Centrum voor Wiskunde en Informatica

REPORT *RAPPORT*

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Probability, Networks and Algorithms (PNA)

PNA-R9715 October 31, 1997

Report PNA-R9715
ISSN 1386-3711

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ABSTRACT

We indicate how granulometries may be useful in the analysis of random sets. We define a suitable size distribution function as a tool in exploratory data analysis and give a new Hanisch-style estimator for it. New Markov random sets are constructed which favour certain sizes above others. Applications and examples on real and simulated data sets are included.

1991 Mathematics Subject Classification: 60G55, 62M30, 68U10.

Keywords and Phrases: contact distribution function, empty space function, granulometry, Hanisch-style estimator, Markov random set, morphological opening and closing, size distribution function.

Note: Work carried out under project PNA4.4 "Stochastic geometry".

1. Introduction

One of the most basic properties of an object is its size. It is no wonder then that size measures have been used for a long time in the empirical sciences, and more recently in the analysis of (binary) images. For instance Serra [29] employs size distributions for shape and texture analysis, Maragos uses them for multiscale shape representation [19], [12, 26] apply size distributions to shape filtering and restoration problems, while Sivakumar [28] gives applications in texture classification and morphological filtering.

In this paper, a sequel to [18], we suggest how size distribution functions may be used in stochastic geometry both as a descriptive tool in exploratory data analysis (cf. Ripley [22]) and to build new models. Section 2 reviews the basic morphological operators and shows how they can be used to define a granulometry to measure size, while Section 3 provides some background in stochastic geometry. In Section 4 we introduce the size distribution function of a stationary random closed set and study basic properties. Section 5 focuses on the estimation of the size distribution function, and proposes a new estimator in the spirit of Hanisch [10, 5, 11]. Applications in exploratory data analysis are the topic of Section 6. New Markov random set models are constructed in Section 7 from a reference Boolean model by biasing towards certain sizes. The discrete analogues have been studied by Sivakumar and Goutsias [27]. Some examples are included as well (Section 8).

2. Morphological granulometries

Perhaps the oldest and most frequently used technique to quantify the size of solid particles in the empirical sciences is to use a series of sieves with varying mesh openings. Clearly, the particles that cannot pass through any given sieve are a subset of the total collection of particles; if the sieve is solid, no particle can pass through it, and if more particles are sieved more will be left over. Moreover, if we sieve the particles successively with two different mesh sizes, the result will be the same as using only the one with the biggest mesh opening. These simple but essential features of sieving (as well as of other 'sizing' methods, see [29]) underly the following definition by Matheron [20].

Definition 1 (Matheron, 1975) *A family of operators* $\psi_r : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ *on the power set* $\mathcal{P}(\mathbb{R}^d)$ *of* $\mathbb{R}^d$*, indexed by* $r \geq 0$ *is a* granulometry *if for all* $X \subseteq \mathbb{R}^d$

(G0) $\psi_0(X) = X$;

(G1) $\psi_r(X) \subseteq X$ *for all* $r \geq 0$;

(G2) *if* $Y \subseteq X$ *then* $\psi_r(Y) \subseteq \psi_r(X)$ *for all* $r \geq 0$;

(G3) $\psi_r(\psi_s(X)) = \psi_s(\psi_r(X)) = \psi_{\max(r,s)}(X)$ *for all* $r, s \geq 0$.

Condition (G3) is sometimes referred to as the *sieving condition*. Intuitively, $\psi_r(X)$ can be thought of as the subset of particles in X that remain after sieving with a mesh size $r \geq 0$.

In this paper, we are interested in a special class of granulometries based on Euclidean openings. Recall that the *Minkowski addition* of a set X with *structuring element* $B \subseteq \mathbb{R}^d$ is defined by

$$X \oplus B = \{h \in \mathbb{R}^d : \check{B}_h \cap X \neq \emptyset\} \tag{2.1}$$

and similarly the *Minkowski subtraction* is given by

$$X \ominus B = \{h \in \mathbb{R}^d : \check{B}_h \subseteq X\}. \tag{2.2}$$

Here we write $\check{B} = \{-b : b \in B\}$ for the reflection of B in the origin and use the subscript h for translation over the vector h. Seen as set operators in X, (2.1)–(2.2) are referred to as *dilation* and *erosion* respectively. The operators are dual, that is dilating the complement X^c of a set X amounts to eroding X itself: $(X^c \oplus B)^c = (X \ominus B)$.

Compositions of Minkowski addition and subtraction define the *opening*

$$X \circ B = (X \ominus \check{B}) \oplus B = \bigcup \{B_h : h \in \mathbb{R}^d, B_h \subseteq X\} \tag{2.3}$$

and the *closing*

$$X \bullet B = (X \oplus \check{B}) \ominus B = (X^c \circ B)^c. \tag{2.4}$$

Both opening and closing are increasing and idempotent (i.e. $(X \circ B) \circ B = X \circ B$). The opening is anti-extensive ($X \circ B \subseteq X$), whereas the closing is extensive ($X \subseteq X \bullet B$). Moreover, if the structuring element B is a non-empty compact set and X is closed or compact, then the opening and closing operators are upper semi-continuous in their two arguments. For details see [13, 20, 29].

The opening (2.3) can be used to define a granulometry. Given a non-empty convex compact structuring element B, let $rB = \{rb : b \in B\}$ ($r \geq 0$) be its homothetics. Then

$$\psi_r(X) = X \circ rB, \quad r \geq 0 \tag{2.5}$$

is a granulometry [20]. To see this, note that $\psi_0(X) = X \circ \{0\} = X$, from which (G0) follows. By the anti-extensivity of the opening, ψ_r satisfies (G1) for all $r \geq 0$ and (G2) is a direct consequence of the fact that the opening is increasing. For the sieving condition, we need the following Lemma to show that for $r \geq s$, rB is open with respect to sB and hence that $(X \circ rB) \circ sB = (X \circ sB) \circ rB = X \circ rB$.

Lemma 1 *Define a set $B \subseteq \mathbb{R}^d$ to be open with respect to another set C if $B \circ C = B$ [20]. Then the following properties hold.*

1. *If B is open with respect to C then for every $X \subseteq \mathbb{R}^d$, $(X \circ B) \circ C = X \circ B$.*
2. *If C is open with respect to B then for every $X \subseteq \mathbb{R}^d$, $(X \circ B) \circ C = X \circ C$.*
3. *Let B be a non-empty compact convex structuring element and let $r \geq s \geq 0$. Then rB is open with respect to sB.*

Sketches of the proofs can be found in [29], but we include them here for completeness' sake.

Proof : For 1.) note that as the opening is anti-extensive, $(X \circ B) \circ C \subseteq X \circ B$. Reversely, let $x \in X \circ B$. Then $\exists h : x \in B_h \subseteq X$. Since B is open with respect to C, $x \in (B \circ C)_h \subseteq X$ and hence $\exists k$ such that $x \in C_k \subseteq B_h \subseteq X$. Thus $x \in C_k \subseteq X \circ B$ for some k, and therefore $x \in (X \circ B) \circ C$.

