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ABSTRACT
We investigate the validity of the bootstrap method for the elementary symmetric polynomials $S_n^{(k)} = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} \ldots X_{i_k}$ of i.i.d. random variables $X_1, \ldots, X_n$. For both fixed and increasing order k, as $n \to \infty$ the cases where $\mu = \mathrm{E}X_1 \neq 0$, the nondegenerate case, and where $\mu = \mathrm{E}X_1 = 0$, the degenerate case, are considered.

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1 Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with common distribution function F and

$$-\infty < \mu = \mathrm{E}X_1 < \infty, \quad 0 < \sigma^2 = \sigma^2(X_1) < \infty. \tag{1}$$

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Let, for any $1 \leq k \leq n$,

$$S_n^{(k)} = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} \ldots X_{i_k} \tag{2}$$

and let

$$F_n^{(k)}(x) = P\left(\frac{n^{1/2}(S_n^{(k)} - \mu^k)}{k\mu^{k-1}\sigma} \leq x \right) \tag{3}$$

for real x.

The statistic (2) is called an elementary symmetric polynomial of order k. It is frequently used as a typical example of a U-statistic of order k. Asymptotic normality for U-statistics, with a fixed order k, has been first derived by Hoeffding (1948). For elementary symmetric polynomials it means that, for k fixed and $\mu \neq 0$, we have

$$\sup_x |F_n^{(k)}(x) - \Phi(x)| \to 0, \quad \text{as} \quad n \to \infty, \tag{4}$$

where $\Phi(x)$ denotes the standard normal distribution function.

For distributions F with $\mu = 0$ the U-statistic $S_n^{(k)}$ is degenerate since it is readily seen that

$$\mathrm{E}(X_1 \ldots X_k | X_i) = 0, \quad i = 1, \ldots, n, \tag{5}$$

whenever $k \geq 2$. The asymptotic distribution of degenerate U-statistics can be found in e.g. Rubin and Vitale (1980). Note that the limit distributions are no longer normal in the degenerate case (cf. also Theorem 1.6 of the present paper).

Now consider the standard nonparametric bootstrap introduced by Efron (1979). Let $\hat{F}_n$ denote the empirical distribution function of the sample $X_1, \ldots, X_n$ from F. Furthermore let $X_1^*, \ldots, X_n^*$ denote a bootstrap resample of size n, i.e. given the values of $X_1, \ldots, X_n$ the random variables $X_1^*, \ldots, X_n^*$ denote a sample of size n from the empirical distribution $\hat{F}_n$. We approximate the distribution $F_n^{(k)}(x)$ of the normalized k-th order elementary symmetric polynomial by its bootstrap counterpart $F_n^{(k)^*}(x)$, where

$$F_n^{(k)^*}(x) = P_n^*\left(\frac{n^{1/2}(S_n^{(k)^*} - \bar{X}_n^k)}{k\bar{X}_n^{k-1}s_n} \leq x | X_1, \ldots, X_n \right), \tag{6}$$

with $\bar{X}_n$ and s_n denoting the sample mean and sample standard deviation of the original sample, and

$$S_n^{(k)^*} = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1}^* \ldots X_{i_k}^*. \tag{7}$$

Here P_n^* refers to probability under $\hat{F}_n$. We have

$$\sup_x |F_n^{(k)^*}(x) - F_n^{(k)}(x)| \to 0, \quad \text{almost surely.} \tag{8}$$

Similarly, for studentized elementary symmetric polynomials, one approximates

$$G_n^{(k)}(x) = P\left(\frac{n^{1/2}(S_n^{(k)} - \mu^k)}{k\bar{X}_n^{k-1}s_n} \le x\right) \tag{9}$$

quite well by

$$G_n^{(k)^*}(x) = P_n^*\left(\frac{n^{1/2}(S_n^{(k)^*} - \bar{X}_n^k)}{k(\bar{X}_n^*)^{k-1}s_n^*} \le x|X_1, \ldots, X_n\right), \tag{10}$$

where $\bar{X}_n^*$ and s_n^* denote the sample mean and sample standard deviation of the bootstrap resample $X_1^*, \ldots, X_n^*$. For fixed order Studentized U-statistics Efron's bootstrap has been shown to work quite well (i.e. better than the classical normal approximation) by Helmers (1991): the asymptotic accuracy of the bootstrap approximation $G_n^{(k)^*}$ to the exact cdf. $G_n^{(k)}$ of a Studentized U-statistic is of order $o(n^{-1/2})$, as $n \to \infty$.

