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# Some Bijective Correspondences Involving Domino Tableaux

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## ABSTRACT

We elaborate on the results in [CaLe]. We give bijective proofs of a number of identities that were established there, in particular between the Yamanouchi domino tableaux and the ordinary Littlewood-Richardson fillings that correspond to the same tensor product decomposition.

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## §1. Introduction.

Domino tableaux are combinatorial objects that take a middle position between ordinary (skew) tableaux, and general  $r$ -ribbon tableaux, of which they are the first non-trivial instance ( $r = 2$ ). It was indicated in [CaLe] that for certain constructions with ordinary tableaux, analogous constructions can be defined for domino tableaux (but apparently not for  $r$ -ribbon tableaux with  $r > 2$ ). In particular, an analogue of the Robinson-Schensted correspondence, in its original formulation [Rob], can be defined for domino tableaux [CaLe, theorem 7.3] (there is no direct relation with the Robinson-Schensted algorithm for hyperoctahedral groups described in [vLee1], which is based on Schensted's insertion procedure). An algorithm is given that provides a bijective correspondence between on one side domino tableaux  $D$  of shape  $\lambda/\mu$  and weight  $\alpha$ , and on the other side pairs  $(Y, T)$  consisting of a so-called Yamanouchi domino tableau  $Y$  of shape  $\lambda/\mu$  and weight  $\nu$  (for some partition  $\nu$ ), and an ordinary tableau  $T$  of shape  $\nu$  and weight  $\alpha$ . It follows that a combinatorial description can be given of the decomposition of products of Schur functions, similar to the Littlewood-Richardson rule, but which counts Yamanouchi domino tableaux instead of ordinary Yamanouchi tableaux. For the special case of the square  $s_\lambda^2$  of a Schur function, a simple statistic on the occurring Yamanouchi domino tableaux enables in addition the determination of the plethysm  $\psi^2(s_\lambda)$  (which is not possible using the tableaux that arise in the Littlewood-Richardson rule); thus, the new rule also describes the decomposition of  $s_\lambda^2$  into parts corresponding to the symmetric and alternating tensor square of the irreducible  $\mathbf{GL}_n$  representation  $V_\lambda$  of which  $s_\lambda$  is the character.

In this paper we define several bijective constructions that complement these results. We define a bijection between semistandard domino tableaux and certain pairs of ordinary semistandard tableaux called self-switching tableau pairs, such that the construction of [CaLe] on the domino tableaux matches that of [Rob] on the ordinary tableaux. Therefore the ordinary tableau  $T$  above can be obtained by *jeu de taquin* from either component of the self-switching tableau pair associated to  $D$ , which shows in particular that the association  $D \mapsto T$  is well defined; a point that was not demonstrated convincingly in [CaLe]. We also construct a weight preserving bijection between the set of domino tableaux of a fixed shape and the set of ordinary tableaux of a related fixed shape, which maps Yamanouchi domino tableaux to ordinary Yamanouchi tableaux. In particular this gives a bijection between the two types of Yamanouchi tableaux describing the decomposition of a product of Schur functions, using the combinatorial rules indicated above. Finally, we describe a correspondence between Yamanouchi domino tableaux, that exhibits the identity of different two combinatorial expressions for the scalar product  $\langle s_\lambda, \psi^2(s_\mu) \rangle$ : one that counts with a fixed sign the Yamanouchi domino tableaux of shape  $\lambda$  and weight  $\mu$  [CaLe,

Corollary 4.3], and another that counts with certain alternating signs the Yamanouchi domino tableaux of shape  $\mu^\square$  and weight  $\lambda$  [CaLe, Theorem 5.3], where  $\mu^\square$  is obtained by scaling up the Young diagram of  $\mu$  by a factor 2 both horizontally and vertically.

Several results and constructions needed in this paper for domino tableaux are formulated in [vLee3] in the more general setting of  $r$ -ribbon tableaux. We shall use much of the terminology and notation introduced there, specialised to the case  $r = 2$ . We mention in particular the following: the directions “inward” and “outward” in the plane, the edge sequence  $\delta(\lambda)$  of a partition  $\lambda \in \mathcal{P}$ , the set  $\mathcal{C}_2 \subset \mathcal{P}$  of 2-cores, the map  $d^2$  giving the parameters of 2-cores, the set  $\text{Tab}_2(\lambda/\mu; A)$  of semistandard domino tableaux of shape  $\lambda/\mu$  with entries in  $A$ , the entry  $D(x)$  of a domino  $x$  in a domino tableau  $D$ , the position  $\text{pos}(x)$ , the spin  $\text{Spin}(D)$ , the 2-sign  $\varepsilon_2(\lambda/\mu)$ , the affine permutation group  $\tilde{\mathbf{S}}_2$  with generators  $s_0$  and  $s_1$ , the concept of open and closed chains of dominoes for  $s_0$  or  $s_1$ , and the action denoted  $\sigma \circ D$  of  $\tilde{\mathbf{S}}_2$  on domino tableaux (defined for  $s \in \{s_0, s_1\}$  by moving all open chains for  $s$ ). For the set of dominoes of a domino tableau  $D$  we shall write  $\text{Dom}(D)$  rather than  $\text{Rib}(D)$ ; the set  $\text{Tab}_1(\lambda/\mu; A)$  of ordinary semistandard tableaux of shape  $\lambda/\mu$  with entries in  $A$  will be denoted simply by  $\text{Tab}(\lambda/\mu; A)$ .

Of our three main bijective correspondences indicated above, the first two will be given by an algorithm. In the first case, the algorithm is a variation of the one described in [vLee1] that defines a bijection between self-dual tableaux (with respect to the Schützenberger involution) and domino tableaux; in the second case the correspondence is defined using the action  $\sigma \circ D$  defined in [vLee3]. The third correspondence is defined more directly, although the validity of the definition is not obvious. There is however another construction, that will play a more central rôle than any of these correspondences, namely that of what we shall call coplactic (raising and lowering) operations. Their definition is implicitly contained in the algorithm [CaLe, 7.1], and they define a graph structure on the set of semistandard domino tableaux, in terms of which both the definition of Yamanouchi domino tableaux and the mentioned algorithm can be easily understood. Similar (and in fact simpler) operations are defined for words and ordinary tableaux, leading to similar graph structures; several combinatorial transformations, including jeu de taquin, have the property that they preserve the structure of these graphs. In the same manner, a crucial property of our first two bijective correspondences is that they preserve the coplactic graph structure: the basic steps of their algorithms commute with the coplactic operations.

This paper is organised as follows. In §2 we define our main bijective constructions, and state their essential properties. In §3 we introduce coplactic operations on words, on ordinary tableaux and on domino tableaux. In studying those operations, it is convenient to consider domino tableaux with some information added to them. We call these augmented domino tableaux, and prove some of their fundamental properties in §4; in particular, our third main correspondence is directly related to them, and the proof that it is well-defined is given. Finally, §5 is devoted to the proofs of mentioned commutation theorems, justifying our claims about the other two constructions.

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I wish to thank Mark Shimozono for calling my attention to coplactic operations and their relation to operations on pictures. I also wish to thank Bernard Leclerc for then many stimulating discussions concerning the constructions of [CaLe], and in particular their relation with coplactic graphs.

## §2. Various bijective correspondences for domino tableaux.

In this section we formulate three results about domino tableaux that complement those found in [CaLe]. We introduce the algorithmic constructions involved, but postpone proofs of most of their properties. We use several concepts and constructions that were introduced in [CaLe], most notably the concept of Yamanouchi domino tableaux (a subclass of the semistandard domino tableaux, for which among other things the weight is always a partition) and the bijection of [CaLe, Theorem 7.3]. Formally our Yamanouchi domino tableaux differ from those of [CaLe] in that their entries start from 0 (to remain consistent with our other definitions), but this is just a trivial renumbering.

### 2.1. Projection from domino tableaux to Young tableaux.

In this subsection we shall define a weight preserving map from semistandard domino tableaux to Young tableaux, which coincides with projection onto the second factor after applying the bijection of [CaLe, Theorem 7.3]. We use a construction given in [vLee1, §2]. Consider a family of partitions  $\lambda^{[i,j]}$  with both  $i$  and  $j$  ranging over some interval of  $\mathbf{Z}$ , such that each of  $\lambda^{[i,j+1]}$  and  $\lambda^{[i+1,j]}$ , when defined, is obtained by adding a square to  $\lambda^{[i,j]}$ , and such the the following rule is obeyed.

**2.1.1. Rule.** *One has  $\lambda^{[i,j+1]} = \lambda^{[i+1,j]}$  if and only if  $\lambda^{[i+1,j+1]} \setminus \lambda^{[i,j]}$  is a domino.*

The number of partitions strictly between  $\lambda^{[i,j]}$  and  $\lambda^{[i+1,j+1]}$  is 1 or 2, according as  $\lambda^{[i+1,j+1]} \setminus \lambda^{[i,j]}$  is a domino or not, so the ‘‘if’’ part of the rule is redundant. If  $\lambda^{[i,j]}$  and  $\lambda^{[i+1,j+1]}$  are given, as well as one of  $\lambda^{[i,j+1]}$  and  $\lambda^{[i+1,j]}$ , then the other is determined. It follows that all  $\lambda^{[i,j]}$  for  $k \leq i \leq l$  and  $m \leq j \leq n$  can be uniquely constructed if  $\lambda^{[i,j]}$  is prescribed on some lattice path from  $[k, m]$  to  $[l, n]$  (i.e., at each step one of the parameters increases by 1) by an arbitrary saturated chain in  $(\mathcal{P}, \subseteq)$ . The importance of this construction lies in the fact that if  $\lambda^{[i+1,m]}, \dots, \lambda^{[i+1,n]}$  is the chain in  $(\mathcal{P}, \subseteq)$  corresponding to some skew tableau  $T$ , then  $\lambda^{[i,m]}, \dots, \lambda^{[i,n]}$  is the chain corresponding to the skew tableau obtained from  $T$  by an inward jeu de taquin slide into the square  $\lambda^{[i+1,m]} \setminus \lambda^{[i,m]}$ . Therefore the whole family  $(\lambda^{[i,j]})_{k \leq i \leq l; m \leq j \leq n}$  describes a sequence of inward jeu de taquin slides from a skew tableau with chain  $\lambda^{[l,m]}, \dots, \lambda^{[l,n]}$  to one with chain  $\lambda^{[k,m]}, \dots, \lambda^{[k,n]}$ ; by symmetry of the construction it also describes a sequence of inward jeu de taquin slides from a skew tableau with chain  $\lambda^{[k,n]}, \dots, \lambda^{[l,n]}$  to one with chain  $\lambda^{[k,m]}, \dots, \lambda^{[l,m]}$ . We shall therefore call this a jeu de taquin family of partitions. We find an involutive relation between pairs of skew tableaux that was named ‘tableau switching’ in [BeSoSt].

**2.1.2. Definition.** *Let  $\lambda \supseteq \mu \supseteq \nu$  be partitions, and  $(T, U)$  a pair of semistandard skew tableaux of respective shapes  $\mu/\nu$  and  $\lambda/\mu$ . The pair  $(U', T') = X(T, U)$  of semistandard skew tableaux obtained from  $(T, U)$  by tableau switching is uniquely determined by the requirements  $\text{wt}(T) = \text{wt}(T')$ ,  $\text{wt}(U) = \text{wt}(U')$ , and the following equivalent conditions:*

- (0) *There is a jeu de taquin family of partitions  $(\lambda^{[i,j]})_{k \leq i \leq l; m \leq j \leq n}$  for which the chains in  $(\mathcal{P}, \subseteq)$  of  $T$  and  $U$  are  $\lambda^{[k,m]}, \dots, \lambda^{[l,m]}$  and  $\lambda^{[l,m]}, \dots, \lambda^{[l,n]}$ , while those of  $U'$  and  $T'$  are  $\lambda^{[k,m]}, \dots, \lambda^{[k,n]}$  and  $\lambda^{[k,n]}, \dots, \lambda^{[l,n]}$ .*
- (1)  *$U'$  is obtained from  $U$  by successive inward jeu de taquin slides into the squares of the standardisation of  $T$ , in order of decreasing entries, while the squares evacuated in these steps are those of the standardisation of  $T'$ , in order of decreasing entries.*
- (2)  *$T'$  is obtained from  $T$  by successive outward jeu de taquin slides into the squares of the standardisation of  $U$ , in order of increasing entries, while the squares evacuated in these steps are those of the standardisation of  $U'$ , in order of increasing entries.*

From the symmetry of rule 2.1.1 we obtain immediately:

**2.1.3. Proposition.** *Tableau switching is an involution:  $X(X(T, U)) = (T, U)$ . □*

The correspondence between domino tableaux and pairs of ordinary tableaux that we shall now state is closely related to the correspondence between domino tableaux of partition shape and self-dual ordinary tableaux (those fixed by the Schützenberger algorithm) given in [vLee1, Proposition 2.3.3].

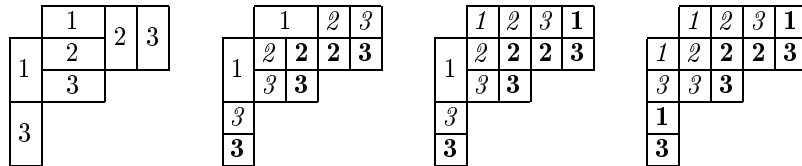
**2.1.4. Definition.** *A self-switching tableau pair of shape  $\lambda/\nu$  and weight  $\omega$  is a pair  $(T, U)$  of semistandard tableaux of weight  $\omega$  and shapes  $\mu/\nu$  and  $\lambda/\mu$  respectively (for some  $\mu$ ), such that  $(T, U) = X(T, U)$ .*

**2.1.5. Proposition.** *There is a shape and weight preserving bijection between semistandard domino tableaux and self-switching tableau pairs. The bijection is such that if  $(\lambda^{[i,j]})_{k \leq i, j \leq l}$  is the matrix of partitions (satisfying rule 2.1.1) exhibiting the fact that  $(T, U)$  is a self-switching tableau pair, then the main diagonal  $\lambda^{[k,k]}, \dots, \lambda^{[l,l]}$  is the chain in  $(\mathcal{P}, \leq_2)$  of the corresponding domino tableau.*

*Proof.* The proof is analogous to that of [vLee1, Proposition 2.3.3]. The fact that  $(T, U)$  is self-switching means that the matrix is symmetric at its borders (for  $i \in \{k, l\}$  or  $j \in \{k, l\}$ ); then rule 2.1.1 ensures that it is symmetric everywhere, and that  $\lambda^{[i+1, i+1]} \setminus \lambda^{[i, i]}$  is always a domino. Conversely, if a saturated chain in  $(\mathcal{P}, \leq_2)$  is given as the main diagonal, then each  $\lambda^{[i, i+1]} = \lambda^{[i+1, i]}$  is uniquely determined by interpolation; thus, the matrix is fully determined and symmetric, establishing the proposition for standard domino tableaux. For the semistandard case we need in addition that two successive dominoes have increasing positions if and only if the corresponding squares in both  $T$  and  $U$  have. For tableaux of two dominoes this is easily verified; as jeu de taquin preserves the condition, this suffices for the general case.  $\square$

This construction can be reformulated in terms of jeu de taquin slides as follows. Given a domino tableau  $D$ , process its dominoes in order of decreasing entries in its standardisation, by the following steps: let  $x$  be the current domino, and  $e = D(x)$ ; remove  $x$ , and fill its outward square with a red entry  $e$ , leaving its inward square empty; apply an inward jeu de taquin slide of the skew tableau of all red entries into the empty square just formed; then fill the final position of the empty square with a blue entry  $e$ . When all dominoes are treated, the red entries form the skew tableau  $T$  and the blue entries form  $U$ . This procedure has a step-by-step inverse, that always succeeds when applied to a self-switching tableau pair  $(T, U)$ ; an attempt to apply it to a pair that is not self-switching will fail at some point because the squares that should combine to a domino are not adjacent. There is also an alternative procedure for computing  $(T, U)$  from  $D$ , which processes the dominoes by increasing entries and uses outwards slides.

Here is an example of such a computation; the red entries are printed in italics (these are the squares that slide), and the blue ones are printed in bold face (these remain in place). Processing of the first 5 dominoes only involves a slide within the domino itself; our first step below shows the combined effect.

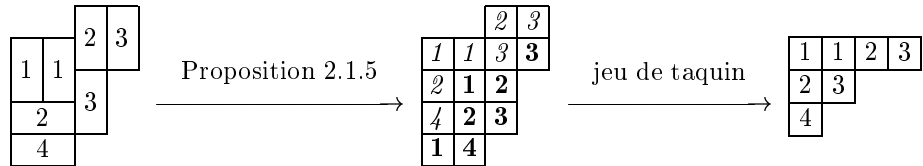


Clearly if  $(T, U)$  is a self-switching tableau pair, then  $T$  and  $U$  are equivalent by jeu de taquin, in fact  $T$  is an inward glissement of  $U$ . Since every equivalence class for jeu de taquin of skew tableaux contains a unique Young tableau, we can now state the following definition, and our first main theorem.

**2.1.6. Definition.** Denote by  $\pi$  the weight preserving map from the set of semistandard skew domino tableaux to the set of semistandard Young tableaux, such that, if a domino tableau  $D$  corresponds to a self-switching tableau pair  $(T, U)$  under the bijection of proposition 2.1.5, then  $\pi(D)$  is the Young tableau equivalent by jeu de taquin to both  $T$  and  $U$ .

**2.1.7. Theorem.** For any domino tableau  $D$ , the second component of the pair associated to it by the bijection of [CaLe, Theorem 7.3] is equal to  $\pi(D)$ . In particular, for any Yamanouchi domino tableau  $Y$  of weight  $\lambda$ , the Young tableau  $\pi(Y)$  is the unique one with shape and weight  $\lambda$ .

As an example, consider the tableau of [CaLe, 7.2, example 2]; we compute  $\pi$  in two steps:



in agreement with the Young tableau computed in [CaLe]. The method by which the Young tableau is obtained is entirely different however, and the proof of theorem 2.1.7 will be a rather indirect one.

**2.2. Yamanouchi domino tableaux and Littlewood-Richardson fillings.**

In [CaLe, Corollary 4.4] it was shown that the numbers of Yamanouchi domino tableaux of given shape and weight are equal to certain structure coefficients of the ring of symmetric functions on the basis of Schur functions. Those structure coefficients are also given by the Littlewood-Richardson rule, i.e., they count

the number of Littlewood-Richardson fillings (or ordinary Yamanouchi tableaux) of a specified shape and weight. We shall construct a natural bijection between such a set of Yamanouchi domino tableaux and the corresponding set of Littlewood-Richardson fillings, settling a question that was left unanswered in [CaLe]. This enables in particular an effective splitting of the set of Littlewood-Richardson fillings describing the square of a Schur function into contributions to the symmetric and alternating part of the square, namely by inspecting the spins of the corresponding Yamanouchi domino tableaux; unfortunately however, the algorithm for determining the Yamanouchi domino tableau is not such that it allows an easy interpretation of (the parity of) its spin in terms of the Littlewood-Richardson filling.

