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Abstract

The present study concerns the analysis of the hitting point identity for a nearest-neighbour random walk of which the one-step transition to the NE, SE, SW and NW are the only transitions with nonzero probabilities. The one-step transition vector has a symmetrical probability distribution with zero drifts. The state space of the random walk is the set of lattice points in the first quarter plane, the point at the coordinate axes are all absorbing states. The distribution of the hitting point with the axes is investigated for the case $-1 < \rho < 0$ and for the case $\rho = 0$, here ρ is the correlation of the components of the one-step transition vector. For $-1 < \rho < 0$ the generating function of this distribution is derived. For $\rho = 0$ the distribution is calculated explicitly.

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1. Introduction

In this study we consider a nearest-neighbour random walk with state space

$$\{0,1,2,\ldots\} \times \{0,1,2,\ldots\}.$$

The interior $\mathcal{S}$ and the boundary $B_{01} \cup B_{00} \cup B_{10}$ are defined by:

$$\begin{aligned}
&\mathcal{S} := \{(x,y) : (x,y) \in \{1,2,\ldots\} \times \{1,2,\ldots\}\}, && (1.1)\\
&B_{10} := \{(x,0) : x \in \{1,2,\ldots\}\},\\
&B_{00} := \{(0,0)\},\\
&B_{01} := \{(0,y) : y \in \{1,2,\ldots\}\},\\
&B := B_{10} \cup B_{00} \cup B_{01}.
\end{aligned}$$

The random walk is supposed to be semi-homogeneous, i.e. the distribution of the one-step displacement vector $(\boldsymbol{\xi}_n, \boldsymbol{\eta}_n)$ out from a point $(x,y) \in \mathcal{S}$ at time n, $n = 1,2,\ldots$, is supposed to be independent of (x,y) and of t_n. For the random walk do be studied we take (see figure 1) with $-1 < \rho < 1$:

$$\mathrm{E}\{p^{\boldsymbol{\xi}} q^{\boldsymbol{\eta}}\} = \frac{1+\rho}{4}(pq + \frac{1}{pq}) + \frac{1-\rho}{4}(\frac{p}{q} + \frac{q}{p}) = 1, \quad |p| = 1,\ |q| = 1, \tag{1.2}$$

with

$$(\boldsymbol{\xi}, \boldsymbol{\eta} \sim (\boldsymbol{\xi}_n, \boldsymbol{\eta}_n),$$

i.e. the vector $(\boldsymbol{\xi}, \boldsymbol{\eta})$ has the same distribution as $(\boldsymbol{\xi}_n, \boldsymbol{\eta}_n)$.

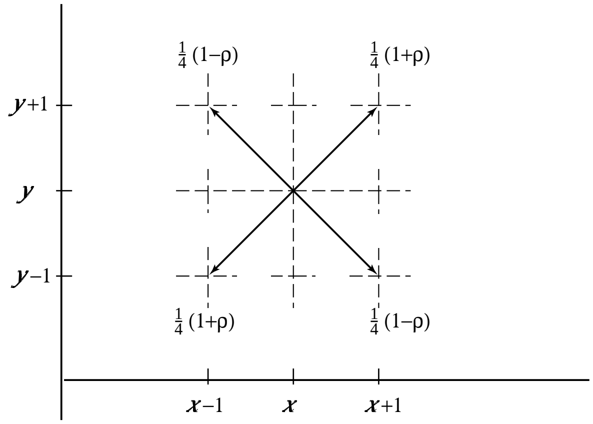

FIGURE 1.

From (1.2) it is readily seen that

$$\begin{aligned} &\mathrm{E}\{\boldsymbol{\xi}\} = \mathrm{E}\{\boldsymbol{\eta}\} = 0, \\ &\mathrm{E}\{\boldsymbol{\xi}^2\} = \mathrm{E}\{\boldsymbol{\eta}^2\} = 1, \\ &\mathrm{E}\{\boldsymbol{\xi}\boldsymbol{\eta}\} = \rho, \end{aligned} \tag{1.3}$$

the first relation in (1.3) shows that the random walk has zero drifts at all points of $\mathcal{S}$.

For the random walk with the one-step transition $(\boldsymbol{\xi}, \boldsymbol{\eta})$ we analyse the hitting point identity, cf. [2], and derive the generating function of the distribution of the first entrance point of the coordinate axes when starting of a point (x_0, y_0) with $x_0 > 0, y_0 > 0$. In Section 2 the kernel of this random walk is analysed. The hitting point identity is discussed in Section 3, and in the subsequent Section 4 an explicit expression for the generating function of the hitting point distribution is obtained for $-1 < \rho < 0$, by solving a standard boundary value problem. The case $\rho = 0$ is analysed in Section 5. The analysis reduces here to the solution of the classical Dirichlet boundary value problem for the circle. As a consequence the distribution of the hitting point of the boundary when starting at the point (1.1) has a simple explicit representation.

2. THE KERNEL

In the analysis of random walks the kernel $K(p, q)$ plays a crucial role, cf. [2]. It is defined by

$$\begin{aligned} K(p,q) :=\ & pq - \mathrm{E}\{p^{\boldsymbol{\xi}+1} q^{\boldsymbol{\eta}+1}\} = pq - \frac{1+\rho}{4}(p^2q^2+1) - \frac{1-\rho}{4}(p^2+q^2) \\ &= \frac{1-\rho}{4}[p - q + \mathrm{i}b(1-pq)][q - p + \mathrm{i}b(1-pq)], \end{aligned} \tag{2.1}$$

here

$$b := \sqrt{\frac{1+\rho}{1-\rho}}. \tag{2.2}$$

Note that

$$-1 < \rho < 0 \iff 0 < b < 1. \tag{2.3}$$

In our analysis we need properties of the zeros of the kernel. It is readily verified that

$$\begin{aligned} K(p,q) = 0 \quad &\to \{p = 1 \leftrightarrow q = 1\}, \\ &\to \{p = -1 \leftrightarrow q = -1\}; \end{aligned} \tag{2.4}$$

and for $p = \pm 1$, the zero $q = \pm 1$ has multiplicity two.
Further we have

LEMMA 2.1. *For $|q| = 1$, $q \neq \pm 1$, the kernel $K(p,q)$ has exactly one zero in $|p| < 1$; similarly, with p and q interchanged.*

PROOF. For $|q| = 1$, $q \neq \pm 1$ put

$$p = sq, \tag{2.5}$$

then

$$K(p,q) = 0 \Longleftrightarrow s = \mathrm{E}\{s^{\boldsymbol{\xi}+1} q^{\boldsymbol{\eta}+\boldsymbol{\xi}}\}. \tag{2.6}$$

