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ABSTRACT

The Boussinesq system arises in Fluid Mechanics when motion is governed by density gradients caused by temperature or concentration differences. In the former case, and when thermodynamical coefficients are regarded as temperature dependent, the system consists of the Navier-Stokes equations and the non linear heat equation coupled through the viscosity, bouyancy and convective terms.

According to the balance between specific heat and thermal conductivity the diffusion term in the heat equation may lead to a singular or degenerate parabolic equation.

In this paper we prove the existence of solutions of the general problem as well as the uniqueness of solutions when the spatial dimension is two.

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1. The model

The Boussinesq system of hydrodynamics equations [3], [26], arises from a zero order approximation to the coupling between the Navier-Stokes equations and the thermodynamic equation [25]. The presence of density gradients in a fluid allows the conversion of gravitational potential energy into motion through the action of buoyant forces. Density differences are induced, for instance, by gradients of temperature arising by non uniform heating of the fluid. In the Boussinesq approximation of a large class of flow problems, thermodynamical coefficients, such as viscosity, specific heat and thermal conductivity, can be assumed constant leading to a coupled system with linear second order operators in the Navier-Stokes and in heat equations (see, e.g., [11], [12], [17], [31]). However, there are some fluids, such as lubrificants or some plasma flow, for which this is no longer an accurate assumption (see, e.g., [15], [29]). In this paper we present results on the existence and uniqueness of weak solutions of this problem. Results on some qualitative properties related with the spatial and time localization of solutions will be published elsewhere ([14]). We start by considering the system derived in [25]

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\ (\mu(\theta)D(\mathbf{u})) + \nabla p = \mathbf{F}(\theta), \\ \operatorname{div}\mathbf{u} = 0, \\ \mathcal{C}(\theta)_t + \mathbf{u}\cdot\nabla\mathcal{C}(\theta) - \Delta\varphi(\theta) = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u}$ is the velocity field of the fluid, θ its temperature, p the pressure, $\mu(\theta)$ the viscosity of the fluid, $\mathbf{F}(\theta)$ the buoyancy force, $D(\mathbf{u}) :=\nabla\mathbf{u}+\nabla\mathbf{u}^T$,

$$\mathcal{C}(\theta) := \int_{\theta_0}^{\theta} C(s)ds \quad \text{and} \quad \varphi(\theta) := \int_{\theta_0}^{\theta} \kappa(s)ds$$

with $C(\tau)$ and $\kappa(\tau)$ being the specific heat and the thermal conductivity of the fluid, respectively. Assuming, as usual, $C > 0$ then $\mathcal{C}$ is inversible, and so $\theta = \mathcal{C}^{-1}(\bar{\theta})$ for some real argument $\bar{\theta}$. Then we can define the functions

$$\bar{\varphi}(\bar{\theta}) := \varphi \circ \mathcal{C}^{-1}(\bar{\theta}), \quad \bar{\mathbf{F}}(\bar{\theta}) := \mathbf{F} \circ \mathcal{C}^{-1}(\bar{\theta}), \quad \bar{\mu}(\bar{\theta}) := \mu \circ \mathcal{C}^{-1}(\bar{\theta}).$$

Substituting these expressions in (1.1) and omitting the bars we get the following formulation of the Boussinesq system

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \text{div } (\mu(\theta)D(\mathbf{u})) + \nabla p = \mathbf{F}(\theta), \\ \text{div}\,\mathbf{u} = 0, \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \Delta\varphi(\theta) = 0. \end{cases} \tag{1.2}$$

We briefly comment some interesting features that characterize this model. There are two paradigmatic situations: the *fast* and the *slow* heat diffusion. These cases mathematically correspond to the singular or degenerate character of the heat equation which may occur according the relative behavior of C and κ. Indeed, since C and κ are non negative, their primitives $\mathcal{C}$ and φ are non decreasing functions. Suppose that a perturbation of a constant temperature θ_0 causes a small gradient of temperature between the boundary (higher temperature) and the interior (lower temperature) in a neighborhood, and assume that the behavior of $\mathcal{C}$ and φ near θ_0 can be approximate as

$$\mathcal{C}(s) \sim c_1\,(s-\theta_0) + c_2\,(s-\theta_0)^p\,, \quad \varphi(s) \sim k_1\,(s-\theta_0) + k_2\,(s-\theta_0)^q\,,$$

for $s > \theta_0$, where $p, q > 0$. From (1) we have

$$\bar{\varphi}'(\mathcal{C}(s)) = \varphi'(s)(\mathcal{C}^{-1})'(\mathcal{C}(s)) = \frac{\varphi'(s)}{\mathcal{C}'(s)} = \frac{k_1 + k_2 q(s-\theta_0)^{q-1}}{c_1 + c_2 p(s-\theta_0)^{p-1}}.$$

So when $s \to \theta_0$ (and therefore $\mathcal{C}(s) \to 0$) we get one of the following behaviors of $\bar{\varphi}'$ close to zero:
(i) if $p, q > 1$ then $\bar{\varphi}'(0) = k_1/c_1$,
(ii) if $1 > q > p$ either $q > 1 > p$ then $\lim\limits_{\mathcal{C}(s)\to 0} \bar{\varphi}'(\mathcal{C}(s)) = 0$,
(iii) if $p > 1 > q$ either $1 > p > q$ then $\lim\limits_{\mathcal{C}(s)\to 0} \bar{\varphi}'(\mathcal{C}(s)) = +\infty$.
In the first case both linear parts dominate: this case arises, for instance, when conductivity and specific heat are taken as constants, leading to the classical heat equation with a linear diffusion term. In the other two cases the non linear parts dominate and this leads to two different behaviors:

if $p < q$ the specific heat dominates over the conductivity, i.e., when temperature approaches θ_0 the fluid stores more heat and this is worstly conduced. It was proved in [6] (see also [14]) that a front of temperature $\theta = \theta_0$ arises. This type of phenomenon is known as *slow diffusion* : heat spends a positive time to spread over the neighborhood,

if $p > q$ the opposite effect arises: the conductivity dominates over the specific heat. In this case the phenomenon is called *fast diffusion*. In [6] (and [14]) was proved that, in fact, $\theta = \theta_0$ in the whole domain when the time is large enough

The outline of the paper is the following. In section 2 we state the main assumptions on the data that will hold through the article and introduce the usual Navier-Stokes functional setting consisting of the variational formulation of these equations introduced by Leray [23] under the framework of functional spaces of free divergence. We also define the notion of weak solution for the heat equation. In section 3 we prove existence of solutions of (2.6) by introducing an iterative scheme to uncouple the system and then we use a modification of the proof of existence of solutions for the Navier-Stokes equations due to J.L. Lions [24] and a regularization technique together with results in [1] to prove the existence of weak solutions of the heat equation. Finally we pass to the limit in the iterative scheme to find a solution of the coupled system. In section 4 we present two results on uniqueness of solutions (in spatial dimension $N = 2$) corresponding to the fast and slow diffusion cases. Proofs of both results are based in a duality technique.

2. Functional setting of the problem

We consider the system of equations given in (1.2) holding on a bounded domain Ω and satisfying the following auxiliary conditions:

$$\begin{cases} \mathbf{u} = \mathbf{0} \quad \text{and} \quad \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{and} \quad \theta(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

with $Q_T := \Omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$, $T > 0$. Before introducing the functional setting we shall give some assumptions that will hold through the article:

Assumptions on the data.

H$_1$**.** $\Omega \subset I\!R^N$, $N = 2,3$, denotes an open, bounded and connected set, with boundary $\partial\Omega$ of class $\mathcal{C}^1$ and finite $(N-1)$-dimensional Haussdorf measure. We suppose that $T > 0$ is arbitrarily fixed.

H$_2$**.** We assume

$$\varphi \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^1((0,\infty)), \quad \varphi(0) = 0, \quad \varphi \text{ non decreasing} \tag{2.2}$$

$$\mathbf{F} \in \mathcal{C}^{0,1}_{loc}([0,\infty); I\!R^N), \tag{2.3}$$

$$\mu \in \mathcal{C}^{0,1}_{loc}([0,\infty)) \quad \text{and satisfies} \tag{2.4}$$

$$m_0 \leq \mu(s) \leq m_1 \quad \forall s \in [0,\infty) \tag{2.5}$$

for some constants $m_1 \geq m_0 > 0$.

H$_3$. The data satisfy: $\mathbf{u}_0 \in L^2_\sigma(\Omega)$, $\theta_0 \in L^\infty(\Omega)$ and $\theta_0 \geq 0$, $\phi_D \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(Q_T)$.

H$_4$. If $\mu' \neq 0$ and $\mathbf{F}' \neq 0$ we assume φ^{-1} is Hölder continuous of exponent α.

Notice that functions $\varphi, \mathbf{F}$ and μ applies on θ. In the following, we shall show that $\theta \in L^\infty(Q_T)$ and, therefore, that local and global Lipschitz continuity will be equivalent. Following [24] (see also [21] and [32]), we introduce the usual Navier-Stokes functional setting by considering functional spaces of free divergence and the variational formulation of these equations. More precisely, we consider

$$\begin{aligned} &\mathcal{C}^\infty_\sigma(\Omega) := \left\{ \mathbf{u} \in \mathcal{C}^\infty_0(\Omega; I\!R^N) : div\mathbf{u} = 0 \right\}, \\ &L^p_\sigma(\Omega) := \text{closure of } \mathcal{C}^\infty_\sigma(\Omega) \text{ in the } L^p(\Omega; I\!R^N) \text{ norm}, \\ &W^{1,p}_\sigma(\Omega) := W^{1,p}_0(\Omega; I\!R^N) \cap L^p_\sigma(\Omega), \\ &L^p_\sigma(Q_T) := L^p(0,T; L^p_\sigma(\Omega)), \end{aligned}$$

and the orthogonal projection

$$P_\sigma : L^2(\Omega; I\!R^N) \to L^2_\sigma(\Omega).$$

Applying P_σ to both parts of the Navier-Stokes equation and taking into account that $P_\sigma \nabla p \equiv 0$ and that $\mathbf{u} = P_\sigma \mathbf{u}$ due to $div\mathbf{u} = 0$ we get

$$\begin{cases} \mathbf{u}_t + P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u} - P_\sigma div\,(\mu(\theta)D(\mathbf{u})) = P_\sigma \mathbf{F}(\theta) & \text{in } Q_T, \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T, \\ \mathbf{u} = 0 \quad \text{and} \quad \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{and} \quad \theta(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \tag{2.6}$$

which is the final formulation of the problem we shall study. We define the usual bilinear and trilinear forms

$$a_\theta(\mathbf{u},\mathbf{v}) := \frac{1}{2}\sum_{i,j=1}^{N}\int_\Omega \mu(\theta)\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\frac{\partial v_j}{\partial x_i} := \int_\Omega \mu(\theta) D\left(\mathbf{u}\right) : \nabla\mathbf{v},$$

for all $\mathbf{u},\mathbf{v} \in W^{1,2}_\sigma(\Omega)$ and with $\theta \in L^\infty(Q_T)$ and

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \sum_{i,j=1}^{N}\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j := \int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{v}\cdot\mathbf{w}, \quad \mathbf{u},\mathbf{v} \in W^{1,2}_\sigma(\Omega),$$

for all $\mathbf{w} \in W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega)$. It is well known that a_θ is continuous and coercive in $W^{1,2}_\sigma(\Omega) \times W^{1,2}_\sigma(\Omega)$ for a.e. $t \in [0,T]$ and that b is anti-symmetric and continuous in $W^{1,2}_\sigma(\Omega) \times W^{1,2}_\sigma(\Omega) \times \left(W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega)\right)$.

