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Abstract

This paper is supplementary to my paper "Multiparameter Quantum Groups and Multiparameter R-Matrices", [5]. Its main purpose is to point out that among the single block solutions of the Yang-Baxter equation given in [5] there occurs an $\binom{n+m}{2}+1$ parameter quantum deformation of the supergroup $GL(m|n)$ for every $n, m \geq 1$.

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1. Introduction

Let K be any field of charcteristic $\neq 2$ and consider the free associative algebra $K<t>$ $= K<t_1^1,\dots,t_n^1;t_1^2,\dots,t_n^2;\dots;t_1^n,\dots,t_n^n>$ over K in n^2 (noncommuting) indeterminates $t_1^1,\dots,t_n^n$.

Let $R=(R_{cd}^{ab})$ be any $n^2\times n^2$ matrix with coefficients in K (and its rows and columns numbered $11,\dots,1n;21\cdots 2n;\dots;n1,\dots,nn$ with lower indices for columns and upper indices for rows). Consider the $n^2\times n^2$ elements making up the $n^2\times n^2$ matrix

$$RT_1T_2 \;-\; T_2T_1R \tag{1.1}$$

where

$$T=\begin{pmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & & \\ t_1^n & \cdots & t_n^n \end{pmatrix},\ T_1=T\otimes I_n,\quad T_2=I_n\otimes T \tag{1.2}$$

Let $I(R)$ be the two-sided ideal in $K<t>$ generated by the elements of (1.1).

On $K<t>$ consider the standard "matrix function algebra" comultiplication

$$t_j^i \mapsto t_k^i\otimes t_j^k \tag{1.3}$$

where the Einstein summation convention is in force, i.e. on the righthand side of (1.3) a sum over k is implied. This makes $K<t>$ a bialgebra and it is always the case that $I(R)$ is a bialgebra ideal; cf e.g. [4] for a proof. Thus

$$K_n(R)=K<t>/I(R) \tag{1.4}$$

inherits a bialgebra structure. This is sometimes called the FRT construction (Faddeev - Reshetikin - Taktadzhyan). It was also indepedently noticed by other authors. Relations like (1.1) first came up in the work of the Faddeev group in what was then called Leningrad (now Sankt Petersburg) in the context of quantum completely integrable dynamical systesm, [3]. In that context they were called "Fundamental Commutation Relations" and they form the basis for the socalled "Algebraic Bethe Ansatz" approach to finding the eigenvalues and vectors of quantum Hamiltonians.

For certain, by now well known, solutions R of the (quantum) Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.5}$$

the bialgebra $H_n(R)$, localized at a quantum determinant, are the Hopf algebras of functions of the one parameter "standard" quantum groups such as $GL_q(n)$. E.g. for $n = 2, GL_q(2)$ (over K) is obtained from the R-matrix

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{1.6}$$

In [5] all solutions of the Yang-Baxter equation (1.5) were determined that satisfy the additional condition

$$R^{ab}_{cd} = 0 \quad \text{unless } \{a,b\} = \{c,d\} \tag{1.7}$$

i.e. an entry of R is zero unless its sets of upper and lower indices are equal. The example (1.6) satisfies (1.7). Among the solutions of the Yang-Baxter equation (1.5) given in [5] are what are there called "single block cell size 1" solutions. These look as follows

"Single block cell size 1" solutions. Choose two elements $y, z \in K,\ z \neq 0$. For each $i > j; i, j \in \{1, \dots n\}$ choose $0 \neq x_{ij} \in K$. Consider the quadratic scalar equation

$$X^2 = yX + z \tag{1.8}$$

with solutions ρ_1, ρ_2 (assumed to exist in K). Now define an $n^2 \times n^2$ matrix R over K as follows

$$\begin{aligned} R^{ii}_{ii} &= \rho_1 \text{ or } \rho_2 && , i = 1, \dots, n \\ R^{ij}_{ji} &= y && \text{if } i < j; i, j \in \{1, \dots, n\} \\ R^{ij}_{ji} &= 0 && \text{if } i > j; i, j \in \{1, \dots, n\} \\ R^{ij}_{ij} &= x_{ij} && \text{if } i > j; i, j \in \{1, \dots, n\} \\ R^{ij}_{ij} &= zx^{-1}_{ji} && \text{if } i > j; i, j \in \{1, \dots, n\} \end{aligned} \tag{1.9}$$

Then R is a solution of the Yang-Baxter equation (1.5).

