that $(\rho(V_n))_n$ is Cauchy in $\hat{B}$. We shall prove: $\lim \rho(V_n) = \rho(\{\lim v_n \mid v_n \in V_n, \text{ for all } n, \text{ and } (v_n)_n \text{ is Cauchy in } B\})$. (It will follow from the proof below that the set on the right is non-empty.) Let $(v_n)_n$, with $v_n \in V_n$ be a Cauchy sequence in B. For all n, $\rho(V_n) \leq v_n$ (in $\hat{B}$ taken with the pointwise extension of the standard ordering on $[0, \infty]$). Therefore $\lim \rho(V_n) \leq \lim v_n$. Because $(v_n)_n$ is arbitrary, this implies

$$\lim \rho(V_n) \leq \rho(\{\lim v_n \mid v_n \in V_n, \text{ for all } n, \text{ and } (v_n)_n \text{ is Cauchy in } B\}).$$

For the converse let $b \in B$ and $\epsilon > 0$. We shall construct a Cauchy sequence $(v_n)_n$ in B such that

$$\lim v_n(b) \leq \lim \rho(V_n)(b) + 2 \cdot \epsilon.$$

Let N be such that for all $n \geq N$,

$$\hat{B}(\rho(V_N), \rho(V_n)) \leq \epsilon, \text{ and } \rho(V_N)(b) \leq \lim \rho(V_n)(b) + \epsilon.$$

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Choose v_i in V_i arbitrarily, for $0 \leq i < N$. Because V_N is finite there exists $v_N \in V_N$ such that $\rho(V_N)(b) = B(b, v_N) = v_N(b)$. Choose v_{N+1} in V_{N+1} such that

$$B(v_N, v_{N+1}) = \min_{w \in V_{N+1}} B(v_N, w).$$

Because, by Theorem 7.3,

$$\hat{B}(\rho(V_N), \rho(V_{N+1})) = \max_{v \in V_N} \min_{w \in V_{N+1}} B(v, w),$$

it follows that

$$B(v_N, v_{N+1}) \leq \hat{B}(\rho(V_N), \rho(V_{N+1})) \leq \epsilon.$$

Continuing this way, we find a sequence $(v_n)_n$ in B which is Cauchy because $(\rho(V_n))_n$ is. Now for all $n \geq N$, $[0, \infty](v_N(b), v_n(b)) \leq \epsilon$, or equivalently, $v_n(b) \leq v_N(b) + \epsilon$. Thus

$$
\begin{aligned}
\lim v_n(b) \quad &\leq \quad v_N(b) + \epsilon \\
&= \quad \rho(V_N)(b) + \epsilon \\
&\leq \quad \lim \rho(V_n)(b) + 2 \cdot \epsilon.
\end{aligned}
$$

$\square$

The reverse inclusion: $Im^+(\rho) \subseteq \mathcal{P}_{gl}(X)$, is a consequence of Theorem 7.8 and the following.

Theorem 7.12 *Let B be countable. For any non-empty subset V of X there exists a sequence $(V_n)_n$ of finite and non-empty subsets of B such that $\rho(V) = \lim \rho(V_n)$ in $\hat{B}$.*

Proof: Let $V \subseteq X$ be non-empty. We shall define a sequence $(V_n)_n$ of finite and (eventually) non-empty subsets of B such that for any $\phi \in \hat{B}$,

$$\hat{B}(\rho(V), \phi) = \lim_{\leftarrow} \hat{B}(\rho(V_n), \phi).$$

The proof proceeds in five steps as follows.

1. Let $b_1, b_2, \ldots$ be an enumeration of B. The sets V_n are defined by induction on n. They will consist of elements of B which are approximations of elements of V. More precisely, they will satisfy, for all $n \geq 1$,

   $$\forall b \in V_n, \ B_{1/n^2}(b) \cap V \neq \emptyset.$$

   (Recall that $B_\epsilon(b) = \{x \in X \mid X(b, x) < \epsilon\}$.) For convenience, we start at $n = 1$. Let

   $$V_1 = \begin{cases} \{b_1\} & \text{if } B_1(b_1) \cap V \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

   Now suppose we have already defined V_n. We assume: for all $b \in V_n$, $B_{1/n^2}(b) \cap V \neq \emptyset$. In the construction of V_{n+1}, we shall include for every element of the previously constructed set V_n again an element (possibly the same), which will be a better approximation of the set V. Moreover, we shall take into account b_{n+1}, the $(n+1)$-th element in the enumeration of B. Let

   $$V_{n+1} = \{\text{improve}(b) \mid b \in V_n\} \cup \{\text{represent}(b_{n+1}) \mid B_1(b_{n+1}) \cap V \neq \emptyset\},$$

   where 'improve(b)' and 'represent(b_{n+1})' are defined as follows:

- If $B_{1/(n+1)^2}(b) \cap V \neq \emptyset$ then put improve$(b) = b$: b is still 'good enough'. Otherwise consider $y \in V$ with $\hat{B}(b, y) < 1/n^2$, which exists by the inductive hypothesis that $B_{1/n^2}(b) \cap V \neq \emptyset$. Let $y = \lim y_k$, with y_k in B for all k. Because b is in B it is finite in $\hat{B}$, whence

$$\hat{B}(b, y) = \lim \hat{B}(b, y_k).$$

  Therefore we can choose a number k big enough such that

$$\hat{B}(y_k, y) < 1/(n + 1)^2 \text{ and } \hat{B}(b, y_k) < 1/n^2.$$

  Define improve$(b) = y_k$. Note that

$$B_{1/(n+1)^2}(\text{improve}(b)) \cap V \neq \emptyset \text{ and } \hat{B}(b, \text{improve}(b)) < 1/n^2.$$