Because

$$\Pr\{\boldsymbol{\xi} = \pm 1\} = 1,$$

it follows that the righthand side in (2.5) is a regular function of s for $|s| < 1$ and that it is continuous for $|s| \leq 1$. By using: for $|q| = 1$, $q \neq \pm 1$,

$$|\mathrm{E}\{s^{\boldsymbol{\xi}+1} q^{\boldsymbol{\eta}+\boldsymbol{\xi}}\}| < 1 \text{ for } |s| = 1,$$

it follows from Rouché's theorem that the equation for s, cf. (2.6), has exactly one zero in $|s| < 1$; and so from (2.5) the statement follows. □

We have, cf. (2.1),

$$\begin{aligned} p - q + \mathrm{i}b(1 - pq) = 0 \quad &\Rightarrow p = \frac{q - \mathrm{i}b}{-\mathrm{i}bq + 1} \quad \text{and} \quad q = \frac{p + \mathrm{i}b}{\mathrm{i}bp + 1}, \\ q - p + \mathrm{i}b(1 - pq) = 0 \quad &\Rightarrow q = \frac{p - \mathrm{i}b}{-\mathrm{i}bp + 1} \quad \text{and} \quad p = \frac{q + \mathrm{i}b}{\mathrm{i}bq + 1}. \end{aligned} \tag{2.7}$$

Put

$$\begin{aligned} p_1(q) &:= \frac{q - \mathrm{i}b}{-\mathrm{i}bq + 1}, \quad & p_2(q) &:= \frac{q + \mathrm{i}b}{\mathrm{i}bq + 1}, \\ q_1(p) &:= \frac{p + \mathrm{i}b}{\mathrm{i}bp + 1}, \quad & q_2(p) &:= \frac{p - \mathrm{i}b}{-\mathrm{i}bp + 1}. \end{aligned} \tag{2.8}$$

Hence from (2.1) we have:

$$K(p_{1,2}(q), q) = 0 \text{ and } K(p, q_{1,2}(p)) = 0. \tag{2.9}$$

Note that the reducibility of the kernel $K(p,q)$ leads to a very simple characterisation of its zero-tuples. Define

$$\begin{aligned} \mathrm{S}_1 &= \{p : p = p_1(q), -\infty < q < \infty\}, \\ \mathrm{S}_2 &:= \{p : p = p_2(q), -\infty < q < \infty\}. \end{aligned} \tag{2.10}$$

Because $q \to p = p_1(q)$ is a linear fractional transformation it follows that for $0 < b < 1$, S_1, and similarly S_2, is a circle. Actually with $p = u + \mathrm{i}v$ we have

$$\begin{aligned} \mathrm{S}_1 &= \{(u,v) : u^2 + \left(v - \frac{1-b^2}{2b}\right)^2 = 1 + \left(\frac{1-b^2}{2b}\right)^2\}, \\ \mathrm{S}_2 &= \{(u,v) : u^2 + \left(v + \frac{1-b^2}{2b}\right)^2 = 1 + \left(\frac{1-b^2}{2b}\right)^2\}. \end{aligned} \tag{2.11}$$

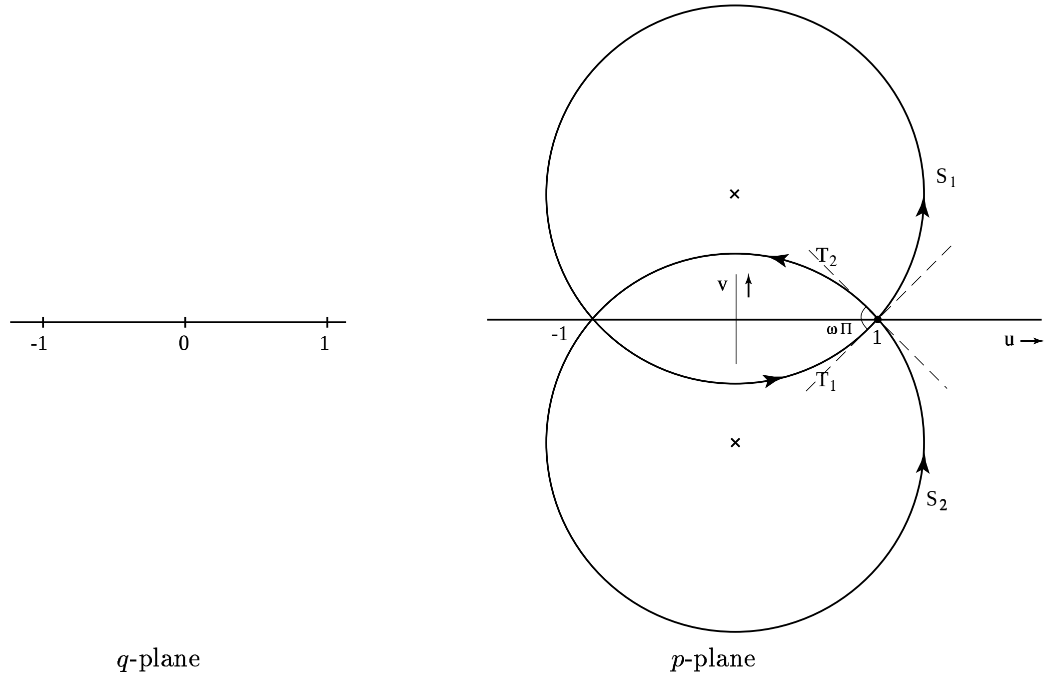

FIGURE 2

Note that for $i = 1, 2$,

$$(1,0) \in \mathrm{S_i} \;\text{ and }\; (-1,0) \in \mathrm{S}_i. \tag{2.12}$$

In Figure 2 these circles S_1 and S_2 are traced for $0 < b < 1$. The angle between the two circles is denoted by $\omega\pi$. From (2.11) we have

$$2u + 2(v - \frac{1-b^2}{2b})\frac{\mathrm{d}v}{\mathrm{d}u} = 0, \tag{2.13}$$

and so for $u = 1$, $v = 0$ we obtain: for $0 < b < 1$,

$$\tan\frac{1}{2}\omega\pi = \frac{2b}{1-b^2} > 0,\; 0 < \frac{1}{2}\omega\pi < \frac{1}{2}\pi. \tag{2.14}$$

From (2.8) and (2.11) it is seen that

$$p_1(0) = -\mathrm{i}b,\; p_2(0) = \mathrm{i}b. \tag{2.15}$$

Denote by

$$\begin{aligned} &\mathrm{T}_1 \text{ the part of } \mathrm{S}_1 \text{ with } \operatorname{Im} p \leq 0, \\ &\mathrm{T}_2 \text{ the part of } \mathrm{S}_2 \text{ with } \operatorname{Im} p \geq 0. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) it seen that

$$q \to p = p_1(q) \text{ maps } (-1,1) \text{ one-to-one onto } \mathrm{T}_1,$$

$$q \to p = p_1(q) \text{ maps } (-1,1) \text{ one-to-one onto } \mathrm{T}_2,$$

and for $0 < b < 1$:

$$S_1 \text{ and } S_2 \text{ are both traversed counterclockwise for } q : -\infty \to \infty. \tag{2.17}$$

Note further that $q \to p = p_1(q)$ maps the halfplane Im $q > 0$ conformally onto S_1^+, the interior of S_1; similarly $q \to p = p_2(q)$ maps the halfplane Im $q > 0$ onto S_2^+.