The result above can be summarized by saying that Efron's bootstrap works well in the case of a fixed order nondegenerate U-statistic. In this paper we study two different ways of departing from the standard case. Firstly in Section 1.1 we investigate the case of *increasing order nondegenerate* elementary symmetric polynomials. In Section 1.2 we investigate the case of *fixed order degenerate* elementary symmetric polynomials. In both cases we show that the bootstrap still works, up to a certain degree and with possible modification of the resampling scheme.

1.1 Nondegenerate polynomials

Suppose that we are in the case where μ does not vanish. For this case, the question how far the standard asymptotic normality, stated in (4), still holds if we allow k to increase with n, has been investigated in van Es and Helmers (1988). It turns out that essentially we have to require $k = o(n^{1/2})$ for the polynomials to remain asymptotically normally distributed, with the standardization given by (4). The case where $k \sim \alpha n^{1/2}$, for some constant $\alpha > 0$ serves as a border case. For results on the asymptotic distribution of the k-th root of the polynomials see Székely (1974, 1982), Halász and Székely (1976), Móri and Székely (1982) and van Es (1986).

Our first theorem states that Efron's bootstrap still works in cases where the order is allowed to increase, as long as asymptotic normality holds .

Theorem 1.1 *The bootstrap works, i.e. (4) holds, with the standard resampling scheme, provided* $k = o(n^{1/2})$.

Theorem 1.2 *If* $\mu = \mathrm{E}X_1 \neq 0,\ 0 < \sigma^2 = \sigma^2(X_1),\ \mathrm{E}|X_1|^3 < \infty$, *and* $k = o(n^{1/2}\log^{-1} n \log_2^{-1} n)$, *then, as both* k *and* $n \to \infty$,

$$\frac{n^{1/2}}{k}\sup_x \left|F_n^{(k)}(x) - F_n^{(k)^*}(x)\right| \to 0, \text{ almost surely,} \tag{11}$$

where $F_n^{(k)}$ *denotes the distribution function of* $n^{1/2}(S_n^{(k)} - \mu)/(k\mu^{k-1}\sigma)$ *and* $F_n^{(k)^*}$ *is its bootstrap counterpart.*

A result, similar to Theorem 1.2 holds true for *studentized* elementary symmetric polynomials. More precisely, we have the following theorem:

Theorem 1.3 *If the conditions of Theorem 1.2 are satisfied and, in addition,* $\mathrm{E}|X_1|^{4+\epsilon} < \infty$*, for some* $\epsilon > 0$ *, then as* $n \to \infty$*,*

$$\frac{n^{1/2}}{k} \sup_x \left| G_n^{(k)}(x) - G_n^{(k)^*}(x) \right| \to 0, \text{ almost surely,} \tag{12}$$

where $G_n^{(k)}$ *denotes the distribution function of* $n^{1/2}(S_n^{(k)} - \mu)/(k\bar{X}_n^{k-1}s_n)$ *and* $G_n^{(k)^*}$ *is its bootstrap counterpart.*

So, the bootstrap approximations $F_n^{(k)^*}$ respectively $G_n^{(k)^*}$ are asymptotically closer to $F_n^{(k)}$ respectively $G_n^{(k)}$ than the normal approximation. Typically, one may expect that the error in these bootstrap approximations is of the exact order kn^{-1}, an improvement by a factor $n^{-1/2}$ over the error $kn^{-1/2}$ in the normal approximation. A proof of this result is feasible, but outside the scope of the present paper. In any case one would need a Cramer type condition for F.

