Centrum voor Wiskunde en Informatica
REPORT*RAPPORT*

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Probability, Networks and Algorithms (PNA)

PNA-R9701 February 28, 1997

Report PNA-R9701
ISSN 1386-3711

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The Netherlands

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SMC is sponsored by the Netherlands Organization for Scientific Research (NWO). CWI is a member of ERCIM, the European Research Consortium for Informatics and Mathematics.

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Frames, Riesz Systems and MRA in Hilbert Spaces

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ABSTRACT

The concept of multiresolution analysis (MRA) is introduced for arbitrary separable Hilbert spaces H. It is put in the general terms of unitary operators U_1 and $U_{2,1},\dots,U_{2,d}$, $d \in \mathbb{Z}$ and a generating element ϕ. Each MRA yields a system $\mathcal{V} = \{U_1^k U_{2,1}^{l_1} \cdots U_{2,d}^{l_d} \psi_n \mid n = 0,\dots,N-1,\ k \in \mathbb{Z}, l \in \mathbb{Z}^d\}$, where the ψ_n are related to ϕ. Necessary and sufficient conditions on U_1, $U_{2,1},\dots,U_{2,d}$, ϕ and ψ_n are given, such that V is a Riesz system or basis in H.

1991 Mathematics Subject Classification: 41A65,42C15,46C99.
Keywords and Phrases: Frames, Riesz systems, multiresolution analysis, approximation theory, wavelet theory
Note: Work carried out under project PNA 4.2 "Wavelets"

1. Introduction

Multiresolution analysis (MRA) is one of the features of wavelet theory, signal processing and approximation theory. Take D to be a $(d \times d)$ matrix, with integer entries and eigenvalues ν_i, $i = 1,\dots,d$, such that $|\nu_i| > 1$, and $k \in \mathbb{Z}^d$. Define the dilation operator Z_D on $L^2(\mathbb{R}^d)$ by

$$(Z_D f)(x) = \sqrt{|\det(D)|}\, f(Dx),$$

and the shift operators $T_1,\dots,T_d$ on $L^2(\mathbb{R}^d)$ by

$$(T_i f)(x) = f(x - e_i),\ i = 1,\dots,d,$$

with e_i the standard basis vectors in $\mathbb{R}^d$. Then an MRA of $L^2(\mathbb{R}^d)$ related to the matrix D is an increasing sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^d)$, such that

- $Z_D(V_j) = V_{j+1},\ \ T_k(V_j) = V_j,\ \ \forall k \in \mathbb{Z}^d,\ j \in \mathbb{Z}$,
- $clos \bigcup_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R}^d),\quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
- $\exists_{\phi \in L^2(\mathbb{R}^d)}\ \{T_k\phi \mid k \in \mathbb{Z}^d\}$ is a Riesz basis for V_0.

Starting from an MRA, a Riesz basis for $L^2(\mathbb{R}^d)$ of the form

$$\{Z_D^j T^k \psi_n \mid n = 1, \ldots, |\det(D)| - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$$

can be constructed, where ψ_n is such that the set

$$\{T^k \psi_n \mid n = 1, \ldots, |\det(D)| - 1,\ k \in \mathbb{Z}^d\}$$

is a Riesz basis of $V_1 \cap V_0^{\perp}$. Here T^k denotes $T_1^{k_1} \cdots T_d^{k_d}$.

In this paper we put the concept of MRA in a functional analytic setting, replacing $L^2(\mathbb{R}^d)$ by a separable Hilbert space H and the operators Z_D and $T_1, \ldots, T_d$ by arbitrary operators U_1 and $U_{2,1}, \ldots, U_{2,d}$ on H satisfying the relation $U_2^k U_1 = U_1 U_2^{Dk}$ for some $k \in \mathbb{Z}^d$. Here we use the multi-index notation $U_2^k = U_{2,1}^{k_1} \cdots U_{2,d}^{k_d}$, for $k \in \mathbb{Z}^d$. This notation will be used also in the sequel of this report. Then an MRA for H can be defined solely in terms of conditions on the tuple $[U_1, U_{2,1}, \ldots, U_{2,d}, \phi]$, where ϕ is an MRA-generator.

The paper is organized as follows. In Section 2 we recall the general concept of frames and Riesz systems and in Section 3 we recall the concept of MRA in terms of a tuple $[U_1, U_{2,1}, \ldots, U_{2,d}, \phi]$. It is shown how MRA is related to the existence problem of a special kind of frames and Riesz bases. In Section 4 some theory on Riesz systems generated by a finite number of mutually commuting unitary operators is developed. In Section 5 necessary and sufficient conditions are presented guaranteeing a positive solution of a related existence problem in $l^2(\mathbb{Z})$. In Section 6 similar conditions are given guaranteeing a solution of the existence problem related to MRA.

2. Riesz Systems, General Concept

Let H be a separable Hilbert space with inner product $(\cdot,\cdot)$. For a countable index set $\mathbb{D}$ by $l^2(\mathbb{D})$ we denote the Hilbert space of all square summable functions $\mathbb{D}$ into $\mathbb{C}$ and by $(\cdot,\cdot)_{\mathbb{D}}$ we denote its inner product. The standard orthonormal basis in $l^2(\mathbb{D})$ is denoted by $\{e_j\}_{j\in\mathbb{D}}$, so $e_j(i) = \delta_{i,j}$ for $i, j \in \mathbb{D}$. The expression $l_0^2(\mathbb{D})$ indicates the linear span of the set $\{e_j \mid j \in \mathbb{D}\}$.

Definition 2.1
A collection $\mathcal{F} = \{v_j \mid j \in \mathbb{D}\}$ in H is called a frame with frame bounds m_F and M_F, real and positive, if the sequence $(x, v_j)_{j\in\mathbb{D}}$ belongs to $l^2(\mathbb{D})$ and

$$m_F\|x\|^2 \le \sum_{j\in\mathbb{D}} |(x, v_j)|^2 \le M_F\|x\|^2, \ \ \forall\, x \in H. \tag{2.1}$$

A frame $\mathcal{F}$ is called tight if the frame bounds satisfy $m_F = M_F$. A frame is called exact if it is no longer a frame whenever any one of its elements is removed

Now let $\{v_j \mid j \in \mathbb{D}\}$ be a frame in H. Then define $S : H \to l_0^2(\mathbb{D})$ by

$$Sx = \sum_{j\in\mathbb{D}} (x, v_j) e_j, \ \ \forall x \in H.$$

According to Definition 2.1, there are positive m_F and M_F such that

$$m_F\|x\|^2 \le \|Sx\|^2 \le M_F\|x\|^2. \tag{2.2}$$

It thus follows that S is a bounded linear operator from H into $l^2(I\!D)$, which is also bounded from below. The constants m_F and M_F are given respectively by

$$m_F = \|(S^*S)^{-1}\|^{-1} \text{ and } M_F = \|S^*S\|.$$

The operator S is called the frame generator associated with the frame $\mathcal{F}$.
A straight forward computation shows that the adjoint of S is given by

$$S^*\alpha = \sum_{j\in I\!D} (\alpha, e_j)v_j, \quad \forall \alpha \in l^2(I\!D),$$

from which it follows that $S^*e_i = v_i$, $i \in I\!D$. We observe that S^* is an injective bounded linear operator from $l^2(I\!D)$ onto H with range, Ran(S^*), closed in H. Equivalently S^*S is a boundedly invertible operator on H and $S(S^*S)^{-1}$ the right inverse of S^* with minimal norm. Now define $\tilde{v}_j$, $j \in I\!D$, in Ran(S^*) by

$$\tilde{v}_j = (S^*S)^{-1}S^*e_j.$$

Then for all $x \in$ Ran(S^*)

$$x = S^*(S(S^*S)^{-1})x = \sum_{j\in I\!D} (x, (S^*S)^{-1}S^*e_j)S^*e_j = \sum_{j\in I\!D} (x, \tilde{v}_j)v_j$$

and a posteriori $x = \sum_{j\in I\!D} (x, v_j)\tilde{v}_j$.
We summarize in the following theorem.