Put

$$T := T_1 \cup T_2, \tag{2.18}$$

so that T is a simple smooth contour except for its corner points $p = \pm 1$. Denote by T^+ the interior of T.

For the analysis in the next section we need the conformal map of $\{z : \mathrm{Im} z > 0\}$ onto T^+ for the case with $0 < b < 1$. Denote this map by

$$z \to p = f(z), \ \mathrm{Im}\ z > 0. \tag{2.19}$$

A result in [1] p. 209 shows that $f(z)$ has the following structure:

$$f(z) = \frac{A(z-\alpha)^\omega + B(z-\beta)^\omega}{C(z-\alpha)^\omega + D(z-\beta)^\omega}, \tag{2.20}$$

with A,B,C and D complex constants satisfying $AD - BC \neq 0$, and α, β the points of the z-plane which correspond to the corner points of T, note that $\omega\pi$ is the angle at the corner points of the contour T.

REMARK 2.1. The Figure 22 in [1] p. 209 contains a misleading misprint; in that Figure α should be replaced by $\alpha\pi$. □

To determine A,B,C and D we require that for $0 < b < 1$:

$$f(1) = 1, \quad f(-1) = 1, \quad f(0) = -\mathrm{i}b. \tag{2.21}$$

The first two conditions in (2.21) lead to

$$A = C, \ B = -D \text{ for } \alpha = -1, \ \beta = 1,$$

so

$$f(z) = \frac{(z+1)^\omega - c(z-1)^\omega}{(z+1)^\omega + c(z-1)^\omega} \quad \text{with} \quad c := \frac{D}{A}. \tag{2.22}$$

From the third condition in (2.21) we obtain

$$c = \frac{1+\mathrm{i}b}{1-\mathrm{i}b}(-1)^\omega. \tag{2.23}$$

We have for $0 < b < 1$.

$$\frac{1+\mathrm{i}b}{1-\mathrm{i}b} = \frac{1-b^2+2\mathrm{i}b}{1+b^2} = \mathrm{e}^{\mathrm{i}\arctan\frac{2b}{1-b^2}}, \tag{2.24}$$

$$(-1)^\omega = \mathrm{e}^{\mathrm{i}\pi\omega} = \mathrm{e}^{2\mathrm{i}\arctan\frac{2b}{1-b^2}}, \text{ cf. (2.14).}$$

Hence from (2.23) and (2.24),

$$c = \mathrm{e}^{-\mathrm{i}\arctan\frac{2b}{1-b^2}} = \frac{1-\mathrm{i}b}{1+\mathrm{i}b}, \tag{2.25}$$

and from (2.22) we obtain

$$f(z) = \frac{(1+\mathrm{i}b)(z+1)^\omega - (1-\mathrm{i}b)(z-1)^\omega}{(1+\mathrm{i}b)(z+1)^\omega + (1-\mathrm{i}b)(z-1)^\omega}. \tag{2.26}$$

A simple calculation shows that

$$f(\infty) = \mathrm{i}b, \tag{2.27}$$

and it is seen from (2.19), (2.21) and (2.27) that whenever z with Im $z = 0$ traverses from $-\infty \to \infty$ then p traverses T counterclockwise.

REMARK 2.1. In the derivations above we have used the principal value of z^ω, i.e. we take in

$$z^\omega = \mathrm{e}^{\omega\{\log|z|+2\pi\mathrm{i}m\}}, \quad m = \dots, -1, 0, 1, \dots,$$

$m = 0$. Obviously this does not influence the expression for $f(z)$. □

Obviously $p = 0 \in \mathrm{T}^+$. Define z_0 by

$$f(z_0) = 0, \quad \mathrm{Im}\ z_0 > 0. \tag{2.28}$$

Because $f(z)$ maps the halfplane Im$z > 0$ conformally onto T^+ it follows that the equation (2.28) has a unique solution. From (2.24), (2.26) and (2.28) it follows that

$$\left(\frac{z_0+1}{z_0-1}\right)^\omega = \frac{1-\mathrm{i}b}{1+\mathrm{i}b} = \mathrm{e}^{-\mathrm{i}\arctan\frac{2b}{1-b^2}} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi\omega},$$

so

$$\frac{z_0+1}{z_0-1} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi} = -\mathrm{i}.$$

from which we obtain

$$z_0 = \mathrm{i}. \tag{2.29}$$

With

$$f(\infty) := \lim_{|z|\to\infty} f(z), \tag{2.30}$$

it is readily verified by using (2.26) that

$$f(\infty) = \mathrm{i}b, \tag{2.31}$$

and a simple calculation shows that: for $|z| \to \infty$,

$$|f(z) - f(\infty)| = \frac{(1+b^2)\omega}{|z|} + \mathrm{O}\left(\frac{1}{|z|^2}\right). \tag{2.32}$$

3. THE HITTING POINT IDENTITY

In this section we consider the random walk with the one-step transition vector as defined in Section 1, cf. (1.2), and with the boundary B, cf. (1.1), a set of absorbing states.

It will *always be assumed* in this section that, cf. (1.3),

$$-1 < \rho < 0. \tag{3.1}$$

Hence from (2.3) we have

$$0 < b < 1. \tag{3.2}$$

Let

$(x_0, y_0) \in \mathcal{S}$.

be the starting point of this random walk.

Denote by $\mathbf{m}(x_0, y_0)$ the first entrance time out from (x_0, y_0) into the boundary B. From Theorem II.2.4.4 of [2] p. 81, we have because of (3.1) that

$$\Pr\{\mathbf{m}(x_0, y_0) < \infty\} = 1. \tag{3.3}$$

$$\begin{aligned} \mathrm{E}\{\mathbf{m}(x_0, y_0)\} &= \frac{x_0 y_0}{-\rho} \quad \text{for} \quad \rho < 0, \\ &= \quad \infty \qquad \text{for} \quad \rho \geq 0. \end{aligned}$$

Note that in Theorem II.2.4.4 of [2] we have to take: $\boldsymbol{\xi}_3 = \boldsymbol{\xi} + 1$, $\boldsymbol{\eta}_3 = \boldsymbol{\eta} + 1$.

Denote by $(\mathbf{k}_1, \mathbf{k}_2)$ the hitting point of B when starting in (x_0, y_0). From Corollary II.2.4.1 of [2] p. 79, it follows that: for every zero-tuple (p, q), $|p| \leq 1$, $|q| \leq 1$, of $K(p, q)$ holds

$$p^{x_0} q^{y_0} = \Phi(p, 0) + \Phi_0 + \Phi(0, q), \tag{3.4}$$

where

$$\begin{aligned} &\Phi(p, 0) := \mathrm{E}\{p^{\mathbf{k}_1}(\mathbf{k}_1 > 0, \mathbf{k}_2 = 0)\}, \quad |p| \leq 1. \\ &\Phi_0 := \mathrm{E}\{(\mathbf{k}_1 = \mathbf{k}_2 = 0)\}, \\ &\Phi(0, q) := \mathrm{E}\{q^{\mathbf{k}_2}(\mathbf{k}_1 = 0, \mathbf{k}_2 > 0)\}, \quad |q| \leq 1; \end{aligned} \tag{3.5}$$

(here we have supressed in the notations in (3.5) the conditioning event $\mathbf{x}_0 = x_0, \mathbf{y}_0 = y_0$).