- Suppose that $B_{1/(n+1)^2}(b_{n+1}) \cap V \neq \emptyset$. Then b_{n+1} is close enough to V, and we define: represent$(b_{n+1}) = b_{n+1}$. Otherwise let i be the maximal natural number with $1 \leq i < n + 1$ such that $B_{1/i^2}(b_{n+1}) \cap V \neq \emptyset$ (if such a number does not exist, i.e., $B_1(b_{n+1}) \cap V = \emptyset$ then the second set in the definition of V_{n+1} is empty). Let $y \in V$ be such that $\hat{B}(b_{n+1}, y) < 1/i^2$. Let $y = \lim y_k$, with y_k in B for all k. As before we can choose a number k such that

$$\hat{B}(y_k, y) < 1/(n + 1)^2 \text{ and } \hat{B}(b_{n+1}, y_k) < 1/i^2,$$

  and put: represent$(b_{n+1}) = y_k$. Note that

$$B_{1/(n+1)^2}(\text{represent}(b_{n+1})) \cap V \neq \emptyset \text{ and } \hat{B}(b_{n+1}, \text{represent}(b_{n+1})) < 1/i^2.$$

For all $b \in V_{n+1}$, $B_{1/(n+1)^2}(b) \cap V \neq \emptyset$. Because V is non-empty there exists N such that for all $n \geq N$, V_n is non-empty.

2. Some properties of $(V_n)_n$: Because $\hat{B}(b, \text{improve}(b)) < 1/n^2$, for all $n \geq 1$ and $b \in V_n$, it follows that

$$
\begin{aligned}
\hat{B}(\rho(V_n), \rho(V_{n+1})) &= \sup_{v \in V_n} \inf_{w \in V_{n+1}} \hat{B}(v, w) \quad \text{[Theorem 7.3]} \\
&< 1/n^2.
\end{aligned}
$$

Because $B_{1/n^2}(b) \cap V \neq \emptyset$, for all $n \geq 1$ and $b \in V_n$, also

$$\hat{B}(\rho(V_n), \rho(V)) < 1/n^2.$$

3. As a consequence, $(\rho(V_n))_n$ is a Cauchy sequence in $\hat{B}$. Since for all $n \geq 1$ and $\phi \in \hat{B}$,

$$
\begin{aligned}
\hat{B}(\rho(V_n), \phi) &\leq \hat{B}(\rho(V_n), \rho(V)) + \hat{B}(\rho(V), \phi) \\
&\leq 1/n^2 + \hat{B}(\rho(V), \phi),
\end{aligned}
$$

it follows that

$$\lim_{\leftarrow} \hat{B}(\rho(V_n), \phi) \leq \hat{B}(\rho(V), \phi).$$

4. Next we shall prove the converse:

$$\hat{B}(\rho(V), \phi) \leq \lim_{\leftarrow} \hat{B}(\rho(V_n), \phi).$$

Note that by completeness of the quasi metric space $\hat{B}$, $\lim \rho(V_n)$ always exists, and that

$$\lim_{\leftarrow} \hat{B}(\rho(V_n), \phi) = \hat{B}(\lim \rho(V_n), \phi).$$

Because $\hat{B}(\rho(V), \phi) = \hat{B}(\inf V, \phi) = \sup_{y \in V} \hat{B}(y, \phi)$ it will be sufficient to prove for all $y \in V$,

$$\hat{B}(y, \phi) \le \hat{B}(\lim \rho(V_n), \phi).$$

Let $\epsilon > 0$ and $y \in V$. We shall show that

$$\hat{B}(y, \phi) \le \hat{B}(\lim \rho(V_n), \phi) + 3 \cdot \epsilon.$$

Consider a Cauchy sequence $(y_m)_m$ in B with $y = \lim y_m$. Let M be a natural number such that

$$\sum_{m=M}^{\infty} \frac{1}{m^2} < \epsilon.$$

Choose m big enough such that

$$
\begin{aligned}
\hat{B}(y, \phi) &= \hat{B}(\lim y_m, \phi) \\
&= \lim_{\leftarrow} \hat{B}(y_m, \phi) \\
&\le \hat{B}(y_m, \phi) + \epsilon,
\end{aligned}
$$

and $\hat{B}(y_m, y) < 1/M^2$. Let $k \ge 1$ be such that $y_m = b_k$. (Recall that $B = \{b_1, b_2, \ldots\}$.) We distinguish between the following two cases:

(i) $k \le M$: Because $1/M^2 \le 1/k^2$ it follows from the construction of $(V_n)_n$ that $b_k \in V_k$, $b_k \in V_{k+1}, \ldots, b_k \in V_M$. Therefore

$$
\begin{aligned}
\hat{B}(y_m, \phi) \\
&= \hat{B}(b_k, \phi) \\
&\le \sup_{b \in V_M} \hat{B}(b, \phi) \\
&= \hat{B}(\inf V_M, \phi) \\
&= \hat{B}(\rho(V_M), \phi) \\
&\le \hat{B}(\rho(V_M), \lim \rho(V_n)) + \hat{B}(\lim \rho(V_n), \phi) \\
&\le \sum_{m=M}^{\infty} \frac{1}{m^2} + \hat{B}(\lim \rho(V_n), \phi) \\
&\le \epsilon + \hat{B}(\lim \rho(V_n), \phi).
\end{aligned}
$$

(ii) $M < k$: If $B_{1/k^2}(b_k) \cap V = B_{1/k^2}(y_m) \cap V \ne \emptyset$ then $\text{represent}(b_k) = b_k$. Otherwise let i be the maximal number below k such that $B_{1/i^2}(b_k) \cap V \ne \emptyset$. Because $\hat{B}(b_k, y) = \hat{B}(y_m, y) < 1/M^2$ it follows that $M \le i$, whence

$$\hat{B}(b_k, \text{represent}(b_k)) < 1/i^2 \le \epsilon.$$

Thus whether $B_{1/k^2}(b_k) \cap V$ is empty or non-empty,

$$\hat{B}(b_k, \text{represent}(b_k)) \le \epsilon.$$

Consequently,

$$
\begin{aligned}
\hat{B}(y_m, \phi) \\
&= \hat{B}(b_k, \phi) \\
&\le \hat{B}(b_k, \text{represent}(b_k)) + \hat{B}(\text{represent}(b_k), \phi) \\
&\le \epsilon + \hat{B}(\text{represent}(b_k), \phi) \\
&\le \epsilon + \sup_{b \in V_k} \hat{B}(b, \phi)
\end{aligned}
$$

$$
\begin{aligned}
&= \quad \epsilon + \hat{B}(\rho(V_k), \phi) \\
&\leq \quad \epsilon + \hat{B}(\rho(V_k), \lim \rho(V_n)) + \hat{B}(\lim \rho(V_n), \phi) \\
&\leq \quad \epsilon + \sum_{m=k}^{\infty} \frac{1}{m^2} + \hat{B}(\lim \rho(V_n), \phi) \\
&\leq \quad \epsilon + \epsilon + \hat{B}(\lim \rho(V_n), \phi) \quad [\text{since } k > M].
\end{aligned}
$$

It follows that in both cases

$$
\begin{aligned}
\hat{B}(y, \phi) & \\
&\leq \quad \hat{B}(y_m, \phi) + 1/M^2 \\
&\leq \quad \hat{B}(y_m, \phi) + \epsilon \\
&\leq \quad \hat{B}(\lim \rho(V_n), \phi) + 3 \cdot \epsilon.
\end{aligned}
$$

5. We have shown:

$$
\hat{B}(\rho(V), \phi) = \varprojlim \hat{B}(\rho(V_n), \phi).
$$

$\square$

Lemma 7.11 and Theorem 7.12, together with Theorem 7.8, imply:

Corollary 7.13 $\mathcal{P}_{gl}(X) = Im^+(\rho)$. $\qquad\square$

All in all, we have:

Theorem 7.14 *For an ω-algebraic complete gms X, $\mathcal{P}_{gl}(X) \cong \mathcal{P}_{gS}^+(X)$.*

Proof: The isomorphism $\mathcal{P}_{gS}(X) \cong Im(\rho)$ restricts to an isomorphism $\mathcal{P}_{gS}^+(X) \cong Im^+(\rho)$. By Corollary 7.13, $\mathcal{P}_{gl}(X) = Im^+(\rho)$. Therefore, $\mathcal{P}_{gl}(X) \cong \mathcal{P}_{gS}^+(X)$. $\qquad\square$

Using the characterization of $\mathcal{P}_{gl}(X)$ as a completion, it follows that $\mathcal{P}_{gl}(X)$ is an ω-algebraic complete quasi metric space with as (countable) basis the set

$$\{cl_S(V) \mid V \in \mathcal{P}_{nf}(B)\}.$$

The collection of closed sets of a given topological space X often comes with the *lower topology* [Mic51, Nad78]. Recall that given a topological space $\langle X, \mathcal{O}(X)\rangle$, the lower topology $\mathcal{O}_L(S)$ on a collection of subset $S \subseteq \mathcal{P}(X)$ is defined by taking the collection of sets of the form

$$L_o = \{V \in S \mid V \cap o \neq \emptyset\},$$

for all $o \in \mathcal{O}(X)$, as a subbasis. This subsection is concluded by showing that for an ω-algebraic complete quasi metric space X, the lower topology on $\mathcal{P}_{gS}(X)$ and the generalized Scott topology on $\mathcal{P}_{gS}(X)$ coincide.