Theorem 2.2
Let $\mathcal{F} = \{v_j \mid j \in I\!D\}$ be a collection in H. Then $\mathcal{F}$ is a frame if and only if the adjoint of the frame generator S associated with $\mathcal{F}$ is surjective. If $\mathcal{F}$ is a frame the collection $\{\tilde{v}_j \mid j \in I\!D\}$, defined by $\tilde{v}_j = (S^*S)^{-1}S^*e_j$, is the frame dual to $\mathcal{F}$.

Definition 2.3
The collection $\mathcal{R} = \{v_j \mid j \in I\!D\}$ in H is called a Riesz system with Riesz bounds m_R and M_R, real and positive, if

$$m_R\|\alpha\|^2_{I\!D} \le \|\sum_{j\in I\!D} (\alpha, e_j)v_j\|^2 \le M_R\|\alpha\|^2_{I\!D}, \quad \forall\, \alpha \in l^2_0(I\!D) \tag{2.3}$$

Now let $\{v_j \mid j \in I\!D\}$ be a Riesz system. Then define $T_0 : l^2_0(I\!D) \to H$ by

$$T_0\alpha = \sum_{j\in I\!D} \alpha(j)v_j.$$

Because of the above definition T_0 extends to a bounded linear operator T from $l^2(I\!D)$ into H, called the Riesz generator of $\mathcal{R}$, satisfying

$$m_R\|\alpha\|^2 \le \|T\alpha\|^2 \le M_R\|\alpha\|^2. \tag{2.4}$$

We conclude that T is injective with range, Ran(T), closed in H. Equivalently T^*T is a boundedly invertible operator on $l^2(I\!D)$ and $(T^*T)^{-1}T^*$ the left inverse of T with minimal norm. Now define $\tilde{v}_j$, $j \in I\!D$, in Ran(T) by

$$\tilde{v}_j = T(T^*T)^{-1}e_j.$$

Then for all $x \in \mathrm{Ran}(T)$

$$x = T(T^*T)^{-1}T^*x = \sum_{j\in\mathbb{D}}((T^*T)^{-1}T^*x, e_j)Te_j = \sum_{j\in\mathbb{D}}(x, \tilde{v}_j)v_j$$

and a posteriori $x = \sum_{j\in\mathbb{D}}(x, v_j)\tilde{v}_j$.
We summarize in the following theorem.

Theorem 2.4
Let $\mathcal{R} = \{v_j | \, j \in \mathbb{D}\}$ be a collection in H. Then $\mathcal{R}$ is a Riesz system if and only if there is a bounded linear injection $T : l^2(\mathbb{D}) \to H$ with closed range such that $Te_j = v_j, \; j \in \mathbb{D}$. If so the collection $\{\tilde{v}_j \mid j \in \mathbb{D}\}$, defined by $\tilde{v}_j = T(T^*T)^{-1}e_j$, is the Riesz system dual to $\mathcal{R}$.

Definition 2.5
A Riesz system which is total is a Riesz basis.

For a Riesz basis the corresponding Riesz generator T is invertible. From this it follows immediately that the frame $\{v_j \mid j \in \mathbb{D}\}$ is a Riesz basis if and only if S^* is invertible. It can be proved, see [1], that an exact frame is equivalent with a Riesz basis. The connection between frames and Riesz systems is given in the following theorem, which results from the previous derivations.

Theorem 2.6
Let $\mathcal{V} = \{v_j | \; j \in \mathbb{D}\}$ be a collection in H. Then define the operator $T : l^2(\mathbb{D}) \to H$ by $Te_j = v_j, \; j \in \mathbb{D}$. Now $\mathcal{V}$ is a frame if and only if TT^* is a boundedly invertible operator on H. Further $\mathcal{V}$ is a Riesz system if and only if T^*T is a boundedly invertible operator on $l^2(\mathbb{D})$. Finally $\mathcal{V}$ is a Riesz basis if and only if T is a boundedly invertible operator on $l^2(\mathbb{D})$, i.e. if both TT^* and T^*T are boundedly invertible operators.

For $\mathcal{R} = \{v_j | \, j \in \mathbb{D}\}$ in H, define its Gram matrix G_R by

$$G_R(i, j) = (v_j, v_i)_H, \; i, j \in \mathbb{D}.$$

Since $G_R(i, j) = (T^*Te_i, e_j)$ we conclude $G_R = T^*T$, which yields together with (2.4) that $\mathcal{R}$ is a Riesz system if and only if

$$m_R I \leq G_R \leq M_R I. \tag{2.5}$$

3. Multiresolution Analysis

In this section the concept of MRA is introduced for the separable Hilbert space H and its relation with typical Riesz systems is discussed.

Definition 3.1
An MRA for a separable Hilbert space H is a tuple $[\phi, U_1, U_{2,1}, \ldots, U_{2,d}]$, where $\phi \in H$, U_1 is a unitary operator and $U_{2,1}, \ldots, U_{2,d}$ are mutually commuting unitary operators on H, on H satisfying the following conditions

(i) $\{U_2^k\phi |\, k \in \mathbb{Z}^d\}$ is a Riesz system in H

(ii) $\phi \in clos\ span\{U_1 U_2^k \phi |\, k \in \mathbb{Z}^d\}$

(iii) $U_2^k U_1 = U_1 U_2^{Dk}$, for all $k \in \mathbb{Z}^d$, where D is a $(d \times d)$ matrix with integer entries and eigenvalues $\nu_i,\ i = 1, \ldots, d$, such that $|\nu_i| > 1$.

In the sequel ϕ is called the MRA generator.
In the literature an MRA is defined mostly in another way [3, 7], namely in terms of a nested sequence of subspaces of a Hilbert space, cf. Section 1. Using Definition 3.1 we can construct such a nested sequence of subspaces for H as we show now. Let $[\phi, U_1, U_{2,1}, \ldots, U_{2,d}]$ be an MRA for H. Define the closed subspaces $V_j,\ j \in \mathbb{Z}$, in H by $V_j = clos\ span\{U_1^j U_2^k \phi \mid k \in \mathbb{Z}^d\}$. Then the conditions on MRA yield

$$U_1(V_j) = V_{j+1},\ U_2^k(V_j) = V_j,\ k \in \mathbb{Z}^d,\ \text{ and } V_j \subset V_{j+1}.$$

Indeed, since $\{U_1^j U_2^k \phi \mid k \in \mathbb{Z}^d\}$ is a Riesz basis for V_j, $V_j = U_1^j(V_0)$ which yields $U_1(V_j) = V_{j+1}$. Furthermore

$$U_2^l(V_j) = clos\ span\{U_2^l U_1^j U_2^k \phi \mid k \in \mathbb{Z}^d\} = clos\ span\{U_1^j U_2^{k+jDl} \phi \mid k \in \mathbb{Z}^d\},$$

for all $l \in \mathbb{Z}^d$, because of condition (iii) on MRA. This yields $U_2(V_j) = V_j$. Finally condition (ii) on MRA yields $\phi \in V_1$ and from the latter result $U_2^k \phi \in V_1$ for all $k \in \mathbb{Z}^d$. Thus

$$V_0 = clos\ span\{U_2^k \phi \mid k \in \mathbb{Z}^d\} \subset V_1$$

which is equivalent with $V_j \subset V_{j+1}$.