Remark 2.1 The main advantage of the formulation of the Navier-Stokes equations in free divergence spaces is that the component $p$is *eliminated.* Thanks to De Rham's Lemma [28] this unknown can be determined by means of the following property: if $\langle \mathbf{q},\mathbf{w}\rangle = 0$ for all $\mathbf{w} \in W^{1,2}_\sigma(\Omega)$ *then there exists* $p \in L^2(\Omega)$ such that $\mathbf{q} = -\nabla p$.

We consider the following notion of solution:
Definition of weak solution. Assume $\mathbf{H}_3$. Then $(\mathbf{u},\theta)$ is a *weak solution* of (2.6) if:
(i) $\mathbf{u} \in L^2(0,T;W^{1,2}_\sigma(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))$, $\varphi(\theta) \in \phi_D + L^2(0,T;H^1_0(\Omega))$, $\theta \in L^\infty(Q_T)$.
(ii)

$$\begin{aligned} &\int_\Omega (\mathbf{u}_t\cdot\mathbf{w})\,d\mathbf{x} + a_\theta(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{w}) = \int_\Omega \mathbf{F}(\theta)\cdot\mathbf{w} \quad \text{a.e. } t \in (0,T), \\ &\mathbf{u}(0) = \mathbf{u}_0, \end{aligned} \tag{2.7}$$

for any test function $\mathbf{w} \in W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega)$.
(iii) $\theta_t \in L^2(0,T;H^{-1}(\Omega))$,

$$\int_0^T \langle\theta_t,\zeta\rangle + \int_0^T\int_\Omega (\nabla\varphi(\theta) - \theta\mathbf{u})\cdot\nabla\zeta = 0, \tag{2.8}$$

for any test function $\zeta \in L^2(0,T;H^1_0(\Omega))$ and

$$\int_0^T \langle\theta_t,\psi\rangle + \int_0^T\int_\Omega (\theta - \theta_0)\,\psi_t = 0, \tag{2.9}$$

for any test function $\psi \in L^2(0,T;H^1_0(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$ with $\psi(T) = 0$.

3. Existence of solutions

Existence of solutions of (2.6) is a consequence of results on Navier-Stokes and non linear diffusion equations. We shall give a proof based on Galerkin's method although other strategies are also possible (see, for instance, the formulation in [29] as a variational inequality in the context of stationary Boussinesq-Stefan problem).

Theorem 3.1 *Assume* $\boldsymbol{H}_1$-$\boldsymbol{H}_4$*. Then problem* (2.6) *has, at least, a weak solution with the following additional regularity:*

$$\mathbf{u} \in \mathcal{C}([0,T], W_\sigma^{-1,2}(\Omega)) \quad \textit{and} \quad \theta \in \mathcal{C}([0,T], H^{-1}(\Omega)).$$

Moreover, if the auxiliary data satisfy

$$k \geq \theta_0 \geq m \geq 0 \quad \textit{a.e. in } \Omega$$

and

$$\varphi(ke^{\lambda_0 t}) \geq \phi_D \geq \varphi(me^{-\lambda_1 t}) \geq 0 \quad \textit{a.e. in } \Sigma_T$$

for some non negative constants $k, m, \lambda_0, \lambda_1$ *then there exists a constant* $\lambda \geq 0$ *independent of* φ *such that*

$$ke^{\lambda t} \geq \theta \geq me^{-\lambda t} \geq 0 \quad \textit{a.e. in } Q_T.$$

Proof. We start by introducing the following iterative scheme to uncouple the system: for each $n \in I\!N$ we set

$$\begin{cases} \mathbf{u}_{nt} + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n - div\,(\mu(\theta_{n-1})D(\mathbf{u}_n)) = \mathbf{F}(\theta_{n-1}) & \text{in } Q_T, \\ \theta_{nt} + \mathbf{u}_{n-1} \cdot \nabla\theta_n - \Delta\varphi(\theta_n) = 0 & \text{in } Q_T, \\ \mathbf{u}_n = 0 \quad \text{and} \quad \varphi(\theta_n) = \phi_D & \text{on } \Sigma_T, \\ \mathbf{u}_n(x,0) = \mathbf{u}_0(x) \quad \text{and} \quad \theta_n(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

with $\theta_0(x,t) = \theta_0(x)$ and $\mathbf{u}_0(x,t) = \mathbf{u}_0(x)$. In (3.1) and in the sequel we suppress the symbol P_σ that makes reference to the projection on free divergence spaces.

3.1 Navier-Stokes problem with non constant viscosity

Consider the problem

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - div\left(\mu(\hat\theta)D(\mathbf{u})\right) = P_\sigma \mathbf{F}(\hat\theta) & \text{in } Q_T, \\ \mathbf{u} = 0 & \text{on } \Sigma_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \text{in } \Omega, \end{cases} \tag{3.2}$$

where with have changed the notation from $\mathbf{u}_n$, θ_{n-1} to $\mathbf{u}$, $\hat\theta$, respectively. This is the usual Navier-Stokes problem but with viscosity depending upon the spatial and time variables.

Lemma 3.1 *Assume that* $\left\|\hat\theta\right\|_{L^\infty(Q_T)} \leq \Theta$*. Then there exists a weak solution of* (3.2) *in the sense of* (2.7)*. Moreover, it holds*

$$\mathbf{u} \in \mathcal{C}([0,T], W_\sigma^{-1,2}(\Omega)),$$

and the norms of $\mathbf{u}$ *in* $L^2(0,T; W_\sigma^{1,2}(\Omega))$, $L^\infty(0,T; L_\sigma^2(\Omega))$ *and* $L^2(0,T; W_\sigma^{-s,2}(\Omega))$ *are bounded only in terms of* $\|\mathbf{u}_0\|_{L_\sigma^2(\Omega)}$, m_0 *and* Θ.

Remark 3.1 The following result is a consequence of Sobolev's theorem (see, e.g., [14]): the imbedding $L^r(Q_T) \subset L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ is continuous for

$$r := 4(1-\frac{1}{2^*}) = \begin{cases} 4-\varepsilon & \text{if } N=2, \text{ for all } \varepsilon > 0, \\ 10/3 & \text{if } N=3. \end{cases} \tag{3.3}$$

Proof of Lemma 3.1. The proof consists of two steps:
(i) We consider an approximation of (3.2) in a sequence of finite dimensional spaces $(\mathcal{V}^m) \subset W^{s,2}_\sigma(\Omega)$ and prove that the system of ordinary differential equations obtained from (3.2) in each of these spaces has a unique solution $\mathbf{u}_m$.
(ii) Thanks to a priori estimates on (3.2) we pass to the limit $m \to \infty$ and identify $\lim_{m\to\infty} \mathbf{u}_m$ as a solution of (3.2).
We start with (i). The spectral problem

$$\langle \mathbf{v}, \mathbf{w} \rangle_{W^{s,2}_\sigma(\Omega)} = \lambda \langle \mathbf{v}, \mathbf{w} \rangle_{L^2_\sigma(\Omega)}, \quad \text{for all } \mathbf{w} \in W^{s,2}_\sigma(\Omega)$$

has a sequence of solutions $\{\mathbf{v}^m\}_{m\in I\!N} \subset W^{s,2}_\sigma(\Omega)$ satisfying

$$\langle \mathbf{v}^m, \mathbf{w} \rangle_{W^{s,2}_\sigma(\Omega)} = \lambda_m \langle \mathbf{v}^m, \mathbf{w} \rangle_{L^2_\sigma(\Omega)}, \quad \text{for all } \mathbf{w} \in W^{s,2}_\sigma(\Omega),$$

with $\lambda_m > 0$, which span $W^{s,2}_\sigma(\Omega)$ (see [24], Corollaire 6.1). Then, for each m, functions $\mathbf{v}^1, ..., \mathbf{v}^m$ are a base for a m-dimensional space $\mathcal{V}^m$. Consider the vector $\mathbf{u}_m$ given by

$$\mathbf{u}_m(t) = \sum_{j=1}^m h_j(t)\mathbf{v}^j, \tag{3.4}$$

and set the problem of finding $\mathbf{u}_m(t) \in \mathcal{V}^m$ such that

$$\begin{cases} \mathbf{u}_{mt} + (\mathbf{u}_m \cdot \nabla)\mathbf{u}_m - div\left(\mu(\hat\theta)D(\mathbf{u}_m)\right) = P_\sigma \mathbf{F}(\hat\theta) & \text{in } Q_T, \\ \mathbf{u}_m = 0 & \text{on } \Sigma_T, \\ \mathbf{u}_m(x,0) = \mathbf{u}_0(x) & \text{in } \Omega, \end{cases} \tag{3.5}$$

is satisfied in the following sense

$$\begin{cases} \displaystyle\int_\Omega (\mathbf{u}_{mt} \cdot \mathbf{w})\, d\mathbf{x} + a_{\hat\theta}(\mathbf{u}_m, \mathbf{w}) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}) = \int_\Omega \mathbf{F}(\hat\theta) \cdot \mathbf{w} \quad \forall \mathbf{w} \in \mathcal{V}^m, \\ \mathbf{u}_m(0) = \mathbf{u}_{m0} := \sum_{j=1}^m u_{j0} \cdot \mathbf{v}^j, \end{cases} \tag{3.6}$$

with $u_{j0} := \displaystyle\int_\Omega \mathbf{u}_0 \cdot \mathbf{v}^j$, and $\{\mathbf{u}_{m0}\}$ satisfying

$$\lim_{m\to\infty} \|\mathbf{u}_{m0} - \mathbf{u}_0\|_{L^2_\sigma(\Omega)} = 0. \tag{3.7}$$