Theorem 7.15 *For an ω-algebraic complete quasi metric space X,*

$$\mathcal{O}_L(\mathcal{P}_{gS}(X)) = \mathcal{O}_{gS}(\mathcal{P}_{gS}(X)).$$

Proof: Let B be a countable basis for X. Let $o \in \mathcal{O}_{gS}(X)$ and consider the sub-basic open set $L_o \in \mathcal{O}_L(\mathcal{P}_{gS}(X))$. A gS-closed set V is in L_o if and only if $V \cap o \neq \emptyset$ or, equivalently, $V \not\subseteq X \setminus o$. Because $X \setminus o$ is gS-closed, it follows from Lemma 7.4 that $\mathcal{P}(X)(V, X \setminus o) \neq 0$. Therefore,

$$L_o = \{W \in \mathcal{P}_{gS}(X) \mid \mathcal{P}(X)(W, X \setminus o) \neq 0\}.$$

But the rightmost set is open in the gS-topology of $\mathcal{P}_{gS}(X)$ because it is the complement of the gS-closed set

$$cl_S(\{X \setminus o\}) = \{W \in \mathcal{P}_{gS}(X) \mid \mathcal{P}(X)(W, X \setminus o) = 0\}$$

(the latter equality being a consequence of Lemma 6.8 and Lemma 7.4). This proves $\mathcal{O}_L(\mathcal{P}_{gS}(X)) \subseteq \mathcal{O}_{gS}(\mathcal{P}_{gS}(X))$.

For the converse, let V be a finite subset of B and consider, for some $\epsilon > 0$, the basic open set $B_\epsilon(cl_S(V))$ of the gS-topology on $\mathcal{P}_{gS}(X)$. For any $W \in \mathcal{P}_{gS}(X)$,

$$W \in B_\epsilon(cl_S(V))$$
$$\iff \quad \mathcal{P}(X)(cl_S(V), W) < \epsilon$$
$$\iff \quad \mathcal{P}(X)(V, W) < \epsilon \quad [\text{Lemma 7.5}]$$
$$\iff \quad \sup_{b \in V} \inf_{x \in W} X(b, x) < \epsilon \quad [\text{Theorem 7.3}, V \subseteq B]$$
$$\iff \quad \forall b \in V, \ \inf_{x \in W} X(b, x) < \epsilon$$
$$\iff \quad \forall b \in V, \ W \cap B_\epsilon(b) \neq \emptyset$$
$$\iff \quad W \in \bigcap_{b \in V} L_{B_\epsilon(b)} \quad [B_\epsilon(b) \text{ is basic open in } \mathcal{O}_{gS}(X)].$$

Since V is finite, the above proves that every basic open set of $\mathcal{O}_{gS}(\mathcal{P}_{gS}(X))$ can be expressed as the intersection of finitely many sub-basic open sets of $\mathcal{O}_L(\mathcal{P}_{gS}(X))$. Thus $\mathcal{O}_{gS}(\mathcal{P}_{gS}(X)) \subseteq \mathcal{O}_L(\mathcal{P}_{gS}(X))$. $\qquad\square$

Generalized upper and convex powerdomains

We briefly sketch the construction of a generalized upper and convex powerdomain. They will be treated in detail elsewhere.

Let X be an algebraic complete gms with basis B. A *generalized upper powerdomain* on X can be defined dually to $\mathcal{P}_{gl}(X)$ as follows. First $[0, \infty]$ is considered again as a semi-lattice, now with $\pi : [0, \infty] \times [0, \infty] \to [0, \infty]$ sending elements r and s in $[0, \infty]$ to (their product) $\max\{r, s\}$. Next let

$$\breve{B} = ([0, \infty]^B)^{op}.$$

It can be turned into a semi-lattice $\langle \breve{B}, \pi \rangle$ by taking the pointwise extension of π. There is the following dual version of the Yoneda embedding:

$$\breve{\mathbf{y}}_B : X \to \breve{B}, \quad x \mapsto B(x, -),$$

where $B(x, -)$ maps b in B to $B(x, b)$. Now the generalized upper powerdomain is given by

$$\mathcal{P}_{gu}(X) = \bigcap \{V \subseteq \breve{B} \mid \breve{\mathbf{y}}_B(X) \subseteq V, \ V \text{ is a complete subspace of } \breve{B}, \text{ and } V \text{ is closed under } \pi\}.$$

Also this powerdomain can be characterized in a number of ways, one of which is via completion: Consider again $\mathcal{P}_{nf}(B)$, this time with distance, for all V and W in $\mathcal{P}_{nf}(B)$,

$$\mathcal{P}_{nf}(B)(V, W) = \sup_{w \in W} \inf_{v \in V} B(v, w).$$

Then the completion of $\mathcal{P}_{nf}(B)$ is isomorphic to $\mathcal{P}_{gu}(X)$. In the special case that X is a preorder, this amounts to the standard definition of the upper, or Smyth, powerdomain.

A *generalized convex powerdomain* is obtained by combining the constructions of the generalized lower and upper powerdomains (thus using both the Yoneda embedding and its dual). It can again be easily described as the completion of $\mathcal{P}_{nf}(B)$, now taken with distance

$$\mathcal{P}_{nf}(B)(V, W) = \max\{\sup_{v \in V} \inf_{w \in W} B(v, w), \ \sup_{w \in W} \inf_{v \in V} B(v, w)\}.$$

For a preorder X, the convex powerdomain coincides with the standard convex, or Plotkin, powerdomain; for an ordinary metric space, it yields the powerdomain of compact subsets.

