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ABSTRACT
In present teletraffic applications of queueing theory service time distributions $B(t)$ with a heavy tail occur, i.e. $1 - B(t) \sim t^{-\nu}$ for $t \to \infty$ with $\nu > 1$. For such service time distributions not much explicit information is available concerning the tail probabilities of the inherent waiting time distribution $W(t)$. In the present study the waiting time distribution is studied for a stable $M/G/1$ model for a class of service time distributions with $1 < \nu < 2$. For $\nu = 1\frac{1}{2}$ the explicit expression for $Q(t)$ is derived. For rational ν with $1 < \nu < 2$, an asymptotic series for the tail probabilities of $W(t)$ is derived.

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1. Introduction

In classical applications of teletraffic theory the service time distributions in queueing models are frequently assumed to have a negative exponential tail; a modelling assumption which has been justified by extensive measurements. Measurements on high-speed communication networks have shown that service time distributions with heavier tails may occur.

The qualitative relation between the tails of the stationary waiting distribution and that of the service time distribution has been extensively investigated for several basic queueing models by applying results from the theory on regular variation of probability distributions, see e.g. [7], [5]. For a fine and more detailed study on tail behaviour, see [12]. However, in several practical applications more quantitative results are much needed. Explicit representations for heavy-tailed distributions are readily available. To derive from such a representation of the service time distribution a manageable expression for the stationary waiting time distribution is not so simple. This problem has been discussed by Abate, Choudhury and Whitt, cf. [6]. These authors introduce a class of long-tail service time distributions. For such distributions they study multiterm asymptotic expansions for the tail of the stationary waiting time distribution of the $M/G/1$ queueing model. Their results are most interesting, in particular the numerical results show the difficulties involved in obtaining a reasonable accuracy for the smaller tail probabilities.

The present study concerns the stationary waiting time distribution $W(t)$ of the $M/G/1$ queueing model for which the service distribution $B(t)$ has the asymptotic behaviour

$$1 - B(t) = \mathrm{O}(t^{-\nu}) \, , \; t \to \infty, \tag{1.1}$$

with

$$1 < \nu < 2. \tag{1.2}$$

This tail-behaviour is mentioned in [6] but there the discussion mainly concerns the case with $\nu \geq 2$. The case $\nu \in (\frac{1}{2}, 1)$ is of special interest because it involves "long range dependence" of the waiting time process, cf. [8]. Norros [8] analyses a storage model with long range dependence by describing the net input as a fractional Brownian process, he derives a lower bound for the tail probabilities of the stationary distribution of the storage level.

In present day queueing literature no explicit results seem to be known for the stationary waiting distribution $W(t)$ for an $M/G/1$ queueing model with traffic load $a < 1$ and service time distributions as characterised by (1.1) and (1.2). Presumably this is due to the fact that the Laplace-Stieltjes transform $\beta(\rho)$ of distributions with heavy tails is difficult to obtain explicitly. Moreover if $\beta(\rho)$ is known then, although the Laplace-Stieltjes transform $\omega(\rho)$ of $W(t)$ can be easily calculated from the Pollaczek-Khintchine formula, the explicit determination of $W(t)$ from $\omega(\rho)$ seems hardly to be possible and so one has to use numerical inversion of the Laplace-Stieltjes transform, cf. [9], [10].

Explicit results for $W(t)$ in particular for $W(t)$ with $t \to \infty$ are very desirable, they may foster the insight in the effects of long range dependence and may also serve to check the quality of approximations.

In the present study we introduce a class of service time distributions $B(t)$ for which (1.1) and (1.2) apply and which have rather simple Laplace-Stieltjes transforms. For such service time distributions and the related waiting time distributions asymptotic series for $t \to \infty$ can be explicitly calculated. These results extend those obtained by applying "slowly variation" analysis.

For this class of distributions also the explicit expression of the service time distribution can be obtained. For the inherent waiting time distribution we have not succeeded in obtaining an explicit relation for general $\nu \in (1, 2)$. For $\nu = 1\frac{1}{2}$, however, a fairly simple explicit relation for $W(t)$ is obtained (the case $\nu = 1\frac{1}{2}$ seems to be important in the applications). For $\nu = 1\frac{1}{4}$ it is also possible to obtain an explixit result for $W(t)$. It is, however, a fairly complicated expression and therefore omitted here.

We next review shortly the sections of the present study.

In section 2 we introduce the class of service time distributions $B(t), t \in [0, \infty)$. They are characterised by

$$B(t) = 1 - \frac{s^{2-\nu}}{\Gamma(2-\nu)}\delta \int_0^\infty \mathrm{e}^{-s\theta} \frac{\theta}{(\theta+t)^\nu} \mathrm{d}\theta, \tag{1.3}$$

with

$$1 < \nu < 2,\ s > 0\ ,\ 0 < \delta \leq 1;$$

here s and δ are constants and $\Gamma(\cdot)$ is the gamma function. Note that

$$B(0+) = 1 - \delta, \tag{1.4}$$

$$\beta := \int_0^\infty t \mathrm{d}B(t) = \frac{2-\nu}{\nu-1}\frac{\delta}{s}.$$

With

$$\beta(\rho) := \int_{0-}^\infty \mathrm{e}^{-\rho t} \mathrm{d}B(t)\ ,\ \mathrm{Re}\,\rho \geq 0,$$

it is shown that for $\mathrm{Re}\,\rho \geq 0$,

$$\begin{aligned} &\frac{1-\beta(\rho)}{\rho\beta} = \frac{\omega}{\omega-1}\{1 - \frac{1}{2-\nu}\frac{\omega^{2-\nu}-1}{\omega-1}\}, \\ &\omega := \frac{s}{\rho}. \end{aligned} \tag{1.5}$$

In section 3 the case $\nu = 1\frac{1}{2}$ is studied. It is shown that for $t \geq 0$,

$$B(t) = 1 + \frac{2\delta}{\sqrt{\pi}}[\sqrt{st} - (1+2st)\mathrm{e}^{st}\ \mathrm{Erfc}(\sqrt{st})], \tag{1.6}$$

with the complementary error function given by:

$$\mathrm{Erfc}\,(x) = \int_x^\infty \mathrm{e}^{-u^2}\mathrm{d}u.$$

The inherent stationary waiting time distribution for the $M/G/1$ model with load $a < 1$ is given by: for $t > 0$,

$$\begin{aligned} W(t) = \quad & 1 - (1+\sqrt{a})(\frac{a}{\pi})^{1/2}\mathrm{e}^{(1-\sqrt{a})^2 st}\ \mathrm{Erfc}((1-\sqrt{a})\sqrt{st}) \\ & +(1-\sqrt{a})(\frac{a}{\pi})^{1/2}\mathrm{e}^{(1-\sqrt{a})^2 st}\ \mathrm{Erfc}((1+\sqrt{a})\sqrt{st}). \end{aligned} \tag{1.7}$$

The asymptotic relations are given by: for $t \to \infty$,

$$1 - B(t) = \frac{2\delta}{\pi}\sum_{n=1}^{M}(-1)^{n+1}\frac{n\Gamma(n+\frac{1}{2})}{(st)^{n+1/2}} + \mathrm{o}(|st|^{-M-\frac{1}{2}}), \tag{1.8}$$

$$\begin{aligned} 1 - W(t) \quad & = \frac{\sqrt{a}}{2\pi}(1-a)\sum_{m=0}^{M}(-1)^m[\frac{1}{(1-\sqrt{a})^{2m+2}} \\ & -\frac{1}{(1+\sqrt{a})^{2m+2}}]\frac{\Gamma(m+\frac{1}{2})}{(st)^{m+1/2}} + \mathrm{o}(|st|^{-M-\frac{1}{2}}), \end{aligned} \tag{1.9}$$

for every finite $M \in \{1, 2, \dots\}$.

In section 4 an explicit expression for $B(t)$, cf. (1.3), is given, viz., cf. (4.7), for $0 < \mu = 2 - \nu < 1, 0 < \delta \leq 1, s > 0$,

$$1 - B(t) = \frac{\mu}{1-\mu}\frac{\delta}{s}[-\frac{(st)^\mu}{\Gamma(1+\mu)} + \frac{1-\mu+st}{\Gamma(1+\mu)}\mathrm{e}^{st}\int_{st}^{\infty}\mathrm{e}^{-u}u^{\mu-1}\mathrm{d}u]. \tag{1.10}$$

This section further contains the asymptotic series expansion of $1 - B(t)$ for $t \to \infty$ as well as for $t \downarrow 0$, see (4.11) and (4.12).

In section 5 the stationary waiting time distribution $W(t)$ of the $M/G/1$ queue with traffic load $a < 1$ and service time distribution $B(t)$ as given in (1.9) is studied for the case that $\mu = 2 - \nu$, is a rational number; $\mu = M/N$ with $0 < M < N$ and g.c.d. $(M, N) = 1$. The L.S-transform $\omega(\rho)$ of $W(t)$ is easily obtained. However, its inversion requires the knowledge of $2N - 2$ zeros of an algebraic equation of the degree $2N - 2$. We omit the explicit expression for $W(t)$, although it can be easily derived. Instead we derive the asymptotic series expansion of $1 - W(t)$ for $t \to \infty$, it still involves the just mentioned $2N - 2$ zeros, but the asymptotic series can be used for the numerical evaluation of the tail probabilities.

The results obtained in section 5 are used to study the tail probabilities for the case that the traffic load $a \uparrow 1$. The result reads: for $a \uparrow 1, t \to \infty$ and every finite $K \in \{1, 2, \dots\}$,

$$\begin{aligned} & 1 - W(\frac{1-\mu}{\mu\delta}(\frac{a}{1-a)\mu})^{\frac{1}{1-\mu}}\beta\tau) = \\ & \frac{1}{n}[\sum_{k=1}^{K}\sin(\mu k\pi)\Gamma((1-\mu)k)\tau^{-(1-\mu)k} + \mathrm{o}(\tau^{-(1-\mu)K})](1 + \mathrm{O}((1-a)^{\frac{\mu}{1-\mu}})), \end{aligned} \tag{1.11}$$

with $0 < \mu = 2 - \nu < 1, 0 < \delta \leq 1$.