Regarding 2.), since the opening is increasing, $(X \circ B) \circ C \subseteq X \circ C$. Reversely, let $x \in X \circ C$. Then $\exists h : x \in C_h \subseteq X$. Now $C_h \circ B \subseteq X \circ B$ and since C is open with respect to B, the left hand side equals $(C \circ B)_h = C_h$. Hence $x \in C_h \subseteq X \circ B$, thus $x \in (X \circ B) \circ C$.

Finally for 3.) we need to prove that $rB \circ sB = rB$ for any $r \geq s$. Since the opening operator is anti-extensive, $rB \circ sB \subseteq rB$. To prove the reverse inclusion, for any $x \in rB$ we need to find an h such that $x \in (sB)_h \subseteq rB$. To do so, write $x = rb$ for some $b \in B$ and set $y = sb, h = x - y = (r - s)b$. Then $x = x - y + y = h + y = h + sb \in (sB)_h$. Furthermore $(sB)_h = \{(r-s)b + sb' : b' \in B\} = r\{(1-\frac{s}{r})b + \frac{s}{r}b' \ : b' \in B\} \subseteq rB$, using the convexity of B. □

In summary, openings with structuring elements of varying 'mesh' can be used to quantify the size of a (random) set X. By duality, the size of the empty space X^c can be measured by the associated *anti-granulometry*

$$\psi_r(X^c)^c = (X^c \circ rB)^c = X \bullet rB,$$

the closings with structuring elements rB, $r \geq 0$.

3. Random sets and contact distributions

Stochastic geometry [30] is concerned with the study of random closed sets (rcs). Roughly speaking, an rcs is a mapping X from a probability space into the family of closed subsets of $\mathbb{R}^d$ such that

$$\{X \cap K \neq \emptyset\}, \quad \{X \cap G = \emptyset\}$$

are measurable for all compact sets K and all open sets G. (The definition is a bit redundant, since only the measurability involving the compact sets is needed). The probability distribution of X is completely specified by its *capacity functional*

$$T(K) = \mathbb{P}(X \cap K \neq \emptyset) \tag{3.1}$$

with K ranging over the class of compact sets, which plays a role comparable to that of the distribution function of a real-valued random variable. However, the collection of test sets K is huge, and lower-dimensional summary statistics are called for. Typically these are obtained from the capacity functional (3.1) by restricting the choice of K. For instance, allowing only singletons results in the coverage probabilities $p(x) = \mathbb{P}(x \in X)$, $x \in \mathbb{R}^d$.

Below we will assume that the random closed set X is stationary, i.e. its distribution is invariant under translations. In that case, the coverage probabilities $p(x)$ do not depend on the argument x, and $p(x) \equiv p = \mathbb{E}|X \cap U|$, the expected volume covered within any set U of unit volume $|U| = 1$. In the remainder of this paper, we will assume that $0 < p < 1$.

A summary statistic for assessing the 'size' of pores left open by a stationary rcs X is the *empty space function* $F_B(\cdot)$ defined by taking $K = (rB)_x$ in (3.1), that is

$$F_B(r) = \mathbb{P}(x \in X \oplus r\check{B}) \tag{3.2}$$

or the related *contact distribution function*

$$H_B(r) = \mathbb{P}(x \in X \oplus r\check{B} \mid x \notin X) = \frac{F_B(r) - F_B(0)}{1 - F_B(0)} \tag{3.3}$$

for $r \geq 0$. By the stationarity of X, these definitions do not depend on the choice of x. Both (3.2) and (3.3) depend on a structuring element B. Typical choices include balls and squares, although non-isotropic structuring elements may be preferred when investigating directional effects. If B is a compact convex set containing a neighbourhood of the origin, then $H_B(\cdot)$ is a distribution function but this is not the case for general B (see [30]). Note that $1 - H_B(r)$ can be interpreted as the conditional probability that a copy of rB placed at a test point 0 lies entirely in the empty space left by X given that the test point itself does not fall in X. A similar interpretation holds for the empty space function $F_B(r)$.

Definitions (3.2)–(3.3) can be seen as the stochastic counterparts of the granulometries defined in Section 2 with particle size measured by

$$\psi_r(Y) = (Y^c \oplus r\check{B})^c = Y \ominus r\check{B}.$$

However, although under our restrictions on B, $\psi_r(\cdot)$ satisfies (G0)–(G2), in general the sieving condition does not hold. Thus, in the remainder of this paper, we will consider replacing the dilation in (3.2)–(3.3) by a closing to obtain a proper granulometry.

4. Size distribution functions

In mathematical morphology, the *size distribution law* of the particles forming a stationary random closed set X is defined using the granulometry (2.5) described in Section 2 as follows [29, p. 335]

$$G_1(r) = 1 - \mathbb{P}(x \in X \circ rB \mid x \in X) = 1 - \frac{\mathbb{P}(x \in X \circ rB)}{\mathbb{P}(x \in X)}, \quad r \geq 0;$$

the size distribution law of the pores is

$$G_0(r) = \mathbb{P}(x \in X \bullet rB \mid x \notin X) = 1 - \frac{1 - \mathbb{P}(x \in X \bullet rB)}{1 - \mathbb{P}(x \in X)}, \quad r \geq 0.$$

By the stationarity of X, the definitions do not depend on the choice of $x \in \mathbb{R}^d$. Intuitively, $G_1(r)$ will be the conditional probability that a point in X is eliminated by opening with rB ($r > 0$), i.e. the probability that the B-size of X is less than r. Similarly, $G_0(r)$ is the conditional probability that a point from X^c is included by closing with rB, hence that the B-size of the complement is less than or equal to r. By allowing r to vary over $(-\infty, \infty)$, the joint size distribution law

$$G(r) = \begin{cases} 1 - \mathbb{P}(x \in X \circ rB) & r \geq 0 \\ 1 - \mathbb{P}(x \in X \bullet |r|B) & r < 0 \end{cases} \tag{4.1}$$

of pores and particles is obtained. However, from a probabilistic point of view, G (and G_1) is not a proper distribution function, as it is semi-continuous from the left rather than from the right. For this reason we prefer the following definition.

Definition 2 *Let X be a stationary random closed set and B a non-empty convex compact structuring element. Define the* size distribution function *of X by*

$$P_B(r) = \begin{cases} \mathbb{P}(x \in X \bullet rB) & r \geq 0 \\ \mathbb{P}(x \in X \circ |r|B) & r < 0 \end{cases}. \tag{4.2}$$

The function $P_B(\cdot)$ is called the *granulométrie bidimensionelle* in metallurgi [6]. It is easily verified that $P_B(0) = p$, the coverage fraction of the stationary random closed set X. Indeed,

$$P_B(0) = \mathbb{P}(0 \in X \bullet \{0\}) = \mathbb{P}(0 \in X) = p.$$

The next Theorem collects some properties of the size distribution function.

Theorem 1 *Let X be a stationary random closed set and B a non-empty convex compact structuring element. Then $P_B(\cdot)$ is well-defined and does not depend on the choice of $x \in \mathbb{R}^d$. Seen as a function of r, $P_B(\cdot)$ takes values in $[0,1]$, is increasing and semi-continuous from the right.*

Proof : First note that since $B \neq \emptyset$ is compact, $X \oplus rB$, $X \ominus rB$ and hence $X \circ rB$, $X \bullet rB$ ($r \geq 0$) are closed sets (see [20, p. 19]). Since X is stationary, so are $X \circ rB$ and $X \bullet rB$ implying that $P_B(r)$ is well-defined as the coverage fraction of the random closed set $X \bullet rB$ (for $r \geq 0$) or $X \circ |r|B$ (for $r < 0$). In particular, $P_B(r)$ does not depend on the choice of x in (4.2).