Note that we did not introduce the conditions

$$clos \bigcup_{j \in \mathbb{Z}} V_j = H \ \text{ and } \bigcap_{j \in \mathbb{Z}} V_j = \{0\},$$

which occur in the traditional concept of a Multiresolution Analysis for $L^2(\mathbb{R}^d)$. We observe that whether or not inserting these conditions will not change any of the further derivations.

Example 3.2: MRA for $L^2(\mathbb{R})$
Take $U_1 = D$, $U_{2,1} = T_{e_1}$ and $\phi = B_n$, the cardinal B-spline of order n, see [2]. From spline theory, see [2], the following results are known

- $\{T^k B_n \mid k \in \mathbb{Z}\}$ is a Riesz system
- $B_n = \sum_{k=0}^{n} 2^{1/2-n} \binom{n}{k} D T^k B_n$

and so the triple $[B_n, D, T]$ satisfies condition (i) and (ii) of Definition 3.1. Further $TD = DT^2$, so that the triple also satisfies the third condition of Definition 3.1 for $D = (2)$.

Let $[\phi, U_1, U_{2,1}, \ldots, U_{2,d}]$ be an MRA for H. We shall use the concept of MRA for H for constructing Riesz systems in H. We start this construction defining the countable set of closed subspaces $W_j,\ j \in \mathbb{Z}$, by $W_j = V_j^{\perp} \cap V_{j+1}$. Observe that there is no guarantee at this point that $W_j \neq \{0\}$. Since U_1 and U_2 are unitary operators on H

$$U_1(W_j) = U_1(V_j^{\perp}) \cap U_1(V_{j+1}) = (U_1(V_j))^{\perp} \cap U_1(V_{j+1}) = V_{j+1}^{\perp} \cap V_{j+2} = W_{j+1}$$

and similarly $U_2(W_j) = W_j$. Further $W_j \subset W_l^{\perp},\ l > j$, since the subspaces V_j are nested. Now by definition of the subspaces W_j, for each $N \in \mathbb{Z},\ V_N$ can be decomposed as

$$V_N = (\bigoplus_{j=M}^{N-1} W_j) \oplus V_M,\ M < N,$$

which yields

$$clos \bigcup_{k\in\mathbb{Z}} V_k = (\bigoplus_{j=-\infty}^{\infty} W_j) \oplus \bigcap_{k\in\mathbb{Z}} V_k.$$

So, by adding the conditions $clos \bigcup_{k\in\mathbb{Z}} V_k = H$ and $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$, H can be written as the orthogonal direct sum

$$H = \bigoplus_{j=-\infty}^{\infty} W_j.$$

Since $W_j = U_1^j(W_0)$ each Riesz basis $\mathcal{R}$ for W_0 yields the Riesz basis $U_1^j(\mathcal{R})$ for $W_j,\ j \in \mathbb{Z}$. From the direct sum decomposition of H it follows that $\bigcup_{j\in\mathbb{Z}} U_1^j(\mathcal{R})$ is a Riesz basis for H.

Now the idea is to construct a Riesz basis for W_0 of the form $\{U_2^k\psi \mid k \in \mathbb{Z}^d\}$. It will turn out that for constructing a Riesz basis of this form in general more than one element ψ will be needed. Our aim is now to prove the existence of a $N \in \mathbb{N}$ and a collection $\{\psi_1, \ldots, \psi_{N-1}\} \subset V_1$, such that

(a) $(\psi_n, U_2^k\phi) = 0,\ n = 1, \ldots, N-1$, for all $k \in \mathbb{Z}^d$, i.e. $\psi_n \in V_0^{\perp}$,

(b) $\{U_2^k\psi_n \mid n = 1, \ldots, N-1,\ k \in \mathbb{Z}^d\}$ is a Riesz basis for W_0.

Since $V_1 = V_0 \oplus W_0$ condition (b) can be rewritten as

(b') $\{U_2^k\phi \mid k \in \mathbb{Z}^d\} \cup \{U_2^k\psi_n \mid n = 1, \ldots, N-1,\ k \in \mathbb{Z}^d\}$ is a Riesz basis for V_1.

The elements ψ_n are uniquely determined by sequences $q_n,\ n = 1, \ldots,\ N-1$ in $l^2(\mathbb{Z}^d)$. Since $V_0 \subset V_1$ and $W_0 \subset V_1$ and since $\{U_1U_2^k\phi \mid k \in \mathbb{Z}^d\}$ is a Riesz basis in V_1, we get

$$\phi = \sum_{k\in\mathbb{Z}^d} p(k)U_1U_2^k\phi, \tag{3.6}$$

$$\psi_n = \sum_{k\in\mathbb{Z}^d} q_n(k)U_1U_2^k\phi,\ n = 1, \ldots, N-1, \tag{3.7}$$

where $p \in l^2(\mathbb{Z}^d)$, known, and the $q_n \in l^2(\mathbb{Z}^d)$, to be determined, are the generating sequences. So the idea is to formulate conditions on the sequences q_n, given the sequence p, such that the conditions (a) and (b') are satisfied. Therefore we reformulate these conditions in terms of the generating sequences. Condition (a) can be put easily in terms of the generating sequences by substituting (3.6) and (3.7) into this condition and using $U_2^kU_1 = U_1U_2^{Dk}$. We get

$$(\psi_n, U_2^k\phi) = (\tau_\phi * q_n, R^{Dk}p)_{\mathbb{Z}^d},$$

with $\tau_\phi(k) = (\phi, U_2^k\phi)_H,\ k \in \mathbb{Z}^d$, and $R^l = R_1^{l_1}\cdots R_d^{l_d}$ for $l \in \mathbb{Z}^d$ a composition of bilateral shift operators on $l^2(\mathbb{Z}^d)$, each one acting along a standard basis vector of $\mathbb{Z}^d$. So condition (a) is equivalent with

$$(\tau_\phi * q_n, R^{Dk}p)_{\mathbb{Z}^d} = 0,\ \forall_{n\in\{1,\ldots,N-1\}}\ \forall_{k\in\mathbb{Z}^d}. \tag{3.8}$$

In order to set condition (b') in terms of the generating sequences we present the following theorem.

Theorem 3.3
Let $[\phi, U_1, U_{2,1}, \ldots, U_{2,d}]$ be an MRA and p the generating sequence of ϕ. Then $\{R^{Dk}p \mid k \in \mathbb{Z}^d\}$ is a Riesz system in $l^2(\mathbb{Z}^d)$. Let $\psi_n,\ n = 1, \ldots, N-1$, be in W_0 with generating sequences q_n. Then

$$\{U_2^k\phi \mid k \in \mathbb{Z}^d\} \cup \{U_2^k\psi_n \mid n = 1, \ldots, N-1,\ k \in \mathbb{Z}^d\}$$

is a Riesz basis for V_1 if and only if

$$\{R^{Dk}p \mid k \in \mathbb{Z}^d\} \cup \{R^{Dk}q_n \mid n = 1, \ldots, N-1, k \in \mathbb{Z}^d\}$$

is a Riesz basis for $l^2(\mathbb{Z}^d)$.