Note that (3.5) implies that

$$\Phi(0, 0) = 0, \tag{3.6}$$

and that (3.3) implies because $K(1, 1) = 0$ that

$$\Phi(1, 0) + \Phi_0 + \Phi(0, 1) = 1. \tag{3.7}$$

Obviously,

$$\begin{aligned} &\Phi(p,0)/\Phi(1,0) \textit{ is the generating function of a probability distribution with support} \\ &\textit{the set } \{1,2,3,\ldots\}, \textit{ similarly for } \Phi(0,q)/\Phi(0,1), \textit{ further } \Phi_0 \geq 0. \end{aligned} \tag{3.8}$$

The problem to be studied in this paper is the determination of the functions defined in (3.5). Note that we have from Theorem II.2.4.4 of [1], p. 81, cf. (1.3),

$$\begin{aligned} &\mathrm{E}\{\mathbf{k}_1\} = x_0, \quad \mathrm{E}\{\mathbf{k}_2\} = y_0, \\ &\mathrm{E}\{(\mathbf{k}_1 - x_0)^2\} = \mathrm{E}\{(\mathbf{k}_2 - y_0)^2\} = \frac{x_0 y_0}{-\rho}. \end{aligned} \tag{3.9}$$

To study this problem we first derive an analytic continuation of the relation (3.4) to be satisfied by $\Phi(p, 0)$, Φ_0 and $\Phi(0, q)$.

From (2.8) and (2.16) it is readily seen that

$$\begin{aligned} &\text{(3.4) holds for } p \in \mathrm{T}_1 \quad \text{with} \quad q = q_1(p) \in [-1, 1], \\ &\text{(3.4) holds for } p \in \mathrm{T}_2 \quad \text{with} \quad q = q_2(p) \in [-1, 1]. \end{aligned} \tag{3.10}$$

Hence we have

$$p^{x_0}q_1^{y_0}(p) = \Phi(p,0) + \Phi_0 + \Phi(0,q_1(p)), \quad p \in \mathrm{T}_1,$$
$$p^{x_0}q_2^{y_0}(p) = \Phi(p,0) + \Phi_0 + \Phi(0,q_2(p)), \quad p \in \mathrm{T}_2. \tag{3.11}$$

The function $q_1(p)$, cf. (2.8), has a simple pole at $p = \mathrm{i}/b$, similarly, $p = -\mathrm{i}/b$ is the only pole of $q_2(p)$. Consequently

$$p^{x_0}q_1^{y_0}(p) \textit{ and } p^{x_0}q_2^{y_0}(p) \textit{ are both regular for } p \in \mathrm{T}^+ \cup \mathrm{T}. \tag{3.12}$$

For $p \in \mathrm{T}^+ \cup \mathrm{T}$ we have $|p| \leq 1$, and $\Phi(p,0)$ is regular for $|p| < 1$, continuous for $|p| \leq 1$. Further T is an analytic arc except for its endpoints, moreover $q_1(p)$, $p \neq \mathrm{i}/b$ is a regular function of p. Consequently the first relation of (3.11) implies that $\Phi(0,q_1(p))$ possesses an analytic continuation in T^+, which is continuous in $\mathrm{T}^+ \cup \mathrm{T}$. Hence it follows from (3.11): for $p \in \mathrm{T}^+ \cup \mathrm{T}$,

$$p^{x_0}q_1^{y_0}(p) = \Phi(p,0) + \Phi_0 + \Phi(0,q_1(p)),$$
$$p^{x_0}q_2^{y_0}(p) = \Phi(p,0) + \Phi_0 + \Phi(0,q_2(p)), \tag{3.13}$$

with $\Phi(p,0)$, $\Phi(0,q_1(p))$ and $\Phi(0,q_2(p))$ regular for $p \in \mathrm{T}^+$, continuous for $p \in \mathrm{T}^+ \cup \mathrm{T}$; actually only the statements in so for they concern $\Phi(0,q_1(p))$ have been proved; however, those for $\Phi(0,q_2(p))$ can be proved similary.

4. The expressions for $\Phi(p,0)$ and $\Phi(0,q)$

In this section we shall derive the expressions for $\Phi(p,0)$ and $\Phi(0,q)$. We start from the relations (3.13) and assume again $0 < b < 1$. Because

$$q_1(p) \in [-1,1] \quad \text{for} \quad p \in \mathrm{T}_1,$$
$$q_2(p) \in [-1,1] \quad \text{for} \quad p \in \mathrm{T}_2. \tag{4.1}$$

it follows from (3.8) that

$$\Phi(0,q_1(p)) \text{ is real for } p \in \mathrm{T}_1,$$
$$\Phi(0,q_2(p)) \text{ is real for } p \in \mathrm{T}_2. \tag{4.2}$$

Consequently from (3.13)

$$\begin{aligned} \mathrm{Im}\Phi(p,0) \quad &= q_1^{y_0}(p)\mathrm{Im}(p^{x_0}) \quad \text{for} \quad p \in \mathrm{T}_1, \\ &= q_2^{y_0}(p)\mathrm{Im}(p^{x_0}) \quad \text{for} \quad p \in \mathrm{T}_2. \end{aligned} \tag{4.3}$$

Put

$$\begin{aligned} \phi_1(p) \quad &:= q_1^{y_0}(p)\mathrm{Im}(p^{x_0}) \quad \text{for} \quad p \in \mathrm{T}_1, \\ &:= q_2^{y_0}(p)\mathrm{Im}(p^{x_0}) \quad \text{for} \quad p \in \mathrm{T}_2, \end{aligned} \tag{4.4}$$

so that we have, note $|\Phi(p,0)| < 1$ for $p \in \mathrm{T}\backslash\{-1,1\}$,

$$\begin{aligned} &\text{i.} \quad \mathrm{Im}\Phi(p,0) = \phi_1(p) \quad \text{for} \quad p \in \mathrm{T}, \\ &\text{ii.} \quad \Phi(p,0) \text{ is regular for } p \in \mathrm{T}^+ \text{ and continous for } p \in \mathrm{T}^+ \cup \mathrm{T}. \end{aligned} \tag{4.5}$$