2. Heavy-tailed distributions

Let $\boldsymbol{\tau}$ be a nonnegative stochastic variable with

$$\Pr\{\boldsymbol{\tau} > t\} \sim C(t/\theta)^{-\nu} \text{ for } t \to \infty,$$

with $C > 0$ a constant and $\nu > 0$. The distribution of $\boldsymbol{\tau}$ is then said to have a heavy tail, this in contrast to distributions which behave for $t \to \infty$ as:

$$Ct^{-\nu}\mathrm{e}^{-\alpha t} \text{ for } t \to \infty \text{ with } \nu \geq 0,\ \alpha > 0.$$

Obviously $\mathrm{E}\{\boldsymbol{\tau}\} < \infty$ if $\nu > 1$ and $\mathrm{E}\{\boldsymbol{\tau}\} < \infty, \mathrm{E}\{\boldsymbol{\tau}^2\} = \infty$ if $1 < \nu \leq 2$.

In our analysis we shall always consider heavy-tailed distributions with

$$1 < \nu < 2. \tag{2.1}$$

Obviously the nonnegative stochastic variable $\boldsymbol{\tau}_\theta$ with Pareto distribution

$$\Pr\{\boldsymbol{\tau}_\theta < t\} \equiv B(t,\theta) := 1 - \delta(\frac{\theta}{\theta + t})^{\nu},\ t \geq 0, \tag{2.2}$$

where δ and θ are parameters satisfying

$$0 < \delta \leq 1\ ,\ \theta > 0, \tag{2.3}$$

possesses a heavy-tailed distribution

$$1 - B(t,\theta) \sim \delta(t/\theta)^{-\nu} \text{ for } t \to \infty.$$

It is readily verified that

$$\beta(\theta) := \mathrm{E}\{\boldsymbol{\tau}_\theta\} = \delta\frac{\theta}{\nu - 1}. \tag{2.4}$$

The Laplace-Stieltjes transform of the distribution $B(t,\theta)$ is defined by: for $\mathrm{Re}\,\rho \geq 0$,

$$\beta(\rho,\theta) := \mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}_\theta}\} = \int_{0-}^{\infty} \mathrm{e}^{-\rho t}\mathrm{d}B(t,\theta),\ \text{with } \tilde{\beta}(\theta) := \int_0^{\infty} t\mathrm{d}B(t,\theta). \tag{2.5}$$

It is wellknown that: for $\mathrm{Re}\,\rho \geq 0$,

$$\frac{1}{\rho}\{1 - \beta(\rho,\theta)\} = \int_0^{\infty} \mathrm{e}^{-\rho t}(1 - B(t,\theta))\mathrm{d}t = \delta\int_0^{\infty} \mathrm{e}^{-\rho t}\frac{\theta^{\nu}}{(\theta + t)^{\nu}}\mathrm{d}t. \tag{2.6}$$

It follows easily that: for $\mathrm{Re}\,\rho \geq 0$,

$$h(\rho,\theta) := \frac{1 - \beta(\rho,\theta)}{\rho\tilde{\beta}(\theta)} = (\nu - 1)\int_0^{\infty} \mathrm{e}^{-\rho t}\frac{\theta^{\nu-1}}{(\theta + t)^{\nu}}\mathrm{d}t, \tag{2.7}$$

and that $h(\rho)$ is the L.S.-transform of the probability distribution

$$\begin{aligned} H(t,\theta) \quad &:= \frac{1}{\tilde{\beta}(\theta)}\int_0^t \{1 - B(t,\theta)\}\mathrm{d}t\ , \quad t \geq 0, \\ &:= 0, \qquad\qquad\qquad\qquad\qquad t < 0. \end{aligned} \tag{2.8}$$

Let $\boldsymbol{\theta}$ be a nonnegative stochastic variable with density

$$\Pr\{\theta \le \boldsymbol{\theta} < \theta + \mathrm{d}\theta\} = \frac{s^{2-\nu}}{\Gamma(2-\nu)}\theta^{1-\nu}\mathrm{e}^{-s\theta}\mathrm{d}\theta\ ,\ \theta > 0,\ 1 < \nu < 2, \tag{2.9}$$

here s is a positive constant. Note that

$$\int_0^\infty \theta^{1-\nu}\mathrm{e}^{-s\theta}\mathrm{d}\theta = \frac{\Gamma(2-\nu)}{s^{2-\nu}}, \tag{2.10}$$

with $\Gamma(\cdot)$ the gamma function; observe that the integral in (2.10) exists at the lower bound since $1 < \nu < 2$, cf. (2.1).

Let $\boldsymbol{\tau}$ be a nonnegative stochastic variable with distribution $B(t)$ defined by:

$$\Pr\{\boldsymbol{\tau} < t\} \equiv B(t) := \frac{s^{2-\nu}}{\Gamma(2-\nu)}\int_0^\infty \theta^{1-\nu}\mathrm{e}^{-s\theta}B(t,\theta)\mathrm{d}\theta; \tag{2.11}$$

evidently $B(t)$ is a mixture of the distributions $B(t,\theta)$, the mixture being described by (2.9).
Put: for $\mathrm{Re}\,\rho \ge 0$,

$$\beta(\rho) := \mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}. \tag{2.12}$$

By partial integration we have from (2.6): for $\mathrm{Re}\,\rho \ge 0$,

$$\frac{1-\beta(\rho,\theta)}{\rho} = \delta\theta^{\nu}[\frac{\theta^{1-\nu}}{\nu-1} - \frac{\rho}{\nu-1}\int_0^\infty \mathrm{e}^{-\rho t}(\theta+t)^{-\nu+1}\mathrm{d}t]. \tag{2.13}$$

Hence from (2.9) and (2.13): for $\mathrm{Re}\,\rho \ge 0, s > 0$,

$$\begin{aligned}\frac{1-\beta(\rho)}{\rho} &= \frac{s^{2-\nu}}{\Gamma(2-\nu)}\frac{\delta}{\nu-1}\int_0^\infty \theta^{2-\nu}\mathrm{e}^{-s\theta}\mathrm{d}\theta\\ &-\frac{s^{2-\nu}}{\Gamma(2-\nu)}\frac{\delta\rho}{\nu-1}\int_0^\infty \theta\mathrm{e}^{-s\theta}[\int_0^\infty \mathrm{e}^{-\rho t}(\theta+t)^{-\nu+1}\mathrm{d}t]\mathrm{d}\theta\\ &= \frac{s^{2-\nu}}{\Gamma(2-\nu)}\frac{\delta}{\nu-1}\frac{\Gamma(3-\nu)}{s^{3-\nu}} - \frac{s^{2-\nu}}{\Gamma(2-\nu)}\frac{\delta\rho}{\nu-1}\int_0^\infty \theta\mathrm{e}^{-(s-\rho)\theta}[\int_0^\infty \mathrm{e}^{-\rho u}u^{-\nu+1}\mathrm{d}u]\mathrm{d}\theta\\ &= \frac{2-\nu}{\nu-1}\frac{\delta}{s}[1-\rho\frac{s^{3-\nu}}{\Gamma(3-\nu)}\int_0^\infty \theta\mathrm{e}^{-(s-\rho)\theta}[\int_\theta^\infty \mathrm{e}^{-\rho u}u^{-\nu+1}\mathrm{d}u]\mathrm{d}\theta].\end{aligned} \tag{2.14}$$

$z\Gamma(z) = \Gamma(z+1)$. Because, cf. (2.4),

$$\beta := \mathrm{E}\{\tilde{\beta}(\boldsymbol{\theta})\} = \frac{2-\nu}{\nu-1}\frac{\delta}{s}, \tag{2.15}$$

we have from (2.14): for $0 < \rho < s$,

$$\begin{aligned}\frac{1-\beta(\rho)}{\rho\beta} &= 1 - \frac{s^{3-\nu}\rho^{\nu-1}}{\Gamma(3-\nu)}\int_0^\infty \theta \mathrm{e}^{-(s-\rho)\theta}[\int_{\theta\rho}^\infty \mathrm{e}^{-w}w^{1-\nu}\mathrm{d}w]\mathrm{d}\theta \\ &= 1 - \frac{(s/\rho)^{3-\nu}}{\Gamma(3-\nu)}\int_0^\infty z\mathrm{e}^{-(\frac{s}{\rho}-1)z}[\int_z^\infty \mathrm{e}^{-u}u^{1-\nu}\mathrm{d}u]\mathrm{d}z, \\ &= 1 + \frac{\omega^{3-\nu}}{\Gamma(3-\nu)}\frac{\mathrm{d}}{\mathrm{d}\omega}\int_0^\infty \mathrm{e}^{-(\omega-1)z}[\int_z^\infty \mathrm{e}^{-u}u^{1-\nu}\mathrm{d}u]\mathrm{d}z,\end{aligned} \tag{2.16}$$

with

$$\omega = \frac{s}{\rho} > 1. \tag{2.17}$$

We have via partial integration: for $\omega > 1$,

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-(\omega-1)z}\int_z^\infty \mathrm{e}^{-u}u^{1-\nu}\mathrm{d}u\mathrm{d}z = \frac{1}{\omega-1}\int_0^\infty \mathrm{e}^{-u}u^{1-\nu}\mathrm{d}u - \frac{1}{\omega-1}\int_0^\infty \mathrm{e}^{-\omega z}z^{1-\nu}\mathrm{d}z = \\ &\{\frac{1}{\omega-1} - \frac{1}{\omega-1}\frac{1}{\omega^{2-\nu}}\}\Gamma(2-\nu).\end{aligned} \tag{2.18}$$

Hence from (2.16) and (2.18): for $\omega > 1$,

$$\begin{aligned}\frac{1-\beta(\rho)}{\rho\beta} &= 1 + \frac{\omega^{3-\nu}}{\Gamma(3-\nu)}\Gamma(2-\nu)\frac{\mathrm{d}}{\mathrm{d}\omega}[\frac{1}{\omega-1}\{1 - \frac{1}{\omega^{2-\nu}}\}] \\ &= \frac{\omega}{\omega-1}[1 - \frac{1}{2-\nu}\frac{\omega^{2-\nu}-1}{\omega-1}].\end{aligned} \tag{2.19}$$

It is readily seen that the righthand side has a finite limit for $\omega \to 1$, and that (2.19) also holds for $0 \le \mathrm{Re}\,\rho < s$. From the definition of $\beta(\rho), \mathrm{Re}\,\rho \ge 0$, it is readily verified that $\{1-\beta(\rho)\}/\rho\beta$ is regular for $\mathrm{Re}\,\rho > 0$, continuous for $\mathrm{Re}\,\rho \ge 0$. Because the righthand side of (2.19) is regular for $\mathrm{Re}\,\rho > 0, \rho \ne s$, continuous for $\mathrm{Re}\,\rho \ge 0, \rho \ne s$, and has a finite limit for $\rho = s$ it follows that: for $s > 0, \mathrm{Re}\,\rho \ge 0, 1 < \nu < 2$,

$$\begin{aligned}\frac{1-\beta(\rho)}{\rho\beta} &= \frac{\omega}{\omega-1}[1 - \frac{1}{2-\nu}\frac{\omega^{2-\nu}-1}{\omega-1}] \quad &&\text{for } \omega = \frac{s}{\rho} \ne 1, \\ &= \frac{1}{2}(\nu-1) &&\text{for } \omega = 1, \\ \beta &= \frac{2-\nu}{\nu-1}\frac{\delta}{s},\end{aligned} \tag{2.20}$$

with $\beta(\rho)$ defined by (2.2), (2.11) and (2.12).