**2.2.1. Definition.** Let  $\lambda^{(0)}, \lambda^{(1)}, \mu^{(0)}, \mu^{(1)} \in \mathcal{P}$  with  $\mu^{(0)} \subseteq \lambda^{(0)}$ ,  $\mu^{(1)} \subseteq \lambda^{(1)}$ , let  $T_0, T_1$  be ordinary semistandard tableaux of respective shapes  $\lambda^{(0)}/\mu^{(0)}$  and  $\lambda^{(1)}/\mu^{(1)}$ , and let  $\gamma \in \mathcal{C}_2$ .

- (1)  $\text{cq}_2(\gamma, \lambda^{(0)}, \lambda^{(1)})$  is the unique partition with 2-core  $\gamma$  and 2-quotient  $(\lambda^{(0)}, \lambda^{(1)})$ ;
- (2)  $\text{cq}_2(\gamma, \lambda^{(0)}/\mu^{(0)}, \lambda^{(1)}/\mu^{(1)}) = \text{cq}_2(\gamma, \lambda^{(0)}, \lambda^{(1)}) / \text{cq}_2(\gamma, \mu^{(0)}, \mu^{(1)})$ ;
- (3)  $\text{cq}_2(\gamma, T_0, T_1)$  is the semistandard domino tableau of shape  $\text{cq}_2(\gamma, \lambda^{(0)}/\mu^{(0)}, \lambda^{(1)}/\mu^{(1)})$  corresponding to  $(T_0, T_1)$  under the bijection of [vLee3, proposition 3.2.2].

We state the relation between Yamanouchi domino tableaux and Littlewood-Richardson coefficients in its most general form.

**2.2.2. Theorem.** [Carré & Leclerc] Let  $\lambda', \mu', \lambda'', \mu'' \in \mathcal{P}$  with  $\mu' \subseteq \lambda'$ ,  $\mu'' \subseteq \lambda''$ , and let  $\gamma \in \mathcal{C}_2$ . For  $\lambda/\mu = \text{cq}_2(\gamma, \lambda'/\mu', \lambda''/\mu'')$ , one has the following decomposition of a product of skew Schur functions:

$$s_{\lambda'/\mu'} \cdot s_{\lambda''/\mu''} = \sum_{Y \in \text{Yam}_2(\lambda/\mu)} s_{\text{wt}(Y)},$$

where  $\text{Yam}_2(\lambda/\mu)$  denotes the set of all Yamanouchi domino tableau of shape  $\lambda/\mu$ .

In fact, only the special case with  $\gamma = \mu' = \mu'' = \emptyset$  is stated explicitly in [CaLe].

*Proof.* By [vLee3, corollary 3.2.3] we have

$$\sum_{D \in \text{Tab}_2(\lambda/\mu; A)} x^{\text{wt}(D)} = s_{\lambda'/\mu'}(x_A) \cdot s_{\lambda''/\mu''}(x_A),$$

while by [CaLe, Theorem 7.3],

$$\sum_{D \in \text{Tab}_2(\lambda/\mu; A)} x^{\text{wt}(D)} = \sum_{Y \in \text{Yam}_2(\lambda/\mu)} \left( \sum_{T \in \text{Tab}(\text{wt}(Y); A)} x^{\text{wt}(T)} \right) = \sum_{Y \in \text{Yam}_2(\lambda/\mu)} s_{\text{wt}(Y)}(x_A). \quad \square$$

To express the Littlewood-Richardson coefficients as numbers of Yamanouchi domino tableaux, one may take  $\mu' = \mu'' = \emptyset$  in the theorem, but not necessarily  $\gamma = \emptyset$ . To express the same coefficients as numbers of Littlewood-Richardson fillings, there is some freedom of choice (see for instance [vLee2, 2.6]). The best choice for our purposes is to interpret the coefficient  $c_{\lambda, \mu}^{\nu}$  of  $s_{\nu}$  in  $s_{\lambda} s_{\mu}$  as the number of Littlewood-Richardson fillings of the shape  $\lambda \uplus \mu$  and weight  $\nu$ , where  $\lambda \uplus \mu$  is the skew diagram obtained by attaching the diagram of  $\lambda$  to the left and below that of  $\mu$  (this is defined up to translations of the connected components that do not affect the definition of a Littlewood-Richardson filling). This interpretation remains valid when  $\lambda$  and  $\mu$  are replaced by arbitrary skew diagrams, and the weight  $\nu$  of the Littlewood-Richardson fillings counted by  $c_{\lambda, \mu}^{\nu}$  is the same as that of the Yamanouchi domino tableaux counted by it. Therefore we can define a bijection corresponding to theorem 2.2.2 in its full generality, by establishing a weight preserving bijection between ordinary Yamanouchi tableaux of shape  $(\lambda'/\mu') \uplus (\lambda''/\mu'')$  and Yamanouchi domino tableaux of shape  $\text{cq}_2(\gamma, \lambda'/\mu', \lambda''/\mu'')$ ; in fact we shall define such a bijection for semistandard tableaux, such that the Yamanouchi property is preserved. A semistandard tableau  $T$  of shape  $(\lambda'/\mu') \uplus (\lambda''/\mu'')$  can be split into a pair  $(T_0, T_1)$  of semistandard tableaux of respective shapes  $\lambda'/\mu'$  and  $\lambda''/\mu''$ , and we shall express this by writing  $T = T_0 \uplus T_1$ .

Now the correspondence  $T_0 \uplus T_1 \mapsto \text{cq}_2(\gamma, T_0, T_1)$  already provides a bijection between the indicated sets of semistandard (domino) tableaux, but it does not in general preserve the Yamanouchi property. That property can be stated as a condition on the word formed by reading the entries of a tableau in a particular order (the ‘‘column reading’’); this order is such that for  $T_0 \uplus T_1$  all entries of  $T_0$  precede those

of  $T_1$ , but in the reading of  $\text{cq}_2(\gamma, T_0, T_1)$  those entries will in general be interleaved. However, if  $\gamma$  is sufficiently large, then interleaving will not occur, and the Yamanouchi property *will* be preserved. Let us denote by  $\gamma_c$  the 2-core with parameters  $d^2(\delta(\gamma_c)) = (c, -c)$ ; more explicitly,  $\gamma_c = (-2c, -2c-1, \dots, 1)$  for  $c \leq 0$  and  $\gamma_c = (2c-1, 2c-2, \dots, 1)$  for  $c > 0$ . According to [vLee3, proposition 3.1.2], a square  $s$  of  $T_0$  with  $\text{pos}(s) = p$  will correspond to a domino  $x$  of  $\text{cq}_2(\gamma_c, T_0, T_1)$  with  $\text{pos}(x) = 2(p+c)$ , while a square  $s'$  of  $T_1$  with  $\text{pos}(s') = p'$  will correspond to a domino  $y$  with  $\text{pos}(y) = 2(p'-c) + 1$ . Therefore, for  $c \ll 0$  one will have  $\text{pos}(x) < \text{pos}(y)$  for all such dominoes; for  $c \gg 0$  the same will be true if  $x$  and  $y$  are taken to be the dominoes of  $\text{cq}_2(\gamma_c, T_1, T_0)$  corresponding to  $s$  and  $s'$  respectively. In practice it is useful to require slightly more, namely  $\text{pos}(y) - \text{pos}(x) \geq 3$ , so that there is a complete diagonal free of dominoes, separating the dominoes corresponding to squares of  $T_0$  from those corresponding to squares of  $T_1$ ; this will also ensure that the former dominoes are all vertical and the latter horizontal. Bounds on  $p$  and  $p'$  can be derived from the shapes of  $T_0$  and  $T_1$ , which leads to the following definition.

**2.2.3. Definition.** Let  $T_0$  and  $T_1$  be ordinary semistandard tableaux of respective shapes  $\lambda'/\mu'$  and  $\lambda''/\mu''$ ; put  $n = \lambda'_0 + (\lambda'')_0^+$ . A segregated domino tableau corresponding to  $(T_0, T_1)$  is a domino tableau of the form  $\text{cq}_2(\gamma_c, T_0, T_1)$  with  $2c \leq -n + 1$ , or of the form  $\text{cq}_2(\gamma_c, T_1, T_0)$  with  $2c \geq n$ .

As an example, consider the tableaux

$$T_0 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 0 & 3 & \\ \hline 2 & & \\ \hline \end{array}, \quad T_1 = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & & \\ \hline \end{array}, \quad \text{with} \quad T_0 \uplus T_1 = \begin{array}{|c|c|c|c|} \hline & & & 0 & 0 & 0 \\ \hline & & & 1 & & \\ \hline & & 1 & 1 & 1 & 2 \\ \hline & & 0 & 3 & & \\ \hline & 2 & & & & \\ \hline \end{array}.$$

We have  $\lambda' = (4, 2, 1)$ , and  $\lambda'' = (3, 1)$ , so  $n = 4 + 2 = 6$ , and the smallest segregated domino tableau corresponding to  $(T_0, T_1)$  occurs for  $\gamma_3 = (5, 4, 3, 2, 1)$ , the next one for  $\gamma_{-3} = (6, 5, 4, 3, 2, 1)$ :

$$\text{cq}_2(\gamma_3, T_1, T_0) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 0 & 0 & 0 & 0 \\ \hline & & & & 1 & & & \\ \hline & & & 2 & & & & \\ \hline & & 1 & & & & & \\ \hline & 1 & & & & & & \\ \hline & 0 & 3 & & & & & \\ \hline & 2 & & & & & & \\ \hline \end{array}, \quad \text{cq}_2(\gamma_{-3}, T_0, T_1) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 0 & 0 & 0 & 0 \\ \hline & & & & 1 & & & \\ \hline & & & 2 & & & & \\ \hline & & 1 & & & & & \\ \hline & 1 & & & & & & \\ \hline & 0 & 3 & & & & & \\ \hline & 2 & & & & & & \\ \hline \end{array}.$$

On the other hand

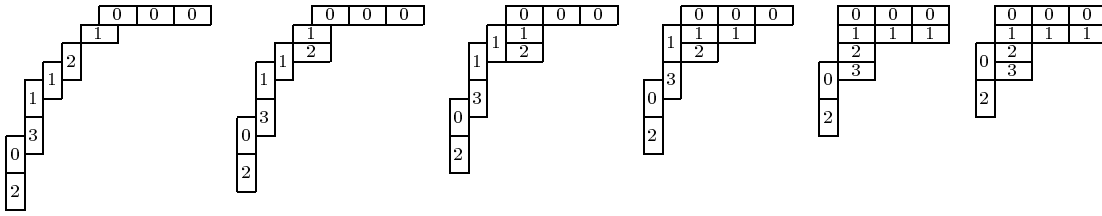
$$\text{cq}_2(\gamma_0, T_0, T_1) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 2 \\ \hline 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 3 & \\ \hline 2 & & & \\ \hline \end{array}$$

is not segregated. In the example  $T_0 \uplus T_1$ ,  $\text{cq}_2(\gamma_3, T_1, T_0)$ , and  $\text{cq}_2(\gamma_{-3}, T_0, T_1)$  are all Yamanouchi (domino) tableaux, but  $\text{cq}_2(\gamma_0, T_0, T_1)$  is not. We have in general:

**2.2.4. Proposition.** A segregated domino tableau corresponding to a pair  $(T_0, T_1)$  of semistandard tableaux is a Yamanouchi domino tableau if and only if  $T_0 \uplus T_1$  is an ordinary Yamanouchi tableau.  $\square$

This proposition provides a first step from the realm of ordinary tableaux into that of domino tableaux. The real work to be done involves domino tableaux of shapes with smaller cores, i.e., those that are not segregated. Our approach will be to, starting from a segregated domino tableau, gradually decrease the size of the core, while preserving the Yamanouchi property. More precisely, we shall transform a segregated domino tableau  $D$  of shape  $\lambda/\mu$  into one of the desired shape  $\sigma(\lambda)/\sigma(\mu)$ , for an appropriately chosen  $\sigma \in \mathfrak{S}_2$ . The second part of [vLee3, proposition 4.3.2] makes it clear that  $\sigma(D)$ , although of the right shape, does not have the required properties (it is just  $\text{cq}_2(\gamma', T_0, T_1)$  or  $\text{cq}_2(\gamma', T_1, T_0)$  for some  $\gamma' \in \mathcal{C}_2$ ); instead we compute  $\sigma \circ D$ , which, as we shall show, does have the required properties. This computation consists of applying the generators  $s_0, s_1$  as they occur in the reduced expression of  $\sigma$ , as was illustrated in [vLee3]; indeed this gradually changes the size of the core. As an example we compute  $\sigma \circ D$  for the segregated tableau  $D = \text{cq}_2(\gamma_3, T_1, T_0)$  of our example above, and  $\sigma = s_1 s_0 s_1 s_0 s_1$ .





Note that these are all Yamanouchi domino tableaux. On the other hand, moving any of the occurring closed chains (there was one in the second step, and two in the final step), while valid for semistandard tableaux, would have destroyed the Yamanouchi property. For open chains, this cannot happen:

**2.2.5. Proposition.** *Moving open chains in domino tableaux preserves the Yamanouchi property.*

In fact we have more generally that moving open chains does not affect the image under the projection  $\pi$  from domino tableaux to Young tableaux. The promised Yamanouchi-preserving bijection between ordinary tableaux of shape  $(\lambda'/\mu') \uplus (\lambda''/\mu'')$  and domino tableaux of shape  $\text{cq}_2(\gamma, \lambda'/\mu', \lambda''/\mu'')$  is now obtained by first transforming to a segregated domino tableau and then reducing the core, as illustrated for  $T_0 \uplus T_1$  above. We state our second main theorem.

**2.2.6. Theorem.** *Let  $\lambda', \mu', \lambda'', \mu'' \in \mathcal{P}$  with  $\mu' \subseteq \lambda', \mu'' \subseteq \lambda''$ , and let  $\gamma \in \mathcal{C}_2$ ; put  $\lambda = \text{cq}_2(\gamma, \lambda', \lambda'')$  and  $\mu = \text{cq}_2(\gamma, \mu', \mu'')$ . The following procedure gives a weight preserving bijection between ordinary semistandard tableaux  $T_0 \uplus T_1$  of shape  $(\lambda'/\mu') \uplus (\lambda''/\mu'')$  and semistandard domino tableaux  $D$  of shape  $\lambda/\mu$ : let  $D'$  be any segregated domino tableau corresponding to  $(T_0, T_1)$ , and let  $\sigma \in \check{\mathfrak{S}}_2$  be such that its action transforms the shape of  $D'$  into  $\lambda/\mu$ ; then  $D = \sigma \circ D'$ . This bijection is such that  $\pi(D)$  is the Young tableau equivalent by jeu de taquin to  $T_0 \uplus T_1$ ; in particular,  $D$  is a Yamanouchi domino tableau if and only if  $T_0 \uplus T_1$  is a Yamanouchi tableau.*

One easily sees that if  $D'$  is a segregated domino tableau corresponding to  $(T_0, T_1)$ , and  $s_j \circ D'$  is also segregated, then it corresponds to  $(T_0, T_1)$  as well; therefore  $D$  is independent of the choice of  $D'$ . An element  $\sigma$  as indicated always exists, because the action of  $\check{\mathfrak{S}}_2$  on  $\mathcal{C}_2$  is transitive, and any 2-core has a stabiliser consisting of 2 elements, that can be used if necessary to interchange the two components of the 2-quotient. In more practical terms this means that if one alternately applies  $s_0$  and  $s_1$  to reduce size of the core, and continues to do so after the empty core  $\gamma_0$  is reached, then one gets  $\gamma_0$  once more, but with  $\lambda'/\mu'$  and  $\lambda''/\mu''$  interchanged. After which the core increases again, so that eventually each core occurs twice; the desired shape  $\lambda/\mu$  is obtained at one of these two occurrences. The case where  $\lambda'/\mu' = \lambda''/\mu''$  is special, in that at the transition between the two occurrences of  $\gamma_0$  nothing happens, because the shape is unchanged, and therefore there are no open chains (for  $s_1$  in this case). This means that the entire sequence of domino tableaux becomes symmetric around this point, and there are two possible choices for  $\sigma$  in the theorem, both of which give the same result.

This shows that the bijection of the theorem is well defined. The proof of the theorem, which will be given later, deals with the crucial property that  $\pi(s_j \circ D) = \pi(D)$ ; it will be shown in fact that  $\pi(D)$  remains unchanged when individual open chains are moved.

**2.3. Matching two expressions for  $\langle s_\lambda, \psi^2(s_\mu) \rangle$ .**

Our third bijective construction involves the relationship between the plethysm operator  $\psi^2$  (or its dual  $\phi^2$ ) and Yamanouchi domino tableaux that was described in [CaLe]; in fact two different relations were indicated. The first relation comes from combining theorem 2.2.2 with the classical fact that for  $\lambda/\mu = \text{cq}_2(\gamma, \lambda'/\mu', \lambda''/\mu'')$  one has  $\varepsilon_2(\lambda/\mu)\phi^2(s_{\lambda/\mu}) = s_{\lambda'/\mu'} \cdot s_{\lambda''/\mu''}$ ; it follows that

$$\varepsilon_2(\lambda/\mu)\phi^2(s_{\lambda/\mu}) = \sum_{Y \in \text{Yam}_2(\lambda/\mu)} s_{\text{wt}(Y)} \tag{1}$$

([CaLe, Corollary 4.3]), which is a nice interpretation of Yamanouchi domino tableaux, as it does not require explicit mention of  $\lambda', \lambda'', \mu', \mu''$  and  $\gamma$ . The second relation comes from an analysis of the spin statistic on  $\text{Tab}_2(\mu^\square; A)$ , where  $\mu^\square = \text{cq}_2(\emptyset, \mu, \mu)$  is the partition obtained from the Young diagram of  $\mu$  by scaling up by a factor 2 both horizontally and vertically. On one hand,

$$\sum_{D \in \text{Tab}_2(\mu^\square; A)} (-1)^{\text{Spin}(D)} x^{\text{wt}(D)} = (-1)^{|\mu|} \psi^2(s_\mu)(x_A); \tag{2}$$

this is deduced from the fact that contributions to the sum of tableaux related to each other by moving of a closed chain for  $s_1$  cancel each other. On the other hand,

$$\sum_{D \in \text{Tab}_2(\mu^\square; A)} (-1)^{\text{Spin}(D)} x^{\text{wt}(D)} = \sum_{Y \in \text{Yam}_2(\mu^\square)} (-1)^{\text{Spin}(Y)} s_{\text{wt}(Y)}(x_A), \quad (3)$$

since the projection of domino tableaux onto Yamanouchi domino tableaux in [CaLe, Theorem 7.3] preserves the spin. It follows that

$$(-1)^{|\mu|} \psi^2(s_\mu) = \sum_{Y \in \text{Yam}_2(\mu^\square)} (-1)^{\text{Spin}(Y)} s_{\text{wt}(Y)} \quad (4)$$

(cf. [CaLe, Theorem 5.3]). Equations (1) and (4) give two distinct combinatorial interpretations for the number  $\langle \phi^2(s_\lambda), s_\mu \rangle = \langle s_\lambda, \psi^2(s_\mu) \rangle$  for  $\lambda, \mu \in \mathcal{P}$ . By (1), it is the cardinality of the set  $\text{Yam}_2(\lambda, \mu)$  of Yamanouchi domino tableaux of shape  $\lambda$  and weight  $\mu$ , multiplied by the sign  $\varepsilon_2(\lambda)$  (which is short for  $\varepsilon_2(\lambda/\emptyset)$ , and by [vLee3, definition 3.3.2] is equal to  $(-1)^{2\text{Spin}(T)}$  for all  $T \in \text{Yam}_2(\lambda, \mu)$ ), while by (4), it is  $\sum_{Y \in \text{Yam}_2(\mu^\square, \lambda)} (-1)^{|\mu| - \text{Spin}(Y)}$  (the exponent is equal to half the number of *horizontal* dominoes in  $Y$ ).