The conditions (4.5) formulate a standard Boundary Value Problem for the domain T^+, cf. [5]. By using the conformal map (2.19) we transform this problem into a boundary value problem with domain a halfplane. Put, cf. (2.19),

$$F_1(z) := \phi_1(f(z)) \quad \text{for} \quad \text{Im } z \geq 0, \tag{4.6}$$

then (4.5) may be rewritten as:

$$\begin{aligned} &\text{i.} \quad \text{Re}\{-\mathrm{i}\Phi(F(z),0)\} = F_1(z) \quad \text{for} \quad \text{Im } z = 0, \\ &\text{ii.} \quad \Phi(f(z)) \text{ is regular for Im } z > 0, \text{ and continuous for Im } z \geq 0. \end{aligned} \tag{4.7}$$

Obviously $F_1(z)$ is continuous as a function of z with Im $z = 0$. It is further readily verified that $F_1(z)$ satisfies on Im $z = 0$ a Hölder condition, cf. (2.19), (2.26) and [3], p. 13 § 6,1°. Further it follows from (2.32) that

$$|F_1(z) - F_1(\infty)| < \frac{\text{constant}}{|z|^{\varkappa_0}} \text{ for } |z| \to \infty. \tag{4.8}$$

Because $F_1(\cdot)$ has the properties just mentioned, we can use the result of Section 43 in [3] to formulate the solution of the boundary value problem stated in (4.7). It results that: for Im $z > 0$,

$$\Phi(f(z),0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t-z} \mathrm{d}t - C_p, \tag{4.9}$$

here C_p is a real constant and the integral is a principal value integral, i.e.

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \dots \mathrm{d}t := \lim_{\tau \to \infty} \frac{1}{\pi} \int_{-\tau}^{\tau} \dots \mathrm{d}t.$$

It is readily verified that (4.9) satisfies (4.7), as we now show. Obviously $\Phi(f(z),0)$ as given by (4.9) is regular for Im $z > 0$, so that (4.7)ii is satisfied. To prove that (4.7)i is fulfilled note that $F_1(z)$ satisfies a Hölder condition, so it follows by applying the Plemelj-Sokhotski formula, cf. [3], p. 42, Section 17, that: for $-\infty < z < \infty$,

$$\lim_{z \to z_1} \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t-z} \mathrm{d}t - \frac{1}{\mathrm{i}} C_p = F_1(z_1) + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t-z_1} \mathrm{d}t - \frac{1}{\mathrm{i}} C_p. \tag{4.10}$$

Because $F_1(z)$ is real for $z \in (-\infty, \infty)$ and C_p is real it follows from (4.9) and (4.10) that (4.9) satisfies (4.7)i. Hence $\Phi(f(z),0)$ as given by (4.10) satisfies (4.7). It is readily shown that apart from the additive constant C_p the conditions (4.7) determine $\Phi(f(z),0)$, Im $z > 0$, uniquely.

From the continuity of $\Phi(z,0)$ for Im $z \geq 0$ we have

$$\Phi(f(z),0) = \lim_{\substack{\zeta \to z \\ \text{Re}\zeta > 0}} \Phi(f(\zeta),0), \quad \text{for} \quad \text{Im } z = 0. \tag{4.11}$$

It follows by using the Plemelj-Sokhotski formula, cf. [3] p. 42, that we obtain from (4.9): for Im $z = 0$,

$$\Phi(f(z),0) = \mathrm{i}F_1(z) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t-z} \mathrm{d}t - C_p. \tag{4.12}$$

We next derive the expression for $\Phi(0,q)$. From (2.8) and (2.16) it seen that we have

$$\begin{aligned} T_1 &= \{q : q = q_1(p), -1 \leq p \leq 1\}, \\ T_2 &= \{q : q = q_2(p), -1 \leq p \leq 1\}. \end{aligned} \tag{4.13}$$

Put

$$\begin{aligned} \phi_2(q) \quad &:= p_1^{x_0}(q)\mathrm{Im}(q^{y_0}) \quad \text{for} \quad q \in \mathrm{T}_1, \\ &:= p_2^{x_0}(q)\mathrm{Im}(q^{y_0}) \quad \text{for} \quad q \in \mathrm{T}_2, \end{aligned} \tag{4.14}$$

and

$$F_2(z) := \phi_2(f(z)). \tag{4.15}$$

The same argumentation as that which has led to the derivation of the expression (4.9) yields: for Im $z > 0$,

$$\Phi(0, f(z)) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_2(t)}{t - z} \mathrm{d}t - C_q. \tag{4.16}$$

with C_q a real constant.

Next we shall determine the constants Φ_0, C_p and C_q.

Because $(p, q) = (-\mathrm{i}b, 0)$ is a zero-tuple of $K(p, q)$, cf. (2.28), and $p = -\mathrm{i}b \in \mathrm{T}$, it follows from (3.11) since $\Phi(0, 0) = 0$, cf. (3.6), that

$$\Phi_0 + \Phi(-\mathrm{i}b, 0) = 0. \tag{4.17}$$

From (2.21) it is seen that we have to calculate $\Phi(f(0), 0)$. From (4.12) we obtain

$$\Phi(-\mathrm{i}b, 0) = \Phi(f(0), 0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t} \mathrm{d}t - C_p, \tag{4.18}$$

because for $(p, q) = (-\mathrm{i}b, 0)$ we have from (4.6),

$$F_1(0) = 0.$$

Hence from (4.17) and (4.18),

$$\begin{aligned} &\Phi_0 - C_p + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t} \mathrm{d}t = 0, \\ &\Phi_0 - C_q + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_2(t)}{t} \mathrm{d}t = 0; \end{aligned} \tag{4.19}$$

the second relation in (4.19) follows from the first one by using the symmetry. To obtain a third relation for Φ_0, C_p and C_q we use (3.7). Because the point $p = 1 \in T$ corresponds to the point $z = 1$, cf. (2.21), and $F_1(1) = 0$, cf. (4.6), we obtain from (3.7) and (4.12) by using again the symmetry

$$\Phi_0 - C_p - C_q + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_1(t)}{t - 1} \mathrm{d}t + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_2(t)}{t - 1} \mathrm{d}t = 1. \tag{4.20}$$

From (4.19) and (4.20) we obtain

$$\Phi_0 = -1 + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F_1(t) + F_2(t)}{t(t - 1)} \mathrm{d}t, \tag{4.21}$$

$$C_p = \Phi_0 + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F_1(t)}{t}\mathrm{d}t,$$

$$C_q = \Phi_0 + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F_2(t)}{t}\mathrm{d}t.$$

REMARK 4.1. From (2.28) and (2.29) we have $f(\mathrm{i}) = 0$, and so from (3.6) and (4.9),

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F_1(t)}{t-\mathrm{i}}\mathrm{d}t - C_p = 0,$$

or

$$-C_p + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{tF_1(t)}{t^2+1}\mathrm{d}t + \frac{\mathrm{i}}{\pi}\int_{-\infty}^{+\infty} \frac{F_1(t)}{t^2+1}\mathrm{d}t = 0. \tag{4.22}$$

Because C_p is real and $F_1(t)$, $t \in (-\infty, \infty)$ is real it follows that