Example 1.4 Consider the situation where the X_i are drawn from a distribution concentrated on zero and one. Let $p = P(X_i = 1) = 1 - P(X_i = 0)$ and let E_n denote the number of ones in the sample. The symmetric polynomial of such X's can be expressed as the quotient of two binomial coefficients

$$S_n^{(k)} = \binom{n}{k}^{-1} \binom{E_n}{k}. \tag{13}$$

Defining

$$g_{k,n}(x) = \prod_{i=0}^{k-1} (x - \frac{i}{n}) \tag{14}$$

we can rewrite this quotient to obtain

$$S_n^{(k)} = \frac{g_{k,n}(E_n/n)}{g_{k,n}(1)}. \tag{15}$$

Using $\log(x+h) = \log x + h\frac{1}{x} + r(x,h)$, where $|r(x,h)| \le Ah^2$, for some constant $A > 0$, uniformly for $x \in [\delta, \infty)$, $\delta > 0$, and $h > -\delta/2$, we get

$$\begin{aligned}
\log g_{k,n}(x) &= \sum_{i=0}^{k-1} \log(x - \frac{i}{n}) \\
&= \sum_{i=0}^{k-1} \Big(\log x - \frac{1}{x}\frac{i}{n} + r(x, -\frac{i}{n}) \Big) \\
&= k \log x - \frac{1}{x}\frac{1}{n} \sum_{i=0}^{k-1} i + \sum_{i=0}^{k-1} r(x, -\frac{i}{n}) \\
&= k \log x - \frac{1}{x}\frac{1}{n} 1/2(k-1)k + \sum_{i=0}^{k-1} r(x, -\frac{i}{n}).
\end{aligned}$$

Furthermore, if $k = O(n^{1/2})$, we have

$$\left|\sum_{i=0}^{k-1} r(x, -\frac{i}{n})\right| \leq A \sum_{i=0}^{k-1} \frac{i^2}{n^2} = O(\frac{k^3}{n^2}) = O(\frac{k}{n}\frac{k^2}{n}) = O(\frac{k}{n}). \tag{16}$$

First we assume $k = o(n^{1/2})$. Then we have the following expansion

$$\begin{aligned} \log g_{k,n}(\frac{E_n}{n}) &= k\log(\frac{E_n}{n}) - \frac{n}{2E_n}O(\frac{k^2}{n}) + o_P(n^{-1/2}) \\ &= \frac{k}{n^{1/2}}\; n^{1/2}\log(\frac{E_n}{n}) + o_P(\frac{k}{n^{1/2}}) \end{aligned}$$

and

$$\log g_{k,n}(1) = O(\frac{k^2}{n}) = o(\frac{k}{n^{1/2}}). \tag{17}$$

We now see that

$$\begin{aligned} \frac{n^{1/2}}{k}(\log S_n^{(k)} - k\log p) &= n^{1/2}\Big(\log(\frac{E_n}{n}) - k\log p\Big) + o_P(1) \\ &\xrightarrow{\mathcal{D}} \frac{\sqrt{p(1-p)}}{p}N, \end{aligned}$$

where N is a standard normal random variable. Taking the exponent we get

$$\frac{n^{1/2}}{k}\left(\frac{S_n^{(k)} - p^k}{p^{k-1}\sqrt{p(1-p)}}\right) \xrightarrow{\mathcal{D}} N, \tag{18}$$

which confirms Theorem 1 in van Es and Helmers (1988).

It is readily seen that the naive bootstrap works in this case because the asymptotics are based on a sample average E_n/n.

Next assume $k \sim \alpha n^{1/2}$. Then we get

$$\log g_{k,n}(\frac{E_n}{n}) = k\log(\frac{E_n}{n}) - \frac{n}{2E_n}(\alpha^2 + o(1)) + O_P(n^{-1/2}) \tag{19}$$

and

$$\log g_{k,n}(1) = -\frac{1}{2}\alpha^2 + o(1). \tag{20}$$

Together this gives

$$p^{-k}S_n^{(k)} \xrightarrow{\mathcal{D}} e^{-\frac{\alpha(p(1-p))^{1/2}}{p}N - \frac{1}{2}\frac{1-p}{p}\alpha^2}, \tag{21}$$

where N is a standard normal random variable.