Our third main result will be bijective construction corresponding to this identity. More precisely, we shall establish a bijection between  $\text{Yam}_2(\lambda, \mu)$  and a subset of  $\text{Yam}_2(\mu^\square, \lambda)$ , such that each  $T \in \text{Yam}_2(\lambda, \mu)$  has half as many vertical dominoes as the corresponding  $Y \in \text{Yam}_2(\mu^\square, \lambda)$  has horizontal dominoes (this will imply  $\varepsilon_2(\lambda) = (-1)^{|\mu| - \text{Spin}(Y)}$ ), and in addition show that  $\sum_Y (-1)^{\text{Spin}(Y)}$  vanishes for  $Y$  ranging over the remainder of  $\text{Yam}_2(\mu^\square, \lambda)$ . The latter part of the construction is in fact quite similar to the argument leading to (2): the remainder of  $\text{Yam}_2(\mu^\square, \lambda)$  consists precisely of those tableaux  $Y$  for which there is at least one closed chain for  $s_1$  that can be moved while preserving the Yamanouchi property, and by [vLee3, proposition 4.4.1(2)] the contributions to the alternating sum of the tableaux before and after such a move cancel each other. This part of the argument does require one non-obvious fact:

**2.3.1. Lemma.** *Let  $Y$  be a Yamanouchi domino tableau,  $s \in \{s_0, s_1\}$ , and let  $S$  be the set of closed chains  $C$  in  $Y$  for  $s$  for which the tableau obtained from  $Y$  by moving  $C$  is again a Yamanouchi domino tableau. Then the tableau obtained from  $Y$  by simultaneously moving the chains of any subset of  $S$  is also a Yamanouchi domino tableau.*

If ‘‘Yamanouchi’’ were replaced by ‘‘semistandard’’, the lemma would be a direct consequence of [vLee3, proposition 4.3.1]; indeed it was that fact that was used in the derivation of (2). Although [CaLe] makes a remark to the effect that the same argument is valid for Yamanouchi domino tableaux (after its Lemma 8.5), that remark is not justified (nor used) there. We shall provide a proof for the lemma below.

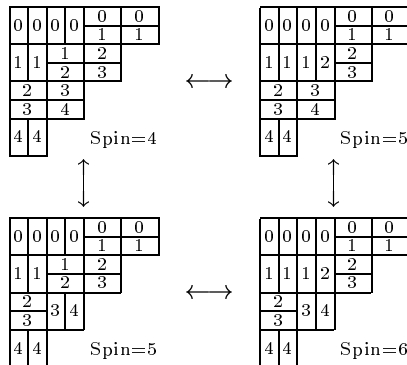
The correspondence between tableaux  $T \in \text{Yam}_2(\lambda, \mu)$  and tableaux  $Y \in \text{Yam}_2(\mu^\square, \lambda)$  for which no (closed) chain for  $s_1$  can be moved without destroying the Yamanouchi property, is constructed as follows. First we define a bijection  $f$  from the set  $\text{Dom}(T)$  of dominoes of  $T$  to the set of squares of  $\mu$ : it is determined by the requirement that for any  $i$ , if  $x$  traverses the set of dominoes of  $T$  with entry  $i$  by increasing value of  $\text{pos}(x)$ , then  $f(x)$  traverses row  $i$  of  $\mu$  from right to left (i.e., the value of  $\text{pos}(f(x))$  *decreases*). Then we divide the Young diagram  $\mu^\square$  into  $2 \times 2$  blocks of squares that correspond to the individual squares of  $\mu$  in the most obvious way: any square  $(i, j) \in \mu^\square$  belongs to the block corresponding to the square  $(\lfloor i/2 \rfloor, \lfloor j/2 \rfloor) \in \mu$ . For each domino  $x$  of  $T$ , the block corresponding to  $f(x)$  is filled in  $Y$  with two dominoes whose entries are the row numbers of the squares of  $x$ : if  $x$  is a horizontal domino in row  $r$  then the block is divided into two vertical dominoes, both with entry  $r$ , and if  $x$  is a vertical domino in rows  $r, r+1$ , then the block is divided into two horizontal dominoes with entries  $r$  and  $r+1$ . Although it is not obvious that this construction actually defines a Yamanouchi domino tableau (or even a semistandard one), it is clear that  $Y$  has twice as many horizontal dominoes as  $T$  has vertical dominoes. We can now formulate our third main theorem.

**2.3.2. Theorem.** *For any  $\lambda, \mu \in \mathcal{P}$ , the construction above defines an injective map from  $\text{Yam}_2(\lambda, \mu)$  to  $\text{Yam}_2(\mu^\square, \lambda)$ , whose image consists of those Yamanouchi domino tableaux  $Y$  for which the set  $S$  of lemma 2.3.1 for  $s = s_1$  is empty.*

As a concrete example consider  $\lambda = (6, 5, 3, 3, 3)$  and  $\mu = (4, 3, 2, 1)$ . Now the set  $\text{Yam}_2(\lambda, \mu)$  has just one element  $T$ , which we display here together with the corresponding element  $Y \in \text{Yam}_2(\mu^\square, \lambda)$ :

$$T = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline & & 1 & 1 \\ \hline 1 & & & \\ \hline 2 & 2 & & \\ \hline & 3 & & \\ \hline \end{array} \qquad Y = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & 1 & 1 & & \\ \hline 1 & 1 & 2 & 2 & & \\ \hline & & 3 & 3 & & \\ \hline 3 & 3 & 4 & & & \\ \hline & & 4 & 4 & & \\ \hline \end{array}$$

For each of the remaining 4 elements of  $\text{Yam}_2(\mu^\square, \lambda)$  the set  $S$  of closed chains for  $s_1$  that can be moved while preserving the Yamanouchi condition (as in lemma 2.3.1) has 2 elements (both of which in fact occupy a  $2 \times 2$  block), and these tableaux form a nice illustration of that lemma:



**§3. Raising and lowering operations.**

For a detailed understanding of the bijection of [CaLe, Theorem 7.3] we must study the algorithm defining it [CaLe, 7.1], and in particular the basic steps of that algorithm, which are shape preserving operations that change the weight of a domino tableau by changing an entry, and possibly rearranging some dominoes. Before doing that however, we shall consider simpler versions of these operations, defined for words over the alphabet  $\mathbf{N}$ , and for ordinary semistandard tableaux. Such operations have been used for a long time and in many contexts; in implicit form they occur already in the algorithm of [Rob, §5] that defines what is now known as the Robinson-Schensted correspondence. However they hardly ever seem to be described explicitly and independently of the constructions in which they are used, and no name appears to have been given to them. We shall call them *coplactic operations*, as they are compatible with, and in a sense dual to, the relations defining the plactic monoid (also known as Knuth transformations).

**3.1. Coplactic operations on words.**

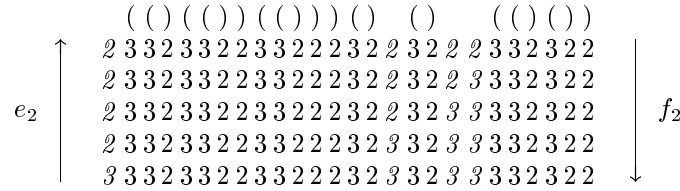
There are two types of coplactic operations: raising operations  $e_i$  and lowering operations  $f_i$ . The index  $i$  can assume any integer value, but when considering words over a finite alphabet  $\{0, \dots, n\}$ , we require  $0 \leq i < n$ . The terms “raising” and “lowering” should be understood in terms of the dominance ordering on weights: application of  $e_i$  replaces an occurrence of  $i + 1$  by  $i$ , while application of  $f_i$  similarly replaces an  $i$  by  $i + 1$ . It is not always possible to apply a given coplactic operation to a given word; however if  $e_i$  can be applied, then its effect can be undone by an application of  $f_i$ , and vice versa.

Although coplactic operations on words are simple to define, one finds various descriptions of them, the equivalence of which is not always immediately obvious. We recall here a transparent formulation that can be found for instance in [LeTh, §3]. In a word  $w$  one associates to each letter  $i + 1$  an opening parenthesis and to each  $i$  a closing parenthesis; any entry that is either unequal to  $i$  and  $i + 1$ , or whose parenthesis matches another one (in the usual sense) within  $w$  is ignored for the purpose of applying  $e_i$  or  $f_i$  to  $w$ . The remaining entries of  $w$  form a subword of the form  $i^r (i + 1)^s$  with  $r, s \in \mathbf{N}$ . If  $s > 0$ , then  $e_i$  can be applied to  $w$ , and its effect is replace the first  $i + 1$  of the subword by  $i$ , leaving all other entries in their original positions; the subword becomes  $i^{(r+1)} (i + 1)^{(s-1)}$ . Similarly  $f_i$  can be applied to  $w$  if  $r > 0$ , and transforms the subword into  $i^{(r-1)} (i + 1)^{(s+1)}$ . Since the only letter that is changed by a coplactic operation has an unmatched parenthesis both before and after the change, we see that indeed  $e_i$  and  $f_i$  undo each other’s effect. Moreover, we see that  $r$  and  $s$  respectively indicate the number of successive times that  $f_i$  and  $e_i$  can be applied to  $w$ ; we shall write  $r = n_i^-(w)$  and  $s = n_i^+(w)$ . For future reference we single out a trivial consequence of the definition.

**3.1.1. Proposition.** When a coplactic operation  $e_i$  or  $f_i$  is applied to a word  $w$ , the letter that is affected by it is neither followed by  $i$  nor preceded by  $i + 1$  in the subword of  $w$  of letters from the alphabet  $\{i, i + 1\}$ .

If an index set  $I$  is fixed, we can construct for each word  $w$  a graph that represents the structure of the set of words that can be obtained from  $s$  by sequences of operations  $e_i, f_i$  for  $i \in I$ . It is a directed graph, with edges labelled by elements of  $I$ , and with a distinguished vertex. As set of vertices we take the closure (within the set of all words over the given alphabet) of  $\{w\}$  with respect to the indicated coplactic operations, and the distinguished vertex is  $w$  itself; there is an edge labelled  $i$  from  $x$  to  $y$  whenever  $y = f_i(x)$  or equivalently  $x = e_i(y)$  (this notation is taken to imply that applications of  $f_i$  to  $x$  and of  $e_i$  to  $y$  are possible). We shall call this the coplactic graph associated to  $w$  (for  $I$ ). In case  $I = \{i\}$  this graph will be a ‘‘ladder’’: a linear graph with all edges pointing in one direction; the distinguished vertex is reached from the start of the ladder after following  $n_i^+(w)$  edges, and there are  $n_i^-(w)$  more edges to the end of the ladder. In general however, not all coplactic operations that can be applied to a word commute with each other, and there can be multiple words in the graph with the same weight. For  $I = \{1, \dots, n - 1\}$  these graphs are in fact isomorphic to the crystal graphs for irreducible integrable  $U_q(\mathfrak{g}_n)$ -modules of [KaNa] (see also [LeTh], which contains some explicit examples of these graphs); this accounts for some remarkable properties, such as the fact that the multiset of weights occurring at the vertices of the graph form the character of an irreducible representation of  $\mathbf{GL}_n$ .

As a concrete example of the application of coplactic operations, we show below a typical ladder, obtained for the index set  $I = \{2\}$ . The only letters relevant to  $e_2$  and  $f_2$  are 2 and 3, so we choose a word  $w = 23323322332223223233332322$  over that alphabet. We have  $n_2^+(w) = n_2^-(w) = 2$ , so there are 5 words in the ladder, and  $w$  is the middle one. The set of matching parentheses, which is the same for all words in the ladder, is displayed at the top; the letters that have unmatched parentheses are italicised.



Another description of coplactic operations is the following. Define  $h_i(w)$  to be the multiplicity in  $w$  of the letter  $i$  minus that of  $i + 1$ . Then it can be seen that  $n_i^-(w) = \max \{ h_i(u) \mid w = uv \}$  (here  $u$  ranges over all prefixes of  $w$ ), and  $n_i^+(w) = \max \{ -h_i(v) \mid w = uv \}$ . If  $n_i^-(w) > 0$  then  $f_i(w)$  is obtained by changing the last letter of the smallest prefix  $u$  of  $w$  for which  $h_i(u)$  obtains its maximum  $n_i^-(w)$ , from  $i$  to  $i + 1$ , and similarly if  $n_i^+(w) > 0$ , then  $e_i(w)$  is obtained by changing the first letter of the smallest suffix  $v$  of  $w$  for which  $-h_i(v)$  obtains its maximum  $n_i^+(w)$ , from  $i + 1$  to  $i$ . It is now easy to see that the basic step of Robinson’s algorithm in [Rob, §5] (with interchange left and right because of different conventions used there), and the basic modification to the word  $w$  in [CaLe, Algorithm 7.1], both amount to applying  $e_i$  for the smallest  $i$  for which this is possible (the ‘‘index’’ associated to an occurrence of  $i + 1$  in  $w$  is just  $-h_i(v)$  for the suffix  $v$  of  $w$  starting at that occurrence of  $i + 1$ ). One can also see that, if one associates to a word a piecewise linear path in  $\mathbf{R}^n$  by traversing the entries from right to left and making for each letter  $i$  a unit step in the direction of the basis vector  $\varepsilon_i$ , then the operations  $e_i$  and  $f_i$  correspond to the operators  $e_\alpha$  and  $f_\alpha$  of [Litt1, §1] on paths, for  $\alpha$  equal to the root  $\varepsilon_i - \varepsilon_{i+1}$ .

There is yet another description, related to the previous one, that we shall be using in this paper. It requires that we first select a shape  $\lambda/\mu$  with  $\lambda, \mu \in \mathcal{P}$ , that is compatible with  $w$  according to the following definition:  $\lambda = \mu + \text{wt}(w)$ , and for all  $i$  one has the equivalent inequalities  $n_i^+(w) \leq \mu_i - \mu_{i+1}$  and  $n_i^-(w) \leq \lambda_i - \lambda_{i+1}$ . It is easy to see that such  $\lambda/\mu$  exists for every  $w$  (one can even achieve that the inequalities become equalities). An alternative formulation of the inequalities is that  $\mu + \text{wt}(v)$  is a partition for each suffix  $v$  of  $w$ ; in the terminology of [Litt1], the path corresponding to  $w$  is  $\mu$ -dominant. The sequence of partitions  $\mu + \text{wt}(v)$ , for all suffixes  $v$  of  $w$  in increasing order, forms an ordinary standard tableau  $S$  of shape  $\lambda/\mu$ ; as set of entries we take  $\{i \in \mathbf{N} \mid i < l(w)\}$ . The word  $w$  can be reconstructed from  $S$ : listing the row numbers of the squares added at the successive steps in the chain of partitions from  $\mu$  to  $\lambda$ , gives the letters of  $w$  from right to left.

**3.1.2. Proposition.** Let  $w, \lambda/\mu$ , and  $S$  be as above, let  $S'$  be obtained from  $S$  by an inward jeu de taquin slide, and let  $w'$  be the word reconstructed from  $S'$  in the same way as  $w$  is reconstructed from  $S$ .

Then  $w'$  can be obtained via a sequence of 0 or more raising operations from  $w$ ; more precisely, if the slide starts in row  $i$  and ends in row  $i' \geq i$  then  $w' = e_{i'-1}(\cdots e_{i+1}(e_i(w))\cdots)$ . Similarly, if  $S'$  is obtained instead from  $S$  by an outward jeu de taquin slide starting in row  $i'$  and ending in row  $i \leq i'$ , one will have  $w' = f_i(f_{i+1}(\cdots f_{i'-1}(w)\cdots))$ . Moreover one can get the case  $i' = i + 1$  of a single application of  $e_i$  or  $f_i$ , provided such an application is possible, by a suitable choice of  $\lambda/\mu$ .

*Proof.* We attach a subsidiary value to each letter of  $w$ , which we shall call its ordinate, so that the combination is unique; the occurrence of  $i$  with ordinate  $j$  shall be denoted by  $i_j$ . The assignment is such that if square  $(i, j)$  of  $S$  has entry  $k$ , then  $i_j$  occurs in  $w$  with  $k$  letters to the right of it; this means that  $i_j$  exists if and only if  $\mu_i \leq j < \lambda_i$ , and among the occurrences  $i_j$  of a fixed letter  $i$ , the ordinate  $j$  increases from right to left by unit steps. Among all  $i_j$  with a fixed ordinate  $j$ , the letter  $i$  will also increase from right to left by unit steps. The changes caused to  $S$  by the basic steps of in inward jeu de taquin slide correspond in  $w$  to a replacement  $i_{j+1} \rightarrow i_j$  (for horizontal moves), or  $(i + 1)_j \rightarrow i_j$  (for vertical moves). If the slide consists only of horizontal moves in row  $i$ , then clearly  $w' = w$ ; this implies that we had a strict inequality  $\mu_i - \mu_{i+1} > n_i^+(w)$ , as the shape  $\lambda'/\mu'$  obtained by decreasing parts  $i$  of  $\lambda$  and  $\mu$  by 1 is also compatible with  $w$ . Otherwise let the first vertical move correspond to a replacement  $(i + 1)_j \rightarrow i_j$ ; we claim that this is exactly the change affected by application of  $e_i$  to  $w$ . Let  $v$  be the suffix of  $w$  starting with  $(i + 1)_j$ . By the definition of jeu de taquin, if  $i_{j+1}$  exists, then it lies to the left of  $(i + 1)_j$ ; consequently, the occurrences of the letters  $i$  and  $i + 1$  within  $v$  are precisely those with ordinates  $j' \leq j$ . Since  $j \geq \mu_i - 1$ , it follows that  $-h_i(v) = \mu_i - \mu_{i+1}$ , so the inequalities  $-h_i(v) \leq n_i^+(w) \leq \mu_i - \mu_{i+1}$  are equalities. Also, no strictly smaller suffix  $v'$  of  $w$  can have  $-h_i(v') = n_i^+(w)$ , since in the subtableau of  $S$  corresponding to  $v'$  the slide involves only horizontal moves in row  $i$ , which as we saw implies  $n_i^+(v') < \mu_i - \mu_{i+1}$ , and hence  $-h_i(v') < n_i^+(w)$ ; therefore  $v$  is the smallest suffix for which  $-h_i(v) = n_i^+(w)$ , proving our claim.