Because $B(\cdot)$ is a probability distribution with support $[0,\infty)$ it is seen that

$$k(\rho) := \frac{1-\beta(\rho)}{\rho\beta}, \ \mathrm{Re}\,\rho \ge 0, \tag{2.21}$$

is the L.S.-transform of the distribution

$$\begin{aligned}K(t) &:= \frac{1}{\beta}\int_0^t \{1 - B(\tau)\}\mathrm{d}\tau, \quad &&t \ge 0, \\ &:= 0, &&t < 0.\end{aligned} \tag{2.22}$$

As above it is readily shown that for $\mathrm{Re}\,\rho \geq 0, s > 0$,

$$h(\rho) := \mathrm{E}\{\frac{1-\beta(\rho,\boldsymbol{\theta})}{\rho\tilde{\beta}(\boldsymbol{\theta})}\} \begin{array}{ll} = \dfrac{\omega}{\omega-1}\{1-\omega^{1-\nu}\} & \text{for } \omega = \dfrac{s}{\rho} \neq 1, \\ = \nu - 1 & \text{for } \omega = 1. \end{array} \tag{2.23}$$

The relations (2.2) and (2.11) describe a class of probability distributions $B(t)$ with support $[0,\infty)$ and heavy tails, i.e.

$$1 - B(t) = \mathrm{O}(t^{-\nu}) \text{ for } t \to \infty. \tag{2.24}$$

An element of this class is characterised by the three parameters ν, s and δ with

$$1 < \nu < 2, 0 < s < \infty \text{ and } 0 < \delta \leq 1. \tag{2.25}$$

Note that, cf. (2.15),

$$\beta = \int_0^\infty t \mathrm{d}B(t) = \frac{2-\nu}{\nu-1}\frac{\delta}{s} > 0, \tag{2.26}$$

so that for given β, ν and δ there exists always an s with $s \in (0,\infty)$.

It is readily verified that the definition of $B(t)$ for $\nu \downarrow 1$ makes also sense and that, cf. (2.19),

$$\begin{aligned} \frac{1-\beta(\rho)}{\rho} &= \lim_{\nu\downarrow 1} \frac{2-\nu}{(\nu-1)}\frac{\omega}{\omega-1}\{1-\frac{1}{2-\nu}\frac{\omega^{2-\nu}-1}{\omega-1}\}\frac{\delta}{s} = \\ &= \frac{\delta}{s}\frac{\omega}{(\omega-1)^2}[1-\omega+\omega\log\omega],\ \omega = \frac{s}{\rho},\ \mathrm{Re}\,\rho > 0. \end{aligned} \tag{2.27}$$

However, the case $\nu = 1$ is for the analysis of the $M/G/1$ queueing model not relevant since $\beta = \infty$ for $\nu = 1$.

For $\nu \uparrow 2$ the definition (2.11) of $B(t)$ degenerates. This is due to the degeneration of the weighing function (2.9) for $\nu \uparrow 2$; note that the integral in (2.10) diverges for $\nu \uparrow 2$.

3. The case $\nu = 1\frac{1}{2}$

In this section we consider the case

$$\nu = 1\frac{1}{2}, \tag{3.1}$$

and we derive an explicit expression for $B(t)$. With this $B(t)$ as the service time distribution of a stable $M/G/1$ queue with traffic load $a < 1$ an explicit expression for the stationary waiting time distribution $W(t)$ as well as for its tail behaviour is derived.

From (2.20) we have for $\nu = 1\frac{1}{2}, \mathrm{Re}\,\rho \geq 0$,

$$\frac{1-\beta(\rho)}{\rho\beta} = \frac{\omega}{\omega-1}[1-2\frac{\omega^{1/2}-1}{\omega-1}] = \frac{\omega}{\omega-1}\frac{\omega^{1/2}-1}{\omega^{1/2}+1} = \frac{\omega}{(\omega^{1/2}+1)^2},\ \omega = \frac{s}{\rho}, \tag{3.2}$$

and so with $\beta = \delta/s$: for $\mathrm{Re}\,\rho \geq 0$,

$$\beta(\rho) = 1 - \delta\frac{\rho}{(\sqrt{s}+\sqrt{\rho})^2}. \tag{3.3}$$

Hence by using the inversion formula for the L.S.-transform, cf. [2], p. 69,

$$B(t) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \frac{\beta(\rho)}{\rho} \mathrm{e}^{\rho t} \mathrm{d}\rho \,,\; t > 0, \tag{3.4}$$

with $\varepsilon > 0$ and the integral defined as a principal value integral, we obtain from (3.3): for $t > 0$,

$$B(t) = 1 - \frac{\delta}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \frac{\mathrm{e}^{\rho t}}{(\sqrt{s}+\sqrt{\rho})^2} \mathrm{d}\rho. \tag{3.5}$$

The integral in (3.5) is the inversion integral for the Laplace transform. Hence the integral can be evaluated by using the tables in [1], actually formula (10) p. 234 of [1]. Unfortunately this formula contains an error. Therefore we calculate directly. We have from (2.2) and (2.9):

$$B(t) = \frac{s^{1/2}}{\Gamma(1/2)} \int_0^\infty \theta^{-1/2} \mathrm{e}^{-s\theta} \{1 - \delta(\frac{\theta}{\theta+t})^{3/2}\} \mathrm{d}\theta = 1 - \frac{\delta s^{1/2}}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-s\theta} \frac{\theta}{(\theta+t)^{3/2}} \mathrm{d}\theta,$$

so

$$\begin{aligned}
1 - B(t) &= -\frac{\delta s^{1/2}}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty \frac{\mathrm{d}\theta}{(\theta+t)^{3/2}} \mathrm{e}^{-s\theta} = -\frac{\delta s^{1/2}}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{st} \int_t^\infty \mathrm{e}^{-su} u^{-3/2} \mathrm{d}u \\
&= -\frac{\delta s^{1/2}}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s} [\mathrm{e}^{st} \{-2u^{-1/2} \mathrm{e}^{-su} |_t^\infty - 2s \int_t^\infty \mathrm{e}^{-su} u^{-1/2} \mathrm{d}u\}] \\
&= 2\frac{\delta s^{1/2}}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s} [\mathrm{e}^{st} s^{1/2} \int_{st}^\infty \mathrm{e}^{-v} v^{-1/2} \mathrm{d}v] = 2\frac{\delta s^{1/2}}{\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{st} s^{1/2} \int_{\sqrt{st}}^\infty 2\mathrm{e}^{-u^2} \mathrm{d}u = \\
&= \frac{4\delta s^{1/2}}{\sqrt{\pi}} [t\mathrm{e}^{st} s^{1/2} \ \mathrm{Erfc}(\sqrt{st}) + \frac{1}{2} \mathrm{e}^{st} s^{-1/2} \ \mathrm{Erfc}(\sqrt{st}) - \frac{1}{2}\sqrt{t}] \\
&= \frac{2\delta}{\sqrt{\pi}} [(1+2st)\mathrm{e}^{st} \ \mathrm{Erfc}(\sqrt{st}) - \sqrt{st}],
\end{aligned}$$

with, cf. [3], vol. 2, p. 147,

$$\mathrm{Erfc}\,(x) := \int_x^\infty \mathrm{e}^{-u^2} \mathrm{d}u. \tag{3.6}$$

So we have for $s > 0, t > 0$,

$$\begin{aligned}
&B(t) = 1 + \frac{2\delta}{\sqrt{\pi}} [\sqrt{st} - (1+2st)\mathrm{e}^{st} \ \mathrm{Erfc}(\sqrt{st})], \\
&B(0+) = 1 - \delta.
\end{aligned} \tag{3.7}$$

To obtain the asymptotic expression for $1 - B(t)$ for $t \to \infty$ we use the asymptotic relation for Erfc x, cf. [3] vol 2, p. 147: for $x \to \infty$,

$$\mathrm{Erfc}\,(x) = \int_x^\infty \mathrm{e}^{-u^2} \mathrm{d}u = \frac{1}{2} \mathrm{e}^{-x^2} \sum_{n=0}^{M+1} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(1/2)} \frac{1}{x^{2n+1}} + \mathrm{o}(x^{-2M+1}), \tag{3.8}$$

for every $M = 1, 2, \ldots$. From (3.7) and (3.8) we obtain for $t \to \infty$ and every $H = 1, 2, \ldots$,

$$1 - B(t) = \frac{2\delta}{\pi} \sum_{n=1}^{H} (-1)^{n-1} \frac{n\Gamma(n+1/2)}{(st)^{n+1/2}} + \mathrm{o}((\frac{1}{st})^{H+1/2}). \tag{3.9}$$

Next we consider the $M/G/1$ model with traffic load $a = \lambda\beta < 1$ and service time distribution $B(t)$ as given by (3.7), here $\beta = \delta/s$, cf. (2.15) with $\nu = 1\frac{1}{2}$. For this $M/G/1$ model the L.S.-transform $\omega(\rho)$ of the stationary waiting time distribution $W(t)$ is given by the Pollaczek-Khintchine formula: for $\mathrm{Re}\,\rho \geq 0$,

$$\omega(\rho) = (1-a)\frac{1}{1 - a\frac{1-\beta(\rho)}{\rho\beta}}. \tag{3.10}$$