For instance the example (1.6) arises by taking $y = q - q^{-1}, z = 1$ (so that the solutions of (1.8) are $\rho_1 = q, \rho_2 = -q^{-1}), x_{21} = 1$, and taking $R^{11}_{11} = R^{22}_{22} = q$.

The "single block cell size 1" solutions described above form $2^n\ \binom{n}{2} + 2$ parameter families indexed by the possible choices ρ_1 or ρ_2 which intersect in points where y and z are such that $\rho_1 = \rho_2$.

Multiplying R with a nonzero scalar in K does not change the ideal $I(R)$. For instance if $\sqrt{z}$ exists in K, the parameter z can be normalized to 1. Thus these solutions give rise to a number of $\binom{n}{2} + 1$ parameter families of "quantum groups". Among these are $\binom{n}{2} + 1$ parameter quantum deformations of the supergroups $GL(l|m), l + m = n,\ m \geq 1$. These I now proceed to describe.

$\binom{n}{2}+1$ *parameter quantum deformation of* $GL(l|m), l+m=n$.

In the "single block" recipe above take $z=1$, so that $\rho_1=q, \rho_2=q^{-1}$. Take $R^{ii}_{ii}=q$ for $1\le i\le l, R^{ii}_{ii}=-q^{-1}$ for $l+1\le i\le l+m$, and the remaining R^{ab}_{cd} as in (1.9). The supergroup $GL(l|m)$ itself sits at the spot where $q=1, x_{ij}=1$ for $i\le l$ or $j\le l$ and $x_{ij}=-1$ for $i,j\ge l+1$.

More precisely the corresponding $H_n(R)$ describes the bialgebras of functions for the matrix algebras of functions from which the corresponding quantum supergroups arise by localization at the appropriate quantum super determinant.

For instance for $l=1, m=2$, the R-matrix for $GL(1|2)$ is given by

$$\left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
\hline
 & & & 1 & & & & & \\
 & & & & -1 & & & & \\
 & & & & & -1 & & & \\
\hline
 & & & & & & 1 & & \\
 & & & & & & & -1 & \\
 & & & & & & & & -1
\end{array}\right) \tag{1.10}$$

and its $4=\binom{3}{2}+1$ parameter quantum deformation is given by the R-matrix

$$\left(\begin{array}{ccc|ccc|ccc}
q & & & & & & & & \\
 & x_{21}^{-1} & & q-q^{-1} & & & & & \\
 & & x_{31}^{-1} & & & & q-q^{-1} & & \\
\hline
 & & & x_{21} & & & & & \\
 & & & & -q^{-1} & & & & \\
 & & & & & x_{32}^{-1} & & q-q^{-1} & \\
\hline
 & & & & & & x_{31} & & \\
 & & & & & & & x_{32} & \\
 & & & & & & & & -q^{-1}
\end{array}\right)$$

(from which (1.10) arises by taking $q=1, x_{21}=x_{31}=1, x_{32}=-1$).

Note that the $\binom{n}{2}+1$ parameter quantum deformation of $GL(n)$ and the $\binom{n}{2}+1$ parameter deformation of $GL|l)m), l+m=n$ intersect at the points where $q=-q^{-1}=\pm i=\pm\sqrt{-1}$.