It remains to show that the remainder of the jeu de taquin slide after the first vertical move corresponds to a jeu de taquin slide starting in row  $i + 1$ , applied to some standard tableau  $S_1$  that corresponds to  $e_i(w)$ . The intermediate state of  $S$  after the first vertical move is not a tableau because of the empty square at  $(i + 1, j)$ ; however, we can make it into a tableau by shifting one place to the right all entries in row  $i$  or above it, as well as the entries in row  $i + 1$  to the left of column  $j$ . The tableau  $S_1$  so obtained clearly corresponds to  $e_i(w)$ , and allows an inward jeu de taquin slide to be started in row  $i + 1$ ; that slide will cause horizontal moves until the empty square is at  $(i + 1, j)$ . At that point the situation differs from the one obtained after the first vertical move in the jeu de taquin slide applied to  $S$  only in the locations of the entries in the rows above row  $i + 1$ , but these entries do not influence the remainder of the slide. The statement about outward slides and lowering operations obviously follows from the fact that these invert inward slides and raising operations. To single out one application of  $e_i$  it, suffices to choose  $\mu$  such that  $\mu_i - \mu_{i+1} = n_i^+(w)$  while  $\mu_{i+1} - \mu_{i+2} > n_{i+1}^+(w)$ ; since we are assuming  $n_i^+(w) > 0$  here,  $\mu$  will have a corner in row  $i$  to start the slide at.  $\square$

As an example we show an inward jeu de taquin slide of a standard tableau  $S$  corresponding to the word  $w$  of in the example above for  $\lambda/\mu = (15, 15, 15, 13)/(15, 15, 2, 0)$  (only rows 2 and 3 are non-empty):

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & & 0 & 1 & 3 & 8 & 10 & 11 & 13 & 14 & 15 & 18 & 19 & 22 & 25 \\ \hline 2 & 4 & 5 & 6 & 7 & 9 & 12 & 16 & 17 & 20 & 21 & 23 & 24 & & & \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & & 0 & 1 & 3 & 7 & 8 & 10 & 11 & 13 & 14 & 15 & 18 & 19 & 22 & 25 \\ \hline 2 & 4 & 5 & 6 & 9 & 12 & 16 & 17 & 20 & 21 & 23 & 24 & & & & \\ \hline \end{array};$$

the result corresponds to  $e_2(w)$ . The corresponding transformation of words with ordinates (written here for better readability as superscripts to letters 2 and as subscripts to letters 3) is

$$\begin{array}{cccccccccccccccccccc} 14 & & 13 & & 12 & 11 & & 10 & 9 & 8 & & 7 & 6 & & 5 & & 4 & & 3 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 2 & 3 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 \\ 12 & 11 & & 10 & 9 & & 8 & 7 & & 6 & & 5 & 4 & 3 & 2 & 1 & & 0 & & & & & \end{array} \quad \longrightarrow \quad \begin{array}{cccccccccccccccccccc} 14 & & 13 & & 12 & 11 & & 10 & 9 & 8 & & 7 & 6 & & 5 & 4 & & 3 & & 2 & 1 \\ 2 & 3 & 3 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 \\ 11 & 10 & & 9 & 8 & & 7 & 6 & & 5 & 4 & 3 & 2 & 1 & & 0 & & & & & & & \end{array}.$$

For a word  $w$  one may choose  $\lambda/\mu = \text{wt}(w)/\emptyset$  if and only if  $n_i^+(w) = 0$  for all  $i \geq 0$ , which means that no raising operation can be applied to  $w$ . Such  $w$  is called a Yamanouchi word; the construction of the tableau  $S$  above defines a bijection between Yamanouchi words of weight  $\lambda$  and standard Young tableaux of shape  $\lambda$ . We shall see that Yamanouchi tableaux and Yamanouchi domino tableau are also characterised by the fact that no raising operation can be applied to them. By repeatedly applying inward jeu de taquin slides to  $S$ , we see that any word can be made into a Yamanouchi word by repeated application of raising operations; moreover, this word is independent of the order of these applications, because of the analogous property of jeu de taquin.

There is a connection between the construction of proposition 3.1.2 and the concept of image-glisement defined in [vLee2]. If we place the letters of  $w$  with their ordinates in the squares of an anti-diagonal skew shape  $\chi$ , retaining their left-to-right order, then the map that sends the square containing  $i_j$  to the square  $(-i, -j)$  is a picture  $\chi \rightarrow -(\lambda \setminus \mu)$ , and the transformation  $w \rightarrow w'$  corresponds to applying an image-glisement to this picture. We shall not stress this point of view here, although we shall refer to the construction of proposition 3.1.2 as computing coplactic operations by image-glissements. Conversely, it is possible to describe the effect of an image-glisement entirely in terms of the coplactic graph.

**3.1.3. Proposition.** *The number  $i' - i$  of coplactic operations corresponding to a jeu de taquin slide in proposition 3.1.2 depends, apart from the shape  $\lambda/\mu$  and the starting square of the slide, only on the coplactic graph of the word  $w$ .*

*Proof.* Consider the case of an inward slide starting in row  $i$ , and let  $\mu'$  be the partition obtained by decreasing the part  $\mu_i$  of  $\mu$  by 1. Then the word  $w'$  obtained after the slide is the first word of the form  $w' = e_{i'-1}(\cdots e_{i+1}(e_i(w)) \cdots)$  for which  $\mu' + \text{wt}(w')/\mu'$  is compatible with  $w'$ . Whether this is the case depends only on  $n_i^+(w')$ , which can be read off from the coplactic graph of  $w$ .  $\square$

### 3.2. Coplactic operations on ordinary tableaux, and Robinson's bijection.

We shall now consider the bijection defined in [Rob, §5], after which the algorithm [CaLe, 7.1] for the domino case was modelled. Robinson's description is not particularly clear; a more practical reference is [Macd, I 9]. We first define coplactic operations on (ordinary skew) semistandard tableaux, as follows. With such a tableau a word is associated by reading its entries in a prescribed order, which is used to determine which coplactic operations can be applied, and if so, which letter is affected; application of the coplactic operation to the tableau then causes the same change to the corresponding entry of the tableau. It must be shown of course that this will always produce another semistandard tableau. For the reading order by columns from left to right, and from bottom to top within each column, this property follows from proposition 3.1.1. For the reading order by rows from bottom to top, and from left to right within each row, the same property follows by a slightly deeper analysis (see [Macd, I (9.6)], noting that there, like in [Rob], left and right are reversed in words, but not in tableaux). In fact it can be seen that the effect of a coplactic operation on any tableau is the same for both reading orders, so it does not matter which one is used in the definition. A Yamanouchi tableau is one to which no raising operation  $e_i$  can be applied, which is equivalent to the fact that reading the entries of the tableau in either of these orders gives a Yamanouchi word. We see that by construction any coplactic graph associated to a tableau is isomorphic to one associated to a word, which word is generally not unique. In the sequel we shall encounter many more cases of isomorphic coplactic graphs.

Robinson's bijection now associates to each semistandard tableau  $T$  a pair  $(Y, P)$  where  $Y$  is a Yamanouchi tableau of the same shape as  $T$ , and  $P$  is a semistandard Young tableau of shape  $\text{wt}(Y)$  and weight  $\text{wt}(T)$ . The tableau  $Y$  is obtained by transformation of  $T$ , where each step consists of applying the raising operator  $e_i$  for the smallest possible  $i$ , and the transformation is complete when the result is a Yamanouchi tableau. The determination of  $P$  is more complicated. Let  $Y = e_{i_1}(\cdots e_{i_1}(e_{i_0}(T)) \cdots)$  record the transformation of  $T$  into  $Y$ , then the sequence  $e_{i_1}, \dots, e_{i_0}$  is first factored into a minimal number of factors of the form  $S_{i,j} = e_i \circ e_{i+1} \circ \cdots \circ e_{j-1}$  for  $i < j$ . It can be seen that in this factorisation the second index  $j$  of the factors  $S_{i,j}$  increases weakly from right to left, since after application of any  $S_{i,j}$  the subtableau of all entries  $< j$  is Yamanouchi. Then for each occurrence of a factor  $S_{i,j}$ , the Young tableau  $P$  has an entry  $j$  in row  $i$ , and any remaining entries  $j$  needed to make  $\text{wt}(P) = \text{wt}(T)$  lie in row  $j$  of  $P$  (if one prefers, one could add the required number of identity operations  $S_{j,j}$  to the factorisation). Thus the tableau  $P$  can be built up in order of increasing entries while the sequence of raising operations is being determined. The fact that  $P$  is well defined is not at all obvious (Robinson completely ignores this point), and requires a detailed analysis of the structure of the coplactic graph (at least, of those edges used by this construction); in particular, one must show that within any sequence of factors  $S_{i,j}$  with fixed  $j$ , the index  $i$  decreases weakly from right to left ([Macd, (9.7)]). Note that these properties of the sequence  $e_{i_1}, \dots, e_{i_0}$  also allow it to be uniquely reconstructed from the tableau  $P$ .

This construction of  $P$ , which is also used identically in [CaLe, Algorithm 7.1], is not very insightful or practical to work with, and it is strongly dependent on the chosen order of applying raising operations. We shall give an alternative method of determining  $P$ , which is based on the following simple observation.

**3.2.1. Proposition.** *If in Robinson's construction the tableau  $T$  is a Young tableau, then  $P = T$ .*

*Proof.* It is not difficult to see that in this case the algorithm proceeds as follows. Each (compound) operation  $S_{i,j}$  has the effect of changing a single entry  $j$  in row  $i$  of  $T$  into  $i$ ; the sequence of consecutive operations  $S_{i,j}$  with fixed  $j$  processes all entries  $j$  of  $T$  from left to right, except those that are already in their proper row  $j$  (these would correspond to identity operations  $S_{j,j}$ ). Therefore, after all these operations  $S_{i,j}$  for some fixed  $j$  are applied, the subtableau of all entries not exceeding  $j$  has each row  $i$  filled with entries  $i$  only. It then follows from the way  $P$  is defined that  $P = T$ .  $\square$

Note that if the shape of  $T$  is  $\lambda$ , then the tableau  $Y$  produced in this case can be characterised as the unique tableau of shape and weight  $\lambda$ , or as the unique Yamanouchi tableau of shape  $\lambda$ ; we shall call this the canonical tableau  $\mathbf{T}_\lambda$  of shape  $\lambda$ . Returning to the situation of an arbitrary tableau  $T$ , we see that the sequence  $e_{i_1}, \dots, e_{i_0}$  of raising operations found must be identical to the sequence found for  $P$  in place of  $T$ , in other words  $\mathbf{T}_\lambda = e_{i_1}(\dots e_{i_1}(e_{i_0}(P))\dots)$ . This gives us our alternative description of  $P$ .

**3.2.2. Proposition.** *When in Robinson's construction  $Y = e_{i_1}(\dots e_{i_1}(e_{i_0}(T))\dots)$ , then  $P$  can be expressed as  $P = f_{i_0}(f_{i_1}(\dots f_{i_1}(\mathbf{T}_\lambda)\dots))$ , where  $\lambda = \text{wt}(Y)$ .*  $\square$

This description of  $P$  is still cumbersome, and we have not yet shown why Robinson's correspondence  $T \mapsto (Y, P)$  is bijective. Indeed, even Macdonald's elaborate arguments only show that it is well defined and injective; the fact that "we can unambiguously trace our steps backward" does not imply that for any sequence  $e_{i_1}, \dots, e_{i_0}$  that corresponds to a Young tableau  $P$  of shape  $\lambda$ , the expression  $f_{i_0}(f_{i_1}(\dots f_{i_1}(Y)\dots))$  is well defined for any (skew) Yamanouchi tableau (or word)  $Y$  with  $\text{wt}(Y) = \lambda$ . However, the proposition suggests that the coplactic graphs of  $T$  and  $P$  are isomorphic, which would imply the mentioned bijectivity. To prove this, we need a direct description of a transformation that sends  $T$  to  $P$ , and a proof that it commutes with coplactic operations. Such a transformation is given by jeu de taquin; this is no surprise, as it is well known that jeu de taquin can be used to compute the  $P$ -tableau of the Robinson-Schensted correspondence (however, we know of no published proof of this in terms of Robinson's construction rather than of Schensted's insertion procedure). In the sequel we shall encounter various other instances of operations that commute with coplactic operations. The precise meaning of the statement that an operation  $f$  commutes with coplactic operations is that any coplactic operation  $g$  can be applied to any object  $x$  if and only if it can be applied to  $f(x)$ , and if so,  $f(g(x)) = g(f(x))$ . Like for the coplactic operations themselves, we allow  $f$  to be conditionally applicable or to depend on some parameters, in which case the conditions and parameters must be the same for the applications of  $f$  to  $x$  and to  $g(x)$ .

**3.2.3. Proposition.**

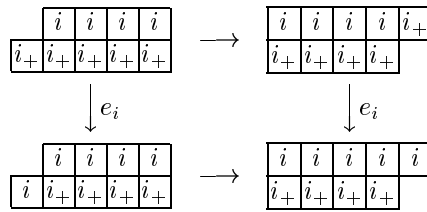
- (1) *Coplactic operations on words commute with elementary Knuth transformations (replacements of triples of consecutive letters of one of the following forms:  $acb \leftrightarrow cab$  with  $a \leq b < c$  or  $bac \leftrightarrow bca$  with  $a < b \leq c$ , cf. [Knuth, (6.6)]).*
- (2) *Coplactic operations on skew tableaux commute with jeu de taquin slides.*

Although these facts are well known to experts, it is hard to find any explicit statements to this effect in the literature. A statement that comes close is [LaSch, 4.5(5)] (however, the proof of the proposition in question does not appear to address this particular issue).

*Proof.* It suffices to consider raising operations  $e_i$ . For (1), commutation is obvious in case  $e_i$  affects a letter outside the triple involved in the Knuth transformation, or if it changes one of these letters but preserves the given inequalities. Of the 8 remaining cases to consider (4 patterns with 2 inequalities each), all are dismissed by proposition 3.1.1, except patterns  $bac$  and  $bca$  when  $a = i$  and  $b = c = i + 1$ ; in those cases we have

$$\begin{array}{ccc}
 i_+ & i & i_+ & \longleftrightarrow & i_+ & i_+ & i \\
 & \downarrow e_i & & & & \downarrow e_i & \\
 i_+ & i & i & \longleftrightarrow & i & i_+ & i
 \end{array}$$

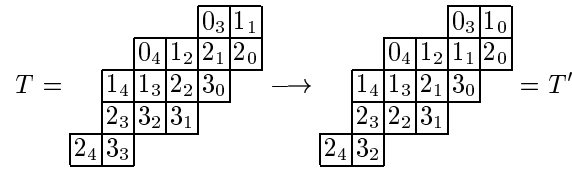
where for clarity we have abbreviated  $i + 1$  to  $i_+$ ; the bottom row is the instance of  $cab \leftrightarrow acb$  with  $a = b = i$  and  $c = i + 1$  (note that a different pair of letters is interchanged than in the top row). The proof for (2) (which does not seem to follow easily from (1), despite the fact that each jeu de taquin slide corresponds to some sequence of elementary Knuth transformations) is similar but more elaborate; we shall omit the details. One needs to consider the possibility that the change of an entry caused by  $e_i$  affects the result of a comparison made to determine the path of the slide; this is possible, but only in the situation illustrated by the following commuting diagram



where we have omitted any entries unequal to  $i$  and  $i + 1$ , or to the left or right of the displayed region.  $\square$

**3.2.4. Corollary.** *For each (skew) tableau  $T$ , any tableau obtainable from it by jeu de taquin slides has a coplactic graph isomorphic to that of  $T$ ; in particular this holds for the unique Young tableau  $P$  among those tableaux, and  $P$  is the second factor of the pair associated to  $T$  by Robinson’s bijection.*  $\square$

Before we proceed to domino tableaux, we reconsider the freedom of choosing a reading order in the case of ordinary tableaux, which is of interest because there is no obvious way to define a good reading order for domino tableaux. The computation of coplactic operations by image-glisserments gives some insight in the reason that the two given reading orders lead to equivalent definitions. Let  $w$  be a reading of a semistandard tableau  $T$  by one of the given orders; we shall call properties reading-independent if they would be the same if the other reading order were used, and reading-dependent otherwise. The compatibility of  $\lambda/\mu$  with  $w$  is reading-independent, and for compatible  $\lambda/\mu$ , the attachment of ordinates to entries of  $T$  corresponding to those attached to the letters of  $w$  in the proof of proposition 3.1.2 is also reading-independent; this is because the ordinate of an entry only depends on its location relative to other entries equal to it. Now  $i_j$  is associated to the square  $(i, j)$  in the standard tableau  $S$  corresponding to  $w$  for  $\lambda/\mu$ , but the entry of  $(i, j)$  in  $S$  gives the location of  $i_j$  within  $w$ , and is therefore reading-dependent. However, for the determination of a jeu de taquin slide of  $S$ , it is not so much the numeric values of its entries that are relevant, but the comparison of the entries of certain pairs of squares  $(i, j + 1)$  and  $(j + 1, i)$ ; this amounts to testing which of  $i_{j+1}$  and  $(i + 1)_j$  comes first in  $w$ , which turns out to be reading-independent in all cases that can arise. It follows that coplactic operations can be computed by image-glisserments without even choosing a reading order. Here is an example of an image-glisserment applied to a skew tableau.



Here we used  $\lambda/\mu = (5, 5, 5, 4)/(3, 1, 0, 0)$ ; the image-glisserment starts at square  $(1, 0)$  and follows the path  $1_0 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 2_2 \rightarrow 3_2 \rightarrow 3_3$ ; hence  $\lambda'/\mu'$  obtained for  $T'$  is  $(5, 5, 5, 3)/(3, 0, 0, 0)$ , and  $T' = e_2(e_1(T))$ . The comparisons involved in determining the slide are  $2_0 < 1_1$ ,  $1_2 < 2_1$ ,  $3_1 < 2_2$ , and  $2_3 < 3_2$ , where  $a < b$  means that  $a$  occurs to the left of  $b$  in  $w$ , independently of the reading. Note that the ordinates in a tableau are strictly decreasing along rows, and weakly decreasing down columns.