So we have by using (3.3): for $\mathrm{Re}\,\rho \geq 0$,

$$\begin{aligned} \omega(\rho) = {} & (1-a)\frac{(1+\sqrt{\frac{\rho}{s}})^2}{(1+\sqrt{\frac{\rho}{s}})^2 - a} \\ & = 1 - a + \frac{1}{2}(1-a)\sqrt{as}[\frac{1}{(1-\sqrt{a})\sqrt{s}+\sqrt{\rho}} - \frac{1}{(1+\sqrt{a})\sqrt{s}+\sqrt{\rho}}]. \end{aligned} \tag{3.11}$$

By applying the inversion integral for the L.S.-transform we have with $\varepsilon > 0$: for $t > 0$,

$$W(t) = 1 - a + \frac{1-a}{2}\sqrt{as}\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon} \mathrm{e}^{\rho t}\frac{1}{\rho}[\frac{1}{(1-\sqrt{a})\sqrt{s}+\sqrt{\rho}} - \frac{1}{(1+\sqrt{a})\sqrt{s}+\sqrt{\rho}}]\mathrm{d}\rho. \tag{3.12}$$

Consider the integral in (2.3) as the inversion integral for the Laplace transform; then, see remark 3.1 below, we obtain for $t > 0$,

$$\begin{aligned} W(t) = {} & 1 - (1+\sqrt{a})(\frac{a}{\pi})^{1/2}\mathrm{e}^{(1-\sqrt{a})^2 st}\ \mathrm{Erfc}((1-\sqrt{a})\sqrt{st}) \\ & +(1-\sqrt{a})(\frac{a}{\pi})^{1/2}\mathrm{e}^{(1+\sqrt{a})^2 st}\ \mathrm{Erfc}((1+\sqrt{a})\sqrt{st}), \end{aligned} \tag{3.13}$$

so we have an explicit expression for $W(t)$.

Remark 3.1. To calculate (3.13) from (3.12) note that we have by using partial integration: for $\mathrm{Re}\,\rho > b$,

$$\int_0^\infty \mathrm{e}^{-\rho t}\mathrm{e}^{bt}\ \mathrm{Erfc}(\sqrt{bt})\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(\rho-b)t}\int_{\sqrt{bt}}^\infty \mathrm{e}^{-u^2}\mathrm{d}u\mathrm{d}t$$

$$= \frac{1/2\sqrt{\pi}}{\rho-b} - \frac{1/2\sqrt{\pi}}{\rho-b}\int_0^\infty \mathrm{e}^{-\rho t}\frac{\mathrm{d}t}{t^{1/2}} = \frac{1}{2}\sqrt{\pi}\frac{1}{\sqrt{\rho}}\frac{1}{\sqrt{\rho}+\sqrt{b}}.$$

Because for $t \to \infty$ the first integral behaves as $\mathrm{e}^{-\rho t}(bt)^{-1/2}$ the integral in the lefthand side exists for $\mathrm{Re}\,\rho > 0$, and hence by analytic continuation it is seen that the relation holds for $\mathrm{Re}\,\rho > 0$. This latter integral is listed in [1] vol. 1, p. 177, form. (7). However, it contains a misprint, viz, the factor $\frac{1}{2}\sqrt{\pi}$ is missing; similarly, on p. 233, form. (5). By noting further that

$$\frac{1}{\rho}\frac{1}{\sqrt{\rho}+\sqrt{b}} = \frac{1}{\sqrt{b}}\{\frac{1}{\rho} - \frac{1}{\sqrt{\rho}}\frac{1}{\sqrt{\rho}+\sqrt{b}}\},$$

the derivation of (3.13) from (3.12) is readily obtained. □

To study the tail of the distribution $W(t)$ we apply a theorem of Doetsch, see the appendix.

From (3.12) it is readily seen by analytic continuation that for

$$|\rho/s| < (1-\sqrt{a})^2, \ -\pi < \arg\rho < \pi, \rho \neq 0, \tag{3.14}$$

holds

$$\frac{1-\omega(\rho)}{\rho} = \frac{1}{2}(1-a)\sqrt{a}\sum_{n=0}^{\infty}(-1)^n\{\frac{1}{(1-\sqrt{a})^{n+2}} - \frac{1}{(1+\sqrt{a})^{n+2}}\}\frac{\rho^{\frac{1}{2}n-1/2}}{s^{\frac{1}{2}n+1/2}}. \tag{3.15}$$

As in appendix A it is verified that $\{1-\omega(\rho)\}/\rho$ satisfies the conditions for the application of the theorem of Doetsch. Applying this theorem and noting that $\{1-\omega(\rho)\}/\rho$ is the Laplace transform of $1-W(t)$ we have: for $t \to \infty$,

$$1-W(t) = \frac{1-a}{2}\sqrt{a}\sum_{\substack{n=0\\ n\,\text{even}}}^{H}(-1)^n\{\frac{1}{(1-\sqrt{a})^{n+2}} - \frac{1}{(1+\sqrt{a})^{n+2}}\}\frac{\Gamma^{-1}(-\frac{1}{2}n+\frac{1}{2})}{(st)^{\frac{1}{2}n+\frac{1}{2}}} + \mathrm{o}((st)^{-\frac{1}{2}H-\frac{1}{2}}),$$

for every finite $H \in \{1,2,\dots\}$,

By using

$$\Gamma^{-1}(-m+\frac{1}{2}) = \frac{1}{\pi}\Gamma(m+\frac{1}{2})\sin(m-\frac{1}{2})\pi, \ m=0,1,2,\dots,$$

we obtain the asymptotic expression for $1-W(t)$, i.e. for $t\to\infty$ and every $H \in \{0,1,2,\dots\}$,

$$\begin{aligned}1-W(t) &= \frac{1}{2}(1-a)\frac{\sqrt{a}}{\pi}\sum_{m=0}^{H}(-1)^m\{\frac{1}{(1-\sqrt{a})^{2m+2}} \\ &-\frac{1}{(1+\sqrt{a})^{2m+2}}\}\frac{\Gamma(m+\frac{1}{2})}{(st)^{m+1/2}} + \mathrm{o}((st)^{-(H+\frac{1}{2})}).\end{aligned} \tag{3.16}$$

REMARK 3.2. For fixed st the term $\Gamma(m+1/2)(st)^{-m-1/2}$ for $m=0,1,2,\dots$, initially decreases, reaches a minimum for $m \sim st$ and then increases. The error involved in (3.16) is in absolute value less than the last term i.e. $m=H$. Hence for given st the smallest error is obtained by taking that H for which the absolute value of the last used term reaches a minimum, see further [11] p. 339. It should be noted that the asymptotic result (3.16) can be obtained also from (3.14) by using the asymptotic relations for the error function, cf. (3.8). □

4. THE EXPLICIT EXPRESSION FOR $B(\cdot)$ WITH $1<\nu<2$.

In this section we shall derive the general expression for $B(t), t>0$. Put

$$\mu := 2-\nu, \ 1<\nu<2, \tag{4.1}$$

so, cf. (2.15),

$$0<\mu<1, \ s\beta = \frac{\mu}{1-\mu}\delta. \tag{4.2}$$

From (2.20) we have: for $s>0, \mathrm{Re}\,\rho>0, \omega = s/\rho$,

$$\frac{1-\beta(\rho)}{\rho\beta} = \frac{\omega}{\omega-1} + \frac{1}{\mu}\frac{\omega}{(\omega-1)^2} - \frac{1}{\mu}\frac{\omega}{(\omega-1)^2}\omega^{\mu}. \tag{4.3}$$

Further: for $\mathrm{Re}\,\rho > s > 0$,

$$\begin{aligned}\frac{\omega}{\omega-1} &= \frac{s}{s-\rho} = -\frac{s}{\rho-s} = -s\int_0^\infty \mathrm{e}^{-\rho t}\mathrm{e}^{st}\mathrm{d}t,\\ \frac{\omega}{(\omega-1)^2} &= \frac{s\rho}{(\rho-s)^2} = s\{\frac{1}{\rho-s} + \frac{s}{(\rho-s)^2}\} = \\ &= s\int_0^\infty \mathrm{e}^{-\rho t}\mathrm{e}^{st}\mathrm{d}t + s^2\int_0^\infty \mathrm{e}^{-\rho t}t\mathrm{e}^{st}\mathrm{d}t,\end{aligned} \tag{4.4}$$

$$\omega^\mu = \int_0^\infty \mathrm{e}^{-\rho t}\frac{s^\mu}{\Gamma(\mu)}t^{\mu-1}\mathrm{d}t.$$

Hence from (4.3) for $t > 0$,

$$\begin{aligned}\frac{1}{\beta}(1-B(t)) &= -s\mathrm{e}^{st} + \frac{1}{\mu}s\mathrm{e}^{st} + \frac{1}{\mu}s^2t\mathrm{e}^{st}\\ &-\frac{1}{\mu}\int_0^\infty \frac{s^\mu}{\Gamma(\mu)}\tau^{\mu-1}\{s\mathrm{e}^{s(t-\tau)} + s^2(t-\tau)\mathrm{e}^{s(t-\tau)}\}\mathrm{d}\tau.\end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned}\frac{1-B(t)}{\beta s} &= \frac{1-\mu}{\mu}\mathrm{e}^{st} + \frac{st}{\mu}\mathrm{e}^{st}\\ &-\frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}\int_0^t \mathrm{e}^{-s\tau}\tau^{\mu-1}(1-s\tau)\mathrm{d}\tau + \frac{s^\mu}{\Gamma(1+\mu)}st\int_0^t \mathrm{e}^{-s\tau}\tau^{\mu-1}\mathrm{d}\tau = \\ &\frac{1-\mu}{\mu}\mathrm{e}^{st} + \frac{st}{\mu}\mathrm{e}^{st} - \frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}(1+st)\int_0^t \mathrm{e}^{-s\tau}\tau^{\mu-1}\mathrm{d}\tau + \frac{s^{\mu+1}}{\Gamma(1+\mu)}\mathrm{e}^{st}\int_0^t \mathrm{e}^{-s\tau}\tau^{\mu}\mathrm{d}\tau\end{aligned}$$

$$\begin{aligned}&= \frac{1-\mu}{\mu}\mathrm{e}^{st} + \frac{st}{\mu}\mathrm{e}^{st} - \frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}(1+st)\int_0^t \mathrm{e}^{-s\tau}\tau^{\mu-1}\mathrm{d}\tau\\ &+\frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}\{-\mathrm{e}^{-st}t^\mu + \int_0^t \mu\mathrm{e}^{-s\tau}\tau^{\mu-1}\mathrm{d}\tau\}\\ &= -\frac{(st)^\mu}{\Gamma(1+\mu)} + \frac{1-\mu}{\mu}\mathrm{e}^{st} + \frac{st}{\mu}\mathrm{e}^{st} - \frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}\{1-\mu+st\}\int_0^t \mathrm{e}^{-s\tau}\tau^{\mu-1}\mathrm{d}\tau\\ &= -\frac{(st)^\mu}{\Gamma(1+\mu)} + \frac{1}{\mu}\{1-\mu+st\}\mathrm{e}^{st} - \frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}\{1-\mu+st\}\frac{\Gamma(\mu)}{s^\mu}\\ &+\frac{s^\mu}{\Gamma(1+\mu)}\mathrm{e}^{st}(1-\mu+st)\int_t^\infty \mathrm{e}^{-st}\tau^{\mu-1}\mathrm{d}\tau\\ &= -\frac{(st)^\mu}{\Gamma(1+\mu)} + \frac{1}{\Gamma(1+\mu)}(1-\mu+st)\mathrm{e}^{st}\int_{st}^\infty \mathrm{e}^{-u}u^{\mu-1}\mathrm{d}u.\end{aligned} \tag{4.6}$$