2. Algebra of functions of super space

Consider a superspace $V=V_0\oplus V_1$ over the field K of dimension $l|m$ so that the summand V_0 of even vectors is of dimension l and the space V_1 of odd vectors is of dimension m. The algebra of (algebraic) functions of V is, [1,2,6], and especially [7],

$$A_{l|m}=K[X^1,\dots,X^l]\otimes K<\zeta^{l+1},\dots,\zeta^{l+m}> \tag{2.1}$$

where the $X^1,\dots,X^l$ are commuting indeterminates, $X^iX^j=X^jX^i$, and the $\zeta^{l+1},\dots,{}^{l+m}$ are anti-commuting indeterminates, $\zeta^{l+1}\zeta^{l+j}=-\zeta^{l+j}\zeta^{l+1}$ (Grassmann indeterminates). In other words $A_{l|m}$ is the quotient of the free associative algebra $K<Z^1,\dots,Z^{l+m}>$ subject to the relations $Z^iZ^j=Z^jZ^i$ if i or $j\le l$ and $Z^iZ^j=-Z^jZ^i$ if both i and j are $\ge l+1$.

3. Algebra of functions on supermatrices

One way to find the algebra of functions for supermatrices is to consider the free algebra $K<t>$ and $K<Z>$ and the standard left matrix coaction

$$Z^i \mapsto t^i_j \otimes Z^j \tag{3.1}$$

Now consider the relations defining $A_{l|m}$ as a quotient of $K < Z >$ and calculate the relations that are needed between the t^i_j in order that (3.1) respects the relations defining $A_{l|m}$. For instance from $X^1\zeta^2 = \zeta^2 X^1$ in $A_{1|2}$ one gets,

$$\begin{aligned}&(t^1_1 \otimes X^1 + t^1_2 \otimes \zeta^2 + t^1_3 \otimes \zeta^3)(t^2_1 \otimes X^1 + t^2_2 \otimes \zeta^2 + t^2_3 \otimes \zeta^3) = \\ &= (t^2_1 \otimes X^1 + t^2_2 \otimes \zeta^2 + t^2_3 \otimes \zeta^3)(t^1_1 \otimes X^1 + t^1_2 \otimes \zeta^2 + t^1_3 \otimes \zeta^3)\end{aligned} \tag{3.2}$$

which, using $X^1\zeta^2 = \zeta^2 X^1, (\zeta^2)^2 = (\zeta^3)^2 = 0, \zeta^2\zeta^3 = -\zeta^3\zeta^2$, gives

$$t^1_1 t^2_1 = t^2_1 t^1_1 \tag{3.3}$$

$$t^1_1 t^2_2 + t^1_2 t^2_1 = t^2_2 t^1_1 + t^2_1 t^1_2 \tag{3.4}$$

$$t^1_1 t^2_3 + t^1_3 t^2_1 = t^2_1 t^1_3 + t^2_3 t^1_1 \tag{3.5}$$

$$t^1_2 t^2_3 - t^1_3 t^2_2 = t^2_2 t^1_3 - t^2_3 t^1_2 \tag{3.6}$$

This gives precisely half of the relations needed. The other half comes from the right coaction on the dual superspace $K[Y_1, \dots, Y_l] \otimes K < \eta_{l+1}, \dots, \eta_{l+m} >$ coming from the right matrix coaction

$$Z_i \mapsto Z_j \otimes t^j_i \tag{3.7}$$

For instance for $l = 1, m = 2$, one of the relations coming from $\eta_2\eta_3 = -\eta_3\eta_2$ is

$$t^1_2 t^2_3 + t^2_2 t^1_3 = -t^2_3 t^1_2 - t^1_3 t^2_2 \tag{3.8}$$

which combined with (3.6) gives (unless $\mathrm{char}(K) = 2$)

$$t^2_2 t^1_3 = -t^1_3 t^2_2, t^1_2 t^2_3 = -t^2_3 t^1_2 \tag{3.9}$$

These calculations are very similar to those in section 2 of [5] and as in appendix 1 of [5] one verifies that the resulting relations generate a bialgebra ideal in $K < t >$. The final result is, see also [7],