The bijection between the set of squares of  $T$  and  $-(\lambda/\mu)$ , in which the square containing  $i_j$  corresponds to  $-(i, j)$ , is a picture; the argument above is just a reflection of the fact that image-glisserments of pictures are natural (see [vLee2, theorem 5.1.1], where these matters are discussed in detail). There is in fact more freedom in choosing a reading order than just reading by columns or by rows; the only restriction on the order is that whenever a square  $t$  lies (weakly) both to the left of and below another square  $t'$  (i.e., if  $t <_{\swarrow} t'$  in the notation of [vLee2]), then  $t$  must be read before  $t'$ . The counterpart of proposition 3.2.3 in the theory of pictures is the fact that domain-glisserment and image-glisserment commute [vLee2, theorem 5.3.1]; by propositions 3.1.2 and 3.1.3, this is in fact equivalent to proposition 3.2.3.

**3.3. Coplactic operations on domino tableaux.**

As shown in [CaLe], it is possible to define coplactic operations for domino tableaux. In general they do not just change the value of some entry, but a rearrangement of the set of dominoes may be involved as well; the definition is essentially contained in [CaLe, Algorithm 7.1]. In that definition, a specific reading order called “column reading” is used; after applying a coplactic operation to the word obtained by that reading, the tableau may cease to be a proper semistandard domino tableau, which is then repaired by



performing one or more transformations that reposition certain dominoes and their entries. We shall give an alternative description, analogous to the computation of coplactic operations for ordinary tableaux by image-glissements; it avoids the choice of a particular reading order. What we do need is a partial ordering to compare locations of dominoes, analogous to ' $<_{\swarrow}$ ' for comparing locations of squares. We shall first define two relations analogous to the refinements of ' $<_{\swarrow}$ ' that hold between successive squares with equal entries in a semistandard tableau, respectively between successive squares with equal ordinates, namely the *horizontal ordering* ( $x <_{\swarrow} y$  and  $x, y$  do not lie in the same column), respectively the *vertical ordering* ( $x <_{\swarrow} y$  and  $x, y$  do not lie in the same row). For dominoes  $x, y$ , we define the horizontal ordering  $x <_h y$  to hold whenever a semistandard domino tableau exists consisting of the dominoes  $x, y$  with equal entries, and  $\text{pos}(x) < \text{pos}(y)$ ; the vertical ordering ' $<_{\swarrow}$ ' is defined by transposition:  $x <_{\swarrow} y \iff y^t <_h x^t$ . When  $x <_h y$  or  $x <_{\swarrow} y$  holds, we shall write  $x <_{\swarrow} y$ .

Since we shall use image-glissements to define coplactic operations on semistandard domino tableaux, we change the order of presentation, with respect to what we did for words and ordinary tableaux, of the concepts of coplactic operations, the quantities  $n_i^+(D)$  and  $n_i^-(D)$ , compatibility of  $\lambda/\mu$  with  $D$ , and the augmentation of  $D$  with ordinates. Nonetheless, the relations between these concepts will be the same as before, e.g.,  $n_i^+(D)$  tells how often  $e_i$  can be successively applied to  $D$ . We start with defining what a valid augmentation of  $D$  with ordinates corresponding to  $\lambda/\mu$  is, and thus when  $\lambda/\mu$  is compatible with  $D$ , which determines the values of  $n_i^+(D)$  and  $n_i^-(D)$ . Then we shall define image-glissements of such augmented tableaux, and their translation into coplactic operations, from which the intended interpretations of  $n_i^+(D)$  and  $n_i^-(D)$  follow.

**3.3.1. Definition.** Let  $D$  be a semistandard domino tableau, and  $\lambda, \mu \in \mathcal{P}$  such that  $\lambda = \mu + \text{wt}(D)$ . The augmentation of  $D$  for  $\lambda/\mu$  is obtained by attaching for each  $i$  the distinct numbers  $j$  with  $\mu_i \leq j < \lambda_i$  as ordinates to the entries  $i$ , so that they increase from right to left. This augmentation is valid if for each  $j$  the set of dominoes with ordinate  $j$  is totally ordered by ' $<_{\swarrow}$ ', and their entries increase from top to bottom; in this case  $\lambda/\mu$  and  $D$  are called compatible. An augmented domino tableau is a semistandard domino tableau provided with a valid augmentation. If  $\text{wt}(D)/\emptyset$  is compatible with  $D$ , then  $D$  is called a Yamanouchi domino tableau, and the augmentation for  $\text{wt}(D)/\emptyset$  its Yamanouchi augmentation.

For instance, here is a domino tableau  $D$  with a valid augmentation for  $\lambda/\mu = (7, 6, 5, 4)/(3, 1, 0, 0)$ .

$$D = \begin{array}{ccccccc} & & & & & & 1_1 \\ & & & & & & 2_0 \\ & & & 0_4 & 0_3 & & \\ & & 0_5 & 1_3 & 1_2 & & \\ & & 1_4 & 2_2 & 2_1 & 3_0 & \\ 0_6 & 1_5 & 2_4 & 2_3 & & 3_1 & \\ 3_3 & 3_2 & & & & & \end{array}$$

Applying an image-glissement to an augmented domino tableau simulates the same operation for ordinary tableaux, as long as all comparisons of domino locations are possible using ' $<_{\swarrow}$ '. This is illustrated by the image-glissement into  $(0, 2)$  of the tableau  $D$  depicted above. The path of the slide is  $0_2 \rightarrow 0_3 \rightarrow 0_4 \rightarrow 0_5 \rightarrow 1_5$ ; the comparisons involved are  $1_2 <_{\swarrow} 0_3$ ,  $1_3 <_{\swarrow} 0_4$ ,  $1_4 <_{\swarrow} 0_5$ , and  $0_6 <_{\swarrow} 1_5$ . The slide corresponds to an application of  $e_0$  affecting the entry  $1_5$  of  $D$ , and results in the following augmented domino tableau.

$$e_0(D) = \begin{array}{ccccccc} & & & & & & 1_1 \\ & & & & & & 2_0 \\ & & & 0_3 & 0_2 & & \\ & & 0_4 & 1_3 & 1_2 & & \\ & & 1_4 & 2_2 & 2_1 & 3_0 & \\ 0_6 & 0_5 & 2_4 & 2_3 & & 3_1 & \\ 3_3 & 3_2 & & & & & \end{array}$$

At some point in the computation of an image-glissement, it may happen however that the partial ordering ' $<_{\swarrow}$ ' does not provide an answer to a required comparison. This is illustrated by the image-glissement of  $D$  into  $(1, 0)$ . The beginning of the slide gives no problems: the comparisons  $2_0 <_{\swarrow} 1_1$ ,  $2_1 <_{\swarrow} 1_2$ , and  $2_2 <_{\swarrow} 1_3$  determine the initial part of the path:  $1_0 \rightarrow 1_1 \rightarrow 1_2 \rightarrow 1_3$ . At that point a comparison is needed between the dominoes  $1_4$  and  $2_3$ , but these are incomparable by ' $<_{\swarrow}$ '. This

indicates that a rearrangement of dominoes, among which the incomparable ones, is required; in the current case this means making the following replacement:

$$\begin{array}{|c|c|} \hline 1_4 & \\ \hline 2_4 & 2_3 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline 1_4 & 1_3 \\ \hline 2_3 & \\ \hline \end{array}.$$

After this, the slide is complete, because there is no entry  $2_5$  or  $3_4$  to change into  $2_4$ ; the result is

$$e_1(D) = \begin{array}{cccccccc} & & & & & & & 1_0 \\ & & & & & & & 2_0 \\ & & & & 0_4 & 0_3 & & \\ & & & 0_5 & 1_2 & 1_1 & & \\ & & & 1_4 & 1_3 & 2_2 & 2_1 & 3_0 \\ & & & & & & 3_1 & \\ & 0_6 & 1_5 & & & & & \\ & & & & & & 2_3 & \\ & & & & & & 3_3 & 3_2 \\ & & & & & & & \end{array}$$

To complete the description of image-glissements for domino tableaux, we consider all configurations that can arise when during an inward slide a comparison of dominoes cannot be made using ‘ $\prec_{\swarrow}$ ’. Let the incomparable dominoes be  $x$  containing  $i_{j+1}$  and  $y$  containing  $(i+1)_j$ . At this point we may ignore all  $i'_{j'}$  with  $i' < i$  or  $j' < j$ , which will not affect the remainder of the slide, and so assume that the incomparability occurs at the beginning of a slide into  $(i, j)$ . The incomparability of  $x$  and  $y$  implies that there is a square not occupied by  $x$  or  $y$ , that is either to the right of  $x$  and above  $y$ , or below  $x$  and to the left of  $y$ ; this square is necessarily occupied by some other domino  $z$ . The former possibility would imply that  $z$  contains  $i_j$ , which contradicts the fact that we are sliding into  $(i, j)$ ; therefore the latter possibility applies,  $z$  contains  $(i+1)_{j+1}$ , and we have two possible configurations:

$$C_0: \begin{array}{|c|c|} \hline x & y \\ \hline z & \\ \hline \end{array} \qquad C_1: \begin{array}{|c|c|} \hline x & \\ \hline z & y \\ \hline \end{array}$$

Configuration  $C_0$  may need to be extended to the left in order to allow a rearrangement of dominoes. We include any horizontal dominoes directly to the left of  $z$  with entries  $i+1$ ; above each such domino containing  $(i+1)_{j'}$  (with  $j' \geq j+2$ ) there is another horizontal domino containing  $i_{j'}$ , which we include as well. The extended configuration will be called  $\overline{C}_0$ , and is rearranged as follows:

$$\overline{C}_0: \begin{array}{|c|c|c|c|c|} \hline i & \dots & i & i & i_+ \\ \hline i_+ & \dots & i_+ & i_+ & \\ \hline \end{array} \longrightarrow \overline{C}'_0: \begin{array}{|c|c|c|c|c|} \hline i & i & \dots & i & i \\ \hline i_+ & \dots & i_+ & i_+ & \\ \hline \end{array}.$$

The ordinates in  $\overline{C}'_0$ , which were omitted for readability, are as follows: the vertical domino has some ordinate  $k > j$ , and each pair of horizontal dominoes to its right has equal ordinates, decreasing from  $k-1$  at the left to  $j$  at the right. It may be verified that the replacement cannot violate the definition a of semistandard domino tableau; in the case under consideration, the application of  $e_i$  consists of this replacement. When it has been done, the image-glissement continues as before, with  $(i+1, k)$  as empty square. For a configuration  $C_1$  the situation is symmetric to that of  $C_0$ , under reflection in an anti-diagonal and interchange of the rôles of entries and ordinates; this means that  $C_1$  may need to be extended downwards by additional pairs of vertical dominoes, and then the extended configuration will be rearranged as a whole. The (equal) entries of the lowest pair of dominoes in the extended configuration are some  $k > i$ , and the replacement corresponds to an application of  $e_{k-1} \circ \dots \circ e_i$ . If, for the purpose of defining  $e_i$ , we choose  $\lambda/\mu$  so that it singles out one raising operation, then no extension of  $C_1$  occurs, and we have the rearrangement

$$C_1: \begin{array}{|c|c|} \hline i & \\ \hline i_+ & i_+ \\ \hline \end{array} \longrightarrow C'_1: \begin{array}{|c|c|} \hline i & i \\ \hline i_+ & \\ \hline \end{array}$$

### 3.4. The bijection of Carré and Leclerc.

Let us relate the definition of coplactic operations above to the original one, which uses the word  $w$  obtained by column reading a domino semistandard domino tableau  $D$ . It can be seen that any  $\lambda/\mu$

is compatible with  $D$  if and only if it is compatible with  $w$ ; consequently,  $w$  can be used to determine  $n_i^+(D)$ ,  $n_i^-(D)$ , and in particular whether  $D$  is a Yamanouchi domino tableau. If no incomparabilities arise during an image-glissement of  $D$ , then the corresponding image-glissement of  $w$  proceeds identically, so the definitions of coplactic operations agree in those cases. If on the other hand a configuration  $C_0$  or  $C_1$  arises in an image-glissement of  $D$ , then the change corresponding to applying the same image-glissement to  $w$  will be to change the entry of the domino  $y$  from  $i + 1$  into  $i$ , after which  $D$  is no longer a semistandard domino tableau. The method used to repair this in [CaLe, Algorithm 7.1] is to apply one or more transformations  $R_2$  in case  $C_0$ , or one transformation  $R_1$  in case  $C_1$ ; the total effect is exactly a replacement  $\bar{C}_0 \rightarrow \bar{C}'_0$  or  $C_1 \rightarrow C'_1$  as indicated above, so the definitions agree here as well.

Besides giving a procedure that repeatedly applies coplactic operations to a domino tableau  $D$  until a Yamanouchi domino tableau  $Y$  is obtained, [CaLe, Algorithm 7.1] also constructs an ordinary semistandard tableau  $t$ , based on  $\text{wt}(D)$  and the sequence of coplactic operations being applied; the construction is identical to that of  $P$  in Robinson's algorithm. Like in that case, the question arises whether this tableau is always well defined, and if so whether the correspondence  $D \mapsto (Y, t)$  is bijective, as claimed in [CaLe, Theorem 7.3]. By what we have seen above, this would follow if one could indicate a word  $w$  (or an ordinary tableau) with a coplactic graph isomorphic to that of  $D$ , and  $\text{wt}(w) = \text{wt}(D)$ ; also,  $t$  could then be identified with the  $P$ -tableau of  $w$ , by proposition 3.2.2. An obvious candidate for  $w$  is the column reading of  $D$ ; however its coplactic graph is not always isomorphic to that of  $D$ : for

$$D = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline & 3 \\ \hline \end{array} \quad \xrightarrow{e_1 \circ e_0} \quad Y = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & 1 \\ \hline 1 & 2 \\ \hline & 3 \\ \hline \end{array} :$$

the Yamanouchi domino tableau  $Y$  has weight  $(2, 2, 1, 1)$ , whereas the column reading  $w = 210321$  of  $D$  leads to the Yamanouchi word  $e_2(e_1(e_0(w))) = 210210$  with weight  $(2, 2, 2)$ . This discrepancy can arise because rearrangement of dominoes causes  $e_i(w)$  to differ from the column reading of  $e_i(D)$ ; therefore, while the column reading provides correct information about the quantities  $n_i^+(D)$  and  $n_i^-(D)$  describing the local structure of the coplactic graph around the distinguished vertex, it fails to do so globally.

These matters are not addressed at all in [CaLe], so the proof of its Theorem 7.3 is deficient in much the same way as Robinson's claims have remained unjustified for a long time. The proof of our theorem 2.1.7 will provide a justification, by giving a direct construction of an ordinary tableau whose coplactic graph is isomorphic to that of  $D$ . In fact the proof of theorem 2.1.7 consists precisely in establishing the isomorphism of the coplactic graph of  $D$  with those of the components  $T, U$  of the corresponding self-switching tableau pair, by showing that the correspondence commutes with coplactic operations; as remarked above, this allows the tableau  $t$  computed by [CaLe, Algorithm 7.1] to be identified with  $\pi(D)$ . Our theorem 2.2.6 associates to  $D$  another ordinary tableau  $T_0 \uplus T_1$  with an isomorphic coplactic graph; again the proof consists of showing that the construction (in this case moving open chains) commutes with coplactic operations. One may define plactic equivalence of semistandard domino tableaux  $D, D'$  by the condition  $\pi(D) = \pi(D')$ , which implies isomorphism of their coplactic graphs; then the operation of moving open chains, which preserves plactic equivalence, is a domino analogue of jeu de taquin. By contrast we note that the construction of [StWhi] does not preserve plactic equivalence. An interesting question that we shall not discuss here is whether moving open chains is sufficient to completely generate this plactic equivalence.

*Remark.* Despite these two constructions of ordinary tableaux with the same coplactic graph as a given domino tableau  $D$ , it would be convenient to have a direct method to find a word with this property, by reading off the entries of  $D$  using some other ordering criterion than column reading (whose definition after all does not seem very natural). We have however not been able to find such a criterion. It is not easy to understand which properties would make a reading order good; some orderings that are compatible with ' $\prec_{\swarrow}$ ' even fail to provide correct local information, such as the row reading (defined analogously to the column reading), which associates the Yamanouchi word 111000 to the non-Yamanouchi domino tableau

$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline & 1 & 1 \\ \hline 1 & & \\ \hline \end{array} .$$

A property that appears to be desirable, and that is always satisfied by at least one reading order (but not always by the column reading), is that whenever two dominoes are adjacent along an edge, then

depending on the orientation of that edge, the left one is read before the right one or the bottom one before the top one. This property can be shown to guarantee, like column reading, that the reading gives the correct values for the local quantities  $n_i^+(D)$  and  $n_i^-(D)$ . Moreover, it forces the tableau  $D$  displayed above to be read as 232101, which has the correct coplactic graph. The property is not sufficient to guarantee this in all cases however: for

0		1	1
1		1	1
	2	2	2
	2	3	

it allows 6 different readings, of which *only* 22321011 has the correct coplactic graph. We do not know whether there is always even at least one reading with this property that has the correct coplactic graph.

#### §4. More about augmented domino tableaux.

As a preliminary to the proofs that the transition from domino tableaux to self-switching tableau pairs and moving open chains in domino tableaux both commute with coplactic operations, we shall show in this section that they are compatible with augmented domino tableaux in the following ways. Firstly, if one identifies individual entries of a semistandard domino tableau before and after these constructions by attaching their ordinates, then the augmentations obtained afterwards will still be valid, and (since the set of entry-ordinate combinations is unchanged) will correspond to the same  $\lambda/\mu$  as before; in particular the Yamanouchi property will be preserved. Secondly, by putting certain orderings on the set of all entry-ordinate combinations, more standard domino tableaux can be associated to an augmented domino tableau than just the standardisation of the semistandard domino tableau; for each such ordering the mentioned operations give the same result for the standard domino tableau as for the semistandard domino tableau. This all amounts to the statement that these operations are well defined and natural for the domino analogues of pictures; we shall not however formally introduce a concept of domino pictures to replace augmented domino tableaux, as such a geometric language is more likely to be confusing than helpful. In passing we shall also complete the proof of theorem 2.3.2.