Proof
For convenience, write $\psi_0 = \phi$ and $q_0 = p$. Further, introduce the boundedly invertible operator $S : V_1 \to l^2(\mathbb{Z}^d)$ by

$$Sf = \alpha \ \text{ if } \ f = \sum_{k\in\mathbb{Z}^d} \alpha(k) U_1 U_2^k \phi.$$

Note that $SU_2^k = R^{Dk}S,\ k \in \mathbb{Z}^d$ and $S\phi = p$. Thus applying S on the Riesz system $\{U_2^k\phi \mid k \in \mathbb{Z}^d\}$ yields $\{R^{Dk}p \mid k \in \mathbb{Z}^d\}$, which is also a Riesz system, since S is a bounded invertible operator with closed range. The second result follows immediately after observing

$$\{R^{Dk}q_n \mid n = 0, \ldots, N-1,\ k \in \mathbb{Z}^d\} = S(\{U_2^k\psi_n \mid n = 0, \ldots, N-1,\ k \in \mathbb{Z}^d\}).$$

□

The previous results yield that if we can construct sequences $q_n \in l^2(\mathbb{Z}^d)$, such that

- $(\tau_\phi * q_n, R^{Dk}p)_{\mathbb{Z}^d} = 0,\ \forall_{n\in\{1,\ldots,N-1\}}\ \forall_{k\in\mathbb{Z}^d}$,
- $\{R^{Dk}p \mid k \in \mathbb{Z}^d\} \cup \{R^{Dk}q_n \mid n = 1, \ldots, N-1, k \in \mathbb{Z}^d\}$ is a Riesz basis for $l^2(\mathbb{Z}^d)$,

then we get elements $\psi_n,\ n = 1, \ldots, N-1$, in V_1 for which condition (a) and (b) are satisfied.

4. Riesz Systems Generated by Unitary Operators

Let $U_1, \ldots, U_d$ be unitary operators on H and let $\phi \in H$. We shall present necessary and sufficient conditions on the tuple $[U_1, \ldots, U_d, \phi]$ such that $\{U^j\phi \mid j \in \mathbb{Z}^d\}$ is a Riesz system, with $U^j = U_1^{j_1} \cdots U_d^{j_d}$. Further its dual Riesz system will be computed.

In Section 2 it has been shown that $\{U^j\phi \mid j \in \mathbb{Z}^d\}$ is a Riesz system if and only if its Gram matrix G,

$$G(i,j) = (U^j\phi, U^i\phi) = (\phi, U^{i-j}\phi) = \tau_\phi(i-j),$$

satisfies (2.5). Observing that T^*T with matrix G acts by convolution on $l^2(\mathbb{Z}^d)$,

$$T^*T\alpha = \tau_\phi * \alpha,$$

we arrive at the following theorem.

Theorem 4.1
For unitary operators $U_1, \ldots, U_d$ on H and $\phi \in H$, the collection $\{U^j\phi \mid j \in \mathbb{Z}^d\}$ is a Riesz system if and only if the sequence τ_ϕ defined by $\tau_\phi(j) = (\phi, U^j\phi),\ j \in \mathbb{Z}^d$, yields a boundedly invertible convolution operator on $l^2(\mathbb{Z}^d)$, or equivalently if and only if

$$0 < \text{ess}\ \inf_{z\in\mathrm{T}^d} \hat{\tau}_\phi(z) \leq \text{ess}\ \sup_{z\in\mathrm{T}^d} \hat{\tau}_\phi(z) < \infty, \tag{4.9}$$

where $\hat{\tau}_\phi$ denotes the d-dimensional discrete Fourier transform of τ_ϕ

$$\hat{\tau}_\phi(z) = \sum_{j\in\mathbb{Z}^d} \tau_\phi(j) z^{-j},\ z \in \mathbf{T}^d,$$

with $z^j = z_1^{j_1} \cdots z_d^{j_d},\ j \in \mathbb{Z}^d$, and $\mathbf{T}$ the d-fold product of the unit circle with normalized Lebesgue measure μ.

Note that from this theorem it follows that for $\tau_\phi \in l^1(\mathbb{Z}^d)$ the collection $\mathcal{R} = \{U^j\phi \mid j \in \mathbb{Z}^d\}$ is a Riesz system if and only if $\hat{\tau}_\phi$ has no zeroes on $\mathbf{T}^d$.

Since $T^*T\alpha = \tau_\phi * \alpha$ we see that the dual Riesz system of $\mathcal{R}$ is of the form $\tilde{\mathcal{R}} = \{U^j\tilde{\phi} \mid j \in \mathbb{Z}^d\}$ with $\tilde{\phi}$ given by

$$\tilde{\phi} = \sum_{j\in\mathbb{Z}^d} \tilde{\tau}_\phi(j) U^j\phi,$$

where $\tilde{\tau}_\phi * \tau_\phi = e_0$.

Next we replace the vector $\phi \in H$ by a finite collection $\{\phi_1, \ldots, \phi_N\}$ and pose the same problem, namely under which conditions

$$\mathcal{R}_N = \{U^j\phi_n \mid n = 1, \ldots, N,\ j \in \mathbb{Z}^d\}$$

is a Riesz system. For this we introduce the index set $\mathbb{D} = \{1, \ldots, N\} \times \mathbb{Z}^d$ and define the unitary operator $\mathcal{S}_N$ from $l^2(\mathbb{D})$ into $L^2(\mathbf{T}^d, \mathbb{C}^N) = L^2(\mathbf{T}^d) \otimes \mathbb{C}^N$ by

$$(\mathcal{S}_N e_{n,j})(z) = z^j \varepsilon_n,$$

with $\{\varepsilon_1, \ldots, \varepsilon_N\}$ the standard orthonormal basis in $\mathbb{C}^N$. Now we see that $\mathcal{R}_N$ is a Riesz system if and only if the Gram matrix G with entries $(U^j\phi_n, U^i\phi_m)_{(n,j),(m,i)}$ represents a bounded invertible operator on $l^2(\mathbb{Z}^d)$. A straight forward computation shows

$$(\mathcal{S}_N G e_{n,j})(z) = z^j \hat{G}(z) \varepsilon_n,$$

where $\hat{G}(z) \in \mathbb{C}^{N\times N}$ is defined a.e. by

$$(\hat{G}(z))_{m,n} = \sum_{j\in\mathbb{Z}^d} (\phi_n, U^j\phi_m) z^{-j}.$$

Hence the Gram matrix G represents a bounded invertible operator on $l^2(\mathbb{D})$ if and only if the matrix valued function $\hat{G}$ from $\mathbf{T}^d$ into $\mathbb{C}^{N\times N}$ satisfies

$$0 < \operatorname{ess\ inf}_{z\in \mathrm{T}^d} \min_{\substack{\xi \in \mathbb{C}^N,\\ \|\xi\|=1}} (\hat{G}(z)\xi,\xi) \le \operatorname{ess\ sup}_{z\in \mathrm{T}^d} \max_{\substack{\xi \in \mathbb{C}^N,\\ \|\xi\|=1}} (\hat{G}(z)\xi,\xi) < \infty,$$

which means

$$\exists_{m,M>0}\forall_{z\in \mathrm{T}^d} \;\; mI_N \le \hat{G}(z) \le MI_N \;\; a.e. \tag{4.10}$$