3.10 THEOREM. The bialgebra of functions $A^{l|m\times l|m}$ for the supermatrices of size $(l|m) \times (l|m)$ over K is the quotient of $K < t >$ given by the relations

$$\begin{aligned}
t^a_b t^r_s &= t^r_s t^b_b && \text{if} \quad 1 \le a, b \le l \\
t^a_b t^r_s &= t^r_s t^a_b && \text{if} \quad 1 \le a, s \le l; l+1 \le b, r \le l+m \\
t^a_b t^r_s &= -t^r_s t^a_b && \text{if} \quad 1 \le a \le l; l+1 \le b, s \le l+m \\
t^a_b t^r_s &= -t^r_s t^a_b && \text{if} \quad 1 \le b \le l; l+1 \le a, r \le +m \\
t^a_b t^r_s &= t^r_s t^a_b && \text{if} \quad l+1 \le a, b, r, s \le l+m
\end{aligned} \tag{3.10}$$

These relations are easily remembered by partitioning the matrix $T = (t^i_j)$ in four parts as indicated

$$\begin{array}{cc} & \begin{array}{cc} l & m \end{array} \\ \begin{array}{c} l \\ m \end{array} & \left(\begin{array}{c|c} I & II \\ \hline III & IV \end{array}\right) = T \end{array}$$

and assigning the bidegrees $(0,0),(1,0),(0,1),(1,1)$ to the variables in the blocks I, II, III, IV respectively. These bidegrees correspond to even to even, odd to even, even to odd, odd to odd maps. Writing d_1 and d_2 for the first and second components of the bidegree respectively, the commutation rules (3.11) become

$$t_b^a t_s^r = (-1)^{d_1(t_b^a)d_1(t_s^r)+d_2(t_b^a)d_2(t_s^r)} t_s^r t_b^a \tag{3.11}$$

It is now straightforward to verify

3.13 PROPOSITION. Let R be the diagonal $n^2 \times n^2$ matrix, $n = l + m$, with $R_{ij}^{ij} = 1$ for $i \le l$ or $j \le l$ and $R_{ij}^{ij} = -1$ if $i > l$ and $j > l$. Then the ideal $I(R)$ is precisely that of the relations (3.11). I.e. $H_n(R)$ for this particular R is the bialgebra of (algebraic) functions for $(l|m \times l|m)$ supermatrices.

4. MULTIPARAMETER QUANTUM SUPERGROUPS

Now consider the R-matrtix already specified in the introduction, to obtain

4.1. THEOREM. Let R be the $n^2 \times n^2$ matrix, $n = l + m$, with $R_{ij}^{ij} = q - q^{-1}$ if $i < j, R_{ii}^{ii} = q$ if $i \le l, R_{ii}^{ii} = -q^{-1}$ if $i > l, R_{ij}^{ij} = x_{ij}$ if $i > j, R_{ij}^{ij} = x_{ij}^{-1}$ if $i < j, R_{cd}^{ab} = 0$ in all other case. Then the $H_n(R)$ are an $\binom{n}{2} + 1$ parameter family of bialgebra deformations of the bialgebra of functions on $l|m \times l|m$ supermatrices. This algebra itself is located at $q = 1, x_{ij} = 1$ for $i \le l$ or $j \le l$, $x_{ij} = -1$ if $i, j \ge l + 1$.

5. CONCLUDING REMARKS

The algebra of functions of the supergroup $GL(l|m)$ arises from $A^{l|m \times l|m}$ by localization at the superdeterminant (polynomial). For the single parameter quantum deformation of $A^{l|m \times l|m}$ defined by $x_{ij} = 1$ for $i \le l$ or $j \le l, x_{ij} = -$ if $i, j \ge l + 1$ there is a corresponding quantum superdeterminant, [7]. This should generalize to the multiparameter case, but that remains to be verified.

For the $\binom{n}{2}+1$ parameter quantum deformation of $GL(n)$ the Poincaré series of each of the deformed algebras is the same as that of the function algebra of $GL(n)$ itself. This is not the case for the quantum deformation of $GL(l|m)$. At $q = -q^{-1}$, i.e. $q = \pm i$, that is precisely the places where the two families can intersect, there is a jump.

Over $\mathbb{C}$ the quantum groups (algebras of functions) and quantum super group (algebras of functions) with $z = 1$ and $z = -1$ are isomorphic. It remains to sort out what is the situation over $\mathbb{R}$ where $z = -1$ cannot be eaesily normalized away to $z = 1$.

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