##### 4.1. Moving chains in domino pictures.

We first consider the operation of moving chains in a domino tableau. Let an augmented domino tableau  $D$  be given, and a chain  $C$  for  $s \in \{s_0, s_1\}$  in  $D$ . If  $C$  is not a forbidden chain, then [vLee3, proposition 4.3.1] tells us that it can be moved in  $D$ , resulting in another semistandard domino tableau; we shall now study the additional requirement that the augmentation retains its validity. So assume that  $C$  is not a forbidden chain, and let  $x, y \in \text{Dom}(D)$  be dominoes containing respectively  $(i+1)_j$  and  $i_j$ ; then  $x <_v y$ , and the question is whether the corresponding dominoes  $x'$  and  $y'$  obtained after moving  $C$  satisfy  $x' <_v y'$ . We may assume that at least one of  $x$  and  $y$  occurs in  $C$ . Let  $x_0$  and  $y_0$  be the squares of  $x$  and  $y$  respectively, that are fixed for  $s$  (so we also have  $x_0 \in x'$  and  $y_0 \in y'$ ). We have  $\text{pos}(x_0) \leq \text{pos}(y_0)$ , and if this inequality is strict, then  $x' <_v y'$  always holds; this needs special argumentation only when  $x_0$  and  $y_0$  lie two rows apart in the same column: the fixed squares to the left and right of the square in between  $x_0$  and  $y_0$  cannot contain  $(i+1)_{j+1}$  respectively  $i_{j-1}$ , lest both  $x$  and  $y$  be part of a forbidden chain. If on the other hand  $\text{pos}(x_0) = \text{pos}(y_0)$ , then we have the following configuration:

$y$
$x$

 ;

the squares  $x_0$  and  $y_0$  lie on the main diagonal of the  $2 \times 2$  square. Since  $(x, y) \neq (x', y')$  we have  $\text{pos}(x') \geq \text{pos}(y')$ , whence  $x' <_v y'$  does not hold in this case. It can be seen moreover that the successor of  $x$  in its chain for  $s$  is  $y$ , unless the square to the right of  $y$  is part of a domino containing  $i_{j-1}$ , in which case there is another  $2 \times 2$  square like  $x \cup y$  to the right of it, containing  $i_{j-1}$  and  $(i+1)_{j-1}$ . Similarly,  $x$  is the successor of  $y$ , unless there is a similar  $2 \times 2$  square containing  $i_{j+1}$  and  $(i+1)_{j+1}$  to the left of  $x \cup y$ . Therefore  $C$  is a closed chain containing both  $x$  and  $y$ , and all of its dominoes are contained in similar pairs. The following definition covers both such pairs of dominoes and the ones forming forbidden chains.

**4.1.1. Definition.** Let  $D$  be an augmented domino tableau. A pair  $x, y \in \text{Dom}(D)$  is called a *blocked pair* for  $s \in \{s_0, s_1\}$ , if  $x \cup y$  is a  $2 \times 2$  square whose main diagonal is fixed for  $s$ , and either the entries or the ordinates of  $x$  and  $y$  are equal. A chain in  $D$  for  $s$  is called *blocked* if one (and hence each) of its dominoes occurs in a blocked pair; if not, it is called *unblocked*.

**4.1.2. Proposition.** Let  $D$  be a semistandard domino tableau with a valid augmentation for  $\lambda/\mu$ , and  $C$  a chain in  $D$  for  $s \in \{s_0, s_1\}$ ; another semistandard domino tableau with a valid augmentation for  $\lambda/\mu$  can be obtained by moving  $C$  in  $D$  if and only if  $C$  is unblocked.  $\square$

Note that the definition of blocked pairs and blocked chains in  $D$  depends on the augmentation of  $D$ . However, open chains are never blocked, so we have obtained in particular a proof of proposition 2.2.5. Since movement of chains disjoint from  $C$  does not affect whether or not  $C$  is blocked, we have also proved lemma 2.3.1. Having characterised the immobile chains in an augmented domino tableau, we also have the necessary ingredients for the proof of theorem 2.3.2, but we postpone that proof to a later subsection, and for now concentrate on chains that do move.

In an augmented domino tableau  $D$ , each combination of an entry and an ordinate is unique, which allows us to view  $D$  as a standard domino tableau, if we provide a proper ordering on the set of all such combinations. The usual ordering is to consider  $i_j < i'_j$  when either  $i < i'$ , or  $i = i'$  and  $j > j'$ ; what this gives us, as far as the chain in  $(\mathcal{P}, \leq_2)$  is concerned, is just the standardisation of  $D$ . However, other total orderings ' $\prec$ ' will produce standard domino tableaux as well, which correspond to different chains in  $(\mathcal{P}, \leq_2)$ ; the only requirement is that  $i_j \prec i'_j$  whenever at least one of the squares of the domino containing  $i_j$  in  $D$  lies inwards of the domino containing  $i'_j$  in  $D$ . Now entries increase weakly along rows and strictly down columns, and ordinates can easily be seen to decrease strictly along rows and weakly down columns, so in the indicated situation we certainly have  $i \leq i'$  and  $j \geq j'$ . These two inequalities define a partial ordering  $i_j \leq_{\succ} i'_j$  between entry-ordinate combinations (the notation is inspired by the fact that  $i_j \leq_{\succ} i'_j$  is equivalent to  $(i', j') \leq_{\prec} (i, j)$  in the notation of [vLee2]); we shall call a total ordering ' $\prec$ ' on entry-ordinate combinations ' $\leq_{\succ}$ -compatible' if  $i_j \leq_{\succ} i'_j$  implies  $i_j \prec i'_j$ . Then each ' $\leq_{\succ}$ -compatible' ordering ' $\prec$ ' on the set of entry-ordinate combinations occurring in  $D$  makes  $D$  into a standard domino tableau; we shall call that standard domino tableau the *specialisation* of  $D$  for ' $\prec$ '. Now if an operation on an augmented domino tableau preserves the validity of the augmentation, then one may ask whether the specialisations for ' $\prec$ ' before and after the operation are related by the same operation; if this is the case for all ' $\leq_{\succ}$ -compatible' orderings ' $\prec$ ', then we call the operation *natural* (in analogy of the terminology for pictures).

**4.1.3. Proposition.** Moving a unblocked chain in an augmented domino tableau is natural.

*Proof.* One easily shows that the fact that two standard domino tableaux  $S, S'$  of respective shapes  $\lambda/\mu$  and  $\lambda'/\mu'$  are related by movement of a number of chains for  $s$  can be characterised as follows: the sets of squares of  $\lambda$  and  $\lambda'$  that are fixed for  $s$  are the same, similarly for  $\mu$  and  $\mu'$ , and for each square  $x \in \lambda \setminus \mu$  that is fixed for  $s$ , the dominoes of  $S$  and  $S'$  that contain  $x$  have equal entries. Now if two augmented domino tableaux  $D, D'$  are related by the movement of a unblocked chain  $C$ , then clearly their specialisations for any ' $\leq_{\succ}$ -compatible' ordering ' $\prec$ ' satisfy this characterisation, and are therefore related by movement of some chains for  $s$ ; by comparing their dominoes one sees that they are in fact related by moving a single chain, which is  $C$ .  $\square$

*Remark.* The essential point of the proof is that  $C$  is a chain for any specialisation of  $D$  (which fails in general for blocked chains), and it is deduced here from the fact that the specialisation of  $D'$  is well defined as a standard domino tableau. The ' $\leq_{\succ}$ -compatibility' of ' $\prec$ ' is not really used; any other ordering that happens to make both  $D$  and  $D'$  into standard domino tableaux would do as well.

## 4.2. Self-switching tableau pairs.

We shall now consider similar questions for the operation of proposition 2.1.5, passing from domino tableaux to self-switching tableau pairs. We shall need the concepts of augmentation and specialisation for ordinary semistandard tableaux, which are entirely analogous to those for semistandard domino tableaux. The basic property of a valid augmentation is that whenever two squares  $x, y$  contain respectively  $(i+1)_j$  and  $i_j$ , then  $x$  lies in a row strictly below and in a column weakly to the left of  $y$ ; like for dominoes we write  $x <_{\vee} y$  in this case, and  $x <_{\text{h}} y$  is also defined similarly. As was mentioned before, augmented ordinary tableaux are essentially the same as pictures, so that their theory can be applied.

**4.2.1. Proposition.** *If a semistandard domino tableau  $D$  has a valid augmentation for  $\lambda/\mu$ , then so do both components of the self-switching tableau pair  $(T, U)$  associated to  $D$  in proposition 2.1.5.*

*Proof.* It will be sufficient to show that for every pair of dominoes  $x, y$  containing  $(i+1)_j$  and  $i_j$  respectively, the squares  $\bar{x}, \bar{y}$  containing those values in  $T$  will satisfy  $\bar{x} <_v \bar{y}$ ; the similar statement for  $U$  follows from it by the symmetry between inward and outward jeu de taquin slides. In fact it is sufficient to show that  $\bar{x}$  lies weakly to the left of  $\bar{y}$ , as the monotonicity conditions for the entries of  $T$  will then imply that it also lies below  $\bar{y}$ . Assume first that the only entries occurring in  $D$  are  $i$  and  $i+1$ ; we use the description of how  $T$  can be determined from  $D$  by jeu de taquin slides given after proposition 2.1.5, ignoring the part that constructs  $U$ . We claim that the square containing  $(i+1)_j$  after all dominoes with entry  $i+1$  have been processed, lies weakly to the left of the final location  $\bar{y}$  of  $i_j$ ; this will be sufficient, since processing the dominoes with entries  $i$  can only move  $(i+1)_j$  further to the left. If  $y$  is either a vertical domino or a horizontal one without any further horizontal dominoes directly to its left, then  $\bar{y}$  is the inward square of  $y$ , so that our claim follows from  $x <_v y$ . Otherwise, if the inward square of  $x$  does not lie both one row below  $y$ , and strictly to the right of the leftmost square in a horizontal domino in the row of  $y$ , then our claim also follows because  $x$  itself lies far enough to the left. In the remaining case we may assume by induction that the claim holds for  $(i+1)_{j+1}$  and  $i_{j+1}$  instead of  $(i+1)_j$  and  $i_j$ ; the claim then follows because at the indicated points of the construction,  $(i+1)_j$  and  $i_j$  are the right neighbours of respectively  $(i+1)_{j+1}$  and  $i_{j+1}$ . Returning to the case of arbitrary entries, this means that, after all dominoes with entries  $i+1$  and  $i$  are processed, we have a valid augmentation for the subtableau consisting of those entries. Since jeu de taquin slides preserve the validity of this augmentation (i.e., they are defined for pictures, [vLee2, Theorem 5.3.1]), we are done.  $\square$

It is convenient to combine the proof of the converse statement with that of the naturality of the bijection of proposition 2.1.5. To that end we first consider an instance  $X(T, U) = (U', T')$  of tableau switching for standard tableaux, and use the characterisation of definition 2.1.2(1). Assume that there are two consecutive entries  $p, q$  of  $U$  such that if one changes their relative ordering, while retaining all other ordering relations among entries,  $U$  is still a standard tableau (this means that the squares of  $p$  and  $q$  are not adjacent in  $U$ ); assume moreover that this remains the case for the tableaux obtained after each of the inward jeu de taquin slides that eventually transform  $U$  into  $U'$ . The latter assumption is equivalent to saying that  $p$  and  $q$  are never compared to each other in the process of performing those jeu de taquin slides. Interchanging  $p$  and  $q$  will change exactly one partition of the chain in  $(\mathcal{P}, \subseteq)$  corresponding to  $U$ , namely the one that includes  $p$  but not  $q$ . If the corresponding partition of the jeu de taquin family of 2.1.2(0) is  $\lambda^{[i, j_0]}$ , then the change to that family caused by interchanging  $p$  and  $q$  affects all partitions  $\lambda^{[i, j_0]}$  for  $k \leq i \leq l$ , and no other ones. For the transformation of  $T$  into  $T'$  of 2.1.2(2), this means that the slides into the squares of  $p$  and  $q$  in  $U$  commute with each other.

Now assume in addition that  $(T, U)$  is a self-switching tableau pair corresponding to a standard domino tableau  $D$ , so one has  $U' = T$  and  $T' = U$ . From the symmetry of the jeu de taquin family it follows that the slides into the squares of  $p$  and  $q$  in  $T$  during the transformation of  $U$  into  $U'$  commute with each other. After interchanging the ordering of  $p$  and  $q$  in  $U$ , the pair is no longer self-switching ( $T$  no longer equals the modified  $U'$ ), but if the mentioned slides into the squares of  $p$  and  $q$  in  $T$  still commute after the change to  $U$ , then after interchanging the ordering of  $p$  and  $q$  in  $T$  as well, the symmetry is restored, and one will again have a self-switching tableau pair. Whether this is the case depends only on the jeu de taquin subfamily of  $(\lambda^{[i, j]})$  for  $j_0 - 1 \leq i, j \leq j_0 + 1$ : commutation holds if and only if the dominoes  $\lambda^{[j_0, j_0]} \setminus \lambda^{[j_0-1, j_0-1]}$  and  $\lambda^{[j_0+1, j_0+1]} \setminus \lambda^{[j_0, j_0]}$  are non-adjacent. If they are, the new self-switching tableau pair will correspond to  $D$ , with the ordering of  $p$  and  $q$  interchanged.

We can apply this to the situation where  $T$  and  $U$  are specialisations for some ' $\leq_{\triangleright}$ '-compatible ordering ' $\prec$ ' of the components  $\hat{T}, \hat{U}$  of a self-switching tableau pair, both of which are augmented for  $\lambda/\mu$ . In an augmented tableau, no values  $i_j$  and  $i'_j$  that are incomparable for ' $\leq_{\triangleright}$ ' can ever share a row or column, so any such pair that is consecutive for ' $\prec$ ' can be taken as  $p$  and  $q$  above. The fact that  $i_j$  and  $i'_j$  cannot be compared to each other in *any* sequence of jeu de taquin slides applied to  $U$ , means by symmetry that the slides into the squares of  $i_j$  and  $i'_j$  in  $T$  will commute, regardless of any modifications to  $U$ . Assuming that we already know for the augmentation for  $\lambda/\mu$  of the semistandard domino tableau  $D$  corresponding to  $(\hat{T}, \hat{U})$  that its specialisation for ' $\prec$ ' is the standard domino tableau corresponding to  $(T, U)$ , we therefore find that the same is true for the specialisations of  $D, \hat{T}$ , and  $\hat{U}$  for the ordering obtained by interchanging the order of  $i_j$  and  $i'_j$  in ' $\prec$ '. Starting with the ordering that gives the standardisation, where this relation holds by definition, one deduces that it holds for all

' $\leq_{\succ}$ '-compatible ' $\prec$ '. A subtle point in the reasoning above is that we have not yet shown that the augmentation of  $D$  for  $\lambda/\mu$  is valid (we shall do so presently); nevertheless, all these specialisations of  $D$  are proper standard domino tableaux.

**4.2.2. Proposition.** *Let  $T, U$  be augmented tableaux for  $\lambda/\mu$ , such that  $(T, U)$  is a self-switching tableau pair, and let  $D$  be the semistandard domino tableau corresponding to it by proposition 2.1.5. Then  $D$  has a valid augmentation for  $\lambda/\mu$ , and for any ' $\leq_{\succ}$ '-compatible ordering ' $\prec$ ', the specialisation for ' $\prec$ ' of  $D$  corresponds to the pair of specialisations for ' $\prec$ ' of  $T$  and  $U$ .*

*Proof.* The only point left to prove is that  $D$  has a valid augmentation for  $\lambda/\mu$ . Consider a pair of dominoes  $x, y$  in  $D$  containing  $(i+1)_j$  and  $i_j$  respectively; we must prove that  $x <_v y$ . We may relate  $D$  to  $(T, U)$  using specialisations for a ' $\leq_{\succ}$ '-compatible ordering ' $\prec$ ' for which  $i_j$  and  $(i+1)_j$  are successive (for instance  $i_j \prec i'_j$ , when either  $j > j'$  or  $j = j'$  and  $i < i'$ ); then using the same argument as used in the proof of proposition 2.1.5 to show that the bijection is well defined for semistandard tableaux, we deduce  $\text{pos}(x) < \text{pos}(y)$ , which implies  $x <_v y$ .  $\square$

### 4.3. Proof of theorem 2.3.2.

We now return to theorem 2.3.2, which claims to construct a bijection between Yamanouchi domino tableaux  $T \in \text{Yam}_2(\lambda, \mu)$ , and  $Y \in \text{Yam}_2(\mu^\square, \lambda)$  for which no (closed) chain for  $s_1$  can be moved without violating the Yamanouchi property; by proposition 4.1.2, this means  $\text{Dom}(Y)$  is a union of blocked pairs for  $s_1$  in the Yamanouchi augmentation of  $Y$ . The given construction is such that for each domino  $x$  containing  $i_j$  in the Yamanouchi augmentation of  $T$  it creates a blocked pair for  $s_1$  in the  $2 \times 2$  block in  $\mu^\square$  corresponding to the square  $(i, j) \in \mu$ , namely two vertical dominoes containing  $r_{c+1}$  and  $r_c$  if  $x = \{(r, c), (r, c+1)\}$ , or two horizontal dominoes containing  $(r+1)_c$  and  $r_c$  if  $x = \{(r, c), (r+1, c)\}$ . Since every blocked pair is of one of these forms, the construction is invertible at the level of individual dominoes of  $T$  and blocked pairs of  $Y$ ; what remains to be shown is that the requirements for an augmented domino tableau are preserved in both directions.