To check that $P_B(\cdot)$ is increasing, first consider $r \geq s \geq 0$. Then using Lemma 1 and the fact that the opening operator is anti-extensive yields

$$X \bullet rB = (X^c \circ rB)^c = ((X^c \circ sB) \circ rB)^c \supseteq (X^c \circ sB)^c = X \bullet sB,$$

hence $P_B(r) \geq P_B(s)$. Also, for $r \leq s < 0$, $\psi_{|r|}(X) = \psi_{|r|}(\psi_{|s|}(X)) \subseteq \psi_{|s|}(X)$ since ψ_t is a granulometry (using the notation of (2.5)). Finally, $X \circ |r|B \subseteq X$ implies $\mathbb{P}_B(r) \leq \mathbb{P}_B(0)$.

The mapping $(r, X) \mapsto X \bullet rB$ is upper semi-continuous [20, 1-5-1, 1-5-2] and increasing in r. Hence as $r_n \downarrow r \geq 0$, $X \bullet r_n B \downarrow X \bullet rB$. Thus $P_B(\cdot)$ is right-continuous on $[0, \infty)$. Similarly the mapping $(r, X) \mapsto X \circ rB$ is upper semi-continuous and decreasing in $r > 0$, hence $0 < r_n \uparrow r$ implies $X \circ r_n B \downarrow X \circ rB$. Thus if $0 > s_n \downarrow s$, then $0 < -s_n \uparrow -s$, and $X \circ |s_n| B \downarrow X \circ |s| B$. We conclude that $P_B(\cdot)$ is right-continuous.

Finally, since $P_B(\cdot)$ is defined in terms of probabilities, it takes its values in $[0, 1]$. □

Summarising, $P_B(\cdot)$ is a proper distribution function. Compared to the empty-space function (3.2), note that the latter is absolutely continuous except in 0 (see Hansen et al. [11]) but that this property does not generally hold for the size distribution which may have countably many discontinuities.

Explicit expressions for size distributions may be hard to find. For instance, consider a *Boolean model*. This is a random closed set defined in two steps: first a Poisson process of *germs* is generated (with intensity $\lambda > 0$); then to each of the germs x_i, a random compact *grain* K_i is assigned according to probability distribution $\mu(\cdot)$, independently of other grains. The union

$$\Xi = \bigcup_i (x_i \oplus K_i)$$

is called a Boolean model.

In order to compute $P_B(r)$, we need $\mathbb{P}(0 \in \Xi \bullet rB) = \mathbb{P}(0 \in (\Xi \oplus r\check{B}) \ominus rB)$. Now $0 \in (\Xi \oplus r\check{B}) \ominus rB$ iff $r\check{B} \subseteq \Xi \oplus r\check{B}$, and thus $P_B(r)$ is related to the covering probabilities of $\Xi \oplus r\check{B}$ (see Hall [9]). Unfortunately, such probabilities are very hard to compute analytically. Similarly for $P_B(-r)$, note that $0 \in \Xi \circ rB$ whenever there exists an h such that $0 \in (rB)_h \subseteq \Xi$ which is again related to covering probabilities.

A more tractable exception is formed by the class of linear granulometries $P_L(\cdot)$ with structuring elements $B = \{\lambda u : -1 \leq \lambda \leq 1\}$ for some unit vector $u \in \mathbb{R}^d$. Then

$$P_L(r) = \mathbb{P}(0 \in X \bullet rB) = F_L(r) - r f_L(r)$$

where $f_L(\cdot)$ is the density of the empty space function $F_L(\cdot)$, see [20, 11].

On the other hand, the contact distribution functions for Boolean models (but not for most other models!) are easy to compute:

$$H_B(r) = 1 - \frac{\mathbb{P}(\Xi \cap rB = \emptyset)}{1 - p} = 1 - \exp[-\lambda(\mathbb{E}|\check{K}_0 \oplus rB| - \mathbb{E}|K_o|)]$$

which by the Steiner formula [30] reduces to

$$H_B(r) = 1 - \exp\left[-\frac{\lambda}{b_d} \sum_{k=1}^{d} \binom{d}{k} \bar{W}_k W_{d-k}(rB)\right] \tag{4.3}$$

if the grains as well as B are convex compact sets containing a neighbourhood of the origin. Here b_d denotes the volume of a $d-$ dimensional unit ball, and $\bar{W}_k$ is the mean k-th order Minkowski mass of the primary grain K_0. For instance in $\mathbb{R}^2$, $W_0(K_0)$ is the area of K_0, $W_1(K_0)$ half its perimeter and $W_2 \equiv \pi$.

From the point of view of statistical inference, (4.3) implies that the contact distribution function (and hence also the empty space function and other derived functionals) depends only on the moments of the primary convex grain. For instance in the planar case, the mean perimeter determines the whole of $H_B(r)$, which may result in poor distinguishing power as an exploratory data analysis tool. See [22] for a discussion of this issue in connection to a particular data set. We will return to this point in Section 6.

5. Estimation and edge effects

In this Section we discuss estimating the size distribution function $P_B(r)$ of a stationary rcs X (cf. Definition 2). As a first step, note that for any $r \geq 0$ and any $A \subseteq \mathbb{R}^d$ of positive volume $|A| > 0$,

$$\frac{\mathbb{E}|(X \bullet rB) \cap A|}{|A|} = \frac{1}{|A|} \int_A \mathbb{P}(a \in X \bullet rB)\, da = P_B(r) \tag{5.1}$$

using Robbins' theorem [16, 24, 25] (essentially Fubini's theorem in a random set context) and the fact that X is stationary. Hence the volume fraction of $X \bullet rB$ (or $X \circ |r|B$ for negative r) in any set A of positive volume yields a pointwise unbiased estimator of $P_B(r)$.

However, in practice, a random set is not observed over the whole space, but within some compact window W of positive volume $|W|$ (typically a square or rectangle). Thus, due to edge effects caused by parts of X outside W, $X \bullet rB$ and $X \circ |r|B$ are not completely observable and the volume fraction estimator (5.1) with $A = W$ may be biased. To overcome this problem, a minus sampling estimator [22, 30]

$$\hat{P}_B(r) = \begin{cases} \frac{|(X \bullet rB) \cap (W \ominus (rB \oplus r\check{B}))|}{|W \ominus (rB \oplus r\check{B})|} & r \geq 0 \\ \frac{|(X \circ |r|B) \cap (W \ominus (|r|B \oplus |r|\check{B}))|}{|W \ominus (|r|B \oplus |r|\check{B})|} & r < 0 \end{cases} \tag{5.2}$$

can be used. This estimator is based on the *local knowledge principle* [29] for openings and closings stating that if the random set X is observed in the compact window W then $X \circ rB$ and $X \bullet rB$ ($r \geq 0$) are observable within $W \ominus (rB \oplus r\check{B})$. More specifically,

$$(X \bullet rB) \cap (W \ominus (rB \oplus r\check{B})) = ((X \cap W) \bullet rB) \cap (W \ominus (rB \oplus r\check{B}));$$

a similar expression holds for the opening.