Introducing in (3.6) the expression of $\mathbf{u}_m$ given in (3.4) we get, for all $\mathbf{w} \in \mathcal{V}^m$

$$\sum_{j=1}^m h_j'(t) \int_\Omega \mathbf{v}^j \cdot \mathbf{w} - \sum_{i,j=1}^m h_i(t)h_j(t) \int_\Omega \mathbf{v}^j \cdot (\mathbf{v}^i \cdot \nabla)\mathbf{w} + \sum_{j=1}^m h_j(t) \int_\Omega \mu(\hat\theta) D(\mathbf{v}^j) : \nabla \mathbf{w} = \int_\Omega \mathbf{F}(\hat\theta) \cdot \mathbf{w},$$

and taking $\mathbf{w} = \mathbf{v}^k$, $k = 1, ...m$ we obtain the following system of ordinary differential equations:

$$\sum_{j=1}^m \langle \mathbf{v}^j, \mathbf{v}^k \rangle_{L^2_\sigma(\Omega)} h_j'(t) + \sum_{i,j=1}^m b(\mathbf{v}^j, \mathbf{v}^i, \mathbf{v}^k) h_i(t)h_j(t) - \sum_{j=1}^m a_{\hat\theta(}\mathbf{v}^j, \mathbf{v}^k)(t)h_j(t) = f(\mathbf{v}^k)(t), \tag{3.8}$$

with $f(\mathbf{v}^k)(t) := \left\langle \mathbf{F}(\hat{\theta}(t)), \mathbf{v}^k \right\rangle_{L^2_\sigma(\Omega)}$, to which we impose the initial condition

$$h_k(0) = \int_\Omega \mathbf{u}_0 \mathbf{v}^k. \tag{3.9}$$

Since we assumed that $\hat{\theta} \in L^\infty(Q_T)$ and that both μ and $\mathbf{F}$ are locally Lipschitz continuous functions we deduce $a_{\hat{\theta}}(\mathbf{v}^j, \mathbf{v}^k)(t)$ and $f(\mathbf{v}^k)(t) \in L^\infty(0,T)$ for all j,k with $1 \le j,k \le m$. Therefore, we can express (3.8) in the form

$$\begin{cases} \mathbf{h}'(t) = \mathbf{g}(t, \mathbf{h}(t)) & t > 0, \\ \mathbf{h}(0) = \mathbf{h}_0, \end{cases}$$

where $\mathbf{g}(t, \mathbf{y})$ is measurable in the first variable and Lipschitz continuous in the second. Hence, we can ensure the existence and uniqueness of a continuous solution of (3.8) in a maximal interval $(0, T_m)$ with $T_m > 0$.

In the second part of the proof we shall show that from the a priori estimates on the approximate problems we can deduce $T_m = T$ for all $m \in \mathbb{N}$ and that the passing to the limit is justified and defines $\lim_{m\to\infty} \mathbf{u}_m$ as a solution of (3.2). Since $\mathbf{u}_m(t) \in \mathcal{V}^m$ we may take $\mathbf{w} = \mathbf{u}_m(t)$ in (3.6), obtaining, due to the anti-symmetry of b that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{u}_m^2(t) + a_{\hat{\theta}}(\mathbf{u}_m, \mathbf{u}_m) = \int_\Omega \mathbf{F}(\hat{\theta}(t)) \cdot \mathbf{u}_m(t)$$

for all $t \in (0, T_m)$. Using that $\mu(s) \ge m_0 > 0$, that $a_{\hat{\theta}}(\mathbf{u}_m, \mathbf{u}_m)$ is a norm in $L^2(0,T; W^{1,2}_\sigma(\Omega))$ and Hölder and Young's inequalities we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{u}_m^2(t) + m_0 \left\|\mathbf{u}_m\right\|^2_{W^{1,2}_\sigma(\Omega)} \le \frac{4}{m_0}\left\|\mathbf{F}(\hat{\theta}(t))\right\|^2_{W^{-1,2}_\sigma(\Omega)},$$

and, integrating in $(0,t)$ we obtain

$$\left\|\mathbf{u}_m\right\|_{L^\infty(0,T;L^2_\sigma(\Omega))} + \left\|\mathbf{u}_m\right\|_{L^2(0,T;W^{1,2}_\sigma(\Omega))} \le c\left(\left\|\mathbf{F}(\hat{\theta})\right\|^2_{L^2(0,T;W^{-1,2}_\sigma(\Omega))} + \left\|\mathbf{u}_{m0}\right\|_{L^2_\sigma(\Omega)}\right), \tag{3.10}$$

with c independent of m. Using that $\mathbf{F}$ is Lipschitz continuous, that $\left\|\hat{\theta}\right\|_{L^\infty(Q_T)} \le \Theta$ and (3.7) we deduce $\{\mathbf{u}_m\}$ is bounded in $L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_\sigma(\Omega))$ with continuity with respect to T. So we can take $T_m = T$. On the other hand, if we denote by P_m to the orthogonal projection of $L^2_\sigma(\Omega)$ in $\mathcal{V}^m$, from (3.8) we obtain

$$\mathbf{u}_{mt} = -P_m(\mathbf{B}(\mathbf{u}_m)) - P_m A_{\hat{\theta}}(t)\mathbf{u}_m + P_m\mathbf{F},$$

where $A_{\hat{\theta}}(t)$ and $\mathbf{B}$ are defined by

$$a_\theta(\mathbf{v}_1, \mathbf{v}_2)(t) = \int_\Omega A_\theta(t)\mathbf{v}_1 \cdot \mathbf{v}_2 \quad \text{y} \quad b(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \mathbf{B}(\mathbf{v}_1) \cdot \mathbf{v}_2,$$

which are continuous from $W^{1,2}_\sigma(\Omega)$ in $W^{-1,2}_\sigma(\Omega)$ and from $W^{s,2}_\sigma(\Omega)$ in $W^{-s,2}_\sigma(\Omega)$, $s = N/2$, respectively, as a consequence of the continuity properties of a_θ and b (see, e.g., [24], Lemme 6.5). From the estimate (3.10) we deduce $A_\theta(t)\mathbf{u}_m$ is bounded in $L^2(0,T; W^{-1,2}_\sigma(\Omega))$ and that $\mathbf{B}(\mathbf{u}_m)$ is bounded in $L^2(0,T; W^{-s,2}_\sigma(\Omega))$, and since $\mathbf{F}(\hat{\theta}) \in L^\infty_\sigma(Q_T) \subset L^2(0,T; W^{-1,2}_\sigma(\Omega))$ (remind that we are denoting $P_\sigma\mathbf{F}$ by $\mathbf{F}$) we conclude, taking into account that $\left\|P_m\right\|_{L(W^{-s,2}_\sigma; W^{-s,2}_\sigma)} \le 1$ due to the choice of the base (see [24], p. 76), that $\mathbf{u}_{mt}$ is also bounded in $L^2(0,T;W_\sigma^{-s,2}(\Omega))$. These bounds allow us to apply Lion's Theorem (see [24], Théorème 5.1) and deduce the existence of a subsequence of $\mathbf{u}_m$ such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{weakly in} \quad L^2(0,T;W_\sigma^{1,2}(\Omega)), \tag{3.11}$$

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{weakly} * \text{ weakly in} \quad L^\infty(0,T;L_\sigma^2(\Omega)),$$

$$\mathbf{u}_m \to \mathbf{u} \quad \text{strongly in} \quad L^2(0,T;L_\sigma^2(\Omega)) \text{ and a.e. in } Q_T, \text{ and} \tag{3.12}$$

$$\mathbf{u}_{mt} \rightharpoonup \mathbf{u}_t \quad \text{weakly in} \quad L^2(0,T;W_\sigma^{-s,2}(\Omega)). \tag{3.13}$$

From (3.11) and (3.13) we deduce $\mathbf{u}_m(0) \rightharpoonup \mathbf{u}(0)$ in $W_\sigma^{-s,2}(\Omega)$ and, therefore, that $\mathbf{u}(0) = \mathbf{u}_0$. Thanks to Remark 3.1 we have that the product of components$(u_i)_m\,(u_j)_m$ is bounded in $L^{r/2}(Q_T)$, so there exists a element v_{ij} in this space such that

$$(u_i)_m(u_j)_m \rightharpoonup v_{ij},$$

but, due to (3.12) it must be $v_{ij} = u_i u_j$. Then we deduce

$$(\mathbf{u}_m \cdot \nabla)\mathbf{u}_m \rightharpoonup (\mathbf{u} \cdot \nabla)\mathbf{u} \quad \text{in } L^{r/2}(Q_T).$$

Also, from (3.11) and $\hat{\theta} \in L^\infty Q_T$ we deduce

$$div\left(\mu(\hat{\theta})D(\mathbf{u}_m)\right) \rightharpoonup div\left(\mu(\hat{\theta})D(\mathbf{u})\right) \quad \text{in } L^2(0,T;W_\sigma^{-1,2}(\Omega)),$$

and therefore that $\mathbf{u}$ satisfies(3.2) in the weak sense. Finally, since $\mathbf{B}(\mathbf{u}) \in L^1(0,T;W_\sigma^{-1,2})$ (see [32], Lemma 3.1), then

$$\mathbf{u}_t = \mathbf{F}(\hat{\theta}) - \mathbf{B}(\mathbf{u}) - A_{\hat{\theta}}\mathbf{u} \in L^1(0,T;W_\sigma^{-1,2})$$

and therefore we obtain $\mathbf{u} \in \mathcal{C}([0,T],W_\sigma^{-1,2}(\Omega))$. □