5. Riesz bases in $l^2(\mathbb{Z}^d)$

In this section we deal with the following problem. Let the sequence γ yield a boundedly invertible convolution operator on $l^2(\mathbb{Z}^d)$ and let $\beta_0 \in l^2(\mathbb{Z}^d)$. Give necessary and sufficient conditions on sequences $\beta_n,\ n = 1,\ldots,N-1$, in $l^2(\mathbb{Z}^d)$ and determine N such that

(i) $(\gamma * \beta_n, R^{Dk}\beta_0)_{\mathbb{Z}} = 0,\ \forall_{n\in\{1,\ldots,N-1\}}\ \forall_{k\in\mathbb{Z}^d}$,

(ii) $\{R^{Dk}\beta_n \mid n = 1,\ldots,N-1, k \in \mathbb{Z}^d\}$ is a Riesz basis for $l^2(\mathbb{Z}^d)$.

We reformulate this into terms of the Hilbert space $L^2(\mathbf{T}^d)$.

Since we deal with a rather arbitrary matrix $D \in \mathbb{Z}^{d\times d}$ we start with some results to relate D to a diagonal matrix. The first result, we present, follows directly from Cramer's rule.

Lemma 5.1
Let $A \in \mathbb{Z}^{d\times d}$. Then there is a unique $A^+ \in \mathbb{Z}^{d\times d}$, such that

$$A^+A = AA^+ = \det(A)I_d.$$

Further we introduce the so-called Smith normal form of a matrix with integer entries, which is given in the following theorem.

Theorem 5.2
Let $A \in \mathbb{Z}^{d\times d}$. Then there are unimodular matrices $U, V \in \mathbb{Z}^{d\times d}$, i.e. $\det(U) = \det(V) = 1$, and a diagonal matrix $\Lambda \in \mathbb{Z}^{d\times d}$, such that

$$A = U\Lambda V. \tag{5.11}$$

This factorisation is not unique.

Proof
Cf. [5]
□

With (5.11) we have

$$\det(\Lambda) = \det(A)$$

and

$$A^+ = V^+\Lambda^+U^+,$$

where $\Lambda^+(i,i) = \prod_{j \neq i} \Lambda(j,j)$. In the sequel we shall use $L = |\det(\Lambda)|$.

It can be proved by some straight forward computations that the problem posed in the beginning of this section is equivalent with the following one. Give necessary and sufficient conditions on sequences $\beta_n,\ n = 1,\dots,N-1$, in $l^2(\mathbb{Z}^d)$ and determine N such that

(i) $(\gamma * \beta_n, R^{\Lambda k}\beta_0)_{\mathbb{Z}} = 0,\ \forall_{n\in\{1,\dots,N-1\}}\ \forall_{k\in\mathbb{Z}^d}$,

(ii) $\{R^{\Lambda k}\beta_n \mid n = 1,\dots,N-1, k \in \mathbb{Z}^d\}$ is a Riesz basis for $l^2(\mathbb{Z}^d)$,

with $\Lambda \in \mathbb{Z}^{d\times d}$ a diagonal matrix involved in the Smith normal form of D.

Let now $\lambda_i = \Lambda(i,i), i = 1,\dots,d$. Define $\omega_{\lambda_i} = e^{2\pi \mathrm{i}/\lambda_i},\ i = 1,\dots,d$, and K_d as the d-fold segment of all $z \in \mathbf{T}^d$ such that

$$\arg(z_i) \in [0, 2\pi/\lambda_i),\ i = 1,\dots,d.$$

We observe, that $\{z^k \mid k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L_2(\mathbf{T}^d)$ and

$$\{\sqrt{L}\, z_1^{k_1\lambda_1}\cdots z_d^{k_d\lambda_d} \mid k \in \mathbb{Z}^d\}$$

is an orthonormal basis for $L_2(K_d)$. So

$$\{\sqrt{L}\, z_1^{k_1\lambda_1}\cdots z_d^{k_d\lambda_d} e_i \mid i = 1,\dots,N,\ k \in \mathbb{Z}^d\}$$

is an orthonormal basis for $L_2(K_d, \mathbb{C}^N)$, the Hilbert space consisting of all $\mathbb{C}^N$-valued Euclidean square integrable functions on K_d. In dealing with this presented problem we present some auxiliary results.

Lemma 5.3
The $(n \times n)$ Fourier matrix F_n with entries

$$F_n(i,j) = 1/\sqrt{n}\, \omega_n^{ij},\ i,j = 0,\dots,n-1.$$

is unitary.

Using this result we prove the following lemma.

Lemma 5.4
Let $g, h \in l^2(\mathbb{Z}^d)$. Then

$$1/L \quad \cdot \quad \int_{\mathbf{T}^d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{g}(\omega_{\lambda_1}^{j_1} z_1,\dots,\omega_{\lambda_d}^{j_d} z_d)\overline{\hat{h}(\omega_{\lambda_1}^{j_1} z_1,\dots,\omega_{\lambda_d}^{j_d} z_d)}\Big) z^r \ d\mu_d(z)$$

$$= \begin{cases} (g, R^{\Lambda k} h)_{\mathbb{Z}^d} & \text{if } r = \Lambda k,\ k \in \mathbb{Z}^d, \\ 0 & \text{if } r \neq \Lambda k,\ k \in \mathbb{Z}^d. \end{cases}$$

Proof
We consider the following computation

$$1/L \int_{\mathbf{T}^d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{g}(\omega_{\lambda_1}^{j_1} z_1,\dots,\omega_{\lambda_d}^{j_d} z_d)\overline{\hat{h}(\omega_{\lambda_1}^{j_1} z_1,\dots,\omega_{\lambda_d}^{j_d} z_d)}\Big)\ z^r\, d\mu_d(z)$$

$$= \quad 1/L \sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \omega_{\lambda_1}^{-j_1 r_1} \cdots \omega_{\lambda_d}^{-j_d r_d} \int_{\mathbf{T}^d} \hat{g}(z) \overline{\hat{h}(z) z^{-r}} \mu_d(z)$$

$$= \quad \begin{cases} \int_{\mathbf{T}^d} \hat{g}(z) \overline{\hat{h}(z) z^{-r}} \mu_d(z) & \text{if } r = \Lambda k, \; k \in \mathbb{Z}^d, \\ 0 & \text{if } r \neq \Lambda k, \; k \in \mathbb{Z}^d. \end{cases}$$

The proof is completed by observing that the d-dimensional discrete Fourier transform of $R^l h$ is given by $z^{-l}\hat{h}$ and that this transform is unitary. □

Lemma 5.5
Let $g, h \in l^2(\mathbb{Z}^d)$. Then for all $k \in \mathbb{Z}^d$

$$\int_{K_d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{g}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \overline{\hat{h}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)} \Big) z_1^{k_1 \lambda_1} \cdots z_d^{k_d \lambda_d} d\mu_d(z)$$
$$= \quad (g, R^{\Lambda k} h)_{\mathbb{Z}^d}.$$