From (5.1) with $A = W \ominus (rB \oplus r\check{B})$ it follows easily that the minus sampling estimator in (5.2) is unbiased whenever $|W \ominus (rB \oplus r\check{B})| > 0$.

Lemma 2 *Consider the set operator $\hat{P}_B(r, X)$ defined by the right hand side of (5.2) as a function of $X \subseteq \mathbb{R}^d$. Then*

$$\hat{P}_B(r, X^c) = 1 - \hat{P}_B(-r, X)$$

for all $r \geq 0$ such that $|W \ominus (rB \oplus r\check{B})| > 0$,

Proof : Using the duality of opening and closing, write

$$\begin{aligned} \hat{P}_B(r, X^c) &= \frac{|(X^c \bullet rB) \cap (W \ominus (rB \oplus r\check{B}))|}{|W \ominus (rB \oplus r\check{B})|} = \frac{|(X \circ rB)^c \cap (W \ominus (rB \oplus r\check{B}))|}{|W \ominus (rB \oplus r\check{B})|} \\ &= \frac{|W \ominus (rB \oplus r\check{B})| - |(X \circ rB) \cap (W \ominus (rB \oplus r\check{B}))|}{|W \ominus (rB \oplus r\check{B})|} = 1 - \hat{P}_B(-r, X). \end{aligned}$$

□

Although unbiased, as both numerator and denominator in (5.2) depend on r, there is no guarantee that the minus sampling estimator is monotone in r. Nor is all available information used. More refined techniques based on survival analysis ideas have been used by Hansen et al. [11] for deriving a Kaplan-Meier type estimator [1] for the empty space function (3.2). Chiu and Stoyan [5] showed that the ideas underlying the Kaplan-Meier approach are very similar to those involved in the Hanisch estimator [10]. In the remainder of this Section, we will derive a Hanisch-type estimator for the size distribution function (Definition 2). To do so, we need three local size measures: with respect to X, its empty spaces and the boundary.

Definition 3 *Let B be a non-empty convex compact structuring element. Define for $A \subseteq \mathbb{R}^d$*

$$\rho(x, A) = \begin{cases} \sup\{r \geq 0 : \exists h \text{ such that } x \in (rB)_h \subseteq A\} & x \in A \\ 0 & x \notin A \end{cases} \tag{5.3}$$

Thus, $\rho(x, A)$ measures the B-size of a point x in A.

Lemma 3 *Let $\rho(\cdot, \cdot)$ be as in Definition 3. Then for any closed set X and $r \geq 0$,*

$$X \circ rB = \{x \in X : \rho(x, X) \geq r\}.$$

Note that for $r > 0$, the restriction to $x \in X$ may be omitted.

Proof : Since $r_n \uparrow r$ implies $X \circ r_n B \downarrow X \circ rB$ (see Section 4), the supremum in (5.3) is attained and moreover $x \in X \circ rB \Leftrightarrow \rho(x, X) \geq r$. □

Definition 4 *Let B be a non-empty convex compact structuring element. Define for $A \subseteq \mathbb{R}^d$*

$$\eta(x, A) = \begin{cases} \inf\{r \geq 0 : x \in A \bullet rB\} & x \notin A \\ 0 & x \in A \end{cases} \tag{5.4}$$

Thus $\eta(x, A)$ measures the B-size of voids at x left by A.

Lemma 4 *Let $\eta(\cdot, \cdot)$ be as in Definition 4. Then for any closed set X and $r \geq 0$,*

$$X \bullet rB = \{x \in \mathbb{R}^d : \eta(x, X) \leq r\}.$$

Proof : Since $r_n \downarrow r$ implies $X \bullet r_n B \downarrow X \bullet rB$ (see Section 4), the infimum in (5.4) is attained and moreover $x \in X \bullet rB \Leftrightarrow \eta(x, X) \leq r$. □

We already saw that observed distances are occluded by the edges of the sampling window. Hence our final function measures the 'distance' from any point in W to the window boundary. That is

$$\zeta(t, W^c) = \begin{cases} \inf\{r \geq 0 : (rB \oplus r\check{B})_t \cap W^c \neq \emptyset\} & t \in W \\ 0 & t \notin W \end{cases} \tag{5.5}$$

Lemma 5 *For any compact window W, $r \geq 0$ and $\zeta(\cdot,\cdot)$ defined above*

$$W \ominus (rB \oplus r\check{B}) = \{t \in W : \zeta(t, W^c) \geq r\}.$$

The infimum in (5.5) is not attained.

Proof : Note that

$$\zeta(t, W^c) = \inf\{r \geq 0 : t \notin W \ominus (rB \oplus r\check{B})\}$$

Let $r_n \uparrow r$. Then by [20, 1-5-1], the mapping $s \mapsto W \ominus (s \oplus sB)$ is upper semi-continuous and decreasing. Hence $W \ominus (r_n B \oplus r_n \check{B}) \downarrow W \ominus (rB \oplus r\check{B})$. Therefore, the infimum in (5.5) is not attained, and $t \in W \ominus (rB \oplus r\check{B}) \Leftrightarrow \zeta(t, W^c) \geq r$. □

We now turn to expressing the minus sampling estimator (5.2) in terms of ρ, η and ζ. First consider the case $r \geq 0$. Then we can write

$$\hat{P}_B(r) = \frac{\int_W 1\{\eta(t,X) \leq r \leq \zeta(t,W^c)\}dt}{\int_W 1\{\zeta(t,W^c) \geq r\}dt}$$

Similarly for $r < 0$,

$$\hat{P}_B(r) = \frac{\int_W 1\{\rho(t,X) \geq |r|; \zeta(t,W^c) \geq |r|\}dt}{\int_W 1\{\zeta(t,W^c) \geq |r|\}dt} = 1 - \frac{\int_W 1\{\rho(t,X) < |r|; \zeta(t,W^c) \geq |r|\}dt}{\int_W 1\{\zeta(t,W^c) \geq |r|\}dt}$$

where we used duality in the last equation above (see Lemma 2). In practice one usually does not compute the integrals, but rather discretises over a sampling grid $T = \{t_i\} \subseteq W$ as follows

$$\hat{P}_B(r) = \begin{cases} \frac{\#\{i:\eta(t_i,X) \leq r \leq \zeta(t_i,W^c)\}}{\#\{i:\zeta(t_i,W^c) \geq r\}} & r \geq 0 \\ 1 - \frac{\#\{i:\rho(t_i,X) < |r|; \zeta(t_i,W^c) \geq |r|\}}{\#\{i:\zeta(t_i,W^c) \geq |r|\}} & r < 0 \end{cases} \tag{5.6}$$

The discretised minus sampling estimator (5.6) is pointwise unbiased for those r for which $\#\{i : \zeta(t_i, W^c) \geq |r|\} > 0$. For instance for positive r,

$$\mathbb{E}\#\{i : \eta(t_i, X \cap W) \leq r \leq \zeta(t_i, W^c)\} = \sum_{t_i \in W \ominus (rB \oplus r\check{B})} \mathbb{P}(t_i \in X \bullet rB)$$

$$= P_B(r)\,\#\{i : \zeta(t_i, W^c) \geq r\}$$

and similarly for $r < 0$.