3.2 The nonlinear diffusion equation with prescribed convection

We pass to analyze the second problem that arise from the uncoupling of (3.1). We shall again use the notation ($\hat{\mathbf{u}}$,θ) instead of $(\mathbf{u}_{n-1},\theta_n)$, so the problem is written as

$$\begin{cases} \theta_t + \hat{\mathbf{u}} \cdot \nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T, \\ \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega. \end{cases} \tag{3.14}$$

LEMMA 3.2 *Assume* $\mathbf{H}_1$-$\mathbf{H}_3$ *and that* $\hat{\mathbf{u}} \in L_\sigma^r(Q_T)$, *with* r *given by* (3.3). *Then problem* (3.14) *has a weak solution in the sense of* (2.8) *and* (2.9) *such that*

$$\theta \in \mathcal{C}([0,T],H^{-1}(\Omega)).$$

Moreover, if the auxiliary data satisfy

$$k \geq \theta_0 \geq m \geq 0 \quad a.e. \ in \ \Omega$$

and

$$\varphi(ke^{\lambda_0 t}) \geq \phi_D \geq \varphi(me^{-\lambda_1 t}) \geq 0 \quad a.e. \ on \ \Sigma_T$$

for some non negative constants $k, m, \lambda_0, \lambda_1$ *then there exists a constant* $\lambda \geq 0$ *independent of* φ *such that*

$$ke^{\lambda t} \geq \theta \geq me^{-\lambda t} \geq 0 \quad a.e. \ in \ Q_T. \tag{3.15}$$

Proof. We proceed by approximation. First we prove the existence of solutions of a sequence of problems in which the convection term is regularized, obtaining solutions with the regularity stated in the above lemma. Then, thanks to the a priori estimates, we shall see that this sequence of solutions converges to a solution of problem (3.14) with the properties stated in the lemma. Consider the problem

$$\begin{cases} \theta_t + \hat{\mathbf{u}}_m \cdot \nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T, \\ \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \tag{3.16}$$

with $\hat{\mathbf{u}}_m \in L^r_\sigma(Q_T)$ satisfying

$$\|\hat{\mathbf{u}}_m\|_{L^\infty_\sigma(Q_T)} \leq m \quad \text{and} \quad \hat{\mathbf{u}}_m \to \hat{\mathbf{u}} \quad \text{in} \quad L^r_\sigma(Q_T).$$

It is a consequence of [1], Theorem 1.7, the existence of a weak solution θ_m of problem (3.16) with $\theta_m \in L^\infty(Q_T)$ and $\varphi(\theta) \in \phi_D + L^2(0,T;H^1_0(\Omega))$ and that satisfies a maximum principle, from where (3.15) is deduced. Moreover, using $\varphi(\theta) - \phi_D$ as a test function we obtain the estimate:

$$\|\Phi(\theta_m)\|_{L^\infty(0,T;L^1(Q_T))} + \|\nabla\varphi(\theta_m)\|_{L^2(Q_T)} \leq \Lambda, \tag{3.17}$$

where $\Phi(s) = \displaystyle\int_0^s \varphi(\sigma)d\sigma$ and Λ is a constant depending only on the auxiliary data. Therefore, since $r > 2$ we have uniform estimate for $\hat{\mathbf{u}}_m$ in $L^2(Q_T)$ and therefore, by using (3.17) we can estimate $\|\theta_{mt}\|_{L^2(0,T;H^{-1}(\Omega))}$ uniformely in m. Hence we can extract subsequences $\varphi(\theta_m)$ and θ_m such that

$$\begin{array}{ll} \theta_m \rightharpoonup \theta & \text{weakly } * \text{ in } L^\infty(Q_T), \\ \varphi(\theta_m) \rightharpoonup \psi & \text{weakly in } L^2(0,T;H^1_0(\Omega)), \\ \theta_{mt} \rightharpoonup \theta_t & \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \end{array}$$

From the compact imbedding $L^\infty(\Omega) \subset H^{-1}(\Omega)$ and Corollary 4 (p. 85) of [30] we have

$$\theta_m \to \theta \quad \text{in} \quad \mathcal{C}([0,T],H^{-1}(\Omega)).$$

Since φ is continuous and non decreasing we have $-\Delta\varphi(\cdot)$ is a maximum monotone graph in $L^2(0,T;H^{-1}(\Omega))$, and therefore, it is strongly weakly closed in such space, from where we deduce $\psi = \varphi(\theta)$.

Finally, since $\mathbf{u}_m \to \mathbf{u}$ in $L^r_\sigma(Q_T)$ and $r > 2$ we get

$$\hat{\mathbf{u}}_m \cdot \nabla\theta_m \rightharpoonup \hat{\mathbf{u}} \cdot \nabla\theta \quad \text{in} \quad L^2_\sigma(Q_T),$$

from where we conclude that θ is a weak solution of problem (3.16) with the additional regularity stated. □

Continuation of the proof of Theorem 3.1 We again consider the problem (3.1). Thanks to Lemmas 3.1 and 3.2 we have that, for each $n \in I\!N$ there exist functions $\mathbf{u}_n$, θ_n, solutions of (3.1), such that

$$\mathbf{u}_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_\sigma(\Omega)), \tag{3.18}$$

$$\mathbf{u}_{nt} \text{ is bounded in } L^2(0,T;W^{-s,2}_\sigma(\Omega)), \tag{3.19}$$

$$\theta_n \text{ is bounded in } L^\infty(Q_T), \text{ and} \tag{3.20}$$

$$\varphi(\theta_n) \text{ is bounded in } L^\infty(Q_T) \cap L^2(0,T;H^1_0(\Omega)), \tag{3.21}$$

with bounds that only depend on the auxiliary data and on the Lipschitz continuity constants of μ and $\mathbf{F}$. We can, then, extract subsequences such that

$$\begin{array}{lll} \mathbf{u}_n & \rightharpoonup \mathbf{u} & \text{weakly in } L^2(0,T;W^{1,2}_\sigma(\Omega)), \\ \mathbf{u}_n & \to \mathbf{u} & \text{strongly in } L^2(0,T;L^2_\sigma(\Omega)) \text{ and a.e. in } Q_T, \\ \mathbf{u}_{nt} & \rightharpoonup \mathbf{u}_t & \text{weakly in } L^2(0,T;W^{-s,2}_\sigma(\Omega)) \end{array}$$

and

$$\begin{aligned} \theta_n &\rightharpoonup \theta \quad \text{weakly } * \text{ in } L^\infty(Q_T), \\ \varphi(\theta_n) &\rightharpoonup \psi \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ \theta_{nt} &\rightharpoonup \theta_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned}$$

As in the above proof we conclude that $\theta_m \to \theta$ in $\mathcal{C}([0,T],H^{-1}(\Omega))$ and that $\psi = \varphi(\theta)$. Assume, now, that $\mu' \neq 0$ and $\mathbf{F}' \neq 0$. In order to pass to the limit on $\mu(\theta_n)$ and $\mathbf{F}(\theta_n)$ we shall prove that $\theta_n \to \theta$ in $L^p(Q_T)$ for all $p < \infty$. To do that we use a modification of the arguments given in [10], [24] (see also [13]). Defining the space

$$\mathcal{H} = \{\theta \in L^{2/\alpha}(0,T;W^{\alpha,2/\alpha}(\Omega)), \theta_t \in L^2(0,T;H^{-1}(\Omega))\},$$

it is easy to see that θ_n is uniformly bounded in $\mathcal{H}$. Then, from the compact imbedding $\mathcal{H} \subset L^{2/\alpha}(Q_T)$ we conclude that there exists a subsequence of θ_n such that

$$\theta_n \to \theta \text{ strongly in } L^{2/\alpha}(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T.$$

This fact together with the weak $*$ convergence of θ_n to θ in $L^\infty(Q_T)$ implies that $\theta_n \to \theta$ in $L^p(Q_T)$ for all $p < \infty$. Then, since μ is locally Lipschitz continuous

$$\mu(\theta_n) \to \mu(\theta) \quad \text{strongly in} \quad L^q(Q_T), \quad \text{for all} \quad q < \infty,$$

and therefore

$$\int_{Q_T} \mu(\theta_n) D(\mathbf{u}_n) : \nabla\zeta \to \int_{Q_T} \mu(\theta) D(\mathbf{u}) : \nabla\zeta,$$

and since $\mathbf{F}$ is locally Lipschitz continuous we get

$$\mathbf{F}(\theta_n) \to \mathbf{F}(\theta) \quad \text{in } L^q(Q_T).$$

Finally, $\mathbf{u}_n \to \mathbf{u}$ in $L^2_\sigma(Q_T)$ and $\theta_n \rightharpoonup \theta$ weakly $*$ in $L^\infty(Q_T)$ implies

$$\int_{Q_T} \theta_n \mathbf{u}_n \cdot \nabla\zeta \to \int_{Q_T} \theta\mathbf{u} \cdot \nabla\zeta,$$

from where the passing to the limit is justified in all the coupling terms of the system. The justification of the convegence for the remaining terms as well as the additional regularity of solutions is analogous as in Lemmas 3.1 and 3.2. □

Remark 3.2 In the case of spatial dimension $N = 2$ and $\mu = const$ it is possible to deduce further regularity of the velocity field. In particular, using Δu as test function we get

$$\mathbf{u} \in L^\infty(0,T;W^{1,2}_\sigma(\Omega)) \cap L^2(0,T;W^{2,2}_\sigma(\Omega)),$$

obtaining then from the Sobolev's Theorem that $\mathbf{u} \in L^\infty(Q_T)$. We also point out that the assumption $\mathbf{H}_4$ could be removed by using time discretization arguments as in [1].

4. Uniqueness of solutions

As it is well known, uniqueness of solutions for Navier-Stokes equations in spatial dimension $N = 3$ is an open problem. We shall, therefore, restrict ourselves to the study of uniqueness of solutions for the Boussinesq system in spatial dimension $N = 2$. When the diffusion term of the heat equation is linear, it has been proved that uniqueness hold in the same class of functions ensured by the existence theorems (see, e.g., [12]). These proofs relies strongly in the fact that natural energy spaces for both unknowns are the same (L^2) and therefore the interchange of information between momentum and heat equations is performable. However, when the diffusion of heat is no longer linear, and specially when it is degenerate, the problem turns to be more involved: natural estimates for the heat equation are obtained in L^1 but proving the well posedness of Navier-Stokes equation in this space seems to be a difficult task. This fact makes difficult to use the L^1 techniques developed in the last years, and still in progress, that have been successfully used to prove uniqueness for degenerate scalar equations as well as for certain systems of equations where comparison principles still hold (see [20], [5], [13], [8], [27]). In this paper we shall approach the problem from a duality technique, i.e., from the search of suitable test functions (perturbations of the sign function) that allows to conclude the uniqueness property. It is worth strengthen here that no comparison principle will hold for the Boussinesq system and this is probably one of the main sources of complexity for the problem.