8 Related work

The thesis that fundamental structures are categories has been the main motivation for Lawvere in his study of generalized metric spaces as enriched categories [Law73]. Lawvere's work together with the more topological perspective of Smyth [Smy88] have been our main source of inspiration for the present paper which continues the work of Rutten [Rut95]. Generalized metric spaces are a special instance of Lawvere's $\mathcal{V}$-categories. The non-symmetric metric for $[0, \infty]$ is also described and studied in his paper. The notion of forward Cauchy sequence for a non-symmetric metric space is from [Smy88] as well as the notion of limit. A purely enriched categorical definition of forward Cauchy sequences and of limits can be found in Wagner's [Wag94, Wag95, Rut95]. In [Rut95] and [Rut96], the definitions of forward limit and backward limit are shown to be special instances of the enriched-categorical notions of weighted limit and weighted colimit. The notion of finiteness and algebraicity for a generalized metric space are from [Rut95].

Clearly we are working in the tradition of domain theory, for which Plotkin's [Plo83] has been our main source of information.

Completion and topology of non-symmetric metric spaces have been extensively studied in [Smy88], seeking to reconcile metric spaces and complete partial orders as topological spaces by considering quasi-uniformities. Smyth gives criteria for the appropriateness of a topology for a quasi-uniform space. Also a completion by means of Cauchy sequences is present in his work. The main difference with our work is the simplicity of the theory of generalized metric spaces obtained by the enriched categorical perspective, in particular by the use of the Yoneda Lemma. Indeed, both the categorical perspective of Lawvere and the topological one of Smyth have been combined in our approach to obtain a reconciliation of complete metric spaces with complete partial orders.

The fact that the Yoneda lemma gives rise to completion is well known for many mathematical structures such as groups, lattices, and categories. In [Wag95], an enriched version of the Dedekind-MacNeille completion of lattices is given. In [SMM95], the Yoneda lemma is used in the definition of a completion of monoidal closed categories. The use of the Yoneda lemma for the completion of generalized metric spaces is new, but it is suggested by an embedding theorem of Kuratowski [Kur35] and the definition of completion as in [Eng89, Theorems 4.3.13-4.3.19] for standard metric spaces. A metric version of the Yoneda lemma also occurs, though not under that name, in [JMP86, Lemma II-2.8].

The comprehension schema as a comparison between predicates and subsets has been studied in the context of generalized metric spaces by Lawvere [Law73] and Kent [Ken88]. The definition of the generalized Scott topology via the Yoneda embedding seems to be new while the direct definition—by specifying the open sets—is briefly mentioned in the conclusion of [Smy88]. Recently, Flagg and Sünderhauf [FS96] have proved that our generalized Scott topology of an algebraic complete qms arises as the sobrification of its basis taken with the generalized Alexandroff topology. A generalized Scott topology is also given in [Wag95]. However his notion of topology does not coincide with the standard one: for example it is not the ϵ-ball topology in the case of standard metric spaces.

Another important topological approach to quasi metric spaces which needs to be mentioned is that of, again, Smyth [Smy91] and Flagg and Kopperman [FK95]. They consider quasi metric spaces equipped with the generalized Alexandroff topology. In order to reconcile metric spaces with complete partial orders they assign to partial orders a distance function which, in general, is not discrete. Their approach to topology, completion and powerdomains is much simpler than ours because many of the standard metric topological theorems can be adapted. The price to be paid for such simplicity is that this approach only works for a restricted class of spaces: they have to be spectral. Hence a full reconciliation between metric spaces and partial orders is not possible (e.g., only algebraic cpo's which are so-called 2/3 SFP are spectral in their Scott topology). Also the work of Sünderhauf on quasi-uniformities [Sün94] is along the same lines.

The study of powerdomains for complete generalized metric spaces is new. Some results on the restricted class of totally bounded quasi metric spaces are due to Smyth [Smy91] and Flagg and Kopperman [FK95]. The lower powerdomain has also been studied by Kent [Ken88] but for generalized metric spaces which need not be complete. Our use of the Yoneda embedding for

defining the powerdomains and for their topological characterization is new. It is inspired by the work of Lawvere [Law73, Law86].

Other papers on reconciling complete partial orders and metric spaces are [WS81, CD85, Mat94]. In [RSV82] seven distinct notions of Cauchy sequences can be found. For one of these notions of Cauchy sequence—but different from ours—completion has been studied in [Doi88].

Acknowledgements

The authors are grateful to the following persons for suggestions, comments, and discussions: Jaco de Bakker, Bob Flagg, Pietro di Gianantonio, Bart Jacobs, Bill Rounds, Philipp Sünderhauf, Erik de Vink, and Kim Wagner. Marcello Bonsangue and Jan Rutten have enjoyed the opportunity to present this work at an early stage at a seminar at LITP (Paris), organized by Paul Gastin and Maurice Nivat.

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A Topological preliminaries

A *topology* $\mathcal{O}(X)$ on a set X is a collection of subsets of X that is closed under finite intersections and arbitrary unions. The pair $\langle X, \mathcal{O}(X)\rangle$ is called a *topological space* and every $o \in \mathcal{O}(X)$ is called an *open set* of the space X. A set is *closed* if its complement is open. A *base* of a topology $\mathcal{O}(X)$ on X is a set $\mathcal{B} \subseteq \mathcal{O}(X)$ such that every open set is the union of elements of $\mathcal{B}$. A *subbase* of $\mathcal{O}(X)$ is a set $\mathcal{S} \subseteq \mathcal{O}(X)$ such that the collection of finite intersections of elements in $\mathcal{S}$ is a basis of $\mathcal{O}(X)$.