Note that (5.2) and (5.6) do not use all information contained in the data. In particular, if $x \notin W \ominus (rB \oplus r\check{B})$, but $\eta(x, X) \leq \zeta(x, W^c)$ the correct void size at x is measured. Using this observation, one can define a Hanisch-type estimator for $P_B(r)$ ($r \geq 0$) by replacing the condition $\zeta(t_i, W^c) \geq r$ by $\zeta(t_i, W^c) \geq \eta(t_i, X)$ with a similar adaptation for $r < 0$:

$$\hat{P}_B^H(r) = \begin{cases} \int_W \frac{1\{x \in X \bullet rB\}\, 1\{x \in W \ominus (\eta(x,X)B \oplus \eta(x,X)\check{B})\}}{|W \ominus (\eta(x,X)B \oplus \eta(x,X)\check{B})|}\, dx & r \geq 0 \\ 1 - \int_W \frac{1\{x \notin X \circ |r|B\}\, 1\{x \in W \ominus (\rho(x,X)B \oplus \rho(x,X)\check{B})\}}{|W \ominus (\rho(x,X)B \oplus \rho(x,X)\check{B})|}\, dx & r < 0 \end{cases} \tag{5.7}$$

$\hat{P}_B^H$ is a bit easier to compute than its Kaplan-Meier counterpart $\hat{P}_B^{KM}$ and, as we shall see below, unbiased rather than ratio-unbiased. However, $\hat{P}_B^H$ may be negative or exceed one; the discretised estimator $\hat{P}_B^{KM}$ may be defective.

Lemma 6 *Consider the set operator $\hat{P}_B^H(r, X)$ defined by the right hand side of (5.7) as a function of $X \subseteq \mathbb{R}^d$. Then*

$$\hat{P}_B^H(r, X^c) = 1 - \hat{P}_B^H(-r, X)$$

for all $r \geq 0$ for which $|W \ominus (rB \oplus r\check{B})| > 0$.

Proof : Note that for $x \in X$,

$$\begin{aligned} \eta(x, X^c) &= \inf\{r \geq 0 : x \in X^c \bullet rB\} \\ &= \inf\{r \geq 0 : x \notin X \circ rB\} \\ &= \sup\{r \geq 0 : x \in X \circ rB\} = \rho(x, X) \end{aligned}$$

and for $x \notin X$

$$\eta(x, X^c) = 0 = \rho(x, X).$$

Furthermore, $1\{x \in X^c \bullet rB\} = 1\{x \notin X \circ rB\}$, yielding the desired result. □

Rewriting (5.7) in terms of ρ, η, ζ and using a sampling grid $T = \{t_i\} \subseteq W$, we obtain the following definition.

Definition 5 *Let X be a realisation of a stationary random closed set observed in a compact window W. Then for all $r \geq 0$ with $\#\{i : \zeta(t_i, W^c) \geq r\} > 0$, define*

$$\hat{P}_B^H(r) = \sum_{s \leq r} \frac{\#\{i : \eta(t_i, X) = s \leq \zeta(t_i, W^c)\}}{\#\{i : \zeta(t_i, W^c) \geq s\}} \tag{5.8}$$

and for $r < 0$ with $\#\{i : \zeta(t_i, W^c) \geq |r|\} > 0$, let

$$\hat{P}_B^H(r) = 1 - \sum_{s < |r|} \frac{\#\{i : \rho(t_i, X) = s \leq \zeta(t_i, W^c)\}}{\#\{i : \zeta(t_i, W^c) \geq s\}} \tag{5.9}$$

Theorem 2 *Let X be a stationary random closed set, observed in a compact window W. Let B be a non-empty convex compact structuring element. Then the Hanisch-type estimator in Definition 5 is pointwise unbiased for $P_B(r)$.*

Proof : First of all, consider the case $r \geq 0$. Then by Lemma 4, Lemma 5 and the stationarity of the random closed set X,

$$\begin{aligned} \mathbb{E}\hat{P}_B^H(r) &= \sum_{t_i} \mathbb{E}\left[\frac{1\{\eta(t_i, X) \leq r\}\, 1\{t_i \in W \ominus (\eta(t_i, X)B \oplus \eta(t_i, X)\check{B})\}}{\#\{j : t_j \in W \ominus (\eta(t_i, X)B \oplus \eta(t_i, X)\check{B})\}}\right] \\ &= \sum_{t_i} \int_{[0,r]} \frac{1\{t_i \in W \ominus (sB \oplus s\check{B})\}}{\#\{j : t_j \in W \ominus (sB \oplus s\check{B})\}} dP_B(s) = P_B(r). \end{aligned}$$

Also, for $r < 0$,

$$\begin{aligned}
\mathbb{E}\hat{P}_B^H(r) &= 1 - \sum_{t_i} \mathbb{E}\left[\frac{1\{\rho(t_i,X) < |r|\}\,1\{t_i \in W \ominus (\rho(t_i,X)B \oplus \rho(t_i,X)\check{B})\}}{\#\{j : t_j \in W \ominus (\rho(t_i,X)B \oplus \rho(t_i,X)\check{B})\}}\right] \\
&= 1 - \sum_{t_i} \int_{[0,|r|)} \frac{1\{t_i \in W \ominus (sB \oplus s\check{B})\}}{\#\{j : t_j \in W \ominus (sB \oplus s\check{B})\}} dQ_B(s) = \mathbb{P}(\rho(0,X) \geq |r|) \\
&= \mathbb{P}(0 \in X \circ |r|B) = P_B(r)
\end{aligned}$$

writing $Q_B(\cdot)$ for the probability distribution of $\rho(0,X)$ and using Lemma 3. The proof for the continuous version (5.7) is similar. □

As we saw before, one of the disadvantages of the minus sampling estimator is that it is not necessarily increasing in r. The Hanisch-type estimator does not suffer from this disadvantage.

Theorem 3 *Let X be a stationary random closed set, observed in a compact window W. Let B be a non-empty convex compact structuring element. Then the Hanisch-type estimator in (5.8)–(5.9) is increasing and semi-continuous from the right.*

This result should be compared to Theorem 1.

Proof : It is clear that both (5.8) and (5.9) are increasing and semi-continuous from the right. Furthermore,

$$\lim_{r\uparrow 0} \hat{P}_B^H(r) = 1 - \frac{\#\{i : \rho(t_i,X) = 0\}}{\#\{i : t_i \in W\}} = \frac{\#\{i : \rho(t_i,X) > 0\}}{\#\{i : t_i \in W\}} \leq \frac{\#\{i : t_i \in X\}}{\#\{i : t_i \in W\}} = \hat{P}_B^H(0).$$

Similarly, the continuous versions are increasing functions of r and – using the proofs of Lemma 3 and Lemma 4 – semi-continuous from the right. □

However, although the Hanisch-type estimator is monotonically increasing, it may be non-negative and exceed 1. If this is undesirable, one can take $R = \sup\{r > 0 : \#\{i : t_i \in W \ominus (rB \oplus r\check{B})\} > 0\}$ and normalise the summands in (5.8) and (5.9) by $\hat{P}_B^H(R)$ and $\hat{P}_B^H(-R)$ respectively. The resulting estimator $\tilde{P}_B^H(r)$ is ratio-unbiased.