Proof
With a straight forward computation we see, that the integrand

$$\Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{g}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \overline{\hat{h}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)} \Big) z_1^{k_1 \lambda_1} \cdots z_d^{k_d \lambda_d}$$

remains unchanged if z_i is replaced by $\omega_{\lambda_i}^{r_i} z_i, \; i = 1, \ldots, d$, for $r \in \mathbb{Z}^d$ and so

$$\int_{K_d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{g}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \overline{\hat{h}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)} \Big) z_1^{k_1 \lambda_1} \cdots z_d^{k_d \lambda_d} d\mu_d(z)$$
$$= \quad \int_{K_d^r} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{g}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \overline{\hat{h}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)} \Big) z_1^{k_1 \lambda_1} \cdots z_d^{k_d \lambda_d} d\mu_d(z),$$

with

$$K_d^r = [r_1 \omega_{\lambda_1}, (r_1+1)\omega_{\lambda_1}) \times \ldots \times [r_d \omega_{\lambda_d}, (r_d+1)\omega_{\lambda_d}).$$

Consequently the result follows from Lemma 5.4.
□

By Lemma 5.5 condition (i) can be written as

$$\int_{K_d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{\gamma}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \hat{\beta}_n(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)$$
$$\overline{\hat{\beta}_0(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)} \Big) z_1^{k_1 \lambda_1} \cdots z_d^{k_d \lambda_d} d\mu_d(z) = 0.$$

Since this relation must hold for every $k \in \mathbb{Z}^d$ and since $\{\sqrt{L} z^{\Lambda k} \mid k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(K_d)$, we find that

$$\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{\gamma}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \hat{\beta}_n(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)$$
$$\cdot \quad \overline{\hat{\beta}_0(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)}) z_1^{k_1\lambda_1} \cdots z_d^{k_d\lambda_d} d\mu_d(z) = 0 \ a.e. \text{ on } K_d, n = 1, \ldots, N-1. \quad (5.12)$$

We proceed by searching necessary and sufficient conditions on $\beta_n \in l^2(\mathbb{Z}^d),\ n = 1, \ldots, N-1$, so that

$$\mathcal{C}_N = \{R^{\Lambda k} \beta_n \mid n = 0, \ldots, N-1,\ k \in \mathbb{Z}^d\}$$

is a Riesz system in $l^2(\mathbb{Z}^d)$. Since $\mathcal{C}_N$ is generated by N vectors $\beta_0, \ldots, \beta_{N-1}$ in the Hilbert space $l^2(\mathbb{Z}^d)$ and the unitary operators $R_1, \ldots, R_d$ we may use result (4.10). Thus $\mathcal{C}_N$ is a Riesz system if and only if

$$m I_N \leq \hat{G}(z) \leq M I_N \ \ a.e.\,, \quad z \in \mathbf{T}^d,$$

with

$$(\hat{G}(z))_{m,n} = \sum_{k \in \mathbb{Z}^d} (\beta_n, R^{\Lambda k} \beta_m)_{\mathbb{Z}^d} z^{-k},\ m, n = 0, \ldots, N-1. \quad (5.13)$$

This result can also be put in terms of the Fourier transforms of β_n. Therefore we derive, using Lemma 5.5,

$$\hat{G}(z_1^{\lambda_1}, \ldots, z_d^{\lambda_d})_{m,n} = \sum_{k \in \mathbb{Z}^d} (\beta_n, R^{\Lambda k} \beta_m) z_1^{-k_1\lambda_1} \cdots z_d^{-k_d\lambda_d}$$
$$= \sum_{k \in \mathbb{Z}^d} \int_{K_d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{\beta}_n(\omega_{\lambda_1}^{j_1} u_1, \ldots, \omega_{\lambda_d}^{j_d} u_d) \overline{\hat{\beta}_m(\omega_{\lambda_1}^{j_1} u_1, \ldots, \omega_{\lambda_d}^{j_d} u_d)}\Big)$$
$$u_1^{k_1\lambda_1} \cdots u_d^{k_d\lambda_d} d\mu_d(u)\ z_1^{-k_1\lambda_1} \cdots z_d^{-k_d\lambda_d}$$
$$= 1/L \sum_{k \in \mathbb{Z}^d} \int_{K_d} \Big(\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{\beta}_n(\omega_{\lambda_1}^{j_1} u_1, \ldots, \omega_{\lambda_d}^{j_d} u_d) \overline{\hat{\beta}_m(\omega_{\lambda_1}^{j_1} u_1, \ldots, \omega_{\lambda_d}^{j_d} u_d)}\Big)$$
$$\sqrt{L} u_1^{k_1\lambda_1} \cdots u_d^{k_d\lambda_d} d\mu_d(u)\ \sqrt{L} z_1^{-k_1\lambda_1} \cdots z_d^{-k_d\lambda_d}.$$

Since $\{\sqrt{L}\ z^{k_1\lambda_1} \cdots z^{k_d\lambda_d} \mid k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L_2(K_d)$, the latter result yields

$$\hat{G}(z_1^{\lambda_1}, \ldots, z_d^{\lambda_d})_{m,n}$$
$$= 1/L \sum_{k_1=0}^{|\lambda_1|-1} \cdots \sum_{k_d=0}^{|\lambda_d|-1} \hat{\beta}_n(\omega_{\lambda_1}^{k_1} z_1, \ldots, \omega_{\lambda_d}^{k_d} z_d) \cdot \overline{\hat{\beta}_m(\omega_{\lambda_1}^{k_1} z_1, \ldots, \omega_{\lambda_d}^{k_d} z_d)}. \quad (5.14)$$

Define $\hat{B}(z) = \hat{G}(z_1^{\lambda_1}, \ldots, z_d^{\lambda_d})$; Then by combining result (4.10) and (5.14),

Theorem 5.6
Let $N \in I\!N$ be fixed and $\{\beta_0, \ldots, \beta_{N-1}\}$ be a subset of $l^2(\mathbb{Z}^d)$. Let further Λ be a $(d \times d)$ diagonal matrix with integer entries. Then the collection

$$\mathcal{C}_N = \{R^{\Lambda k} \beta_n \mid n = 0, \ldots, N-1,\ k \in \mathbb{Z}^d\}$$

is a Riesz system if and only if for the $(N \times N)$ matrix valued function $z \mapsto \hat{B}(z)$, $z \in K_d$, with entries

$$\begin{aligned}&(\hat{B}(z))_{m,n}\\ = \ & 1/L \sum_{k_1=0}^{|\lambda_1|-1} \cdots \sum_{k_d=0}^{|\lambda_d|-1} \hat{\beta}_n(\omega_{\lambda_1}^{k_1} z_1, \ldots, \omega_{\lambda_d}^{k_d} z_d) \cdot \overline{\hat{\beta}_m(\omega_{\lambda_1}^{k_1} z_1, \ldots, \omega_{\lambda_d}^{k_d} z_d)},\\ & j, l = 0, \ldots, N-1, \end{aligned} \tag{5.15}$$

admits real positive constants m and M, such that

$$mI_N \leq \hat{B}(z) \leq MI_N, \ a.e.\,, \ z \in K_d. \tag{5.16}$$

So Theorem 5.6 presents necessary and sufficient conditions on $\hat{\beta}_0, \ldots, \hat{\beta}_{N-1}$, so that $\mathcal{C}_N$ is a Riesz system. We proceed by searching for similar conditions on the Fourier transforms of these sequences, such that $\mathcal{C}_N$ is a Riesz basis for $l^2(\mathbb{Z}^d)$.