So we have: for $t > 0, s > 0$,

$$1 - B(t) = \frac{\mu}{1-\mu}\delta[-\frac{(st)^{\mu}}{\Gamma(1+\mu)} + \frac{1-\mu+st}{\Gamma(1+\mu)}\mathrm{e}^{st}\int\limits_{st}^{\infty}\mathrm{e}^{-u}u^{\mu-1}\mathrm{d}u]. \tag{4.7}$$

The integral in the righthand side of (4.7) is actually the incomplete gamma function

$$\Gamma(\mu, st) = \int\limits_{st}^{\infty}\mathrm{e}^{-u}u^{\mu-1}\mathrm{d}u. \tag{4.8}$$

We have, cf. [3], vol II, p. 133,

$$\Gamma(\mu, st) = \Gamma(\mu) - \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(st)^{\mu+n}}{\mu+n},\ t > 0, \tag{4.9}$$

and for $t \to \infty$, and every finite $H \in \{0, 1, 2, \dots\}$,

$$\Gamma(\mu, st) = (st)^{\mu-1}\mathrm{e}^{-st}[\sum_{m=0}^{H}(-1)^m\frac{\Gamma(1-\mu+m)}{\Gamma(1-\mu)}\frac{1}{(st)^m} + \mathrm{o}(st)^{-H})]. \tag{4.10}$$

Consequently, from (4.7) and (4.10): for $t \to \infty$,

$$\begin{aligned} 1 - B(t) &= \frac{\mu}{1-\mu}\delta[\frac{1-\mu}{\Gamma(1+\mu)}(st)^{\mu-1} + \\ &+ \frac{1-\mu+st}{\Gamma(1+\mu)}\{\sum_{m=1}^{H}(-1)^m\frac{\Gamma(1-\mu+m)}{\Gamma(1-\mu)}\frac{1}{(st)^{m+1-\mu}} + \mathrm{o}(\frac{1}{(st)^{H+1-\mu}})\}], \end{aligned} \tag{4.11}$$

i.e. (4.14) represents an asymptotic relation for $1 - B(t)$ with $t \to \infty$, for every finite $H \in \{1, 2, \dots\}$.

Insertion of (4.9) into (4.7) yields a convergent series expansion for $1 - B(t)$ in $t > 0$.

$$\begin{aligned} 1 - B(t) = \frac{\mu}{1-\mu}\delta[&\frac{1}{\mu}(1-\mu+st)\mathrm{e}^{st} - \frac{(st)^{\mu}}{\Gamma(1+\mu)} \\ &- \frac{1-\mu+st}{\Gamma(1+\mu)}\mathrm{e}^{st}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(st)^{\mu+n}}{\mu+n}]. \end{aligned} \tag{4.12}$$

5. On the tail of the waiting time distribtution

In this section we investigate the tail of the stationary waiting time distribution $W(t)$ of the $M/G/1$ queuing model with traffic load $a < 1$ and service time distribution $B(t)$, cf. section 4, for the case

$$0 < \mu = 2 - \nu < 1. \tag{5.1}$$

With

$$\omega(\rho) := \int\limits_{0}^{\infty}\mathrm{e}^{-\rho t}\mathrm{d}W(t),\ \mathrm{Re}\,\rho \geq 0, \tag{5.2}$$

we have

$$\frac{1-\omega(\rho)}{\rho} = \int\limits_{0}^{\infty}\mathrm{e}^{-\rho t}\{1 - W(t)\}\mathrm{d}t,\ \mathrm{Re}\,\rho \geq 0,\ \rho \neq 0. \tag{5.3}$$

The Pollaczek-Khintchine formula reads: for $\mathrm{Re}\,\rho \geq 0$,

$$\omega(\rho) = (1-a)\frac{1}{1-a\frac{1-\beta(\rho)}{\rho\beta}}, \tag{5.4}$$

from which it follows that

$$1-\omega(\rho) = \frac{a}{1-a}\frac{1-\frac{1-\beta(\rho)}{\rho\beta}}{1+\frac{a}{1-a}\{1-\frac{1-\beta(\rho)}{\rho\beta}\}}. \tag{5.5}$$

With (2.20) and (5.1) we have:

$$0<\mu<1,\ \ \omega = s/\rho\,,\ \beta = \frac{\mu}{1-\mu}\frac{\delta}{s}\,,\ 0<\delta\leq 1, s>0, \tag{5.6}$$

and for $\mathrm{Re}\,\rho \geq 0$,

$$\frac{1-\beta(\rho)}{\rho\beta} = \frac{1}{1-\rho/s} + \frac{1}{\mu}\frac{\rho/s}{(1-\rho/s)^2} - \frac{1}{\mu}\frac{(\rho/s)^{1-\mu}}{(1-\rho/s)^2}; \tag{5.7}$$

note that $\{1-\beta(\rho)\}/\rho\beta$ has in $\mathrm{Re}\,\rho \geq 0$ only one singularity, cf. (2.20), viz. $\rho = 0$ is an algebraic singularity, actually a branch point if and only if μ is a rational number. For μ rational $\{1-\beta(\rho)\}/\rho\beta$ can be written as a convergent series of terms $\rho^{n+m\mu}, n = 0,1,2,\dots; m = 0,1,\dots$, for $|\rho/s|<1$, and hence (5.5) shows that such a series expansion also exists for $1-\omega(\rho)$. Once such a series expansion has been obtained the asymptotics of $1-W(t)$ for $t\to\infty$ can be readily obtained by using a theorem in [4], vol. II, see appendix A.

In this section we assume that μ is *rational*, so we put

$$\mu = \frac{M}{N}\,,\ M<N, \tag{5.8}$$

with

$$M,N\in\{1,2,\dots\} \text{ and g.c.d. } (M,N)=1. \tag{5.9}$$

Put

$$y = (\frac{\rho}{s})^{1/N}\,,\ s>0,\ \mathrm{Re}\,\rho\geq 0, \tag{5.10}$$

with the principal value of y positive for $\rho>0$, so y is well defined by (5.10).
From (5.7) we have: for $\mathrm{Re}\,\rho \geq 0$,

$$\begin{aligned} 1-\frac{1-\beta(\rho)}{\rho\beta} &= \frac{\rho/s}{(1-\rho/s)^2}[\frac{\rho}{s}-\frac{1+\mu}{\mu}+\frac{1}{\mu}(\frac{\rho}{s})^{-\mu}] \\ &= \frac{y^{N-M}}{(1-y^N)^2}[y^{N+M}-\frac{M+N}{M}y^M+\frac{N}{M}]. \end{aligned} \tag{5.11}$$

Hence from (5.5): for $\mathrm{Re}\,\rho \geq 0$,

$$1-\omega(\rho) = \frac{a}{1-a}\frac{y^{N-M}}{A(y)}[y^{N+M}-\frac{M+N}{M}y^M+\frac{N}{M}], \tag{5.12}$$

with

$$A(y) := (1-y^N)^2+\frac{a}{1-a}y^{N-M}[y^{N+M}-\frac{M+N}{M}y^M+\frac{N}{M}]. \tag{5.13}$$

Note that

$$y = 1 \text{ is a double zero of } y^{N+M} - \tfrac{M+N}{M} y^M + \tfrac{N}{M}, \qquad y = 1 \text{ is a double zero of } A(y). \tag{5.14}$$

$A(y)$ is a polynomial of degree $2N$. Because $y = 1$ is a double zero of $A(y)$, we denote its other zeros by

$$y_n,\ n = 1, 2, \dots, 2N - 2, \tag{5.15}$$

and assume for the present that

$$\text{multiplicity of } y_n \text{ is one for all } n = 1, \dots, 2N - 2. \tag{5.16}$$

If (5.16) does not hold then the following analysis requires only minor adaptations (see remark 5.1).

The number of zeros of $A(y)$ is finite and so we can write for (5.12) (note $1 \le M \le N - 1$),

$$1 - \omega(\rho) = \frac{a}{1-a} y^{N-M} \sum_{n=1}^{2N-2} \frac{c_n}{y - y_n}. \tag{5.17}$$

Hence it is seen that the term between brackets in (5.17) can be written as an absolutely convergent power series in y for

$$|y| < Y := \min(|y_1|, |y_2|, \dots, |y_{2N-2}|). \tag{5.18}$$

So we may put

$$1 - \omega(\rho) = \frac{a}{1-a} y^{N-M} \sum_{k=0}^{\infty} w_k y^k. \tag{5.19}$$

To determine the w_k we apply Cauchy's theorem with

$$C_\xi := \{y : |y| < \varepsilon\},\ 0 < \varepsilon < Y, \tag{5.20}$$

i.e. for $k = 0, 1, 2, \dots,$

$$w_k = \frac{1}{2\pi \mathrm{i}} \int_{C_\xi} \frac{\mathrm{d}y}{y^{k+1}} \frac{y^{N+M} - \frac{M+N}{M} y^M + \frac{N}{M}}{A(y)}, \tag{5.21}$$

the direction of integration is counterclockwise.