6. Exploratory data analysis

Usually a statistical analysis of a binary image of a random set begins with plotting summary statistics such as the estimated empty space (3.2) or contact distribution function (3.3). Indeed, Ripley [22] proposed to look at the empirical plots of the opening and closing distributions as well to get a better feel for the data. Here we present some further examples using the size distribution function (4.2).

For ease of computation, we consider a simple square structuring element of three by three pixels. Then $\eta(\cdot, X)$, the size measure of voids, can be computed using the distance transform algorithm [3] for the "square" metric on $\mathbb{R}^2$ defined by

$$d((p_1,p_2),(q_1,q_2)) = \max\{|p_1 - q_1|, |p_2 - q_2|\}.$$

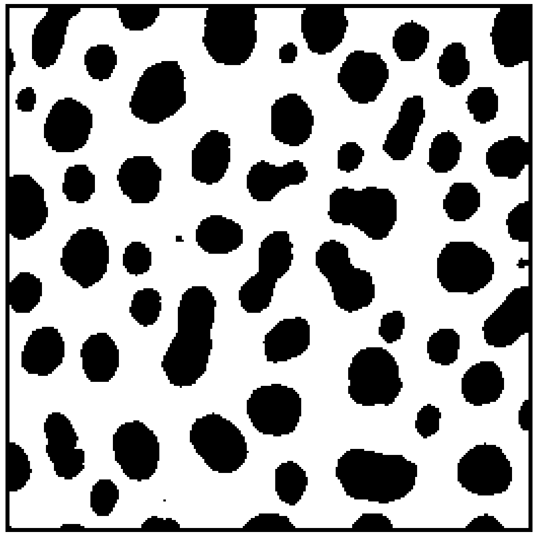

Figure 1: Image `cermet` from the Scilimage database.

Using duality, $\rho(\cdot, X)$ can be computed by reversing the fore- and background.

Figure 1 shows the 256x256 image `cermet` from the Scilimage [14] database after thresholding. We computed the distance transforms of the fore- and background in Figure 2 (top) as well as the $\eta(\cdot, X)$ and $\rho(\cdot, X)$ transforms (bottom of Figure 2). Finally, the Hanisch-type estimator of the size distribution function is plotted in Figure 3. It can be seen from the graph that most of the voids do not exceed size 10, while the foreground components may be slightly larger.

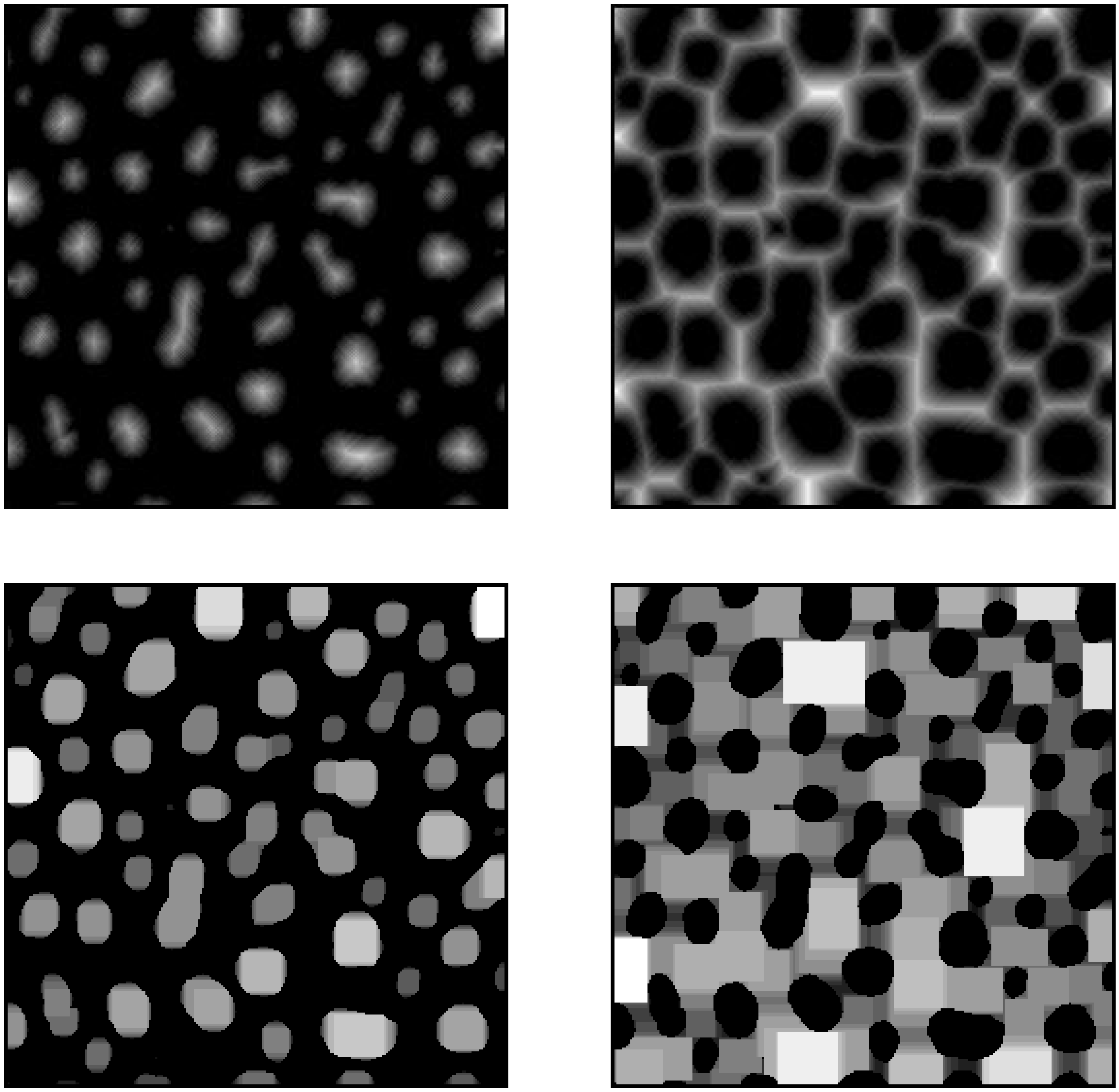

Figure 2: Distance transform (top left), dual distance transform (top right), closing (bottom left) and opening transform (bottom right) of `cermet`.

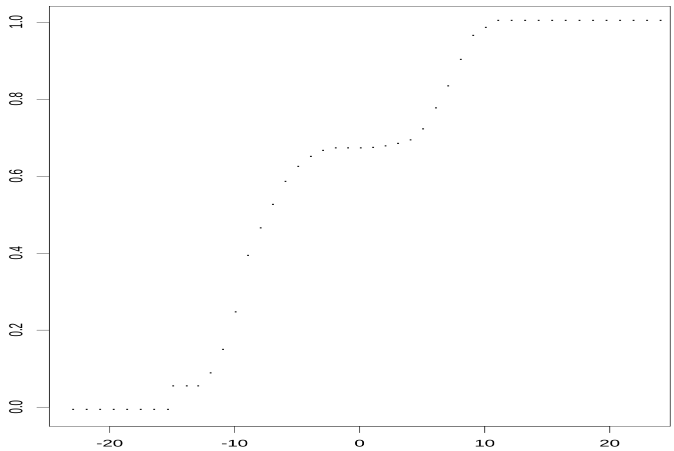

Figure 3: Estimated size distribution of `cermet`.