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F_1(t)}{t^2+1}\mathrm{d}t = 0, \quad C_p = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{tF_1(t)}{t^2+1}\mathrm{d}t, \\ &\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F_1(t)}{t^2+1}\mathrm{d}t = 0, \quad C_q = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{tF_2(t)}{t^2+1}\mathrm{d}t. \end{aligned} \tag{4.23}$$

The last two relations in (4.23) follow from the first two by symmetry. Further from (4.21) and (4.23)

$$\Phi_0 = \frac{-1}{\pi}\int_{-\infty}^{+\infty} \frac{F_1(t)}{t(t^2+1)}\mathrm{d}t = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F_2(t)}{t(t^2+1)}\mathrm{d}t. \tag{4.24}$$

□

REMARK 4.2. Note that for $x_0 = y_0$ we have from the symmetry

$$\Phi(0,p) = \Phi(p,0), \ F_1(z) = F_2(z). \tag{4.25}$$

□

From the relations derived above the expressions for $\Phi(f_1(z),0)$ and $\Phi(0, f_2(z))$, Im $z \geq 0$, are readily obtained, and hence also those for $\Phi(p,0)$ and $\Phi(0,q)$ for all $x_0 > 0, y_0 > 0$, note that $f_1(z)$ and $f_2(z)$, Im $z \geq 0$, both have a unique inverse.

5. THE CASE $\rho = 0$, $x_0 = y_0 = 1$

In this section we consider the case

$$\rho = 0, \tag{5.1}$$

so that (2.2) implies

$$b = 1. \tag{5.2}$$

From (2.11) it is then seen that the circles $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide and

$$\mathcal{S}_1 = \mathcal{S}_2 =: \{(u,v) : u^2 + v^2 = 1\}. \tag{5.3}$$

Further, cf. (2.8) and (2.16),

$$\begin{aligned} p_1(q) &= \frac{q-\mathrm{i}}{1-\mathrm{i}q}, \quad p_2(q) = \frac{q+\mathrm{i}}{1+\mathrm{i}q}, \\ q_1(p) &= \frac{q+\mathrm{i}}{1+\mathrm{i}q}, \quad q_2(p) = \frac{p-\mathrm{i}}{1-\mathrm{i}p}. \end{aligned} \tag{5.4}$$

$$\begin{aligned} T_1 &= \{p : p = \mathrm{e}^{\mathrm{i}\phi}, \pi \leq \phi \leq 2\pi\}, \\ T_2 &= \{p : p = \mathrm{e}^{\mathrm{i}\phi}, 0 \leq \phi \leq \pi\}. \end{aligned} \tag{5.5}$$

Because $q_2(p)$ and $q_n(p)$ are both real for $p \in \mathcal{S}_1$, we obtain from (3.11),

$$\begin{aligned} \text{i.} \qquad \mathrm{Re}\{-\mathrm{i}\Phi(p,0)\} &= q_1^{y_0}(p)\mathrm{Im}(p^{x_0}) \quad \text{for} \quad p \in \mathrm{T}_1, \\ &= q_2^{y_0}(p)\mathrm{Im}(p^{x_0}) \quad \text{for} \quad p \in \mathrm{T}_2; \end{aligned} \tag{5.6}$$

ii. $\Phi(p,0)$ is regular for $|p| < 1$, continuous for $|p| \leq 1$.

The relations (5.6) again formulate a standard Boundary Value Problem, actually it is the Dirichlet problem for the unit circle, cf. [3], Section 41, p. 107.

Put

$$\begin{aligned} F(p) &:= q_1(p)[\mathrm{Im}\ p] \quad \text{for} \quad p \in \mathrm{T}_1, \\ &:= q_2(p)[\mathrm{Im}\ p] \quad \text{for} \quad p \in \mathrm{T}_2, \end{aligned} \tag{5.7}$$

i.e. we consider from now on the case with

$$x_0 = y_0 = 1. \tag{5.8}$$

The solution of the boundary value problem reads, cf. [3], p. 108, with C a real constant:

for $|p| < 1$,

$$\Phi(p,0) = \frac{1}{2\pi} \int\limits_{z \in S} F(z) \frac{z+p}{z-p} \frac{\mathrm{d}z}{z} - C; \tag{5.9}$$

for $|p| = 1$,

$$\Phi(p,0) = \mathrm{i}F(p) + \frac{1}{2\pi} \int\limits_{z \in S} F(z) \frac{z+p}{z-p} \frac{\mathrm{d}z}{z} - C,$$

the direction of integration along $\mathcal{S}$ is counterclockwise.

Note that (5.8) implies symmetry, so that

$$\Phi(p,0) = \Phi(0,p), \quad |p| \leq 1. \tag{5.10}$$

Because $\Phi(0,0) = 0$ we obtain from (5.9),

$$C = \frac{1}{2\pi} \int\limits_{z \in \mathcal{S}} f(z) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi} \int\limits_0^{2\pi} F(\mathrm{e}^{\mathrm{i}\phi}) \mathrm{i} \mathrm{d}\phi. \tag{5.11}$$

Because $F(z)$ is real for $|z|=1$ and $\mathcal{C}$ is real it follows that

$$C=0. \tag{5.12}$$

Hence

$$\frac{1}{2\pi}\int\limits_{z\in\mathcal{S}} F(z)\frac{\mathrm{d}z}{z}=0. \tag{5.13}$$

A direct proof of (5.13) is as follows. From (5.4) we have

$$\begin{aligned}
q_1(\mathrm{e}^{\mathrm{i}\phi}) = \frac{\mathrm{e}^{\mathrm{i}\phi}+\mathrm{i}}{1+\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}} &= \mathrm{i}\frac{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}}{1+\mathrm{i}\mathrm{e}^{i\phi}} &&= \frac{\cos\phi}{1-\sin\phi},\\
q_1(\mathrm{e}^{\mathrm{i}(\phi+\pi)}) &= -\mathrm{i}\frac{1+\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}}{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}} &&= \frac{\cos\phi}{1+\sin\phi},\\
q_2(\mathrm{e}^{\mathrm{i}\phi}) = \frac{\mathrm{e}^{\mathrm{i}\phi}-\mathrm{i}}{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}} &= -\mathrm{i}\frac{1+\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}}{1-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}} &&= \frac{\cos\phi}{1+\sin\phi}.
\end{aligned} \tag{5.14}$$

Hence from (5.7),

$$\begin{aligned}
F(\mathrm{e}^{\mathrm{i}\phi}) &= \frac{\sin\phi\cos\phi}{1-\sin\phi}, \quad \pi<\phi<2\pi,\\
F(\mathrm{e}^{\mathrm{i}(\phi+\pi)}) &= \frac{\sin\phi\cos\phi}{1+\sin\phi}, \quad 0<\phi<\pi,\\
F(\mathrm{e}^{\mathrm{i}\phi}) &= \frac{\sin\phi\cos\phi}{1+\sin\phi}, \quad 0<\phi<\pi.
\end{aligned} \tag{5.15}$$