In this section we shall consider the case when the second order coupling in the viscosity term of the Navier-Stokes equations is no longer present. In the case of fast diffusion, this coupling is not difficult to deal with when suitable assumptions on the regularity of the velocity field are made. In the slow diffusion case, the most interesting feature of the model is the parabolic degeneracy of the heat equation, and the difficulties introduced by the coupling in the viscosity term are no longer tractable without unrealistic assumptions on the regularity of the velocity field. We shall therefore study the following problem

$$\left\{\begin{array}{lr}\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\Delta\mathbf{u}=\mathbf{F}(\theta) & \text{in } Q_T,\\ \theta_t+\mathbf{u}\cdot\nabla\theta-\Delta\varphi(\theta)=0 & \text{in } Q_T,\\ \mathbf{u}=0 \quad \text{and} \quad \varphi(\theta)=\phi_D & \text{on } \Sigma_T,\\ \mathbf{u}(x,0)=\mathbf{u}_0(x) \quad \text{and} \quad \theta(x,0)=\theta_0(x) & \text{in } \Omega,\end{array}\right. \tag{4.1}$$

We first present the result on the fast reaction case:

THEOREM 4.1 *Let $N=2$ and $\mu(\theta)\equiv 1$. Assume $\varphi^{-1}\in\mathcal{C}^{0,1}(I\!R)$. Then, under conditions of Theorem* 3.1 *there exists a unique weak solution of* (4.1).

Proof. Suppose there exist two weak solutions $(\mathbf{u}_1,\theta_1)$, $(\mathbf{u}_2,\theta_2)$ and define $(\mathbf{u},\theta):=(\mathbf{u}_1-\mathbf{u}_2,\theta:=\theta_1-\theta_2)$ and $\mathbf{F}_i:=\mathbf{F}(\theta_i)$. Then $(\mathbf{u},\theta)$ satisfies:

$$\left\{\begin{array}{lr}\mathbf{u}_t+(\mathbf{u}_1\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}_2-\Delta\mathbf{u}=\mathbf{F}_1-\mathbf{F}_2 & \text{in } Q_T,\\ \theta_t+\mathbf{u}_1\cdot\nabla\theta+\mathbf{u}\cdot\nabla\theta_2-\Delta\left(\varphi(\theta_1)-\varphi(\theta_2)\right)=0 & \text{in } Q_T,\\ \mathbf{u}=0 \quad \text{and} \quad \varphi(\theta_1)-\varphi(\theta_2)=0 & \text{on } \Sigma_T,\\ \mathbf{u}(\mathbf{x},0)=0 \quad \text{and} \quad \theta(\mathbf{x},0)=0 & \text{in } \Omega.\end{array}\right.$$

Consider smooth test functions $\mathbf{w}(t),\xi$, with $div\mathbf{w}=0$ and $\mathbf{w}(T)=\mathbf{0}$. Integrating by parts and adding the resulting integral identities we get

$$\begin{aligned}\int_\Omega\theta(T)\xi(T) \;=\; & \int_{Q_T}\mathbf{u}\cdot[\mathbf{w}_t+(\mathbf{u}_1\cdot\nabla)\mathbf{w}+\Delta\mathbf{w}]+\int_{Q_T}\mathbf{u}_2\cdot(\mathbf{u}\cdot\nabla)\mathbf{w}+\\ &+\int_{Q_T}(\mathbf{F}_1-\mathbf{F}_2)\cdot\mathbf{w}+\int_{Q_T}\theta\left(\xi_t+\mathbf{u}_1\cdot\nabla\xi\right)-\int_{Q_T}\xi\mathbf{u}\cdot\nabla\theta_2+\\ &+\int_{Q_T}\left(\varphi(\theta_1)-\varphi(\theta_2)\right)\Delta\xi.\end{aligned} \tag{4.2}$$

We define the differential operator $\mathcal{L}:L^2(0,T;W^{1,2}_\sigma(\Omega))\to L^2(0,T;L^2_\sigma(\Omega))$ by

$$\mathbf{u}\cdot(\mathcal{L}\mathbf{w}:\mathbf{u}_2):=\mathbf{u}_2\cdot(\mathbf{u}\cdot\nabla)\mathbf{w}=\mathbf{u}\cdot\left(\frac{\partial\mathbf{w}}{\partial x}\cdot\mathbf{u}_2,\frac{\partial\mathbf{w}}{\partial y}\cdot\mathbf{u}_2\right), \tag{4.3}$$

where $\mathbf{x} := (x, y)$. It is straightforward to check that $\mathcal{L}$ is linear and continuous. Adding and substracting in (4.2) the term θh_L, where, for $L > 0$ we define

$$h_L(\mathbf{x},t) := \begin{cases} h(\mathbf{x},t) & \text{if } h(\mathbf{x},t) \le L, \\ L & \text{if } h(\mathbf{x},t) > L, \end{cases}$$

and

$$h(\mathbf{x},t) := \begin{cases} \dfrac{\varphi(\theta_1(\mathbf{x},t)) - \varphi(\theta_2(\mathbf{x},t))}{\theta(\mathbf{x},t)} & \text{if } \theta(\mathbf{x},t) \neq 0, \\ 0 & \text{if } \theta(\mathbf{x},t) = 0, \end{cases}$$

and using (4.3) we obtain

$$\begin{aligned} \int_\Omega \theta(T)\xi(T) &= \int_{Q_T} \mathbf{u}\cdot [\mathbf{w}_t + (\mathbf{u}_1 \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2 - \xi\nabla\theta_2 + \Delta\mathbf{w}] + \\ &\quad + \int_{Q_T} \theta\left(\xi_t + \mathbf{u}_1 \cdot \nabla\xi + \mathbf{f}\cdot\mathbf{w} + h_L\Delta\xi\right) + \int_{Q_T} (h - h_L)\,\theta\Delta\xi \end{aligned} \tag{4.4}$$

with

$$\mathbf{f}(\mathbf{x},t) := \begin{cases} \dfrac{\mathbf{F}(\theta_1(\mathbf{x},t)) - \mathbf{F}(\theta_2(\mathbf{x},t))}{\theta(\mathbf{x},t)} & \text{if } \theta(\mathbf{x},t) \neq 0, \\ 0 & \text{if } \theta(\mathbf{x},t) = 0. \end{cases}$$

Notice that since $\mathbf{F}$ and φ^{-1} are Lipschitz continuous then $\mathbf{f} \in L^\infty(Q_T)$ and there exist a $h_0 > 0$ such that

$$h(\mathbf{x},t) > h_0 \quad \text{a.e.} \quad (\mathbf{x},t) \in Q_T. \tag{4.5}$$

We set the following problem to choose the test functions:

$$\begin{cases} \mathcal{L}_1(\mathbf{w},\xi) := \mathbf{w}_t + (\mathbf{u}_1 \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2 - \xi\nabla\theta_2 + \Delta\mathbf{w} = 0 & \text{in } Q_T, \\ \mathcal{L}_2(\mathbf{w},\xi) := \xi_t + \mathbf{u}_1 \cdot \nabla\xi + \mathbf{f}\cdot\mathbf{w} + h_L\Delta\xi = 0 & \text{in } Q_T, \\ \mathbf{w} = 0 \quad \text{and} \quad \xi = 0 & \text{on } \Sigma_T, \\ \mathbf{w}(\mathbf{x},T) = \mathbf{0} \quad \text{and} \quad \xi(\mathbf{x},T) = \theta_m(\mathbf{x},T) & \text{in } \Omega, \end{cases} \tag{4.6}$$

with $\theta_m \in L^\infty(0,T;H_0^1(\Omega))$ satisfying

$$\theta_m \to \theta \quad \text{strongly in} \quad L^\infty(0,T;L^2(Q_T)). \tag{4.7}$$

We state here the result on existence, uniqueness and regularity of solutions of problem (4.6) and prove it at the end of this section.

Lemma 4.1 *Problem* (4.6) *has a unique weak solution with the regularity of test functions of* (4.1) (*see* (2.7), (2.8), (2.9)). *Moreover,*

$$\begin{aligned} &\mathbf{w} \in H^1(0,T;L^2_\sigma(\Omega)) \cap L^\infty(0,T;W^{1,2}_\sigma(\Omega)) \cap L^2(0,T;W^{2,2}_\sigma(\Omega)), \\ &\xi \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega)) \end{aligned}$$

and there exists a positive constant k independent of L and m such that

$$\int_{Q_T} |\Delta\xi|^2 \le e^{kT}\left(\int_\Omega \mathbf{u}^2(T) + \int_\Omega |\nabla\theta_m(T)|^2\right). \tag{4.8}$$

Continuation of the proof of Theorem 4.1. Using these test functions we get from (4.4) that

$$\int_\Omega \theta(T)\theta_m(T) = \int_{Q_T} (h - h_L)\,\theta\Delta\xi. \tag{4.9}$$

Now we shall pass to the limit first in L and afterwards in m: since h_L converges pointwise to h and $|\theta(h - h_L)| \le 2\,|\varphi(\theta_1) + \varphi(\theta_2)| \le$ const., we get from the Theorem of Lebesgue that $\|(h - h_L)\theta\|_{L^2(Q_T)} \to 0$ as $L \to \infty$, and due to the uniform estimate (4.8) we deduce

$$\int_{Q_T} \theta(h - h_L)\Delta\xi \to 0 \quad \text{as} \quad L \to \infty,$$

and therefore, from (4.9)

$$\int_\Omega \theta(T)\theta_m(T) \to 0 \quad \text{as} \quad L \to \infty.$$

Letting $m \to \infty$ and using (4.7) we get $\|\theta(T)\|_{L^2(\Omega)} = 0$, and since T is arbitrarily fixed we conclude $\theta_1 = \theta_2$ a.e. in Q_T. Finally, $\mathbf{F}(\theta_1) = \mathbf{F}(\theta_2)$ a.e. in Q_T and standard arguments for the Navier-Stokes equations in space dimension two (see [32]) implies $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in Q_T. □

Our second result uses a method due to Kalashnikov [18] which consists on making a comparison between an arbitrary weak solution of (4.1) and the weak solution constructed as the limit of a sequence of solutions of regularized problems. Our result is strongly based on the technique introduced by Díaz and Kersner [7] to study a one dimensional scalar equation and, to achieve it, we generalized a comparison argument introduced in [7] to handle some singular boundary integrals. See also [16] for some improvements to [7].