The integrand in (5.21) is regular in y, except at $y = 0$ and at the simple poles $y = y_n, n = 1, \dots, 2N - 2$, and it tends to zero as

$$|y|^{-[N-M+k+1]} \text{ for } |y| \to \infty.$$

It follows from (5.21) that for $k = 0, 1, 2, \dots,$

$$w_k = - \sum_{n=1}^{2N-2} y_n^{-(k+1)} [y_n^{N+M} - \frac{M+N}{M} y_n^M + \frac{N}{M}] \lim_{y \to y_n} \frac{y - y_n}{A(y)}. \tag{5.22}$$

REMARK 5.1 If (5.16) does not apply so that the integrand in (5.21) has some poles with multiplicity larger than one then the calculation of the residues at such poles requires the appropriate adaptation. We leave that to the reader and proceed here with the assumption (5.16). □

Put

$$A^{(1)}(y) = \frac{\mathrm{d}}{\mathrm{d}y} A(y),$$

so that

$$Y_n := \lim_{y \to y_n} \frac{y - y_n}{A(y)} = [A^{(1)}(y_n)]^{-1}. \tag{5.23}$$

Note that

$$(1 - y_n^N)^2 + \frac{a}{1-a} y_n^{N-M} \{ y_n^{N+M} - \frac{M+N}{M} y_n^M + \frac{N}{M} \} = 0; \tag{5.24}$$

hence from (5.29) and (5.25) for $k = 0, 1, 2, \ldots,$

$$w_k = \frac{1-a}{a} \sum_{n=1}^{2N-2} (1 - y_n^N)^2 y_n^{-N+M} Y_n y_n^{-(k+1)}. \tag{5.25}$$

Consequently from (5.13), (5.19) and (5.22): for $y < Y$,

$$1 - \omega(\rho) = \sum_{n=1}^{2N-2} Z_n \sum_{k=0}^{\infty} (\frac{y}{y_n})^{N-M+k}, \tag{5.26}$$

with

$$Z_n := \frac{(1 - y_n^N)^2}{y_n} Y_n, \; n = 1, \ldots, 2N-2. \tag{5.27}$$

Hence from (5.10) and (5.26): for $|\rho/s| < Y^N, |\arg \frac{\rho}{s}| < \pi$,

$$\frac{1 - \omega(\rho)}{\rho} = \frac{1}{s} \sum_{n=1}^{2N-2} Z_n \sum_{k=0}^{\infty} y_n^{-(N-M+k)} (\frac{\rho}{s})^{(k-m)/N}. \tag{5.28}$$

We next apply theorem 2 of [4], vol. II, p. 159 (see also appendix A). In appendix A it is shown that the conditions of this theorem are fulfilled by (5.28). It follows from (5.3), (5.28) and the quoted theorem that: for $s > 0, t \to \infty$ and every finite $H \in \{0, 1, 2, \ldots\}$,

$$1 - W(t) \;\; = \sum_{n=1}^{2N-2} Z_n \sum_{k=0}^{H} \Gamma^{-1}(\frac{M-k}{N})(y_n^N st)^{-\frac{N-M+k}{N}} + \mathrm{o}((st)^{-\frac{N-M+H}{N}}), \tag{5.29}$$

with

$$\begin{aligned} &\Gamma(-m) = 0 \quad \text{for } m = 0, 1, 2, \ldots\,, \\ &\Gamma(-m)\Gamma(1+m) = -(\frac{\sin m\pi}{\pi})^{-1}, \; m \text{ not an integer.} \end{aligned} \tag{5.30}$$

Note that in the righthand side of (5.29) the terms with $k = M + hN, k = 0, 1, 2, \ldots$, disappear. Further, cf. (5.6) and (5.8),

$$\beta = \frac{M}{N-M} \frac{\delta}{s}, \quad 0 < \delta \leq 1, \tag{5.31}$$

here β is the average service time.

The relation (5.31) is the asymptotic series for the tail probabilities of the stationary waiting time distribution of the $M/G/1$ model with traffic load a and service time distribution $B(t)$ given by (4.5) with $\mu = \frac{M}{N}$.

6. THE TAIL PROBABILITIES FOR $1-a<<1$.

In section 3 we have considered the case $\nu = 1\frac{1}{2}$, it corresponds to the case $M=1, N=2$, of the preceding section. The relation (3.16) describes the tail probabilities of the waiting time distribution $W(t)$. From (3.16) it is seen that for $1-a<<1$ the term between curled brackets in (3.16) may be replaced by

$$(\frac{1}{1-\sqrt{a}})^{2m+2}$$

so that for $1-a<<1$ we may write: for $t\to\infty$ and every $H\in\{1,2,\dots\}$ for the case $\nu=1\frac{1}{2}$,

$$1-W(t)=\frac{1}{2\pi}(1+\sqrt{a})\sqrt{a}\sum_{m=0}^{H}(-1)^m\frac{\Gamma(m+\frac{1}{2})}{\{(1-\sqrt{a})^2st\}^{m+\frac{1}{2}}}+\mathrm{o}\{(st)^{-(H+\frac{1}{2})}\}. \tag{6.1}$$

In the present section we shall investigate the tail probabilities of $1-W(t)$ for the case that $1-a<<1$ and, cf. (5.8) and (5.9),

$$\mu=\frac{M}{N}. \tag{6.2}$$

We start from (5.29): for $s>0, t\to\infty$ and every finite $H\in\{0,1,2,\dots\}$,

$$1-W(t)=\sum_{n=1}^{2N-2}Z_n\sum_{k=0}^{H}\Gamma^{-1}(\frac{M-k}{N})(y_n^N st)^{-\frac{N-M+k}{N}}+\mathrm{o}((st)^{-\frac{N-M+H}{N}}). \tag{6.3}$$

From appendix B, cf. (b.10), it is seen that the $2N-2$ zeros $y_n\equiv y_n(a), n=1,\dots,2N-2$, can be divided into three classes with respect to their asymptotic behaviour for $a\uparrow 1$. Viz. cf. (b.4), (b.8), (b.9), for $a\uparrow 1$,

$$\text{i.}\qquad y_n(a)=(\frac{1-a}{a}\frac{M}{N})^{\frac{1}{N-M}}\mathrm{e}^{\frac{2n+1}{N-M}\pi\mathrm{i}}+\mathrm{o}(1-a),\ n=1,\dots,N-M, \tag{6.4}$$

$$\text{ii.}\qquad y_n(a)\sim(\frac{N}{N+M})^{\frac{1}{M}},\ n=N-M+1,\dots,N,$$

$$\text{iii.}\qquad |y_n(a)|\sim|\frac{c_n}{1-a}|^{\frac{1}{2N}},\ n=N+1,\dots,2N-2,$$

here the c_n are constants.

Put, cf. (5.32),

$$b:=(\frac{1-a}{a}\frac{M}{N})^{\frac{N}{N-M}},\ e_n:=\mathrm{e}^{\frac{2n+1}{N-M}\pi\mathrm{i}},$$
$$\tau:=bst=b\delta\frac{M}{N-M}\frac{t}{\beta}. \tag{6.5}$$

Then (6.3) may be rewritten as: for $\tau\to\infty$ and every finite $H\in\{0,1,2,\dots\}$,

$$\begin{aligned}1-W(\tfrac{N-M}{b\delta M}\beta\tau)\ &=\sum_{n=1}^{N-M}Z_n\sum_{k=0}^{H}\Gamma^{-1}(\frac{M-k}{N})\tau^{-\frac{N-M+k}{N}}e_n^{-(N-M+k)}\\ &+\sum_{n=N-M+1}^{N}Z_n\sum_{k=0}^{H}\Gamma^{-1}(\frac{M-k}{N})(\frac{b}{y_n^N})^{\frac{N-M+k}{N}}\tau^{-\frac{N-M+k}{N}}\\ &+\sum_{n=N+1}^{2N-2}Z_n\sum_{k=0}^{H}\Gamma^{-1}(\frac{M-k}{N})(\frac{b}{y_n^N})^{\frac{N-M+k}{N}}\tau^{-\frac{N-M+k}{N}}+\mathrm{o}(\tau^{-\frac{N-M+H}{N}}).\end{aligned} \tag{6.6}$$

We consider the expression (6.6) for $a \uparrow 1$. It is noted that (6.4) and (6.5) describe the asymptotic behaviour of b and $y_n(a)$ for $a \uparrow 1$. Hence we need the behaviour of Z_n, cf. (5.27), for $a \uparrow 1$ for the three cases

$$\text{i.} \quad n = 1, \dots, N-M, \tag{6.7}$$

$$\text{ii.} \quad n = N-M+1, \dots, N,$$

$$\text{iii.} \quad n = N+1, \dots, 2N-2.$$

In appendix C the behaviour of Z_n is derived and it is seen that: for $a \uparrow 1$,

$$\text{i.} \quad Z_n = -\tfrac{1}{N-M} + \mathrm{O}((1-a)^{\frac{M}{N-M}}) \quad \text{for } n = 1, 2, \dots, N-M, \tag{6.8}$$

$$\text{ii.} \quad Z_n = \gamma_3 + \mathrm{O}(1-a) \quad \text{for } n = N-M+1, \dots, N,$$

$$\text{iii.} \quad Z_n = \mathrm{O}(\tfrac{1}{1-a}) \quad \text{for } n = N+1, \dots, 2N-2 \; .$$

From (6.4), (6.5), (6.6) and (6.8) it is seen that: for $a \uparrow 1, t \to \infty$ and every finite $H \in \{0, 1, 2, \dots\}$,

$$\begin{aligned} 1 - W(\tfrac{N-M}{b\delta M}\beta\tau) \;&= [\frac{-1}{N-M} \sum_{n=1}^{N-M} \sum_{k=0}^{H} \Gamma^{-1}(\frac{M-k}{N})\tau^{-\frac{N-M+k}{N}} e_n^{-(N-M+k)} \\ &\quad + \mathrm{o}(\tau^{-\frac{N-M+k}{N}})](1 + \mathrm{O}((1-a)^{\frac{M}{N-M}})). \end{aligned} \tag{6.9}$$