Corollary 5.7
If $\mathcal{C}_N$ is a Riesz system, then $N \leq L$.

Proof
Define the $(L \times N)$ matrix valued function $z \mapsto \hat{H}(z), \ z \in K_d$, with entries

$$(\hat{H}(z))_{r,n} = L^{-1/2} \hat{\beta}_n(\omega_{\lambda_1}^{(\pi(r))(1)} z_1, \ldots, \omega_{\lambda_d}^{(\pi(r))(d)} z_d), \tag{5.17}$$

$n = 0, \ldots, N-1, \ r = 0, \ldots, L-1$, where π is an arbitrary bijection from the set$\{0, 1, \ldots, L-1\}$ onto $\{l \in \mathbb{Z}^d | \, 0 \leq l_i \leq |\lambda_i| - 1, i = 1, \ldots, d\}$. Since $\hat{B}(z) = \hat{H}^*(z)\hat{H}(z)$, $z \in K_d$, $\hat{B}$ is invertible a.e. if and only if $\hat{H}$ is injective a.e. If therefore $\hat{B}$ satisfies (5.16), i.e. $\hat{B}(z)$ is invertible for almost all $z \in K_d$, then $N \leq L$.
□

Consider the special case $N = L$ and assume $\hat{B}$ satisfies (5.16). Now Corollary 5.7 yields that $\hat{B}$ is invertible a.e. if and only if $\hat{H}$, as introduced in the above proof, is invertible a.e. So (5.16) is equivalent with $\hat{H}$ being invertible a.e. on K_d. Further let $\mathcal{B}$ and $\mathcal{H}$ denote bounded linear operators on $L^2(K_d; \mathbb{C}^L)$ corresponding to $\hat{B}$ and $\hat{H}$, respectively, i.e.

$$(\mathcal{B}\eta)(z) = \hat{B}(z)\eta(z) \ \text{ and } \ (\mathcal{H}\eta)(z) = \hat{H}(z)\eta(z), \ a.e.\,, \ z \in K_d$$

for all $\eta \in L^2(K_d; \mathbb{C}^L)$. Then $\mathcal{B} = \mathcal{H}^*\mathcal{H}$ and $\mathcal{H}^{-1} = \mathcal{B}^{-1}\mathcal{H}^*$. So $\mathcal{H}$ is a boundedly invertible operator, since $\mathcal{B}$ is a boundedly invertible operator and

$$(\mathcal{H}^{-1}\eta)(z) = \hat{H}(z)^{-1}\eta(z).$$

Theorem 5.8
Let $N \in \mathbb{N}$ be fixed and Λ be a $(d \times d)$ diagonal matrix with integer entries. Let further $\{\beta_0, \ldots, \beta_{N-1}\}$ be a subset of $l^2(\mathbb{Z}^d)$, such that the collection

$$\mathcal{C}_N = \{R^{\Lambda k}\beta_n \mid n = 0, \ldots, N-1, \ k \in \mathbb{Z}^d\}$$

is a Riesz system. Then this collection is a Riesz basis if and only if $N = L$.

Proof
Let $\mathcal{C}_L$ be a Riesz system in $l^2(\mathbb{Z}^d)$. Then $\hat{H}(z)$ is invertible for almost all $z \in K_d$ and $\mathcal{H}$ is invertible, since $\hat{B}$ satisfies (5.16). Define $\tilde{\varepsilon}_{n,k} \in L^2(K_d; \mathbb{C}^L)$ for $n = 0, \ldots, L-1$, and $k \in \mathbb{Z}^d$ by

$$\tilde{\varepsilon}_{n,k}(z) = \sqrt{L}\, z^{\Lambda k} \varepsilon_n, \; z \in K_d.$$

Furthermore introduce $\mathcal{V}_\Lambda : l^2(\mathbb{Z}^d) \to L^2(K_d; \mathbb{C}^L)$ by

$$(\mathcal{V}_\Lambda h)(z) = \begin{pmatrix} \hat{h}(\omega_{\lambda_1}^{(\pi(0))(1)} z_1, \ldots, \omega_{\lambda_d}^{(\pi(0))(d)} z_d) \\ \vdots \\ \hat{h}(\omega_{\lambda_1}^{(\pi(L-1))(1)} z_1, \ldots, \omega_{\lambda_d}^{(\pi(L-1))(d)} z_d) \end{pmatrix}$$

where π is an arbitrary bijection from the set $\{0, 1, \ldots, L-1\}$ onto the collection $J = \{l \in \mathbb{Z}^d |\, 0 \leq l_i \leq |\lambda_i| - 1, i = 1, \ldots, d\}$. With this definition

$$\begin{aligned} \|\mathcal{V}_\Lambda h\|^2_{L^2(K_d;\mathbb{C}^L)} &= \int_{K_d} \|(\mathcal{V}_\Lambda h)(z)\|^2_{\mathbb{C}^L} d\mu_d(z) \\ &= \int_{K_d} \Big(\sum_{r_1=0}^{|\lambda_1|-1} \cdots \sum_{r_d=0}^{|\lambda_d|-1} |\hat{h}(\omega_{\lambda_1}^{r_1} z_1, \ldots, \omega_{\lambda_d}^{r_d} z_d)|^2 \Big) d\mu_d(z) = \|h\|^2_{\mathbb{Z}^d}, \end{aligned}$$

for all $h \in l^2(\mathbb{Z}^d)$, so that $\mathcal{V}_\Lambda$ is an isometry. Define $h_{n,k} \in l^2(J \otimes \mathbb{Z}^d)$by

$$\hat{h}_{n,k}(z) = \begin{cases} \sqrt{L} z_1^{\lambda_1 k_1} \cdots z_d^{\lambda_d k_d} & \text{if } (\omega_{\lambda_1}^{-n_1} z_1, \ldots, \omega_{\lambda_d}^{-n_d} z_d) \in K_d, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{h_{n,k} \,|\, n \in J, \; k \in \mathbb{Z}^d\}$ is an orthonormal basis for $l^2(\mathbb{Z}^d)$. With this we get $\mathcal{V}_\Lambda h_{\pi(n),k} = \tilde{\varepsilon}_{n,-k}$ and so the operator $\mathcal{V}_\Lambda$ is unitary. Applying $\mathcal{V}_\Lambda$ on $R^{\Lambda k}\beta_n$ yields

$$\begin{aligned} \mathcal{V}_\Lambda(R^{\Lambda k}\beta_n)(z) &= z^{\Lambda k} \begin{pmatrix} \hat{\beta}_n(\omega_{\lambda_1}^{(\pi(0))(1)} z_1, \ldots, \omega_{\lambda_d}^{(\pi(0))(d)} z_d) \\ \vdots \\ \hat{\beta}_n(\omega_{\lambda_1}^{(\pi(L-1))(1)} z_1, \ldots, \omega_{\lambda_d}^{(\pi(L-1))(d)} z_d) \end{pmatrix} \\ &= \sqrt{L} z^{\Lambda k} \hat{H}(z) \varepsilon_n = \hat{H}(z) \tilde{\varepsilon}_{n,k}(z) = (\mathcal{H}\tilde{\varepsilon}_{n,k})(z), \end{aligned}$$