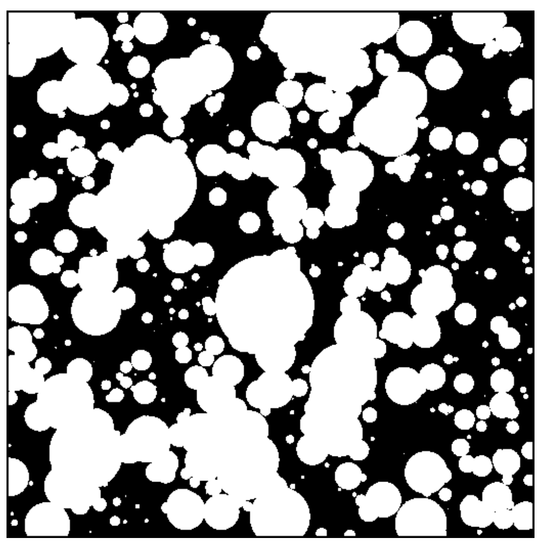

Figure 4: Realisation `boolean1` of a Boolean model of balls of random radius, digitised within a 512x512 sampling window. The intensity is 1000, the radius is exponentially distributed (see text).

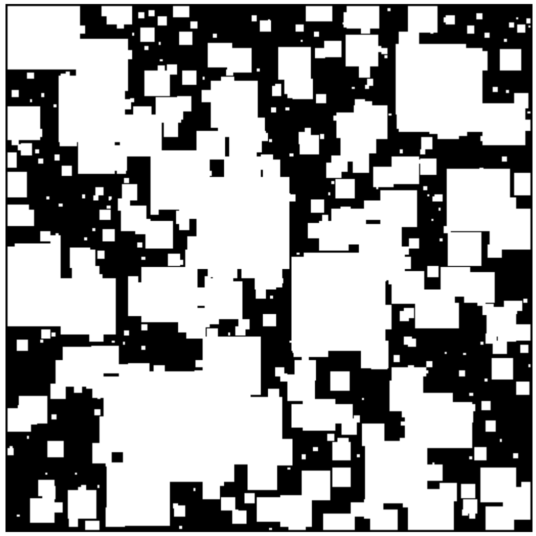

Figure 5: Realisation `boolean2` of a Boolean model of squares of random side length, digitised within a 512x512 sampling window. The intensity is 1000, the side length is gamma distributed (see text).

It was noted in Section 3 that the empty space function of a Boolean model with convex grains depends only on the moments of the primary convex grain. To illustrate this issue, we simulated two Boolean models with different shapes. The grains in the first model are balls with a random exponentially distributed radius; the second model assumes square grains with a side length that is gamma distributed. In order to have identical mean area and mean perimeter, we have to tune the parameters of the second model to μ, the parameter of the exponential distribution in the first Boolean model. Thus, choose $\alpha = \frac{\pi}{8-\pi}$ for the shape parameter of the gamma distribution and $\beta = \frac{2\mu}{8-\pi}$ for the scale parameter. Typical realisations on the unit square with intensity $\lambda = 1000.0$ and $\mu = 75.0$ are given in Figures 4 (`boolean1`) and 5 (`boolean2`), both digitised within an image of 512x512 pixels.

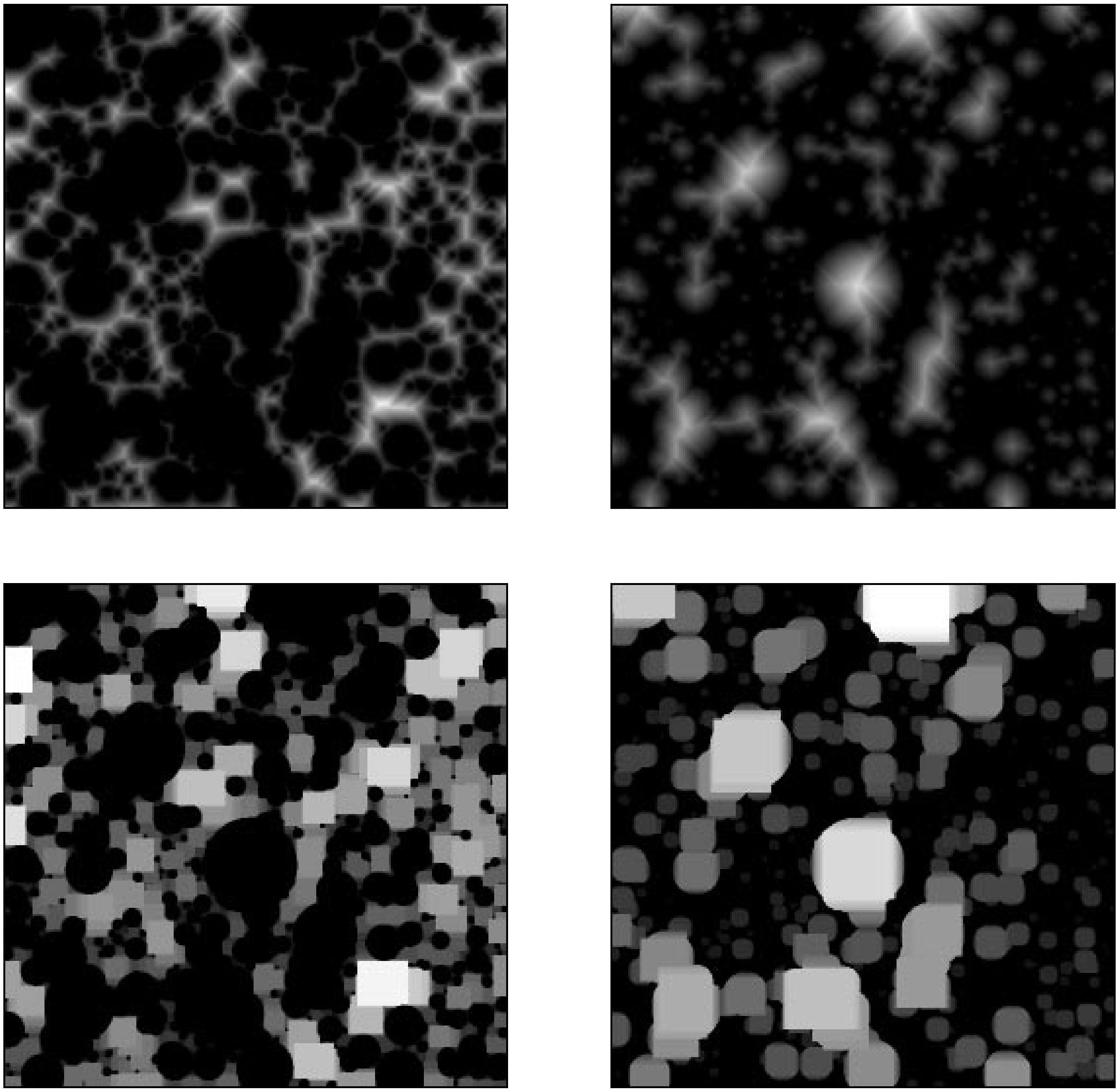

Figure 6: Distance transform (top left), dual distance transform (top right), closing (bottom left) and opening transform (bottom right) of `boolean1`.