*Proof.* The verification is somewhat elaborate, but straightforward; we provide only a few details, leaving the remainder to the reader. Consider for instance two dominoes  $d, e$  of  $Y$  respectively containing  $r_{c+1}$  and  $r_c$ ; we must show that  $d <_h e$ , to ensure that  $Y$  will be a semistandard domino tableau. If  $d \cup e$  is one of the constructed  $2 \times 2$  blocks, this is immediate from the definition. Otherwise the squares  $(r, c)$  and  $(r, c+1)$  belong to different dominoes  $x, y$  of  $T$ , containing  $i_j$  and  $i'_j$ , say, with  $i \leq i'$  and  $j > j'$  because entries increase weakly and ordinates decrease strictly along rows of  $T$ . Since  $d$  and  $e$  are contained in the  $2 \times 2$  blocks of  $\mu^\square$  corresponding respectively to the squares  $(i', j')$  and  $(i, j)$  of  $\mu$ , the only remaining possibilities not to have  $d <_h e$  occur when  $i = i'$ , but these are also dismissed by the monotonicity conditions in  $T$ : for instance if  $d$  is horizontal in row  $2i$  and  $e$  is vertical in rows  $2i$  and  $2i+1$ , then one would have a configuration  $\begin{array}{|c|c|} \hline x & y \\ \hline \end{array}$ , which is impossible because the squares directly below  $x$  must have an entry both  $> i$  and  $\leq i$ . Similarly, for dominoes  $d', e'$  of  $Y$  containing  $(r+1)_c$  and  $r_c$ , the monotonicity conditions in  $T$  imply  $d' <_v e'$ . Conversely, it is easy to see that these monotonicity conditions are implied by the relations of the types  $d <_h e$  and  $d' <_v e'$  in  $Y$  considered here. Finally, one shows by similar means that monotonicity conditions in  $Y$  are equivalent to the proper horizontal and vertical orderings among dominoes with equal entries respectively equal ordinates in  $T$ .  $\square$

From the proof we can deduce a somewhat more general statement than theorem 2.3.2:

**4.3.1. Corollary.** *For any  $\lambda, \mu, \lambda', \mu' \in \mathcal{P}$  with  $\mu \subseteq \lambda$  and  $\mu' \subseteq \lambda'$ , there is a bijection between semistandard domino tableaux  $T$  of shape  $\lambda/\mu$  that are compatible with  $\lambda'/\mu'$ , and semistandard domino tableaux  $U$  of shape  $\lambda'^\square/\mu'^\square$ , that are compatible with  $\lambda/\mu$ , and all of whose chains for  $s_1$  are blocked. If  $T$  corresponds to  $U$  in this bijection, then  $U$  has twice as many horizontal dominoes as  $T$  has vertical dominoes, and similarly with 'horizontal' and 'vertical' interchanged.  $\square$*

In the particular case that  $\lambda/\mu$  is a horizontal strip, every semistandard domino tableau  $U$  of the right weight  $\lambda - \mu$  is compatible with  $\lambda/\mu$ ; moreover, the only type of blocked chains for such an augmentation are the forbidden ones. In this case the set of all  $U$  of the corollary is in bijection not only with the indicated set of domino tableaux  $T$ , but more directly (by shrinking each forbidden chain to a single square) with the set of ordinary semistandard tableaux of shape  $\mu$  and weight  $\frac{1}{2}(\lambda - \mu)$ ; it is the latter correspondence that was used in the derivation of equation (2).

### §5. Commutation theorems.

As was indicated in subsection 3.4, the proofs of theorems 2.1.7 and 2.2.6 are reduced to showing that constructions they refer to commute with coplactic operations, which implies that these theorems induce isomorphisms of coplactic graphs. In this section we shall state and prove these commutation theorems.

#### 5.1. Coplactic operations and self-switching pairs.

Before we can state that the bijection of proposition 2.1.5 commutes with the coplactic operations, we must define such operations on self-switching tableau pairs. This can be done by independently acting on both components of the pair.

**5.1.1. Proposition.** *Let  $g$  be a coplactic operation and  $(T, U)$  a self-switching tableau pair, then  $g$  can be applied to  $T$  if and only if it can be applied to  $U$ ; if so,  $(g(T), g(U))$  is a self-switching tableau pair.*

*Proof.* The first statement follows from proposition 3.2.3(2) and the fact that  $T$  and  $U$  are related by jeu de taquin. Now assume that  $g$  can be applied to  $T$  and  $U$ , then again by proposition 3.2.3(2) we have  $X(T, g(U)) = (g(T), U)$ , and by a similar argument  $X(g(T), g(U)) = (g(T), g(U))$ .  $\square$

Using proposition 2.1.5, these coplactic operations on self-switching tableau pairs can be translated to semistandard domino tableaux. The following theorem states that this implicit definition coincides with the explicit definition of coplactic operations on semistandard domino tableaux given earlier.

**5.1.2. Theorem.** *The bijection of proposition 2.1.5 commutes with the coplactic operations.*

*Proof.* We may restrict ourselves to applications of  $e_i$ . The proof is based on explicit computations for a number of basic domino tableaux, and a reduction of the case of a general domino tableau to that of a subtableau that contains all dominoes affected by  $e_i$ , and that is an instance of one of these basic cases. We start with presenting the explicit computations; we shall display self-switching tableau pairs using the same conventions as before:  $T$  is displayed in italics,  $U$  in boldface, and  $i + 1$  is abbreviated to  $i_+$ . The first case is the trivial one of a domino tableau with a single domino; we have either

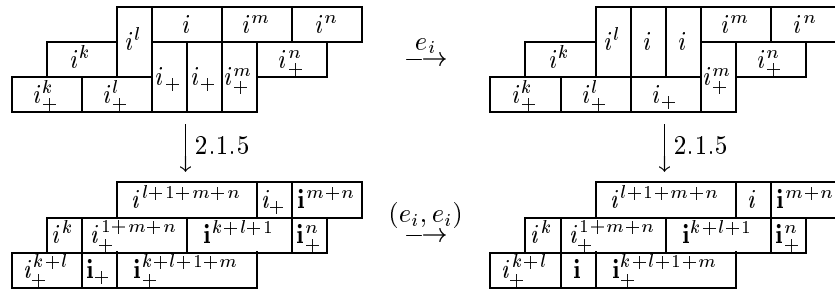
$$\begin{array}{ccc}
 \boxed{i_+} & \xrightarrow{e_i} & \boxed{i} \\
 \downarrow 2.1.5 & & \downarrow 2.1.5 \\
 \boxed{i_+} \boxed{i_+} & \xrightarrow{(e_i, e_i)} & \boxed{i} \boxed{i}
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ccc}
 \boxed{i_+} & \xrightarrow{e_i} & \boxed{i} \\
 \downarrow 2.1.5 & & \downarrow 2.1.5 \\
 \boxed{i_+} \\
 \boxed{i_+} & \xrightarrow{(e_i, e_i)} & \boxed{i} \\
 \boxed{i} & & \boxed{i}
 \end{array}$$

The next case is that of a tableau that is equal to a configuration  $\overline{C}_0$ , which is then transformed by  $e_i$  into  $\overline{C}'_0$ ; we obtain

$$\begin{array}{ccc}
 \begin{array}{cccccccc}
 i & i & . & . & . & i & & i_+ \\
 i_+ & i_+ & . & . & . & i_+ & & i_+
 \end{array} & \xrightarrow{e_i} & \begin{array}{cccccccc}
 i & i & . & . & . & i & i & \\
 i_+ & i_+ & . & . & . & i_+ & i_+ & 
 \end{array} \\
 \downarrow 2.1.5 & & \downarrow 2.1.5 \\
 \begin{array}{cccccccc}
 i & i & . & . & . & i_+ & \mathbf{i} & \mathbf{i} & . & . & . & \mathbf{i} \\
 i_+ & i_+ & . & . & . & i_+ & \mathbf{i}_+ & \mathbf{i}_+ & . & . & . & \mathbf{i}_+ & \mathbf{i}_+
 \end{array} & \xrightarrow{(e_i, e_i)} & \begin{array}{cccccccc}
 i & i & . & . & . & i & i & \mathbf{i} & . & . & . & \mathbf{i} & \mathbf{i} \\
 i_+ & . & . & . & i_+ & i_+ & \mathbf{i} & \mathbf{i}_+ & . & . & . & \mathbf{i}_+ & \mathbf{i}_+
 \end{array}
 \end{array}$$

The third and final case is more complicated. The application of  $e_i$  involves a transformation  $C_1 \rightarrow C'_1$  here, but for reasons that will become apparent below, we allow additional dominoes with entries  $i$  and  $i + 1$  subject to the following conditions: all dominoes with entry  $i$  to the right of  $C_1$  are horizontal and lie in the top row of  $C_1$ , all dominoes with entry  $i + 1$  to the left of  $C_1$  are horizontal and lie in the bottom row of  $C_1$ , and at each side of  $C_1$  there is an equal number of entries  $i$  and  $i + 1$ . The set of such domino tableaux can be parametrised by four parameters  $k, l, m, n \geq 0$  that control how often certain dominoes are repeated horizontally; indicating such repetition by attaching an exponent to the entry of the domino, the computation is as follows:





This diagram gives only a schematic indication of the shape of the tableaux involved, but note that in the self-switching tableau pairs there is exact vertical alignment at the two points where the illustration suggests it. Careful inspection shows that for all parameter values, each application of  $e_i$  in the bottom row of the diagram indeed affects the unique entry written without exponent.

We now proceed to reduce the case of a general semistandard domino tableau  $D$  to one of these special cases. By symmetry between inward and outward jeu de taquin slides, it is sufficient to consider only the first tableau  $T$  of each self-switching pair. The first step of the reduction is to dismiss all dominoes whose entries are not  $i$  or  $i + 1$ . Split each of  $D$  and  $T$  into three subtableaux, consisting of the dominoes with entries  $< i$ , in  $\{i, i + 1\}$ , and  $> i + 1$  respectively; then each such subtableau of  $T$  is obtained by jeu de taquin from the first component of the self-switching pair of the corresponding subtableau of  $D$ , of which only the one with entries in  $\{i, i + 1\}$  is affected by  $e_i$ . If the theorem holds for that subtableau of  $D$ , then proposition 3.2.3(2) allows us to conclude it for  $D$  itself: the first subtableau of  $T$  is unaffected by  $e_i$ , the second is obtained by a fixed sequence of jeu de taquin slides from a tableau to which  $e_i$  has been applied, and is therefore itself affected by  $e_i$ , and for the third subtableau it is the tableau determining the sequence of jeu de taquin slides that is affected by  $e_i$ , but this does not alter the result of that sequence of slides.

Assuming from now on that  $D$  has entries in  $\{i, i + 1\}$  only, the effect of applying of  $e_i$  to  $D$  is the same as that of applying it to a subtableau  $M$  of  $D$ , which is of one of the types for which we did an explicit computation. If applying  $e_i$  involves no rearrangement of dominoes, we take for  $M$  the domino whose entry changes; if a transformation  $\overline{C}_0 \rightarrow \overline{C}'_0$  is involved, we take  $M = \overline{C}_0$ ; if a transformation  $C_1 \rightarrow C'_1$  is involved, we take the for  $M$  largest subtableau of the type occurring in our final explicit computation. The problem in reducing the statement of the theorem from  $D$  to  $M$ , is that the chain in  $(\mathcal{P}, \leq_2)$  corresponding to the standardisation of  $M$  is not a subchain of the one corresponding to the standardisation of  $D$ , so we cannot immediately relate the tableau  $T$  computed from  $D$  to the one computed similarly from  $M$ . Proposition 4.2.2 allows us to work with different specialisations of  $D$  and  $T$ ; although this will not be sufficient to completely resolve our difficulty, it will bring us to a point where the proof can be completed with a few considerations specific to our situation.

Fix some  $\lambda/\mu$  compatible with  $D$  for which the application of  $e_i$  to  $D$  corresponds to an image-glissement of the associated augmentation of  $D$ , and view  $D$  correspondingly as an augmented domino tableau. Let the rightmost occurrence of  $i + 1$  within the subtableau  $M$  of  $D$  be  $(i + 1)_k$ , and let the leftmost occurrence of  $i$  in  $M$  be  $i_l$ ; the restriction of the image-glissement to  $M$  is a slide into  $(i, k)$  that ends in  $(i + 1, l)$ . The dominoes of  $D$  with ordinates  $> l$  form a subtableau  $L$  to the left of  $M$ , and the domino containing  $i_k$ , if it exists, together with all dominoes with ordinates  $< k$ , form a subtableau  $R$  to the right of  $M$ . Define the augmented domino tableau  $D'$  as the result applying the indicated image-glissement to  $D$ ; it has subtableaux  $L'$ ,  $M'$ , and  $R'$ , whose shapes coincide with those of  $L$ ,  $M$ , and  $R$ , respectively. Within  $R$  and  $L$  the effect of the image-glissement is to decrease the ordinates of all dominoes with entry  $i$  respectively  $i + 1$ , so as domino tableaux  $L$  and  $L'$  coincide, as do  $R$  and  $R'$ .

Define a total ordering ' $\prec_*$ ' on the entry-ordinate combinations occurring in  $D$  by putting  $x \prec_* y$  whenever  $x$  occurs in  $L$  and  $y$  in  $M$ , or  $x$  occurs in  $M$  and  $y$  in  $R$ ; when  $x$  and  $y$  occur in the same of these three subtableaux, then the ordering that gives the standardisation is used ( $x \prec_* y$  if either  $x$  has a smaller entry than  $y$  or they have equal entries and  $x$  has the larger ordinate). This ordering is *not* ' $\leq_{\succ}$ '-compatible if  $i_k$  occurs; define another ordering ' $\prec$ ' that differs from ' $\prec_*$ ' only if  $i_k$  occurs, and then only in the fact that  $i_k \prec (i + 1)_k$  (whereas  $(i + 1)_k \prec_* i_k$ ). Since ' $\prec$ ' is ' $\leq_{\succ}$ '-compatible, proposition 4.2.2 tells us that the specialisations of  $D$  and  $T$  for ' $\prec$ ' are related by the construction of proposition 2.1.5, just like  $D$  and  $T$  themselves are; we shall show below that the same is true if ' $\prec$ ' is replaced by ' $\prec_*$ '. In a similar fashion, we shall show that the relation between  $D'$  and the first factor  $T'$  of the self-switching tableau pair corresponding to it is retained when we pass to specialisations for an

ordering that puts the entry-ordinate combinations occurring in  $L'$  before the ones occurring in  $M'$ , and those before the ones occurring in  $R'$ , even though this ordering is not necessarily ' $\leq_{\succ}$ '-compatible. Once this is done, the proof is completed in a way similar to our earlier reduction the case of entries in  $\{i, i+1\}$  only. The tableaux  $T$  and  $T'$  can be provided with augmentations matching those of  $D$  and  $D'$  by proposition 4.2.1, and each is divided into three subtableaux as was done for  $D$  and  $D'$ . The first and last of these subtableaux of  $T'$  match those of  $T$  as semistandard tableaux (only the ordinates are changed), and the middle subtableau of  $T'$  is the result of applying  $e_i$  to the middle subtableau of  $T$ ; therefore the augmented domino tableau  $T'$  is an image-glissemment of the augmented domino tableau  $T$ , and we are done.

Let  $(T_*, U_*)$  be the self-switching tableau pair corresponding to the specialisation  $D_*$  of  $D$  for ' $\prec_*$ '; we must show that  $T_*$  coincides with the specialisation of  $T$  for ' $\prec_*$ '. Since the analogous statement for ' $\prec$ ' is true, we may assume that  $i_k$  occurs, and it suffices to study the effect of interchanging the ordering of  $i_k$  and  $(i+1)_k$ . Recall from the previous section that such a change will not affect the self-switching tableau pair  $(T_*, U_*)$ , provided that during the sequence of jeu de taquin slides that transform  $U_*$  into  $T_*$ , the entries  $i_k$  and  $(i+1)_k$  never lie in the same row or column, and that in  $D$  the dominoes with entries  $i_k$  and  $(i+1)_k$  are not adjacent. The latter property is easy to check from the way  $M$  was defined in each of the cases. To check the former property, we may trace  $i_k$  and  $(i+1)_k$  during the computation of  $T_*$  by inward jeu de taquin slides starting from  $D_*$ , and also during the analogous computation of  $U_*$  by outward slides; together this covers all relevant parts of the transformation of  $U_*$  into  $T_*$ . Because  $(i+1)_k \prec_* i_k$  and  $(i+1)_k$  lies below and to the left of  $i_k$  in  $D$ , it will suffice to show that  $i_k$  and  $(i+1)_k$  never lie in the same row. Now in all cases the inward square of the domino containing  $i_k$  in  $D$  lies in a row above any square of  $M$ , which makes it impossible for  $(i+1)_k$  to ever reach a row as high as  $i_k$  (this is the reason we extended the configuration  $C_1$  by all horizontal dominoes with entry  $i$  in its top row; if we had not done so, the properties being proved here would not hold). For the outward slides computing  $U_*$ , we shall also trace  $(i+1)_{k-1}$ , which exists because the image-glissemment changes  $i_k$  into  $i_{k-1}$ . Now  $(i+1)_{k-1}$  cannot move any lower than the outward square of the domino containing it in  $D$ , which is not lower than the outward square of the domino containing  $(i+1)_k$  in  $D$ ; therefore we can show that  $i_k$  stays in a row above  $(i+1)_k$  by showing that it stays in a row above the square containing  $(i+1)_{k-1}$  in  $U_*$ . For this we need to consider the subtableau  $R$  only, and by reassignment of ordinates we may equivalently consider the augmented domino tableau  $R'$ . The reassignment changes  $i_k$  into  $i_{k-1}$  and leaves  $(i+1)_{k-1}$  unchanged; proposition 4.2.1 ensures that in the result of the computation  $i_{k-1}$  lies in a row above  $(i+1)_{k-1}$ , as desired.

Finally, we must do a similar verification for a pair the augmented domino tableau  $D'$  instead of  $D$ , using a total ordering defined like ' $\prec_*$ ' but using  $L', M', R'$  instead of  $L, M, R$ , and a ' $\leq_{\succ}$ '-compatible ordering differing from it by the interchange of  $i_l$  and  $(i+1)_l$ . This proceeds in the same way as the previous verification, but with inward and outward slides interchanged; in particular, one uses the fact that outward square of the domino containing  $(i+1)_l$  in  $D'$  lies in a row below any square of  $M$ .  $\square$

What we have in fact shown is that the bijection of proposition 2.1.5, when applied to augmented domino tableaux (which is possible by proposition 4.2.1), commutes with those image-glissemments that correspond to application of a single coplactic operation. Using proposition 3.1.3, it follows that the same is true for arbitrary image-glissemments.

**5.1.3. Corollary.** *The bijection of proposition 2.1.5 applied to augmented domino tableaux commutes with image-glissemments.*  $\square$

## 5.2. Coplactic operations and moving open chains.

We have left to the end our most fundamental theorem, which is unfortunately also the most difficult one to prove. Unlike the previous theorem, which may be considered just as a verification that we chose the proper definition for coplactic operations on semistandard domino tableaux, we can see no *a priori* reason why one would expect a statement like the current theorem to hold at all, and its simple formulation hides some intricate combinatorial details, as witnessed by the numerous special cases that will turn up in the proof. From a combinatorial point of view, it is a small miracle that all the cases check out as they do. If some algebraic interpretation of the operation of moving open chains could be found, this might lead to more transparent explanation of the theorem, and a shorter proof.

**5.2.1. Theorem.** *Moving of open chains in domino tableaux commutes with coplactic operations.*

*Proof.* The identification of open chains before and after a coplactic operation is done by means of their starting and ending squares; the theorem claims in particular that the correspondence between starting and ending squares is not affected by coplactic operations. As before, we may take the coplactic operation to be  $e_i$ . The objects pertinent to the theorem will be named as follows:  $D$  is a semistandard domino tableau,  $D' = e_i(D)$ ,  $C$  is an open chain in  $D$  for  $s \in \{s_0, s_1\}$ ,  $C'$  is the open chain in  $D'$  for  $s$  starting in the same square as  $C$ , and  $E$  is the semistandard domino tableau obtained from  $D$  by moving  $C$ . We shall consider  $D$  to be an augmented domino tableau in such a way that the application of  $e_i$  to  $D$  corresponds to an image-glissement, and correspondingly also view  $D'$  and  $E$  as augmented domino tableaux.