In this case it is readily seen that the naive bootstrap works because the asymptotics are based on a sample average E_n/n, i.e. with p_n^* equal to the fraction of ones in the sample, defining

$$H_n^{(k)}(x) = P\left(p^{-k}S_n^{(k)} \leq x\right), \tag{22}$$

and

$$H_n^{(k)^*}(x) = P_n^*\left((p_n^*)^{-k} S_n^{(k)^*} \le x\right), \tag{23}$$

we have uniformly in x,

$$H_n^{(k)}(x) \to H(x) \quad \text{and} \quad H_n^{(k)^*}(x) \to H(x), \quad \text{almost surely}, \tag{24}$$

where $H(x)$ is the distribution function of

$$e^{-\frac{\alpha(p(1-p))^{1/2}}{p}N - \frac{1}{2}\frac{1-p}{p}\alpha^2}.$$

Let us now consider convergence in the norming (3). Note that for this example we have

$$\begin{aligned} F_n^{(k)}(x) &= P\left(p^{-k}S_n^{(k)} \le \frac{k}{n^{1/2}}\frac{(p(1-p))^{1/2}}{p}x + 1\right) \\ &= H_n^{(k)}\left(\frac{k}{n^{1/2}}\frac{(p(1-p))^{1/2}}{p}x + 1\right), \end{aligned} \tag{25}$$

and

$$F_n^{(k)^*}(x) = H_n^{(k)^*}\left(\frac{k}{n^{1/2}}\frac{(p_n^*(1-p_n^*))^{1/2}}{p_n^*}x + 1\right). \tag{26}$$

¿From (24) it now follows that (8) holds true, which means that the bootstrap also works in this norming.

Example 1.5 In our second example we consider the distribution $p = P(X_i = a) = 1 - P(X_i = b)$, for $0 < a < b < \infty$. Note that in this case we have

$$\mu = ap + b(1-p) \quad \text{and} \quad \sigma^2 = (a-b)^2 p(1-p). \tag{27}$$

Let E_n denote the number of a's in the sample. Then

$$S_n^{(k)} = \binom{n}{k}^{-1} \sum_{j=0}^{k} \binom{E_n}{j}\binom{n-E_n}{k-j} a^j b^{k-j}. \tag{28}$$

We rewrite $S_n^{(k)}$ as a function of E_n/n. Define the function $h_{k,n}$ by

$$h_{k,n}(x) = \sum_{j=0}^{k} \binom{n}{k}^{-1} \binom{nx}{j}\binom{n(1-x)}{k-j} a^j b^{k-j}, \tag{29}$$

then we have $S_n^{(k)} = h_{k,n}(E_n/n)$. To simplify this function we approximate the hypergeometric probabilities in (29) by binomial ones. Using the approximation

$$\sum_{j=0}^{k} \left| \frac{\binom{nx}{j}\binom{n(1-x)}{k-j}}{\binom{n}{k}} - \binom{k}{j} x^j (1-x)^{k-j} \right| \le \frac{k(k-1)}{n}, \tag{30}$$

which follows from a result of Freedman (1977), we get

$$h_{k,n}(x) = (ax + b(1-x))^k + O\Big(\frac{k(k-1)}{n} b^k\Big). \tag{31}$$

The difference between

$$\frac{n^{1/2}(S_n^{(k)} - \mu^k)}{k\mu^{k-1}\sigma} \tag{32}$$

and

$$\frac{n^{1/2}}{k} \frac{((a-b)E_n/n + b)^k - \mu^k}{\mu^{k-1}\sigma} \tag{33}$$

is bounded almost surely by

$$\frac{n^{1/2}}{k} \frac{b^k}{\sigma\mu^{k-1}} \frac{k(k-1)}{n} = o((b/\mu)^{k-1}), \tag{34}$$

provided $k = o(n^{1/2})$. By Taylor expansion it now follows that (33) converges in distribution to a standard normal distribution, for any sequence of orders k. Since (34) vanishes as $n \to \infty$ and k is fixed, asymptotic normality of the standardized polynomials, as stated in (4) follows. It is not hard to check that the standard bootstrap works in this case, thus confirming Theorem 1.1. For $k \to \infty$ the bound (34) does not vanish, so apparently the technique employed here is not subtle enough. However Theorem 1.1 states that the bootstrap also works in this case, provided $k = o(n^{1/2})$.