So

$$\frac{1}{2\pi}\int\limits_{z\in\mathcal{S}} F(z)\frac{dz}{z} = \frac{1}{2\pi}\int\limits_{\phi=0}^{2\pi} F(\mathrm{e}^{\mathrm{i}\phi})\mathrm{i}\mathrm{d}\phi = \frac{1}{2\pi}\int\limits_{\phi=0}^{\pi} F(\mathrm{e}^{\mathrm{i}\phi})\mathrm{i}\mathrm{d}\phi + \frac{1}{2\pi}\int\limits_{\phi=\pi}^{2\pi} F(\mathrm{e}^{\mathrm{i}\phi})\mathrm{i}\mathrm{d}\phi =$$
$$\frac{2}{2\pi}\int\limits_{\phi=0}^{\pi}\frac{\sin\phi\cos\phi}{1+\sin\phi}\mathrm{i}\mathrm{d}\phi = \frac{\mathrm{i}}{\pi}\int\limits_{\phi=0}^{\frac{1}{2}\pi}\frac{\sin\phi\cos\phi}{1+\sin\phi}\mathrm{d}\phi + \frac{\mathrm{i}}{\pi}\int\limits_{\phi=0}^{\frac{1}{2}\pi}\frac{-\sin\phi\cos\phi}{1+\cos\phi}\mathrm{d}\phi =$$
$$\frac{\mathrm{i}}{\pi}\int\limits_{0}^{1}\frac{x}{1+x}\mathrm{d}x + \frac{\mathrm{i}}{\pi}\int\limits_{1}^{0}\frac{x}{1+x}\mathrm{d}x = 0,$$

and so (5.12) is again obtained.

From (5.9), (5.12) and (5.13) it follows that

$$\begin{aligned}
\Phi(p,0) &= \frac{1}{\pi}\int\limits_{z\in S} F(z)\frac{\mathrm{d}z}{z-p}, && |p|<1,\\
&= \mathrm{i}F(p) + \frac{1}{\pi}\int\limits_{z\in S} F(z)\frac{\mathrm{d}z}{z-p}, && |p|=1.
\end{aligned} \tag{5.16}$$

It remains to determine Φ_0. From (3.7) we have

$$\Phi_0 = 1-2\Phi(1,0) = 1-\frac{2}{\pi}\int\limits_{z\in S} F(z)\frac{\mathrm{d}z}{z-1}, \tag{5.17}$$

since $F(1)=0$, cf. (5.7). We have

$$\frac{2}{\pi}\int\limits_{z\in S} F(z)\frac{\mathrm{d}z}{z-1} = \frac{2}{\pi}\int\limits_{0}^{\pi} F(\mathrm{e}^{\mathrm{i}\phi})\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}}{\mathrm{e}^{\mathrm{i}\phi}-1}\mathrm{d}\phi + \frac{2}{\pi}\int\limits_{0}^{\pi} F(\mathrm{e}^{\mathrm{i}\phi})\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}}{\mathrm{e}^{\mathrm{i}\phi}-1}\mathrm{d}\phi =$$
$$\frac{2}{\pi}\int\limits_{0}^{\pi}\frac{\sin\phi\cos\phi}{1+\sin\phi}\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\frac{2\mathrm{e}^{\mathrm{i}\phi}}{\mathrm{e}^{2\mathrm{i}\phi}-1}\mathrm{d}\phi = \frac{2}{\pi}\int\limits_{0}^{\pi}\frac{\cos\phi}{1+\sin\phi}(\cos\phi+\mathrm{i}\sin\phi)\mathrm{d}\phi$$
$$=\frac{2}{\pi}\int\limits_{0}^{\pi}(1-\sin\phi)\mathrm{d}\phi = 2-\frac{4}{\pi}.$$

Hence

$$\Phi_0 = -1+\frac{4}{\pi}\sim 0.2732, \tag{5.18}$$

$$\Phi(1,0) = 1-\frac{2}{\pi}\sim 0,3634.$$

From (5.16) we have: for $|p|<1$,

$$\Phi(p,0) = \frac{1}{\pi}\int\limits_{z\in S} F(z)\sum_{n=0}^{\infty} p^n\frac{\mathrm{d}z}{z^{n+1}}, \tag{5.19}$$

and hence since $|F(z)|$ is bounded we obtain from (3.5) for $n=1,2,\ldots$,

$$\begin{aligned}\Pr\{\mathbf{k}_1=n,\mathbf{k}_2=0\} = {} & \frac{1}{\pi}\int\limits_{z\in S} F(z)\frac{\mathrm{d}z}{z^{n+1}} = \frac{1}{\pi}\int\limits_{\phi=0}^{\pi} F(\mathrm{e}^{\mathrm{i}\phi})\frac{\mathrm{i}\mathrm{d}\phi}{\mathrm{e}^{\mathrm{i}n\phi}} + \frac{1}{\pi}\int\limits_{\phi=\pi}^{2\pi} F(\mathrm{e}^{\mathrm{i}\phi})\frac{\mathrm{i}\mathrm{d}\phi}{\mathrm{e}^{\mathrm{i}n\phi}} = \\ & \frac{\mathrm{i}}{\pi}\int\limits_{\phi=0}^{\pi}\frac{\sin\phi\cos\phi}{1+\sin\phi}[\mathrm{e}^{-\mathrm{i}n\phi}+(-1)^n\mathrm{e}^{-\mathrm{i}n\phi}]\mathrm{d}\phi.\end{aligned}$$

Hence

$$\Pr\{\mathbf{k}_1=n,\ \mathbf{k}_2=0\}=0 \qquad \text{for } n \text{ odd}. \tag{5.20}$$

For $n=2,4,6,\ldots$.

$$\begin{aligned}\Pr\{\mathbf{k}_1=n,\ \mathbf{k}_2=0\} &= \frac{\mathrm{i}}{\pi}\int\limits_{\phi=0}^{\pi}\frac{\sin 2\phi}{1+\sin\phi}\mathrm{e}^{\mathrm{i}n\phi}\mathrm{d}\phi \\ &= \frac{1}{\pi}\int\limits_{\phi=0}^{\pi}\frac{\sin 2\phi}{1+\sin\phi}\sin n\phi\mathrm{d}\phi.\end{aligned} \tag{5.21}$$

To calculate this integral put: for $m=1,2,\ldots$,

$$J_m := \frac{1}{\pi}\int\limits_{\phi=0}^{\pi}\frac{\sin 2\phi}{1+\sin\phi}\sin 2m\phi\mathrm{d}\phi. \tag{5.22}$$

By writing the integral in (5.22) as the sum of the integrals over $[0,\frac{1}{2}\pi]$ and $[\frac{1}{2}\pi,\pi]$ it is readily seen by substituting in the second one $\phi=\pi-\omega$ that

$$
\begin{aligned}
J_m &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin 2\phi}{1+\sin\phi}\sin 2m\phi \mathrm{d}\phi = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin\phi(1-\sin\phi)}{\cos\phi}\sin 2m\phi \mathrm{d}\phi \\
&= \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin\phi - 1}{\cos\phi}\sin 2m\phi \mathrm{d}\phi + \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \cos\phi\sin 2m\phi \mathrm{d}\phi.
\end{aligned} \tag{5.23}
$$