First consider the case that  $C' = C$  (this does not require the entries of their dominoes to be the same in  $D'$  and  $D$ ). Clearly this implies that if application of  $e_i$  involves rearrangement of dominoes, then none of those dominoes appears in  $C$ . We shall prove that in this case the image-glissement of  $D$  and the corresponding image-glissement of  $E$  proceed in the same manner, i.e., they involve the same changes of entries and ordinates (although the dominoes containing them may have moved), and possibly an identical rearrangement of dominoes. We prove this by showing that the comparisons of locations of dominoes of  $D$  involved in the image-glissement are unchanged by moving  $C$ . The comparisons are for dominoes containing  $i_{j+1}$  and  $(i+1)_j$  in  $D$ , and we may assume them to be comparable by ' $<_{\swarrow}$ ', since otherwise they are rearranged by the image-glissement, and therefore not part of  $C$ . There are two possibilities to worry about: the corresponding dominoes in  $E$  may have become incomparable by ' $<_{\swarrow}$ ', or they may still be comparable, but with opposite ordering. The latter case is easily dismissed: reversal of the ordering by moving of a chain is only possible if that movement converts one of the following two configurations into the other:

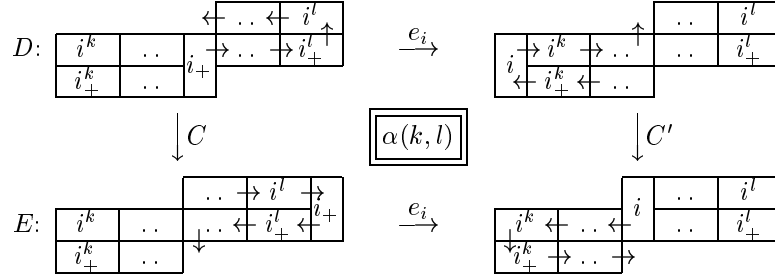
$$\begin{array}{|c|} \hline i \\ \hline i_+ \\ \hline \end{array} \longleftrightarrow \begin{array}{|c|c|} \hline i & i_+ \\ \hline \end{array},$$

but then these two dominoes form a closed chain, contradicting the fact that they occur in the open chain  $C$ . To dismiss the former case, we can use our classification of the situations where incomparability of dominoes can arise during an image-glissement: such dominoes occur as the ones marked  $x$  and  $y$  in the configurations  $C_0$  and  $C_1$  of subsection 3.3. Assuming that the dominoes containing  $i_{j+1}$  and  $(i+1)_j$  in  $E$  occur like that, then by moving a chain (reversing the movement of  $C$ ) they must be brought to locations that are comparable by ' $<_{\swarrow}$ '. In both  $C_0$  and  $C_1$  there are various possibilities for such moves, but in each case it turns out that applying  $e_i$  to  $D$  alters the successor relations for dominoes of  $C$ , contradicting the assumption that  $C' = C$ . For instance, in  $C_0$  the dominoes  $x$  and  $z$  might form a blocked pair for  $s$ , while the domino  $y$  in  $E$  was moved from a domino  $y'$  in  $D$  with  $x <_h y'$ ; in that case however, application of  $e_i$  to  $D$  changes the contents of  $y'$  from  $(i+1)_j$  to  $i_j$ , after which  $x$  is the successor of  $y'$  in its chain for  $s$ , so  $C' \neq C$ . The other possibilities are similar, and we leave them to the reader; we do not even list them, as they can be found among the cases with  $C' \neq C$  below.

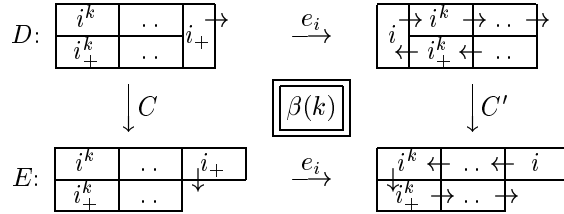
For the case  $C' \neq C$ , we start with displaying a number of special instances where the theorem can be shown to hold by an explicit computation, and then we shall show that whenever  $C' \neq C$ , one of these configurations occurs within  $D$  (which we take to mean also that the effects of moving the chains and applying  $e_i$  on the configuration inside  $D$  is as displayed). Like in the previous proof, we have some parametrised families of configurations, in which one or more parts can be repeated arbitrarily often. Even so, the number of different cases to be considered is quite large; we shall reduce their number in two ways. Firstly, the families are made to include the cases where a repeated part does not occur at all, whenever this makes sense; for instance, if application of  $e_i$  only changes the entry of a vertical domino, this is considered as a special case of the transformation  $\overline{C}_0 \rightarrow \overline{C}'_0$ . Secondly, we shall not separately consider configurations that can be obtained from each other by symmetries. There are two basic symmetries: one is called chain symmetry, and is obtained by reversal of the movement of  $C$ , which interchanges the rôles of  $D$  and  $E$ ; the other symmetry called negation symmetry is more complicated, and involves simultaneously rotating the tableaux by a half-turn, renumbering their entries in the opposite order so that  $i$  and  $i+1$  are interchanged, and interchanging  $D$  and  $D'$ . The composition of these two commuting symmetries will be called combined symmetry.

In displaying the special instances of the theorem, we indicate as before possible horizontal repetition of dominoes by writing their entries with an exponent, and moreover add dominoes without entries next to them, to better suggest the repetition. For clarity we add arrows to indicate the chains  $C$  and  $C'$ , and the ones corresponding to them after they are moved. When the tableaux displayed are viewed as configurations that can occur within some larger tableau, it is assumed that they cannot be extended to larger members of the same family; this assumption is used in determining the chains and the operation

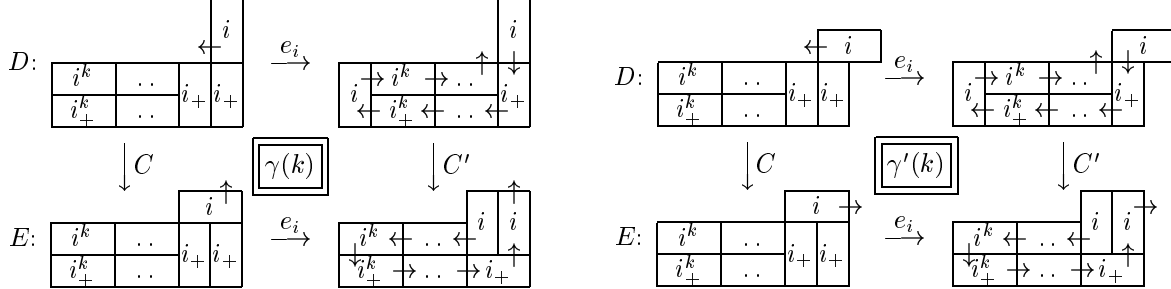
of  $e_i$ . The dominoes within each configuration that are not part of the indicated open chain are grouped into one or more closed chains for  $s$  (in fact blocked ones), so regardless of the context, they cannot be part of  $C$  and  $C'$ . We start with a family  $\alpha(k, l)$  with parameters  $k, l \geq 0$ ; the case  $k = l = 0$  is excluded because it has  $C = C'$ , but otherwise it is a valid instance of the theorem.



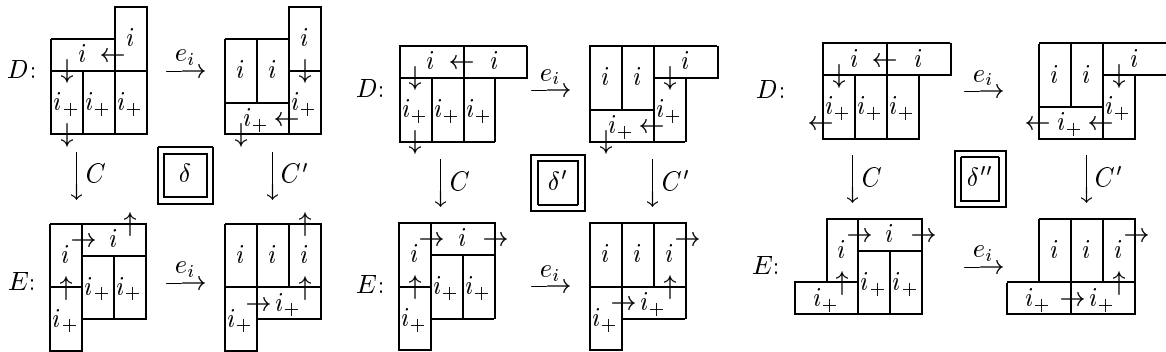
Note that under combined symmetry,  $\alpha(k, l)$  is mapped to  $\alpha(l, k)$ . The next family is  $\beta(k)$  with  $k > 0$ . It is a variation of  $\alpha(k, 0)$ , namely whereas in  $\alpha(k, 0)$  the chains  $C$  and  $C'$  end with upward arrows, they end with rightward arrows in  $\beta(k)$ ; this does not affect any essential characteristic of the configuration.



When  $\beta(k)$  occurs in a larger context, an assumption is made that is not explicitly visible, namely that the square immediately above and to the right of  $D$  does not belong to a domino with entry  $\geq i$ . The third family is  $\gamma(k)$  for  $k \geq 0$  (even  $\gamma(0)$  has  $C \neq C'$ ). Here both a transformation  $\overline{C}_0 \rightarrow \overline{C}'_0$  (for  $e_i$  applied to  $D$ ) and a transformation  $C_1 \rightarrow C'_1$  (for  $e_i$  applied to  $E$ ) occur. There is another very similar family  $\gamma'(k)$ , that differs from  $\gamma(k)$  only in the direction of the beginning of the chains  $C$  and  $C'$ ; we display the two variants side by side.



The final configurations have no parameters and involve transformations  $C_1 \rightarrow C'_1$  in the applications of  $e_i$  to both  $D$  and  $E$ , but at different (overlapping) places. Here the direction of the chains  $C$  and  $C'$  can be varied at both ends, so there are 4 variants of this configuration; the 3 configurations  $\delta$ ,  $\delta'$  and  $\delta''$  displayed below, and a fourth obtained by combined symmetry from  $\delta'$ .



To complete the proof, we show that  $C' \neq C$  implies that one of these configurations occurs within  $D$ . Distinguish the following cases that can arise in the application of  $e_i$  to  $D$ : case  $a$  is when the entry of a horizontal domino changes from  $i + 1$  to  $i$ ; case  $b_k$  (with  $k \geq 0$ ) is when there is a transformation  $\overline{C}_0 \rightarrow \overline{C}'_0$  with  $k$  pairs of horizontal dominoes in  $\overline{C}_0$  (so in case  $b_0$  the entry of a vertical domino changes from  $i + 1$  to  $i$ ); case  $c$  is when there is a transformation  $C_1 \rightarrow C'_1$ . In all cases the part of the domino tableau  $D$  that is altered by  $e_i$  is a rectangular region, and of its top-left and bottom-right corner squares exactly one is fixed for  $s$ ; by applying negation symmetry if necessary, we may assume that it is the top left square.

Tracing the successive dominoes of  $C$  and  $C'$ , there must be a first point of divergence. There are two possibilities: (1) at some point there is a domino of  $C$  that does not occur in  $\text{Dom}(D')$ , due to rearrangement of dominoes, (2) there is some domino  $x$  common to  $C$  and  $C'$  whose successor  $y$  in  $C$  does occur in  $\text{Dom}(D')$ , but is not the successor of  $x$  in  $C'$ . The successor of  $x$  is determined after comparing the entry of  $x$  with the entry of the domino containing the discriminant square of  $x$ . Therefore, possibility (2) requires that either the entry of  $x$  itself, or of the domino containing the discriminant square is changed by  $e_i$ , and in such a way that the result of the comparison changes. Now in each of the cases there is a unique square  $t_0$  that is fixed for  $s$  and for which application of  $e_i$  changes the entry of its domino: in case  $a$  this is the left square of the domino whose entry changes, in case  $b_k$  it is the top-right corner of  $\overline{C}_0$ , and in case  $c$  it is the right square of the middle row of  $C_1$ . Now  $t_0$  is the fixed-end square of a domino  $x_0 \in \text{Dom}(D)$  containing  $(i + 1)_j$ , say; let  $t_1$  be its discriminant square, and  $x_1 \in \text{Dom}(D)$  the domino containing  $t_1$ , if any (so  $t_1$  lies to the bottom-left of  $t_0$  in case  $a$ , and to the top-right of  $t_0$  otherwise). One easily checks that for possibility (2), the mentioned comparison must be the one between the entries of  $x_0$  and  $x_1$ , and the domino of  $t_1$  must contain  $(i + 1)_{j+1}$  in case  $a$ , and  $i_{j-1}$  otherwise. For case  $c$  however, even if this condition is met, the successor  $y$  of the domino of  $t_1$  in its chain in  $D$  for  $s$  is the top domino of  $C_1$ , so  $y \notin \text{Dom}(D')$ , and we have possibility (1) anyway.

In case  $a$  we must have possibility (2) as just described, and there must be another domino  $x_2$  containing  $i_{j+1}$ ; since  $x_1 <_v x_2 <_h x_0$  and  $x_0$  is horizontal, it must be that  $x_1$  and  $x_2$  are both also horizontal, and form a blocked pair for  $s$ . We already knew that either  $x_0$  or  $x_1$  occurs in  $C$ , and now clearly it must be  $x_0$ . By including any further blocked pairs for  $s$  with entries  $i$  and  $i + 1$  that occur to the left of  $x_1$  and  $x_2$ , we find an occurrence of a configuration obtained by chain symmetry from some  $\beta(k)$ . In case  $b_k$ , all horizontal dominoes of the configuration  $\overline{C}_0$  are part of a blocked pair for  $s$ . We distinguish whether  $x_0$  forms a blocked pair for  $s$  with a domino containing  $(i + 1)_{j-1}$  in  $D$ , or not. In the former case,  $C$  cannot contain any dominoes occurring in  $\overline{C}_0$ , so we must have possibility (2), and we find a configuration  $\gamma(k)$  or  $\gamma'(k)$ . In the latter case,  $t_1$  cannot be part of a vertical domino containing  $i_{j-1}$ , and if it is part of a horizontal domino containing  $i_{j-1}$ , then that domino forms a blocked pair for  $s$  with another horizontal domino containing  $(i + 1)_{j-1}$ ; after possibly including this blocked pair, and any further such blocked pairs for  $s$  with entries  $i$  and  $i + 1$  to its right, we find a configuration  $\beta(k)$  or  $\alpha(k, l)$  (we always have possibility (1) here). Finally, in case  $c$  we similarly distinguish whether  $x_0$  forms a blocked pair for  $s$  with a domino containing  $(i + 1)_{j-1}$  in  $D$ , or not. If it does,  $t_1$  must contain  $i_{j-1}$  and we have configuration  $\delta$  or one of its variants. Otherwise we include, in the same way as for case  $b_k$ , 0 or more blocked pairs for  $s$  with entries  $i$  and  $i + 1$  to the right of the two top rows of  $C_1$ , and find a configuration obtained by combined symmetry from some  $\gamma(l)$  or  $\gamma'(l)$ .  $\square$

Like for the previous theorem, we have in fact shown is that moving an open chain in an augmented domino tableau commutes with image-glissements that correspond to application of a single coplactic operation. Again proposition 3.1.3 allows us to generalise to arbitrary image-glissements.

**5.2.2. Corollary.** *Moving of open chains commutes with applying image-glissements.*  $\square$

We mention a consequence of our commutation theorems whose statement does not refer to coplactic operations at all. Attempts to find a more direct proof have so far been unsuccessful.

**5.2.3. Corollary.** *Let  $D$  be a domino tableau,  $C$  an open chain in  $D$ , and  $D'$  the domino tableau obtained by moving  $C$  in  $D$ ; let the self-switching tableau pairs corresponding to  $D$  and  $D'$  by proposition 2.1.5 be respectively  $(T, U)$  and  $(T', U')$ . Then  $T, T', U,$  and  $U'$  can all be obtained from one another by a sequence of jeu de taquin slides.*

*Proof.* By theorems 5.2.1, 5.1.2, and proposition 3.2.3(2), we can reduce to the case that  $D$  is a Yamanouchi domino tableau, of weight  $\lambda$ , say. But then  $T, T', U,$  and  $U'$  are all ordinary Yamanouchi tableaux of weight  $\lambda$ , and therefore related by jeu de taquin to the canonical domino tableau  $\mathbf{T}_\lambda$ .  $\square$

## §6. Concluding remarks.

We have settled some questions left open in [CaLe], clarifying somewhat the combinatorial context of its constructions. One other question that arises is why it is not possible to perform similar constructions using  $r$ -ribbon tableaux with  $r > 2$ . If one could define sets  $\text{Yam}_r(\lambda/\mu)$  of Yamanouchi  $r$ -ribbon tableaux such that the obvious generalisation of equation (1) holds, then this would give a combinatorial description of  $\psi^r(s_\lambda)$ . It would be even better if one could also define coplactic operations on semistandard  $r$ -ribbon tableaux such that the coplactic graphs are isomorphic to the ones we have considered; this would give a decomposition analogous to [CaLe, Theorem 7.3]. However, all our attempts to do either of these have failed. In fact none of the constructions of this paper generalise well to  $r$ -ribbon tableaux.

Proposition 2.1.5 cannot be generalised because tableau switching has no analogues with more than 2 tableaux. The definition of augmented domino tableaux has a straightforward generalisation to  $r$ -ribbon tableaux, but using that to define Yamanouchi  $r$ -ribbon tableaux would admit too few of them, e.g., there would be no Yamanouchi 3-ribbon tableau of shape  $(4, 4, 4)$  and weight  $(2, 2)$ . One might content oneself with defining Yamanouchi  $r$ -ribbon tableaux only in cases where the proper definition seems obvious, and postulate that moving open chains, which is defined in [vLee3] for general  $r$ -ribbon tableaux, should preserve the Yamanouchi condition, in order to extend the definition to more complicated cases. Examples of cases without complications are tableaux  $T$  with  $\text{Spin}(T) = 0$ , or whose ribbons are segregated into  $r$  disconnected subtableaux, similarly to segregated domino tableaux. However, this approach also fails, because the extensions using movement of different sequences of open chains are inconsistent with each other; the fact mentioned in [vLee3] that moving all open chains for  $s_i$  does not define an action of  $\tilde{S}_r$  on semistandard  $r$ -ribbon tableaux indicates the kind of problems that arise. On the positive side, all this suggests that there is something quite special about domino tableaux, possibly because of some application other than that of computing plethysms, that is yet to be discovered.

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