1.2 Degenerate polynomials

If the order k is fixed then the limit distribution of $S_n^{(k)}$ is not normal anymore. It follows from the results of Rubin and Vitale (1980), who investigated the asymptotic distribution of symmetric statistics, that the limit distribution is given by the following theorem:

Theorem 1.6 *Let $X_1, ..X_n$ be i.i.d. random variables with zero mean and variance σ^2. Then, as $n \to \infty$ we have*

$$\sup_x \Big| P\Big(S_n^{(k)} \le x\sigma(S_n^{(k)})\Big) - P\Big(\sqrt{k!} H_k(Z) \le x\Big)\Big| \to 0 \tag{35}$$

where $H_k(.)$ is the Hermite polynomial of order k and Z is a random variable with a $N(0,1)$ distribution.

The theorem says that $S_n^{(k)}/\sigma(S_n^{(k)})$ has the same limit distribution as

$$T_k = \sum_{*} \frac{\sqrt{k!}}{j! 2^{(k-j)/2} (\frac{k-j}{2})!} Z^j (-1)^{(k-j)/2} \tag{36}$$

where Z has a $N(0,1)$ distribution and $\sum_*$ extends over all indices $0 \le j \le k$ such that $(k-j)/2$ is an integer. It can be checked by straightforward calculations that

$$\sigma^2(S_n^{(k)}) = \sigma^{2k} \binom{n}{k}^{-1}. \tag{37}$$

Since the form of the distribution is not explicitly known a bootstrap approximation of the distribution of $S_n^{(k)}$ is really needed.

It is known that in this case the usual bootstrap does not work for this situation (see Bickel and Freedman(1981) for $k = 2$ and Arcones and Giné (1992) for general fixed k). The latter authors showed that if the bootstrap version $S_n^{(k)**}$ of the statistic $S_n^{(k)}$ is defined as

$$S_n^{(k)**} = \binom{n}{k}^{-1} \sum_{1 \le i_1 < \ldots < i_k \le n} (X_{i_1}^* - \overline{X}_n)\ldots(X_{i_k}^* - \overline{X}_n) \tag{38}$$

the bootstrap works. It says that we should simply copy the original model, where the mean is zero, in the bootstrap world.

Applying the results of Arcones and Giné (1992) to our situation we get the following result.

Theorem 1.7 *Let the assumptions of Theorem 1.6 be satisfied. Then as $n \to \infty$ we have*

$$\sup_x |P^*(S_n^{(k)**} \le x s_n^k \sqrt{\binom{n}{k}} |X_1, ..., X_n) - P(\sqrt{k!} H_k(Z) \le x)| \to 0, \tag{39}$$

almost surely, where s_n^2 is the sample variance based on $X_1, ..., X_n$. Hence the Arcones-Giné bootstrap is consistent in estimating the distribution of $S_n^{(k)}/\sigma(S_n^{(k)})$.

Remark 1.8 The question is what the limit behavior of $S_n^{(k)}$ is, when k increases together with n in such a way that $k = o(n^{1/2})$, and whether some resampling scheme works in this situation also. Let us assume additionally that $\mathrm{E}|X_i|^{2+\delta}$ is finite for some $\delta > 0$. According to the Appendix in Rubin and Vitale (1980), we have

$$\begin{aligned} S_n^{(k)} &= \binom{n}{k}^{-1} \sum_+ \frac{(-1)^{k-j_1-\ldots-j_k}}{2^{j_2}\ldots k^{j_k}} \frac{1}{j_1!\ldots j_k!} \\ &\qquad (\sum_{i=1}^n X_i)^{j_1} (\sum_{i=1}^n X_i^2)^{j_2} \ldots (\sum_{i=1}^n X_i^k)^{j_k}, \end{aligned} \tag{40}$$