A simple calculation shows that for $h = 0, 1, 2, \dots,$

$$\begin{aligned} \sum_{n=1}^{N-M} e_n^{-k} \;&= \sum_{n=1}^{N-M} e^{-\frac{2n+1}{N-M}k\pi \mathrm{i}} \\ &= 0 \qquad\qquad\qquad \text{for } k \neq h(N-M), \\ &= (-1)^h (N-M) \quad \text{for } k = h(N-M). \end{aligned} \tag{6.10}$$

Further, cf. (5.30), for $k = h(N-M),\ h = 0, 1, 2, \dots$

$$\begin{aligned} \Gamma^{-1}(\tfrac{M-k}{N}) \;&= -\Gamma(1 + \frac{k-M}{N})\frac{1}{\pi} \sin \frac{k-M}{N}\pi \\ &= \frac{(-1)^h}{\pi}\Gamma((h+1)(1 - \frac{M}{N})) \sin(h+1)\frac{M}{N}\pi. \end{aligned} \tag{6.11}$$

Insert (6.10) and (6.11) into (6.9) then, for $a \uparrow 1, t \to \infty$, and every finite $K \in \{1, 2, \dots\}$,

$$\begin{aligned} 1 - W(\frac{N-M}{b\delta M}\beta\tau) \;&= [\frac{1}{\pi} \sum_{k=1}^{K} \Gamma(k(1 - \frac{M}{N})) \sin(k\frac{M}{N}\pi)\tau^{-k\frac{N-M}{N}} \\ &\quad + \mathrm{o}(\tau^{-K\frac{N-M}{N}}](1 + \mathrm{O}((1-a)^{\frac{M}{N-M}})). \end{aligned} \tag{6.12}$$

With $\mu = M/N$, cf. (5.8), we have from (6.5)

$$b = ((1-a)\frac{\mu}{a})^{\frac{1}{1-\mu}}, 0 < \mu < 1,\ \mu \text{ rational.} \tag{6.13}$$

Put

$$c := \frac{1-\mu}{\mu\delta} b^{-1}, \tag{6.14}$$

then (6.12) may be rewritten as: for $0 < \mu < 1, 0 < \delta \leq 1$,

$$1 - W(c\beta\tau) =$$

$$[\frac{1}{\pi}\sum_{k=1}^{K}((1-\mu)k)\sin(\mu k\pi)\tau^{-(1-\mu)k} + \mathrm{o}(\tau^{-(1-\mu)K})]\{1 + \mathrm{O}((1-a)^{\frac{\mu}{1-\mu}})\}, \tag{6.15}$$

for $a \uparrow 1, \tau \to \infty$ and every finite $K \in \{1, 2, \dots\}$.
This relation (6.15) is the asymptotic expression for the tail probabilities of the stationary waiting time distribution $W(t)$ of the $M/G/1$ queue with traffic load a and service time distribution $B(t)$ as given by (4.7).

Remark 6.1. The condition that μ should be rational is not essential. Actually (6.15) applies for every μ with $0 < \mu < 1$. The proof of this statement is simple. Suppose $\mu = 2-\nu$ is rational. Then a sequence $\mu_j, j = 1, 2, \dots$, exists with all μ_j rational and $\mu_j \to \mu$. Denote by $W_j(t)$ the stationary distribution of the $M/G/1$ queue with traffic load a_j and service time distribution $B_j(\cdot)$ with $B_j(\cdot)$ given by (4.7) where $\mu = \mu_j$. Obviously $B_j(\cdot)$ converges completely to $B(\cdot)$ for $j \to \infty$. Consequently $\omega_j(\rho)$, the L.S.-transform of $W_j(t)$, converges to $\omega(\rho)$, cf. (5.5), for $\mathrm{Re}\,\rho \geq 0$, in particular $\omega_j(0) \to \omega(0)$. Hence by the continuity theorem for the L.S.-transform of probability distributions it follows that $W_j(t) \to W(t)$ for all points of continuity of $W(t)$. From (6.15) with $\mu = \mu_j, j = 1, 2, \dots$, it is hence seen that (6.15) also holds for $\mu = \lim_{j\to\infty} \mu_j$. □

For μ rational we shall rewrite the relation (6.15) is a slightly different form.
Put

$$k = hN + m,\ h \in \{0, 1, 2, \dots\},\ m \in 0, 1, \dots, N-1, \tag{6.16}$$

so that

$$\sin \mu k\pi = (-1)^{hM} \sin m\frac{M}{N}\pi,$$

then (6.15) may be rewritten as:

$$1 - W(c\beta\tau) = [\frac{1}{\pi}\sum_{h=0}^{H}\frac{(-1)^{hm}}{\tau^{(1-\mu)h}}\sum_{m=1}^{N-1}\Gamma((1-\frac{M}{N})(hN+m))\sin(m\frac{M}{N}\pi)\tau^{-(1-\mu)m\frac{M}{N}} + \mathrm{o}(\tau^{-(1-\mu)[H+m\frac{M}{N}}))](1 + \mathrm{O}((1-a)^{\frac{M}{N-M}})), \tag{6.17}$$

for $a \uparrow 1, \tau \to \infty$ and every finite $H \in \{0, 1, 2, \dots\}$.

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Appendix A

Theorem 2. Doetsch [4] vol II, p. 159.

Let $f(s)$ be the Laplace transform of $F(t)$,

$$f(s) = \int_0^\infty \mathrm{e}^{-st} F(t)\mathrm{d}t, \tag{a.1}$$

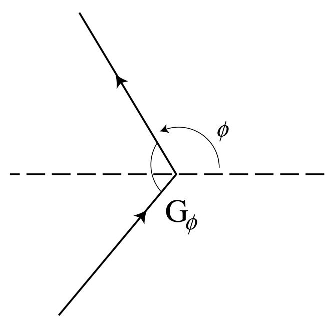
and suppose that holds:

$$F(t) = \frac{1}{2\pi\mathrm{i}} \int_{G_\phi} \mathrm{e}^{ts} f(s)\mathrm{d}s, \tag{a.2}$$

with

$$G_\phi := \{s : s = r\mathrm{e}^{\pm\mathrm{i}\phi}, 0 \le r < \infty, \frac{1}{2}\pi < \phi < \pi\},$$

the integral in (a.2) being a principal value integral, i.e.



$$F(t) = \lim_{R\to\infty} \quad \{\int_0^R \mathrm{e}^{tr\mathrm{e}^{-\mathrm{i}\phi}} f(r\mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}r + \\ + \int_{-R}^0 \mathrm{e}^{tr\mathrm{e}^{-\mathrm{i}\phi}} f(r\mathrm{e}^{-\mathrm{i}\phi}\}\mathrm{e}^{-\mathrm{i}\phi}\mathrm{d}r. \tag{a.3}$$

Theorem 2. Let $f(s)$ be regular in the angular section $|\arg(s-s_0)| \le \psi$, $\frac{\pi}{2} < \psi < \pi$, of a neighbourhood s_0 with exception of the point s_0. In this angular domain holds for $s \to s_0$ uniformly with respect to $\arg(s-s_0)$ an asymptotic representation

$$f(s) = \sum_{k=0}^n c_k(s-s_0)^{\lambda_k} + \mathrm{o}((s-s_0)^{\lambda_n}), \quad \mathrm{Re}\ \lambda_0 < \mathrm{Re}\ \lambda_1 < \dots,$$

for every $n = 0, 1, 2, \dots$.

On the half lines $s = s_0 r e^{\pm i\phi}$, $f(s)$ is in every finite interval absolutely integrable, however not necessarily regular. The integral in (a.2) exists for $t \geq T$. Then $F(t)$ has for $t \to \infty$ the asymptotic representation

$$F(t) = e^{s_0 t}[\sum_{k=0}^{m} \frac{c_k}{\Gamma(-\lambda_k)} t^{-\lambda_k - 1} + o(t^{-\lambda_m - 1}], \tag{a.4}$$

for every $m = 0, 1, 2, \ldots$.

We have to show that this theorem can be applied in the derivation of the result in section 5, cf. (5.34).

We have

$$\frac{1-\omega(\rho)}{\rho} = \int_0^\infty e^{-\rho t}\{1 - W(t)\}dt \ , \ \mathrm{Re}\,\rho \geq 0, \tag{a.5}$$

and since $1 - W(t) > 0$ for $t \geq 0$ it is seen that the integral in (a.5) converges absolutely for $\mathrm{Re}\,\rho \geq 0$. The function $W(t)$ is a bounded monotone function on $(0, \infty)$ and hence has bounded variation. From the representation

$$W(t) = (1-a)\sum_{n=0}^{\infty} a^n H^{n*}(t) \ , \ t \geq 0, \tag{a.6}$$

with

$$\begin{aligned} &H^{0*}(t) = 1 \text{ for } t > 0, \\ &H^{1*}(t) := \frac{1}{\beta}\int_0^t \{1 - B(\tau)\}d\tau, \ t > 0, \\ &H^{(n+1)*}(t) = \int_0^t H^{n*}(t-\tau)dH^*(\tau), \ t \geq 0, \end{aligned} \tag{a.7}$$

and $B(t)$ as given in section 4 it is readily seen that $W(t)$ is continuous for $t \in (0, \infty)$. From theorem 2, Doetsch [4], vol. I, p. 212, it follows that

$$1 - W(t) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\rho t}\frac{1-\omega(\rho)}{\rho}d\rho, \tag{a.8}$$

for every $\varepsilon \geq 0$, the integral in (a.8) is again a principal value integral.

We have, cf. (5.5) and (5.11), for $\mathrm{Re}\,\rho \geq 0$,

$$\begin{aligned} &\frac{1-\omega(\rho)}{\rho} = \frac{a}{1-a}\frac{1 - \frac{1-\beta(\rho)}{\rho\beta}}{1 + \frac{a}{1-a}\{\frac{1-\beta(\rho)}{\rho\beta}\}}, \\ &1 - \frac{1-\beta(\rho)}{\rho\beta} = \frac{\rho/s}{(1-\rho/s)^2}\{\frac{\rho}{s} - \frac{1+\mu}{\mu} + \frac{1}{\mu}(\frac{\rho}{s})^{-\mu}\}, \end{aligned} \tag{a.9}$$

with μ rational and $0 < \mu < 1$, cf. (5.4). It is known, cf. section 5, that $\rho = s > 0$ is not a pole of $\{1 - \beta(\rho)\}/\rho\beta$ neither of $\{1 - \omega(\rho)\}/\rho$, it is a removable singularitiy.