Theorem 4.2 *Let $N = 2$ and $\mu(\theta) \equiv 1$. Suppose that $\varphi \in \mathcal{C}^2(\mathbb{R})$, $\varphi'(0) = 0$, $\varphi'(s) > 0$ and $\varphi''(s) > 0$ if $s > 0$. Then, under the conditions of Theorem* 3.1 *there exist a unique solution of* (4.1) *in the class of weak solutions such that $\nabla\theta \in L^2(Q_T)$.*

Proof. Consider the sequence of problems $(4.1)_\varepsilon$ in which we approximate solutions of the degenerate problem (4.1) by perturbing the initial and boundary data of θ in the following way:

$$\begin{cases} \varphi(\theta_{D\varepsilon}) = \phi_D + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_T, \\ \theta_{0\varepsilon} = \theta_0 + \varepsilon & \text{in } \Omega, \end{cases}$$

for some $\lambda_1 > 0$. Applying Theorem 3.1 we have that for each $\varepsilon > 0$ problem $(4.1)_\varepsilon$ has, at least, one solution $(\theta_\varepsilon, \mathbf{u}_\varepsilon)$ satisfying $\theta_\varepsilon \ge \varepsilon e^{-\lambda t}$ a.e. in Q_T, with $\lambda > 0$ independent of φ and ε. Following the same scheme than in subsection 3.2 it is possible to prove that $(\mathbf{u}_\varepsilon, \theta_\varepsilon) \to (\mathbf{u}, \theta)$ strongly en $L^2(Q_T) \times L^2_\sigma(Q_T)$ where $(\theta, \mathbf{u})$ is a weak solution of (4.1). Now let us suppose that there exists another weak solution $(\theta_2, \mathbf{u}_2)$ of (4.1) and let us define $(\mathbf{U}_\varepsilon, \Theta_\varepsilon) := (\mathbf{u}_\varepsilon - \mathbf{u}_2, \theta_\varepsilon - \theta_2)$. Then $(\mathbf{U}_\varepsilon, \Theta_\varepsilon)$ satisfies

$$\begin{cases} \mathbf{U}_{\varepsilon t} + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{U}_\varepsilon + (\mathbf{U}_\varepsilon \cdot \nabla)\mathbf{u}_2 - \Delta\mathbf{U}_\varepsilon = \mathbf{F}(\theta_\varepsilon) - \mathbf{F}(\theta_2) & \text{in } Q_T, \\ \Theta_{\varepsilon t} + \mathbf{u}_\varepsilon \cdot \nabla\Theta_\varepsilon + \mathbf{U}_\varepsilon \cdot \nabla\theta_2 - \Delta(\varphi(\theta_\varepsilon) - \varphi(\theta_2)) = 0 & \text{in } Q_T, \\ \mathbf{U}_\varepsilon = 0 \quad \text{and} \quad \varphi(\theta_\varepsilon) = \phi_D + \varphi(\varepsilon e^{-\lambda_1 t}), \quad \varphi(\theta_2) = \phi_D & \text{on } \Sigma_T, \\ \mathbf{U}_\varepsilon(x, 0) = 0 \quad \text{and} \quad \Theta_\varepsilon(x, 0) = \varepsilon & \text{in } \Omega. \end{cases} \tag{4.10}$$

Taking smooth test functions $\mathbf{w}(t), \xi$, with $div\mathbf{w} = 0$ and $\mathbf{w}(T) = \mathbf{0}$, integrating by parts and adding the resulting integral identities we get

$$\begin{aligned}\int_\Omega \Theta_\varepsilon(T)\xi(T) &= \int_{Q_T} \mathbf{U}_\varepsilon \cdot [\mathbf{w}_t + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2 - \xi\nabla\theta_2 + \Delta\mathbf{w}] + \\ &+ \int_{Q_T} \Theta_\varepsilon (\xi_t + \mathbf{u}_\varepsilon \cdot \nabla\xi + \mathbf{f}_\varepsilon \cdot \mathbf{w} + h_\varepsilon \Delta\xi) - \\ &- \int_{\Sigma_T} (\varphi(\theta_\varepsilon) - \varphi(\theta_2))\nabla\xi \cdot \nu + \varepsilon \int_\Omega \xi(0) \end{aligned} \tag{4.11}$$

with $\mathcal{L}$ defined in (4.3),

$$h_\varepsilon := \begin{cases} \dfrac{\varphi(\theta_\varepsilon) - \varphi(\theta_2)}{\Theta_\varepsilon} & \text{if } \Theta_\varepsilon \neq 0, \\ 0 & \text{if } \Theta_\varepsilon = 0, \end{cases} \quad \text{and} \quad \mathbf{f}_\varepsilon := \begin{cases} \dfrac{\mathbf{F}(\theta_\varepsilon) - \mathbf{F}(\theta_2)}{\Theta_\varepsilon} & \text{if } \Theta_\varepsilon \neq 0, \\ 0 & \text{if } \Theta_\varepsilon = 0. \end{cases}$$

Since $\mathbf{F}$ is Lipschitz continuous, φ is convex and $\theta_\varepsilon \geq \varepsilon e^{-\lambda T}$ there exist positive constants k_0 and

$$k(\varepsilon) := \varepsilon^{-1} e^{\lambda T} \varphi(\varepsilon e^{-\lambda T}) \tag{4.12}$$

such that

$$0 < k(\varepsilon) \leq h_\varepsilon \leq k_0 \quad \text{and} \quad |\mathbf{f}_\varepsilon| \leq k_0. \tag{4.13}$$

We consider regularizing sequences in $\mathcal{C}^\infty(Q_T)$ and $\mathcal{C}^\infty(0,T;\mathcal{C}^\infty_\sigma(\Omega))$ for the coefficients of equations in (4.10):

$$\begin{array}{llll} h^n_\varepsilon \to h_\varepsilon & \text{and} & \theta^n_2 \to \theta_2 & \text{strongly in } L^2(Q_T), \\ \mathbf{u}^n_\varepsilon \to \mathbf{u}_\varepsilon & \text{and} & \mathbf{u}^n_2 \to \mathbf{u}_2 & \text{strongly in } L^2_\sigma(Q_T), \\ \mathbf{f}^n_\varepsilon \to \mathbf{f}_\varepsilon & & & \text{strongly in } L^2(0,T;L^2(\Omega)^N) \end{array}$$

where h^n_ε is taken monotone increasing. From (4.13) and the regularity $\mathbf{u}_\varepsilon, \mathbf{u}_2 \in L^\infty_\sigma(Q_T)$ and $\theta_2 \in L^\infty(Q_T)$ we deduce (for a new constant k_0)

$$0 < k(\varepsilon) \leq h_\varepsilon \leq k_0 \quad \text{and} \quad \max\{|\mathbf{f}^n_\varepsilon|, |\mathbf{u}^n_\varepsilon|, |\mathbf{u}^n_2|, |\theta^n_2|\} \leq k_0.$$

We rewrite (4.11) as

$$\begin{aligned}\int_\Omega \Theta_\varepsilon(T)\xi(T) &= \int_{Q_T} \mathbf{U}_\varepsilon \cdot [\mathbf{w}_t + (\mathbf{u}^n_\varepsilon \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}^n_2 - \xi\nabla\theta^n_2 + \Delta\mathbf{w}] + \\ &+ \int_{Q_T} \mathbf{U}_\varepsilon \cdot [((\mathbf{u}_\varepsilon - \mathbf{u}^n_\varepsilon) \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : (\mathbf{u}_2 - \mathbf{u}^n_2) - \xi\nabla(\theta_2 - \theta^n_2)] + \\ &+ \int_{Q_T} \Theta_\varepsilon (\xi_t + \mathbf{u}^n_\varepsilon \cdot \nabla\xi + \mathbf{f}^n_\varepsilon \cdot \mathbf{w} + h^n_\varepsilon \Delta\xi) + \\ &+ \int_{Q_T} \Theta_\varepsilon ((\mathbf{u}_\varepsilon - \mathbf{u}^n_\varepsilon) \cdot \nabla\xi + (\mathbf{f}_\varepsilon - \mathbf{f}^n_\varepsilon) \cdot \mathbf{w} + (h_\varepsilon - h^n_\varepsilon)\Delta\xi) - \\ &- \int_{\Sigma_T} (\varphi(\theta_\varepsilon) - \varphi(\theta_2))\nabla\xi \cdot \nu + \varepsilon \int_\Omega \xi(0), \end{aligned} \tag{4.14}$$

and choose the test functions as solutions of

$$\begin{cases} \mathcal{L}_1(\mathbf{w},\xi) := \mathbf{w}_t + (\mathbf{u}^n_\varepsilon \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}^n_2 - \xi\nabla\theta^n_2 + \Delta\mathbf{w} = 0 & \text{in } Q_T, \\ \mathcal{L}_2(\mathbf{w},\xi) := \xi_t + \mathbf{u}^n_\varepsilon \cdot \nabla\xi + \mathbf{f}^n_\varepsilon \cdot \mathbf{w} + h^n_\varepsilon \Delta\xi = 0 & \text{in } Q_T, \\ div\mathbf{w} = 0 & \text{in } Q_T, \\ \mathbf{w} = 0 \quad \text{and} \quad \xi = 0 & \text{on } \Sigma_T, \\ \mathbf{w}(x,T) = \mathbf{0} \quad \text{and} \quad \xi(x,T) = \chi_\delta(T) & \text{in } \Omega, \end{cases} \tag{4.15}$$

with $\chi_\delta \in \mathcal{C}_0^\infty(\Omega)$ satisfying $dist\,\{\Sigma_D, sop(\chi_\delta)\} \geq \delta$ and

$$\chi_\delta \rightharpoonup sign_+\{\varphi(\theta(T)) - \varphi(\theta_2(T))\} \quad \text{weakly in} \quad L^1(\Omega) \quad as \quad \delta \to 0.$$

Notice the similarity between problems (4.15) and (4.6). We again state the result on existence, uniqueness and regularity of (4.15) and prove it at the end of this section (together with Lemma 4.1).