where $\sum_+$ denotes the summation over $j_1, ..., j_k$ such that $j_v = 0, 1.., k$ and $\sum_{v=1}^k v j_v = k$ Now, using the Marcinkiewicz- Zygmund strong of large numbers we find that

$$\frac{1}{n} \sum (X_i^2 - 1) \to 0, \text{ a.s.} \tag{41}$$

and

$$n^{-k/(2+\delta)} \sum |X_i|^k \to 0, \text{ a.s.}, \ k = 3, 4, ... \tag{42}$$

(This is valid even for $k \to \infty$. The reason is the following. Instead of treating $n^{-k/(2+\delta)} \sum |X_i|^k$ it suffices to treat $n^{-k/(2+\delta)} \sum |X_i|^k I\{|X_i| \le n^{1/2}\}$ and it can be easily checked that

$$n^{-k/(2+\delta)} \sum |X_i|^k I\{|X_i| \le n^{1/2}\} \le n^{-k_o/(2+\delta)} \sum |X_i|^{k_o} I\{|X_i| \le n^{1/2}\} \tag{43}$$

for any $2 + \delta < k_o \le k \le n$). Then we carefully treat all terms in (40) to find that only the terms with $j_1 + 2j_2 = k$ need not be negligible. All other terms have no influence on the limit distribution. This means that $S_n^{(k)}(\sigma(S_n^{(k)}))^{-1}$ asymptotically has the same limit distribution as T_k (cf.(36)), even for $k \to \infty$.

Further careful calculations give that the only influential terms are those with $j_1+2j_2 = k$ and $d_{n1}\sqrt{k} \le j_1 \le d_{n2}\sqrt{k}$, where d_{n1} and d_{n2} are arbitrary sequences of positive numbers such that $d_{n1} \to 0$ and $d_{n2} \to \infty$. Concerning the validity of the Arcones-Giné bootstrap under the mentioned stronger assumptions above, Theorem 1.7 remains true even for this situation. The reason is that if we replace the arguments based on the strong law of large numbers by the weak law of large numbers arguments everything goes through.

2 Proofs

Proof of Theorem 1.1. We adapt the proof of Theorem 1 in van Es and Helmers (1988) to the bootstrap world. The proof is based on the Hoeffding decomposition of elementary symmetric polynomials, as given by Karlin and Rinott (1982). For the bootstrap statistic $S_n^{(k)^*}$ we have

$$S_n^{(k)^*} - \bar{X}_n^k = \sum_{r=1}^{k} H_r(X_1^*, \ldots, X_n^*), \tag{44}$$

where

$$H_r(X_1^*, \ldots, X_n^*) = \binom{n}{k}^{-1} \binom{n-r}{k-r} \bar{X}_n^{k-r} \sum_{1 \le j_1 < \ldots < j_r \le n} \prod_{i=1}^{r} (X_{j_i}^* - \bar{X}_n). \tag{45}$$

Next define

$$q_r = \frac{\sigma^2(H_{r+1})}{\sigma^2(H_r)}, \quad r = 1, 2, \ldots, k-1 \tag{46}$$

This gives

$$q_r = \frac{s_n^2}{\bar{X}_n^2}(k-r)^2/((r+1)(n-r)), \quad r = 1, 2, \ldots, k-1. \tag{47}$$

Conditional on $X_1, \ldots, X_n$, the summands of (44) are uncorrelated. Hence we find, given $X_1, \ldots, X_n$

$$\begin{aligned} \sigma^2(S_n^{(k)^*}) &= \sum_{r=1}^{k} \sigma^2(H_r(X_1^*, \ldots, X_n^*)) \\ &= \sigma^2(H_1(X_1^*, \ldots, X_n^*))(1 + q_1 + q_1 q_2 + \ldots + q_1 q_2 \ldots q_{k-1}). \end{aligned}$$