To every zero y_n of $A(y)$, cf. (5.13) and (5.15), corresponds a ρ_n defined by, cf. (5.10),

$$\rho_n = sy_n^N \ , \ n = 1, \dots , 2N-2. \tag{a.10}$$

Obviously such a ρ_n is a singularity of the righthand side in (a.9), i.e this righthand side is not regular at ρ_n. Because $\{1-\omega(\rho)\}/\rho$ is regular for $\mathrm{Re}\,\rho > 0$, continuous for $\mathrm{Re}\,\rho \geq 0$ it follows that $\mathrm{Re}\,\rho_n \leq 0$. However $\{1-\omega(\rho)\}/\rho$ exists for $\mathrm{Re}\,\rho = 0$. Since the ρ_n are zeros of the denominator in (a.9) it follows that

$$\mathrm{Re}\,\rho_n < 0, \ n = 1, \dots , 2N-2. \tag{a.11}$$

So

$$\frac{1}{2}\pi < \arg \rho_n < 1\frac{1}{2}\pi \ , \ n = 1, \dots , 2N-2. \tag{a.12}$$

Hence there exists a ψ with $0 < \psi < \frac{1}{2}\pi$ such that

$$\frac{1}{2}\pi + \psi < \arg \rho_n < 1\frac{1}{2}\pi - \psi \text{ for all } n = 1, \dots , 2N-2. \tag{a.13}$$

For a ψ so defined it is readily shown that

$$\frac{1}{2\pi \mathrm{i}}\{\int\limits_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\psi} + \int\limits_{1\frac{1}{2}\pi-\psi}^{1\frac{1}{2}\pi}\}\mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\phi}}\frac{1-\omega(r\mathrm{e}^{\mathrm{i}\phi})}{r\mathrm{e}^{\mathrm{i}\phi}}\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi \ , \ t > 0,$$

tends to zero for $r \to \infty$. Because $\{1-\omega(\rho)\}/\rho$ is regular for all $\rho = r\mathrm{e}^{\mathrm{i}\phi}$, with $r > 0$ and $\frac{1}{2}\pi + \psi \geq \phi \geq \frac{1}{2}\pi$ and also for $\frac{1}{2}\pi - \psi < \phi < 1\frac{1}{2}\pi$, it follows from (a.8) by applying Cauchy's theorem that: for every $t > 0$,

$$\begin{aligned} 1 - W(t) = \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \quad &\{\int\limits_{r=0}^{R} \mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\psi}}\frac{1-\omega(r\mathrm{e}^{\mathrm{i}\psi})}{r\mathrm{e}^{\mathrm{i}\psi}}\mathrm{e}^{\mathrm{i}\psi}\mathrm{d}r \\ &+ \int\limits_{-R}^{0} \mathrm{e}^{tr\mathrm{e}^{-\mathrm{i}\psi}}\frac{1-\omega(r\mathrm{e}^{-\mathrm{i}\psi})}{r\mathrm{e}^{-\mathrm{i}\psi}}\mathrm{e}^{-\mathrm{i}\psi}\mathrm{d}r\}. \end{aligned} \tag{a.14}$$

Consequently the integral as mentioned in the theorem exists.

It is readily verified that $\{1-\omega(\rho)\}/\rho$ is regular in the angular section $-\frac{1}{2}\pi - \psi < \phi < \frac{1}{2}\pi + \psi$ of a neighbourhood of $\rho = 0$ except at the point $\rho = 0$. Further the singularities $\rho_n, n = 1, \dots , 2N-2$ do not lie on the halflines $\rho = r\mathrm{e}^{\pm\mathrm{i}\psi}, r \geq 0$, and so it is easily seen from (a.9) that $\{1-\omega(\rho)\}/\rho$ is absolutely integrable on every finite interval of these halflines, note that $0 < \mu < 1$, so $0 < 1-\mu < 1$. Finally we have to show that $\{1-\omega(\rho)\}/\rho$ permits for $\rho \to 0$ a uniformly asymptotic representation as mentioned in the theorem, but this is evident because the series representation of (5.29) converges uniformly for $|\rho/s| < Y^N$, cf. (5.18).

Hence it has been shown that theorem 2 of [4], vol. II, p. 159 can be applied.

Appendix B

In this appendix we derive some properties of the zeros $y_1, \dots , y_{2N}$ of the polynomial

$$A(y) = (1-y^N)^2 + \frac{a}{1-a}y^{N-M}[y^{N+M} - \frac{M+N}{M}y^M + \frac{N}{M}], \tag{b.1}$$

with M and N positive integers and

$$1 \leq M < N \text{ and g.c.d. } (M,N) = 1.$$

Note that

$$y^{N+M} - \frac{M+N}{M} y^M + \frac{N}{M} = y^M(y^N - 1) - \frac{N}{M}(y^M - 1),$$

$$\frac{\mathrm{d}}{\mathrm{d}y}[y^{N+M} - \tfrac{M+N}{M} y^M + \tfrac{N}{M}] = (N+M) y^{M-1}(y^N - 1),$$

so that $y = 1$ is a double zero of $A(y)$. We take

$$y_{2N-1} = y_{2N} = 1. \tag{b.2}$$

Rewrite $A(y) = 0$, cf. (b.1), as:

$$y^{N-M} = -\frac{1-a}{a}\frac{M}{N}\frac{(1-y^N)^2}{1 - \frac{M+N}{N} y^M + \frac{M}{N} y^{N+M}}, \quad y \neq 1, \tag{b.3}$$

and consider the the $2N-2$ zeros $y_k(a)$ of (b.3) as a function of a.

Obviously there are exactly $N - M$ zeros $y_k(a), k = 1, \dots, N-M$, say, for which holds: for $a \uparrow 1$,

$$y_k(a) = \left(\frac{1-a}{a}\frac{M}{N}\right)^{\frac{1}{N-M}} \mathrm{e}^{\frac{2k+1}{N-M}\pi \mathrm{i}} + \mathrm{o}(1-a). \tag{b.4}$$

For the other zeros holds: for $a \uparrow 1$,

$$y_k(a) \to y_k(1) \neq 0\ ,\ k = N-M+1, \dots, 2N-2, \tag{b.5}$$

with for some k

$$(1-a)(1 - y_k^N(a))^2 \to c_k, \tag{b.6}$$

and for the other k's

$$1 - \frac{M+N}{N} y_k^M + \frac{M}{N} y_k^{N+M} = (1-a) d_k + \mathrm{o}(1-a), \tag{b.7}$$

with

$$c_k, d_k,\ k = N-M+1, \dots, 2N-2,$$

constants. Hence a zero $y_k, k \in \{N-M+1, \dots, 2N-1\}$ behaves for $a \uparrow 1$ as

$$y_k \sim \left|\frac{c_k}{1-a}\right|^{\frac{1}{2N}} \tag{b.8}$$

or as

$$y_k \sim \left[\frac{N}{M+N}|1 - (1-a)d_k|\right]^{\frac{1}{M}}. \tag{b.9}$$

It is seen that there are:

(b.10)

i. $N - M$ zeros for which (b.4) applies,

ii. M zeros for which (b.9) applies,

iii. $N - 2$ zeros for which (b.8) applies;

note (b.10)i follows from (b.4), (b.10)ii from (b.7), and (b.10)iii from (b.10)i and ii since the total number of zeros is $N - 2$.

Appendix C

The behaviour of Z_n for $a \to 1$.

To study the behaviour of $Z_n, n = 1, \dots, 2N-2$, with, cf. (5.27),

$$Z_n = (1-y_n^N)^2 y_n^{-1} Y_n, \tag{c.1}$$

$$Y_n^{-1} = A^{(1)}(y_n) = [-2(1-y_n^N) + \frac{a}{1-a}[2y_n^N - \frac{M+N}{M} + \frac{N-M}{M} y_n^{-M}]]N y_n^{N-1},$$

for $a \uparrow 1$ we have to consider the three cases of (6.7).

ad. (6.7)i, Z_n for $n = 1, 2, \dots, N-M$.

From (6.4)i we have for $a \uparrow 1$,

$$(1-y_n^N)^2 = 1 + \mathrm{O}((1-a)^{\frac{2N}{N-M}}), \tag{c.2}$$

and

$$A^{(1)}(y_n) = \frac{a}{1-a}(N-M)\frac{N}{M}(\frac{1-a}{a}\frac{M}{N})^{\frac{N-M-1}{N-M}} e_n^{N-M-1} + \frac{a}{1-a}\mathrm{O}(y_n^{(N-1)}),$$

so that

$$y_n A^{(1)}(y_n) \sim -(N-M) + \mathrm{O}((1-a)^{\frac{M}{N-M}}). \tag{c.3}$$

Hence, from (c.1), (c.2) and (c.3), for $a \uparrow 1$,

$$Z_n = -\frac{1}{N-M} + \mathrm{O}((1-a)^{\frac{M}{N-M}}) \text{ for } n = 1, \dots, N-M. \tag{c.4}$$

ad. (6.7)ii, Z_n for $n = N-M+1, \dots, N$.

From (6.4)ii

$$(1-y_n^N)^2 \to (1-(\frac{N}{N+M})^{\frac{N}{M}})^2 \text{ for } a \to 1,$$

and

$$y_n A^{(1)}(y_n) \sim \gamma_1 + \frac{a}{1-a}\gamma_2, \ \gamma_1 \neq 0, \gamma_2 \neq 0,$$

with γ_1 and γ_2 constants. Hence

$$Z_n = \gamma_3 + \mathrm{O}(1-a) \text{ for } a \uparrow 1 \text{ for } n = N-M+1, \dots, N, \tag{c.5}$$

with γ_3 a constant.

ad (6.7)iii, Z_n for $n = N+1, \dots, 2N-2$.

From (6.4)iii we have

$$(1-y_n^N)^2 \sim \frac{c_n}{1-a} \text{ for } a \uparrow 1,$$

and

$$y_n A^{(1)}(y_n) = \mathrm{O}(1).$$

Hence: for $a \uparrow 1$,

$$Z_n = \mathrm{O}((1-a)^{-1}) \quad \text{for } n = N+1, \dots, 2N-2.$$

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