Every topology $\mathcal{O}(X)$ on a set X induces a preorder on X called the *specialization preorder*: for any x and y in X, $x \leq_{\mathcal{O}} y$ if and only if

$$\forall o \in \mathcal{O}(X), \;\; x \in o \Rightarrow y \in o.$$

A topology is called $\mathcal{T}_0$ if the specialization preorder is a partial order.

A *closure operator* on a set X is a function $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ such that, for all S and S' in $\mathcal{P}(X)$,

(i) $S \subseteq cl(S)$ (ii) $cl(S) = cl(cl(S))$

(iii) if $S \subseteq S'$ then $cl(S) \subseteq cl(S')$

A closure operator is strict if $cl(\emptyset) = \emptyset$. A *topological closure operator* is a strict closure operator cl that moreover is finitely additive: $cl(S \cup S') = cl(S) \cup cl(S')$. Every topological closure operator induces a topology: the closed sets are the fixed points of the closure operator. Conversely, every topology $\mathcal{O}(X)$ on X defines a topological closure operator, which maps a subset S of X to the intersection of all closed sets containing S. This closure operator can also be characterized as follows: Let S be a subset of X. An element x in X is a *cluster point* of S if for every open set $o \in \mathcal{O}(X)$, $x \in o$ implies $o \cap (S \setminus \{x\}) \neq \emptyset$; that is, x cannot be separated from S using open sets. Let S^d be the collection of all cluster points of S (it is called the *derived set*). Then

$$cl(S) = S \cup S^d.$$

Let $\langle X, \mathcal{O}(X)\rangle$ be a topological space. A non-empty subset $\mathcal{F} \subseteq \mathcal{O}(X)$ is a *filter* if it satisfies

1. if $o_1 \in \mathcal{F}$ and $o_1 \subseteq o_2$ then $o_2 \in \mathcal{F}$; and

2. if $o_1 \in \mathcal{F}$ and $o_2 \in \mathcal{F}$ then $o_1 \cap o_2 \in \mathcal{F}$.

For instance, every element x in X induces a filter $\mathcal{N}(x) = \{o \in \mathcal{O}(X) \mid x \in o\}$. More generally, any sequence $(x_n)_n$ in X induces a filter

$$\mathcal{N}((x_n)_n) = \{o \in \mathcal{O}(X) \mid \exists N \geq 0 \, \forall n \geq N, \;\; x_n \in o\}.$$

A filter $\mathcal{F}$ *converges* to an element x, denoted by $\mathcal{F} \to x$, if $\mathcal{N}(x) \subseteq \mathcal{F}$. A sequence $(x_n)_n$ is said to converge to an element x if $\mathcal{N}((x_n)_n) \to x$.

A function $f : X \to Y$ between two topological spaces X and Y is *topologically continuous* if the inverse image $f^{-1}(o) = \{x \in X \mid f(x) \in o\}$ of any o in $\mathcal{O}(Y)$ is in $\mathcal{O}(X)$. If $f : X \to Y$ is topologically continuous then for every sequence $(x_n)_n$ in X and $x \in X$

$$\mathcal{N}((x_n)_n) \to x \;\; \Rightarrow \;\; (\mathcal{N}((f(x_n)_n) \to f(x).$$

The standard topology associated with an ordinary metric space X is the ϵ-*ball* topology: a set $o \subseteq X$ is open if

$$\forall x \in o \, \exists \epsilon > 0, \;\; B_\epsilon(x) \subseteq o,$$

where $B_\epsilon(x) = \{y \in X \mid X(x,y) < \epsilon\}$. The set $\{B_\epsilon(x) \mid x \in X \;\&\; \epsilon > 0\}$ is a basis for ϵ-*ball* topology.

The standard topology associated with a preorder X is the *Alexandroff* topology, for which a set $o \subseteq X$ is open if, for x and y in X,

$$x \in o \text{ and } x \leq y \;\Rightarrow\; y \in o,$$

that is, o is upper-closed. If the preorder has a least upper bound for every ω-chain, then the *Scott* topology is more appropriate. It consists of those upper closed subsets $o \subseteq X$ that moreover satisfy, for any ω-chain $(x_n)_n$ in X,

$$\bigsqcup x_n \in o \;\Rightarrow\; \exists N \,\forall n \geq N, \;\; x_n \in o.$$

Clearly, every Scott open set is also Alexandroff open. The converse is generally not true if the preorder X is not finite. If X is an ω-algebraic cpo with basis B_X then the set $\{b\!\uparrow \mid b \in B_X\}$, with $b\!\uparrow = \{x \in X \mid b \leq x\}$, is a basis for the Scott topology.

B Sequences of sequences

The following two lemmas express that the limit of a Cauchy sequence which consists of the limits of Cauchy sequences of finite elements, can be obtained as the limit of a (kind of) diagonal sequence of finite elements.