Since on a set of probability one we have $s_n \to \sigma$ and $\bar{X}_n \to \mu$, by the assumption $k = o(n^{1/2})$, with probability one we have, for fixed k and n and for n sufficiently large ,

$$q_r < c\frac{k^2}{rn}, \quad r = 1, 2, \dots, k-1, \tag{48}$$

for some constant $c > 0$. This implies

$$0 \leq \sum_{r=2}^{k} q_1 \dots q_{r-1} \leq \sum_{1=2}^{\infty} \frac{1}{(r-1)!}\left(\frac{ck^2}{n}\right)^{r-1} = e^{c\frac{k^2}{n}} - 1 = o(1), \quad \text{almost surely,} \tag{49}$$

as $n \to \infty$, which shows that the linear term

$$H_1(X_1^*, \dots, X_n^*) = kn^{-1}\bar{X}_n^{k-1}\sum_{i=1}^{n}(X_i^* - \bar{X}_n) \tag{50}$$

is the dominant term in the expansion (44). The result now follows from the central limit theorem for triangular arrays. □

Proof of theorem1.2. It is proved in the appendix of van Es and Helmers (1988) that

$$F_n^{(k)}(x) = \Phi(x) + \frac{1}{6}n^{-1/2}\phi(x)(1-x^2)\{\sigma^{-3}\mathrm{E}(X_1-\mu)^3 \\ +3(k-1)\sigma\mu^{-1}\} + o(\frac{k}{n^{1/2}}) \tag{51}$$

uniformly in all real x. Here ϕ of course denotes the standard normal density. Note that there is no need for the usual requirement that F is non–lattice, when $k \to \infty$, as $n \to \infty$. (However, if k is fixed, we must of course add the assumption that F is non–lattice, in order to guarantee that our expansion is valid uniformly). It is now easy to check that the argument leading to the expansion for $F_n^{(k)}$ can be repeated to find that, quite similarly, also

$$F_n^{(k)^*}(x) = \Phi(x) + \frac{1}{6}n^{-1/2}\phi(x)(1-x^2)\{s_n^{-3}m_3 \\ +3(k-1)s_n\bar{X}_n^{-1}\} + o(\frac{k}{n^{1/2}}) \tag{52}$$

holds true almost surely. Here m_3 of course denotes the sample third central moment $n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^3$ of the original sample. Comparing (51) with (52) we easily conclude that, because almost surely $\bar{X}_n \to \mu,\ s_n^2 \to \sigma^2,\ m_3 \to \mathrm{E}(X_1-\mu)^3$ by the strong law, the theorem is proved. □

Proof of Theorem1.3. It can be proved by a slight adaptation of the proof given in Maesono (1995) (cf also Helmers (1991) and van Es and Helmers (1988)) that

$$G_n^{(k)}(x) = \Phi(x) + \frac{1}{6}n^{-1/2}\phi(x)\{(2x^2+1)\sigma^{-3}\mathrm{E}(X_1-\mu)^3 \\ +3(k-1)(x^2+1)\sigma\mu^{-1}\} + o(\frac{k}{n^{1/2}}) \tag{53}$$

uniformly in all real x. The main new ingredient in the present proof is to verify that the Studentization we employ - which simply amounts to replacing the scaling factor $k\mu^{k-1}\sigma$ by the plug-in estimate $k\bar{X}_n^{k-1}s_n$ - will yield exactly the same Edgeworth expansion (cf.(53)) as Studentization by means of the delete-one-jackknife method, which is applied in Helmers(1991) and Maesono (1995). Combination of this fact with an argument like the one described in the appendix of van Es and Helmers (1988) will then complete our proof. We omit further details. Similarly, one can also show that

$$G_n^{(k)^*}(x) = \Phi(x) + \frac{1}{6}n^{-1/2}\phi(x)\{(2x^2+1)s_n^{-3}m_3 \\ +3(k-1)(x^2+1)s_n\bar{X}_n^{-1}\} + o(\frac{k}{n^{1/2}}) \quad (54)$$

holds true almost surely. Comparing (53) with (54), we easily conclude that, because almost surely $\bar{X}_n \to \mu$, $s_n^2 \to \sigma^2$, $m_3 \to \mathrm{E}(X_1-\mu)^3$ by the strong law, the theorem is proved. □

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