for all $n = 0, \ldots, L-1$. So $R^{\Lambda k}\beta_n = (\mathcal{V}_\Lambda)^* \mathcal{H} \tilde{\varepsilon}_{n,k}$.
Since $\{\tilde{\varepsilon}_{n,k} \mid n = 0, \ldots, L-1, \; k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(K_d; \mathbb{C}^L)$ and $(\mathcal{V}_\Lambda)^* \mathcal{H}$ is boundedly invertible, we get that $\mathcal{C}_N$ is a Riesz basis for $l^2(\mathbb{Z}^d)$.
For proving the converse we assume $\mathcal{C}_L$ to be a Riesz basis for $l^2(\mathbb{Z}^d)$. Then $\mathcal{H}$ has to be invertible, since $R^{\Lambda k}\beta_n = \mathcal{V}_\Lambda^* \mathcal{H} \tilde{\varepsilon}_{n,k}$. It follows that the matrix valued function $\hat{H}$ has to be invertible a.e. on K_d and thus $N = L$.
□

Similar results as Theorem 5.6 and Corollary 5.7 can now be given rather easily for

$$\mathcal{C}_N = \{R^{\Lambda k}\beta_n \mid n = 0, \ldots, N-1, \; k \in \mathbb{Z}^d\}$$

being a frame. We can write

$$R^{\Lambda k}\beta_n = \mathcal{V}_\Lambda^* \mathcal{H}\tilde{\varepsilon}_{n,k}) = \mathcal{V}_\Lambda^* \mathcal{H} U e_{n,k},$$

with $U : l^2(\{0,\dots,N-1\} \otimes \mathbb{Z}^d) \to L^2(K_d;\mathbb{C}^N)$ the unitary operator given by $Ue_{n,k} = \tilde{\varepsilon}_{n,k}$. So $T^* = U^*\mathcal{H}^*\mathcal{V}_\Lambda$ is the frame generator of $\mathcal{C}_N$ if $\mathcal{C}_N$ is a frame. Now Theorem 2.6 yields immediately the following theorem.

Theorem 5.9
Let $N \in \mathbb{N}$ be fixed and $\{\beta_0,\dots,\beta_{N-1}\}$ be a subset of $l^2(\mathbb{Z}^d)$. Then the collection

$$\{R^{\Lambda k}\beta_n \mid n = 0,\dots,N-1,\ k \in \mathbb{Z}^d\}$$

is a frame if and only if for the $(N \times N)$ matrix valued function $z \mapsto \hat{H}(z)\hat{H}^*(z)$, $z \in K_d$, with $\hat{H}$ defined as in (5.17) there exists real positive constants m and M, such that

$$mI_N \le \hat{H}(z)\hat{H}^*(z) \le MI_N,\ a.e.,\ z \in K_d. \tag{5.18}$$

So similar to Theorem 5.6 we have presented necessary and sufficient conditions on $\hat{\beta}_0,\dots,\hat{\beta}_{N-1}$, so that $\mathcal{C}_N$ is a frame. Finally we present a corollary of Theorem 5.9, analogous to Corollary 5.7.

Corollary 5.10
If $\mathcal{C}_N$ is a frame, then $N \ge L$.

6. Riesz bases in H

In the third section we used MRA to construct Riesz bases of the form

$$\{U_1^j U_2^k \psi_n \mid n = 1,\dots,N-1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$$

for the separable Hilbert space H. The vectors ψ_n were uniquely determined by (3.7). The generating sequences q_n of ψ_n had to be determined such that

- $(\tau_\phi * q_n, R^{Dk}p)_{\mathbb{Z}^d} = 0,\ \forall_{n\in\{1,\dots,N-1\}}\ \forall_{k\in\mathbb{Z}^d}$,
- $\{R^{Dk}p \mid k \in \mathbb{Z}^d\} \cup \{R^{Dk}q_n \mid n = 1,\dots,N-1, k \in \mathbb{Z}^d\}$ is a Riesz basis for $l^2(\mathbb{Z}^d)$.

By taking $\gamma = \tau_\phi$, $\beta_0 = p$ and $\beta_n = q_n$, $n = 1,\dots,N-1$, in (5.12), Theorem 5.4 and 5.6, we arrive at the following result.

Result 6.1
Given a sequence $p \in l^2(\mathbb{Z}^d)$. Then the following two problems are equivalent.
Problem 1:
Construct sequences q_n, $n = 1,\dots,|\det(D)|-1$, in $l^2(\mathbb{Z})$ such that

1.1 $(\tau_\phi * q_n, R^{Dk}p)_{\mathbb{Z}^d} = 0,\ \forall_{n\in\{1,\dots,|\det(D)|-1\}}\ \forall_{k\in\mathbb{Z}^d}$,

1.2 $\{R^{Dk}p \mid k \in \mathbb{Z}^d\} \cup \{R^{Dk}q_n \mid n = 1,\dots,|\det(D)|-1, k \in \mathbb{Z}^d\}$ is a Riesz basis for $l^2(\mathbb{Z}^d)$.

Problem 2:
Construct $\hat{q}_n$, $n = 1,\dots,|\det(D)|-1$, in $L^2(\mathbf{T}^d)$, the d-dimensional discrete Fourier transforms of $q_n \in l^2(\mathbb{Z}^d)$, such that

2.1

$$\sum_{j_1=0}^{|\lambda_1|-1} \cdots \sum_{j_d=0}^{|\lambda_d|-1} \hat{r}(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d) \hat{q}_n(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)$$
$$\cdot \, \overline{\hat{p}_0(\omega_{\lambda_1}^{j_1} z_1, \ldots, \omega_{\lambda_d}^{j_d} z_d)}) z_1^{k_1 \lambda_1} \cdots z_d^{k_d \lambda_d} d\mu_d(z) = 0 \ a.e. \text{ on } K_d,$$
$$n = 1, \ldots, |\det(D)| - 1.$$

2.2 The matrix-valued function $z \mapsto \hat{H}(z),\ z \in K_d$, with entries

$$(\hat{H}(z))_{r,j} = |\det(\Lambda)|^{-1/2} \hat{q}_j(\omega_{\lambda_1}^{(\pi(r))(1)} z_1, \ldots, \omega_{\lambda_d}^{(\pi(r))(d)} z_d),$$

$j, r = 0, \ldots, |\det(D)| - 1$, where π is an arbitrary enumeration from $\{0, 1, \ldots, |\det(D)| - 1\}$ onto $\{l \in \mathbb{Z}^d \mid 0 \leq l_i \leq |\lambda_i| - 1, i = 1, \ldots, d\}$ and with $p = q_0$, is invertible for almost all $z \in K_d$.

The matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d)$ used in Problem 2 is a diagonal matrix appearing in a factorization (5.11).

Observe that in the one-dimensional case, if we start with $p \in l^1(\mathbb{Z})$ and then search for sequences $q_n, n = 1, \ldots, |\det(D)| - 1$, in $l^1(\mathbb{Z})$ such that Problem 1 is solved, condition 2.2 is a generalization of the condition as presented in [2], p.142, where multi resolution analysis for $L^2(\mathbb{R})$, see Section 1, is discussed. We notice also that possible solutions for Problem 2 are given in [4, 7] and [6].

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