Lemma B.1 *Let X be a subspace of a complete qms Y. Let all elements of X be finite in Y. For every n, let $(u_n^m)_m$ be a Cauchy sequence in X with limit*

$$\lim_m u_n^m = y_n. \tag{19}$$

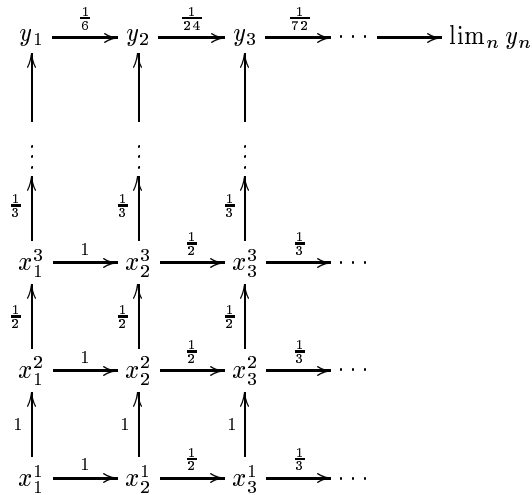*Assume that $(y_n)_n$ is a Cauchy sequence in Y satisfying*

$$\forall n, Y(y_n, y_{n+1}) \leq \tfrac{1}{3n2^n}. \tag{20}$$

Then there exist subsequences $(x_n^m)_m$ of $(u_n^m)_m$ in X satisfying

$$\forall m \forall n, X(x_n^m, x_{n+1}^m) \leq \tfrac{1}{n} \tag{21}$$
$$\forall n \forall m, X(x_n^m, x_n^{m+1}) \leq \tfrac{1}{m} \tag{22}$$
$$\forall n, \lim_m x_n^m = y_n \tag{23}$$

Proof: Because the sequences $(u_n^m)_m$ are Cauchy, there exist subsequences $(v_n^m)_m$ of $(u_n^m)_m$ satisfying

$$\forall n \forall m, X(v_n^m, v_n^{m+1}) \leq \tfrac{1}{m2^m}. \tag{24}$$

We will construct subsequences $(x_n^m)_m$ of $(v_n^m)_m$ satisfying

$$\forall m \forall n, X(x_n^m, x_{n+1}^m) \leq \tfrac{1}{n2^n}. \tag{25}$$

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Since, for all n,

$$\varprojlim_m Y\left(v_n^m, y_{n+1}\right)$$
$$= \quad Y\left(\lim_m v_n^m, y_{n+1}\right)$$
$$= \quad Y\left(y_n, y_{n+1}\right) \quad [(19)]$$
$$\leq \quad \frac{1}{3n2^n} \quad [(20)]$$

we can conclude that

$$\forall n \exists M_n \forall m \geq M_n, Y\left(v_n^m, y_{n+1}\right) \leq \frac{2}{3n2^n}.$$

By removing from each sequence $(v_n^m)_m$ the first M_n elements we obtain the subsequences $(w_n^m)_m = (v_n^{M_n+m})_m$ satisfying

$$\forall n \forall m, Y\left(w_n^m, y_{n+1}\right) \leq \frac{2}{3n2^n}. \tag{26}$$

Since, for all n and m,

$$\lim_k Y\left(w_n^m, w_{n+1}^k\right)$$
$$= \quad Y\left(w_n^m, \lim_k w_{n+1}^k\right) \quad [w_n^m \text{ is finite in } Y]$$
$$= \quad Y\left(w_n^m, y_{n+1}\right) \quad [(19)]$$
$$\leq \quad \frac{2}{3n2^n} \quad [(26)]$$

we have that

$$\forall n \forall m \exists K_n^m \forall k \geq K_n^m, Y\left(w_n^m, w_{n+1}^k\right) \leq \frac{1}{n2^n}.$$

Without loss of generality we can assume that the sequences $(K_n^m)_m$ are strictly increasing. The subsequences $(x_n^m)_m = (w_n^{L_n^m})_m$ where

$$L_n^m = \begin{cases} m & \text{if } n = 1 \\ K_{n-1}^{L_{n-1}^m} & \text{if } n > 1 \end{cases}$$

satisfy (25).

Because the subsequences $(x_n^m)_m$ of the Cauchy sequences $(u_n^m)_m$ are again Cauchy and have the same limits, these subsequences also satisfy (23). Since for all m, n, and i, with $i \geq n$,

$$X\left(x_n^m, x_i^m\right)$$
$$\leq \quad \sum_{h=n}^{i-1} X\left(x_h^m, x_{h+1}^m\right)$$
$$\leq \quad \sum_{h=n}^{i-1} \frac{1}{h2^h} \quad [(25)]$$
$$\leq \quad \frac{1}{n}.$$

Hence the subsequences $(x_n^m)_m$ satisfy (21). Similarly we can show that (24) implies (22). $\quad \square$