LEMMA 4.2 *Problem* (4.15) *has a unique solution with the regularity of test functions of* (4.1) (*see* (2.7), (2.8), (2.9)). *Moreover,*

$$\begin{aligned} \mathbf{w} &\in H^1(0,T;L^2_\sigma(\Omega)) \cap L^\infty(0,T;W^{1,2}_\sigma(\Omega)) \cap L^2(0,T;W^{2,2}_\sigma(\Omega)) \cap L^\infty_\sigma(Q_T), \\ \xi &\in L^\infty(0,T;H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^\infty(Q_T), \end{aligned}$$

with estimates in the norms of these spaces that are independent of n. *Finally, there exists a* $C > 0$ *independent of* ε *and* n *such that*

$$\max\left\{\|\mathbf{w}\|_{L^\infty_\sigma(Q_T)}, \|\xi\|_{L^\infty(Q_T)}\right\} \leq C. \tag{4.16}$$

Continuation of the proof of Theorem 4.2. Using these test functions we get from (4.14) that

$$\begin{aligned} \int_\Omega \Theta_\varepsilon(T)\chi_\delta(T) &= \varepsilon \int_\Omega \xi(0) - \int_{\Sigma_T} (\varphi(\theta_\varepsilon) - \varphi(\theta_2))\nabla\xi\cdot\nu + \\ &+ \int_{Q_T} \mathbf{U}_\varepsilon \cdot [((\mathbf{u}_\varepsilon - \mathbf{u}^n_\varepsilon)\cdot\nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : (\mathbf{u}_2 - \mathbf{u}^n_2) + (\theta_2 - \theta^n_2)\nabla\xi] + \\ &+ \int_{Q_T} \Theta_\varepsilon\left((\mathbf{u}_\varepsilon - \mathbf{u}^n_\varepsilon)\cdot\nabla\xi + (\mathbf{f}_\varepsilon - \mathbf{f}^n_\varepsilon)\cdot\mathbf{w} + (h_\varepsilon - h^n_\varepsilon)\Delta\xi\right). \end{aligned}$$

Letting $n \to \infty$ and taking into account the uniform (in n) estimates for the test functions we get

$$\int_\Omega \Theta_\varepsilon(T)\chi_\delta(T) = \varepsilon \int_\Omega \xi(0) - \int_{\Sigma_T} \varphi(\varepsilon e^{-\lambda_1 t})\nabla\xi\cdot\nu. \tag{4.17}$$

The following lemma (that will be proved at the end of this section) will allow us to control the boundary integral when $\varepsilon \to 0$.

LEMMA 4.3 *Let* A_ε, $\mathbf{B}_\varepsilon$, $g_\varepsilon \in L^\infty(Q_T)$ *with*

$$k(\varepsilon) < A_\varepsilon,$$

where $k(\varepsilon)$ *is given by* (4.12). *Consider the problem*

$$\begin{cases} \psi_t + A_\varepsilon\Delta\psi + \mathbf{B}_\varepsilon\cdot\nabla\psi + g_\varepsilon = 0 & in\ Q_T, \\ \psi = 0 & on\ \Sigma_{DT}, \\ \nabla\psi\cdot\nu = 0 & on\ \Sigma_{NT}, \\ \psi(T,x) = \chi_\delta(x) & in\ \Omega, \end{cases}$$

with $\delta > 0$. *Then, there exist a* $\delta(\varepsilon) > 0$ *and a positive constant* c, *independent of* ε, *such that if* $\delta < \delta(\varepsilon)$ *then*

$$\nabla\psi\cdot\nu \geq -c\frac{\|\mathbf{B}_\varepsilon\|_{L^\infty(Q_T)}\|\psi\|_{L^\infty(Q_T)}}{k(\varepsilon)} \quad a.e.\ in \quad \Sigma_{DT}.$$

Continuation of the proof of Theorem 4.2. Applying Lemma 4.3 with $A_\varepsilon := h_\varepsilon$, $\mathbf{B}_\varepsilon := \mathbf{u}_\varepsilon$ and $g_\varepsilon := \mathbf{f}_\varepsilon \cdot \mathbf{w}$, and using (4.13) and the regularity $\mathbf{u}_\varepsilon \in L^\infty(0,T;L^\infty_\sigma(\Omega))$ together with (4.16) we obtain

$$\nabla\xi \cdot \nu \geq -\frac{c}{k(\varepsilon)} \quad \text{a.e. in} \quad \Sigma_T,$$

and by (4.12) we get $\dfrac{\varphi(\varepsilon e^{-\lambda_1 t})}{k(\varepsilon)} \leq \hat{c}\varepsilon$ for some $\hat{c} > 0$. Then from (4.17) we deduce

$$-\int_{\Sigma_T} \varphi(\varepsilon e^{-\lambda_1 t})\nabla\xi \cdot \nu \leq c\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We also have that (4.16) implies

$$\varepsilon \int_\Omega \xi(0) \to 0,$$

and therefore, letting $\varepsilon, \delta \to 0$ in (4.17) we get

$$\int_\Omega |\Theta(T)| = \lim_{\varepsilon\to 0} \int_\Omega (\mathbf{U}_\varepsilon(T) \cdot \mathbf{U}(T) + \Theta_\varepsilon(T)\chi_\delta(T)) \to 0,$$

and the assertion follows. □

Proof of Lemmas 4.1 and 4.2. The proofs of existence and uniqueness of solutions for problems (4.6) and (4.15) are identical. The only difference between both problems is estimate (4.8). We shall therefore prove existence and uniqueness of solutions of (4.15) and comment whether (4.8) necessarily holds or not.

1. A priori estimates. Multiplying first equation of (4.15) by $\mathbf{w}_t$ and using Hölder and Young's inequalities we get

$$\|\mathbf{w}_t\|_{L^2_\sigma(Q_T)} \leq c_1 \|\nabla\mathbf{w}\|_{L^2_\sigma(Q_T)} + \|\xi\|_{L^\infty(Q_T)} \|\nabla\theta_2^n\|_{L^2(Q_T)}, \tag{4.18}$$

with c_1 depending only on $\|\mathbf{u}_2^n\|_{L^\infty(Q_T)}$ and $\|\mathbf{u}_\varepsilon^n\|_{L^\infty(Q_T)}$. Defining $\mathbf{v} := \mathbf{w}e^{-\beta t}$, with $\beta > 0$ large enough (and only depending on $\|\mathbf{u}_2^n\|_{L^\infty(Q_T)}$) and multiplying now first equation of (4.15) by $\Delta\mathbf{v}$ we obtain

$$\beta \|\nabla\mathbf{w}\|_{L^\infty(0,T;L^2_\sigma(\Omega))} + \|\Delta\mathbf{w}\|_{L^2_\sigma(Q_T)} \leq e^{\beta T} \|\xi\|_{L^\infty(Q_T)} \|\nabla\theta_2^n\|_{L^2(Q_T)}. \tag{4.19}$$

Adding (4.18) to (4.19), using the continuous imbedding $L^\infty(0,T;L^2_\sigma(\Omega)) \subset L^2_\sigma(Q_T)$ and fixing $\beta := \max\{1, 2(c_1 + c_2)\}$ we get

$$\|\mathbf{w}_t\|_{L^2_\sigma(Q_T)} + \|\nabla\mathbf{w}\|_{L^2(0,T;L^2_\sigma(\Omega))} + \|\Delta\mathbf{w}\|_{L^2_\sigma(Q_T)} \leq C(T) \|\xi\|_{L^\infty(Q_T)} \tag{4.20}$$

with $C(T) := \left(1 + e^{\beta T}\right) \|\nabla\theta_2^n\|_{L^2(Q_T)}$. Notice that $C(T)$ does not depend on n or ε because the regularity $\mathbf{u}_2, \mathbf{u}_\varepsilon \in L^\infty_\sigma(Q_T)$ and $\nabla\theta_2 \in L^2(Q_T)$ implies uniform estimates for $\|\mathbf{u}_2^n\|_{L^\infty_\sigma(Q_T)}$ and $\|\nabla\theta_2^n\|_{L^2(Q_T)}$. Moreover, $C(T) \to 0$ as $T \to 0$. Due to the space dimension $N = 2$ and to the Sobolev's theorem we have

$$\|\mathbf{w}\|_{L^\infty_\sigma(Q_T)} \leq C(T) \|\xi\|_{L^\infty(Q_T)}. \tag{4.21}$$

Now we proceed to get a priori estimates for ξ. We use the notation $h_0 := \inf_{Q_T} h_\varepsilon^n$. Notice that in the case of problem (4.6) (with the obvious changes in notation) $h_0 > 0$ (see (4.5)) meanwhile for problem (4.15) it only holds $h_0 \geq k(\varepsilon)$ (see (4.13)). These facts together with estimate (4.23) below allow us to ensure that estimate (4.8) holds for problem (4.6) but, in general, it does not hold for problem (4.15). Multiplying the second equation of (4.15) by $e^{-\lambda(T-t)}\Delta\xi$, with $\lambda > 0$ large enough and proceeding as for the estimates of $\mathbf{w}$ we find

$$\|\nabla\xi\|_{L^\infty(0,T;L^2(\Omega))} \leq \tilde{C}(T)\left(\|\chi_\delta\|_{L^2(\Omega)} + h_0^{-1}\|\mathbf{f}^u\|_{L^\infty_\sigma(Q_T)}\|\mathbf{w}\|_{L^2_\sigma(Q_T)}\right) \tag{4.22}$$

and

$$h_0\|\Delta\xi\|_{L^2(Q_T)} \leq \tilde{C}(T)\left(\|\chi_\delta\|_{L^2(\Omega)} + h_0^{-1}\|\mathbf{f}^u\|_{L^\infty_\sigma(Q_T)}\|\mathbf{w}\|_{L^2_\sigma(Q_T)}\right). \tag{4.23}$$