We have

$$
\begin{aligned}
&\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos\phi\sin 2m\phi \mathrm{d}\phi = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} [\sin(2m-1)\phi + \sin(2m+1)\phi]\mathrm{d}\phi \\
&= -\frac{1}{\pi}\{\frac{\cos(2m-1)\phi}{2m-1}\} + \{\frac{\cos(2m+1)\phi}{2m+1}\}\Big|_0^{\frac{1}{2}\pi} = \frac{1}{\pi}\{\frac{1}{2m-1} + \frac{1}{2m+1}\}.
\end{aligned} \tag{5.24}
$$

Hence from (5.23) and (5.24),

$$
\begin{aligned}
J_m &= -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin 2m\phi}{\cos\phi}\mathrm{d}\phi \\
&+\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \{\frac{\cos(2m-1)\phi}{\cos\phi} - \frac{\cos(2m+1)\phi}{\cos\phi}\}\mathrm{d}\phi + \frac{2}{\pi}\{\frac{1}{2m-1} + \frac{1}{2m+1}\}.
\end{aligned} \tag{5.25}
$$

By using the formulas (2.53)6 and (2.53)7 from [6] we have for $m = 1, 2, \ldots$,

$$
\begin{aligned}
J_m &= -\frac{8}{\pi}(-1)^{m+1}\sum_{k=1}^{m}(-1)^k\frac{\cos(2k-1)\phi}{2k-1}\Big|_0^{\frac{1}{2}\pi} + \frac{2}{\pi}\{\frac{1}{2m-1} + \frac{1}{2m+1}\} \\
&+\frac{4}{\pi}(-1)^{m-1}\sum_{k=1}^{m-1}(-1)^k\frac{\sin 2k\phi}{2k}\Big|_0^{\frac{1}{2}\pi} + (-1)^{m-1}\phi\Big|_0^{\frac{1}{2}\pi}\cdot\frac{2}{\pi} \\
&-\frac{4}{\pi}(-1)^{m}\sum_{k=1}^{m}(-1)^k\frac{\sin 2k\phi}{2k}\Big|_0^{\frac{1}{2}\pi} - (-1)^{m}\phi\Big|_0^{\frac{1}{2}\pi}\cdot\frac{2}{\pi} \\
&= (-1)^{m-1}\{2 - \frac{8}{\pi}\sum_{k=1}^{m}(-1)^{k-1}\frac{1}{2k-1}\} + \frac{2}{\pi}\{\frac{1}{2m-1} + \frac{1}{2m+1}\}.
\end{aligned} \tag{5.26}
$$

Hence form (5.20), (5.21) and (5.26) we obtain for $n = 1, 2, \ldots$,

$$
\begin{aligned}
&\Pr\{\mathbf{k}_1 = 2n-1,\ \mathbf{k}_2 = 0\} = 0, \\
&\Pr\{\mathbf{k}_1 = 2n,\ \mathbf{k}_2 = 0\} = (-1)^{n-1}2\{1 - \frac{4}{\pi}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}\} + \frac{2}{\pi}\{\frac{1}{2n-1} + \frac{1}{2n+1}\}.
\end{aligned} \tag{5.27}
$$

Since $\Phi_0 = \Pr\{\mathbf{k}_1 = \mathbf{k}_2 = 0\}$ is given by (5.18) and because of the symmetry

$$\Pr\{\mathbf{k}_1 = n,\ \mathbf{k}_2 = 0\} = \Pr\{\mathbf{k}_1 = 0,\ \mathbf{k}_2 = n\},$$

it is seen that (5.18) and (5.27) describe the distribution of the hitting point completely for the case $\rho = 0$ with starting point $(x_0, y_0) = (1, 1)$.

Next we shall calculate $\Pr\{\mathbf{k}_1 > n,\ \mathbf{k}_2 = 0\}$. A simple calculation shows that: for $|p| < 1$,

$$\sum_{n=0}^{\infty} p^n \Pr\{\mathbf{k}_1 > n, \mathbf{k}_2 = 0\} = \frac{1}{1-p}[\Phi(1,0) - \Phi(p,0)]$$
$$= \frac{1}{\pi} \int_{z \in S} F(z) \frac{\mathrm{d}z}{(z-1)(z-p)}, \tag{5.28}$$

so that

$$\Pr\{\mathbf{k}_1 > n,\ \mathbf{k}_2 = 0\} = \frac{1}{\pi} \int_{z \in S} \frac{F(z)}{z-1} \frac{dz}{z^{n+1}}. \tag{5.29}$$

From (5.29) we obtain, (cf. the derivation of (5.25) from (5.17)): for $m = 1, 2, \ldots$, with $n = 2m - 1$

$$\Pr\{\mathbf{k}_1 > n,\ \mathbf{k}_2 = 0\} = \frac{1}{\pi} \int_0^{\pi} \frac{\cos(2m-1)\phi}{\cos\phi} \mathrm{d}\phi - \frac{1}{2\pi} \int_0^{\pi} \frac{\sin 2m\phi - \sin(2m-1)\phi}{\cos\phi} \mathrm{d}\phi. \tag{5.30}$$

By using again the relations in [6] p. 175, we obtain: for $m = 1, 2, \ldots$,

$$\Pr\{\mathbf{k}_1 > 2m-1, \mathbf{k}_2 = 0\} = (-1)^{m-1}[1 - \frac{4}{\pi} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{2k-1}] + \frac{1}{\pi} \frac{2}{2m-1}, \tag{5.31}$$

note that $\Pr\{\mathbf{k}_1 = 2m-1, \mathbf{k}_2 = 0\} = 0$, cf. (5.27).

Remark 5.1. It is wellknown that

$$1 - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = 0.$$

So it follows from (5.18), (5.27) and (5.31), for $n = 1, 2, \ldots$,

$$\Pr\{\mathbf{k}_1 = 2n-1, \mathbf{k}_2 = 0\} = (-1)^{n-1} \frac{8}{\pi} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} + \frac{2}{\pi} \{\frac{1}{2n-1} + \frac{1}{2n+1}\}, \tag{5.32}$$

$$\Pr\{\mathbf{k}_1 > 2n-1, \mathbf{k}_2 = 0\} = (-1)^{n-1} \frac{4}{\pi} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} + \frac{2}{\pi} \frac{1}{2n-1},$$

$$\Pr\{\mathbf{k}_1 = \mathbf{k}_2 = 0\} = -1 + \frac{4}{\pi} \sim 0.2374.$$

Hence the analysis of the present section shows that:

For the nearest-neighbour random walk with one-step transition probabilities to the four diagonal points all equal to 1/4. (see fig. 1 with $\rho = 0$) the distribution of the hitting point $\mathbf{k}_1$ with the horizontal axis when starting at $(x_0, y_0) = (1, 1)$ is given by (5.32).

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