The above proof shows some resemblance with the proof of Theorem 2 of [Smy88]. The completeness of Y ensures the existence of the limits of the Cauchy sequences $(u_n^m)_m$. If we drop the condition that all elements of X are finite in Y, then the above lemma does not hold any more.
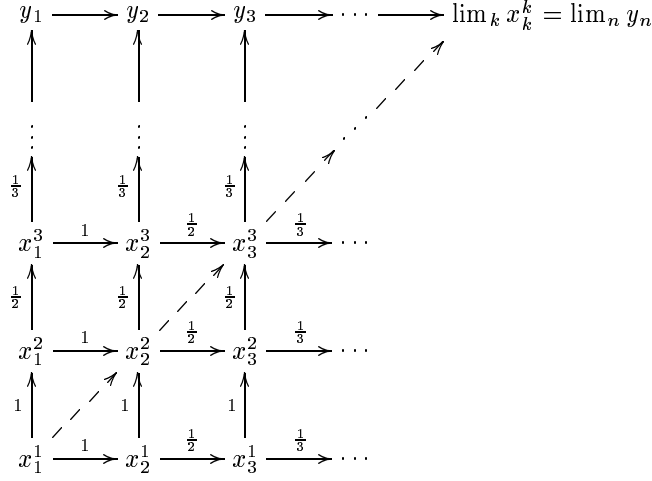
Lemma B.2 *Let X be a subspace of a complete qms Y. Let $(y_n)_n$ be a Cauchy sequence in Y. Let $(x_n^m)_m$ be Cauchy sequences in X satisfying*

$$\forall m \forall n \forall i \geq n, X\left(x_n^m, x_i^m\right) \leq \tfrac{1}{n} \tag{27}$$

$$\forall n \forall m \forall j \geq m, X\left(x_n^m, x_n^j\right) \leq \tfrac{1}{m} \tag{28}$$

$$\forall n, \lim_m x_n^m = y_n \tag{29}$$

Then $(x_k^k)_k$ is a Cauchy sequence in X and $\lim_k x_k^k = \lim_n y_n$.

Proof: Because, for all n and m, with $m \geq n$,

$$X\left(x_n^n, x_m^m\right)$$
$$\leq \quad X\left(x_n^n, x_m^n\right) + X\left(x_m^n, x_m^m\right)$$
$$\leq \quad \tfrac{2}{n} \quad [(27) \text{ and } (28)]$$

the sequence $(x_k^k)_k$ is Cauchy.

For all n, m, and k, with $k \geq n$ and $k \geq m$,

$$Y\left(x_n^m, x_k^k\right)$$
$$\leq \quad Y\left(x_n^m, x_k^m\right) + Y\left(x_k^m, x_k^k\right)$$
$$\leq \quad \tfrac{1}{n} + \tfrac{1}{m} \quad [(27) \text{ and } (28)]$$

Consequently,

$$Y\left(\lim_n y_n, \lim_k x_k^k\right)$$
$$= \quad \varprojlim_n Y\left(y_n, \lim_k x_k^k\right)$$
$$= \quad \varprojlim_n Y\left(\lim_m x_n^m, \lim_k x_k^k\right) \quad [(29)]$$
$$= \quad \varprojlim_n \varprojlim_m Y\left(x_n^m, \lim_k x_k^k\right)$$
$$\leq \quad \varprojlim_n \varprojlim_m \lim_k Y\left(x_n^m, x_k^k\right) \quad [\text{Proposition 3.4}]$$
$$\leq \quad \varprojlim_n \varprojlim_m \lim_k \tfrac{1}{n} + \tfrac{1}{m} \quad [\text{see above}]$$
$$= \quad 0.$$

For all n, m, and k, with $n \geq k$ and $m \geq k$,

$$Y\left(x_k^k, x_n^m\right)$$
$$\leq \quad Y\left(x_k^k, x_k^m\right) + Y\left(x_k^m, x_n^m\right)$$
$$\leq \quad \tfrac{2}{k} \quad [(27) \text{ and } (28)]$$

Hence,

$$
\begin{aligned}
Y\left(\varprojlim_k x_k^k, \lim_n y_n\right) \\
&= \quad \varprojlim_k Y\left(x_k^k, \lim_n y_n\right) \\
&\leq \quad \varprojlim_k \lim_n Y\left(x_k^k, y_n\right) \quad [\text{Proposition 3.4}] \\
&= \quad \varprojlim_k \lim_n Y\left(x_k^k, \lim_m x_n^m\right) \quad [(29)] \\
&\leq \quad \varprojlim_k \lim_n \lim_m Y\left(x_k^k, x_n^m\right) \quad [\text{Proposition 3.4}] \\
&\leq \quad \varprojlim_k \lim_n \lim_m \frac{2}{k} \quad [\text{see above}] \\
&= \quad 0.
\end{aligned}
$$

From the above we can conclude that $\lim_k x_k^k = \lim_n y_n$. $\qquad\square$

From the above two lemmas we can conclude the following.

Proposition B.3 *Let X be a subspace of a complete qms Y. Let all elements of X be finite in Y. Then*

$$
\lim CS\left(X\right) = \left\{\, \lim_n x_n \mid \left(x_n\right)_n \text{ is a Cauchy sequence in } X \,\right\}
$$

is a complete subspace of Y.

Proof: Clearly $\lim CS\left(X\right)$ is a subspace of Y. Let $\left(y_n\right)_n$ be a Cauchy sequence in $\lim CS\left(X\right)$. We have to show that its limit $\lim_n y_n$ is an element of $\lim CS\left(X\right)$. Without loss of generality we can assume that $\forall n, Y\left(y_n, y_{n+1}\right) \leq \frac{1}{3n2^n}$. From Lemma B.1 and B.2 we can conclude that there exists a Cauchy sequence $\left(x_k^k\right)_k$ in X satisfying $\lim_k x_k^k = \lim_n y_n$. Consequently, $\lim_n y_n$ is an element of $\lim CS\left(X\right)$. $\qquad\square$