Finally, by the Alexandrov's Maximum Principle (see [19]) we get

$$\|\xi\|_{L^\infty(Q_T)} \leq c\|\mathbf{w}\|_{L^\infty_\sigma(Q_T)}, \tag{4.24}$$

with c independent of ε and n (notice that this estimate can be obtained also as a consequence of the Sobolev's Theorem due to the space dimension $N = 2$, (4.22) and (4.23)).
2. Existence of solutions. Define

$$K := \left\{h \in L^2(Q_{T^*}) : \|h\|_{L^\infty(Q_{T^*})} < R\right\}$$

with R and $T^* < T$ to be fixed and the operator $Q : K \to L^2(Q_{T^*})$ by

$$Q(\hat{\xi}) := \xi,$$

with ξ solution of

$$\begin{cases} \mathcal{L}_2(\hat{\mathbf{w}}, \xi) = 0 & \text{in } Q_{T^*}, \\ \xi = 0 & \text{on } \Sigma_{T^*}, \\ \xi(0) = \chi_\delta(T^*) & \text{in } \Omega. \end{cases} \tag{4.25}$$

and $\hat{\mathbf{w}}$ solution of

$$\begin{cases} \mathcal{L}_1(\hat{\mathbf{w}}, \hat{\xi}) = 0 & \text{in } Q_{T^*}, \\ \hat{\mathbf{w}} = 0 & \text{on } \Sigma_{T^*}, \\ \hat{\mathbf{w}}(0) = \mathbf{U}(T) & \text{in } \Omega, \end{cases} \tag{4.26}$$

Since $\hat{\xi} \in L^\infty(Q_{T^*})$ we can justify the a priori estimates (4.18), (4.19), (4.20) and (4.21) for solutions of problem (4.26) (that does exist, see e.g., [22], Theorems 4.1 and 5.2), obtaining the regularity stated in Lemma 4.2 for $\mathbf{w}$. Due to this regularity we can justify the a priori estimates (4.22), (4.23) and (4.24) for solutions of (4.25) (that again are proved to exist by applying the results in [22]) and therefore also the regularity stated in Lemma 4.2 for ξ. We shall deduce the existence of a fixed point of Q (that will be a solution of (4.15) with the regularity inherenced from solutions of problems (4.25) and (4.26)) by applying a version of the fixed point theorem given in [2]. For this purpose we must show that K is convex and weakly compact in $L^2(Q_{T^*})$, which is a straightforward consequence of the definition of K, that $Q(K) \subset K$ and that Q is weakly-weakly sequentially continuous in $L^2(Q_{T^*})$. Let us first prove that $Q(K) \subset K$: given $\hat{\xi} \in K$ (independent of ε and n) we have by (4.21) that the solution of (4.26) satisfies

$$\|\hat{\mathbf{w}}\|_{L^\infty_\sigma(Q_{T^*})} \leq C(T^*)\left\|\hat{\xi}\right\|_{L^\infty(Q_{T^*})}, \tag{4.27}$$

and for this $\hat{\mathbf{w}}$ we get, by (4.24), that the solution of (4.25) verifies

$$\left\|Q(\hat{\xi})\right\|_{L^\infty(Q_{T^*})} := \|\xi\|_{L^\infty(Q_{T^*})} \leq c\|\hat{\mathbf{w}}\|_{L^\infty_\sigma(Q_{T^*})}, \tag{4.28}$$

and then, since $C(t) \to 0$ when $t \to 0$, it suffices to choose T^* small enough to get $cC(T^*) \leq 1$ and then from (4.27) and (4.28) and the definition of K

$$\left\| Q(\hat{\xi}) \right\|_{L^\infty(Q_{T^*})} \leq R.$$

Notice that these estimates are independent of ε and n.

To show the continuity we take a sequence $\hat{\xi}_j \in K$ with $\hat{\xi}_j \rightharpoonup \hat{\xi}$ in $L^2(Q_{T^*})$ and prove that $Q(\hat{\xi}) \rightharpoonup Q(\xi)$ in $L^2(Q_{T^*})$. Since $\hat{\xi}_j$ is bounded in $L^\infty(Q_{T^*})$ it follows from (4.20) and (4.21) that the sequence of solutions $\hat{\mathbf{w}}_j$ of (4.26) associated to $\hat{\mathbf{w}}_j$ is bounded in $L^\infty_\sigma(Q_{T^*}) \cap L^2(0,T^*;W^{2,2}_\sigma(\Omega))$ and therefore there exists a subsequence and a $\hat{\mathbf{w}}$ in such space with $\hat{\mathbf{w}}_j \rightharpoonup \hat{\mathbf{w}}$ weakly $*$ in $L^\infty_\sigma(Q_{T^*})$, strongly in $L^2(0,T^*;W^{1,2}_\sigma(\Omega))$ and a.e. in Q_{T^*}. Linearity of the problem and smoothness of the coefficients allows us to identify $\hat{\mathbf{w}}$ as the weak solution of (4.26) associated to $\hat{\xi}$. On the other hand, since $\hat{\mathbf{w}}_j$ is bounded in $L^\infty_\sigma(Q_{T^*}) \cap L^2(0,T^*;W^{1,2}_\sigma(\Omega))$ it follows from (4.24) and (4.23) that the sequence of solutions $\hat{\xi}_j$ of (4.25) associated to $\mathbf{w}_j$ (that, by definition, is $Q(\hat{\xi}_j)$) is bounded in $L^\infty(Q_{T^*}) \cap L^2(0,T^*;H^1_0(\Omega))$ and therefore converges weakly in $L^2(Q_{T^*})$ to an element ξ of K. Again the linearity allows us to identify the limit as the solution of problem (4.25) associated to $\hat{\mathbf{w}}$. Hence, the continuity of Q is established. Notice that all a priori estimates are continuous with respect to time so we can take $T^* = T$. Finally, uniqueness of solutions is a consequence of the linearity of problem (4.15) and the regularity of solutions.□

Proof of Lemma 4.3. Since $\partial\Omega$ is regular, Ω has the property of the exterior sphere, i.e., for all $x_0 \in \partial\Omega$ there exists a $R_1 > 0$ and a $x_1 \in I\!R^N \backslash \bar{\Omega}$ such that

$$B(x_1, R_1) \cap \bar{\Omega} = \{x_0\},$$

where $B(x_1,R_1) := \left\{ x \in I\!R^N : |x - x_1| < R_1 \right\}$. Consider $\delta > 0$ small enough such that, by defining $R_2 := \delta + R_1$, it holds $B(x_1,R_2) \cap \partial\Omega \neq \emptyset$. Since $dist(\partial\Omega, sop(\chi_\delta)) \geq \delta$ we have $\chi_\delta \equiv 0$ in $\omega := \Omega \cap B(x_1,R_2)$. We shall use the notation $k_0(\varepsilon) := \|g\|_{L^\infty(Q_T)}$, $k_1(\varepsilon) := \left(\frac{N-1}{R_1} + 1\right) \|\mathbf{B}\|_{L^\infty(Q_T)}$ and $k_2(\varepsilon) := \|\psi\|_{L^\infty(Q_T)}$. We define

$$\mathcal{L}(\psi) := \psi_t + A_\varepsilon \Delta\psi + \mathbf{B} \cdot \nabla\psi \quad \text{and} \quad w(x,t) := \psi(x,t) + \sigma(r),$$

where $(x,t) \in \omega \times (0,t)$, $r := |x - x_0|$ and $\sigma \in C^2([R_1,R_2])$ will be chosen such that the maximum of w in $\bar{\omega} \times [0,T]$ is attained in $\{x_0\} \times [0,T]$, and such that $\sigma''(r) \geq 0$ and $\sigma'(r) \leq 0$. Assuming these properties we get, due to (4.3), that w satisfies

$$\mathcal{L}(w) = -g + A_\varepsilon \Delta\sigma + \mathbf{B} \cdot \nabla\sigma \geq k(\varepsilon)\sigma''(r) + k_1(\varepsilon)\sigma'(r) - k_0(\varepsilon).$$

Choosing $\sigma(r) := \frac{k_0(\varepsilon)}{k_1(\varepsilon)} r + C_2 e^{-\frac{k_1(\varepsilon)}{k(\varepsilon)} r}$, with C_2 an arbitrary constant, we obtain

$$k(\varepsilon)\sigma''(r) + k_1(\varepsilon)\sigma'(r) - k_0(\varepsilon) = 0, \quad \sigma''(r) \geq 0 \quad \text{and}$$

$$\text{if} \quad C_2 \geq k(\varepsilon) \frac{k_0(\varepsilon)}{k_1^2(\varepsilon)} e^{\frac{k_1(\varepsilon)}{k(\varepsilon)} R_2} \quad \text{then} \quad \sigma'(r) \leq 0. \tag{4.29}$$

Taking C_2 with this restriction we have $\mathcal{L}(w) \geq 0$ in $\bar{\omega} \times [0,T]$ and therefore, by the Maximum Principle we deduce w attains its maximum on the parabolic boundary of $\omega \times [0,T]$. In this boundary the values of w may be estimated as follows:

$$\begin{cases} w(x,t) = \sigma(r) \leq \sigma(R_1) & \text{on } (\Gamma_D \cap \partial\omega) \times [0,T], \\ w(x,t) = \psi(x,t) + \sigma(r) \leq k_2(\varepsilon) + \sigma(R_2) & \text{on } (\partial B(x_1,R_2) \cap \partial\omega) \times [0,T], \\ w(x_0,t) = \sigma(R_1) & \text{on } [0,T], \\ w(x,T) = \sigma(r) + \chi_\delta(x) \leq \sigma(R_1) & \text{in } \omega, \end{cases}$$

where we have used that $\chi_\delta \equiv 0$ in ω. It is a straightforward computation to see that we can choose C_2 (by making δ small enough) such that (4.29) and $\sigma(R_1) = k_2(\varepsilon) + \sigma(R_2)$ hold. As a consequence we obtain $\nabla w(x_0, t) \cdot \nu \geq 0$ and by the definition of w and taking δ suitably we obtain

$$\nabla \psi(x_0, t) \cdot \nu \geq -c \frac{k_1(\varepsilon) k_2(\varepsilon)}{k(\varepsilon)} \quad \text{in } [0, T]\,.$$

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