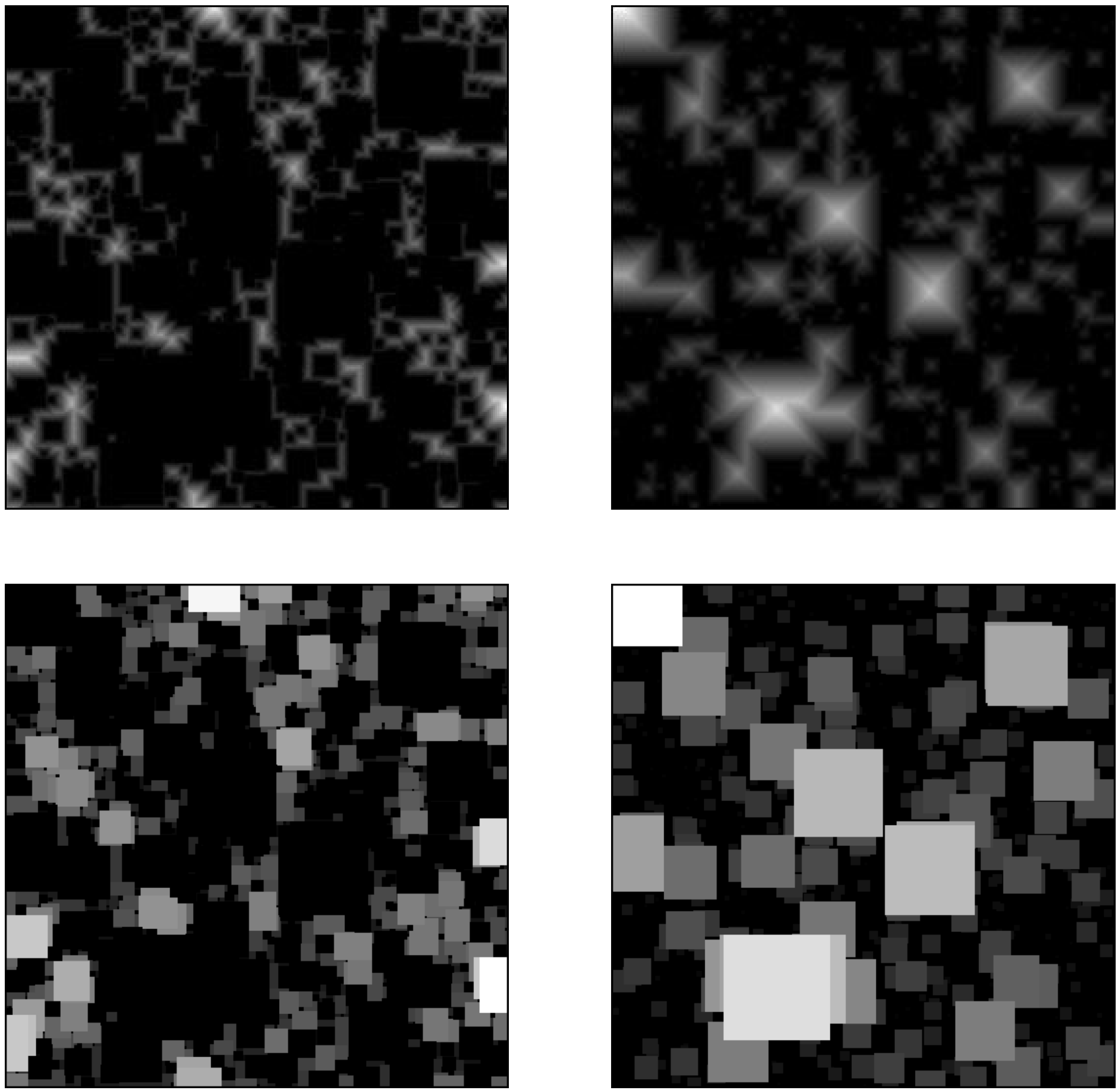

Figure 7: Distance transform (top left), dual distance transform (top right), closing (bottom left) and opening transform (bottom right) of `boolean2`.

The distance transforms and size measures for a square 3×3 structuring element are given in Figures 6 and 7. Note that the $\rho(\cdot, X)$ transform measuring the particle sizes seem to differ more than the distance transform images. This is reflected in the estimated size distribution and empty space functions plotted in Figure 8. As expected, the graphs of the estimated empty space functions (3.2) are rather similar, but the estimated size distributions for the two Boolean models are rather different, especially for $r < 0$. In particular, the sizes of particles in `boolean2` are larger than those in `boolean1`. An intuitive explanation is that balls do fit inside squares, but not the other way round.

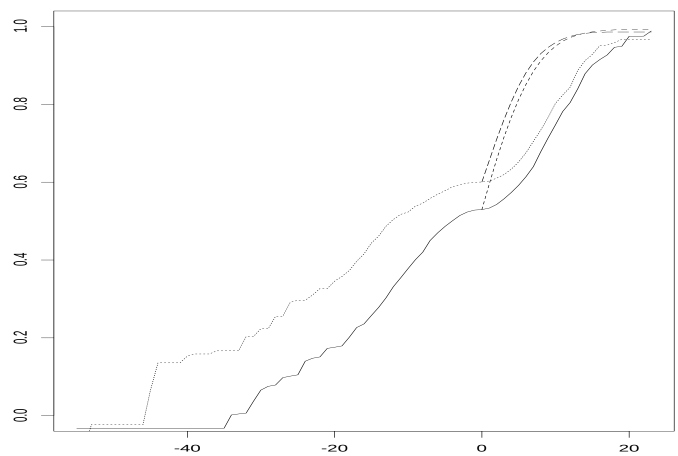

Figure 8: Estimated size distribution and empty space functions of `boolean1` and `boolean2`. From top to bottom: the empty space function of `boolean2`, the empty space function of `boolean1`, the size distribution function of `boolean2` and the size distribution function of `boolean1`.

7. Size-biased Markov random sets

Except from being helpful statistics in exploratory data analysis, size distributions can also be used to define new random set models, by biasing a reference Boolean model towards certain sizes.

On a discrete lattice, Sivakumar and Goutsias [27] defined *morphologically constrained random fields* by

$$p(X) = \frac{1}{Z} \exp[-\sum_{i=0}^{I} \beta_i |X \circ iB \setminus X \circ (i+1)B| - \sum_{j=1}^{J} \gamma_j |X \bullet jB \setminus X \bullet (j-1)B|] \qquad (7.1)$$

where X is the set of foreground pixels in a binary image W and for any $A \subseteq W$, $|A|$ denotes the number of pixels in A. The special case $\beta_i, \gamma_j \equiv 1$ was studied by Chen and Kelly [4].

The probability $p(X)$ in (7.1) can be rewritten as $p(X) = \frac{1}{Z} \exp[-\int_{[-J+1,I+1]} f(s) d\hat{G}_X(s)]$

where $\hat{G}_X(s)$ is the empirical size estimator

$$\hat{G}_X(r) = \begin{cases} 1 - \frac{|W \cap (X \circ rB)|}{|W|} & r \geq 0 \\ 1 - \frac{|W \cap (X \bullet |r| B)|}{|W|} & r < 0 \end{cases}$$

and $f(\cdot)$ is a step function taking values β_i, γ_j. Note that $\hat{G}_X(s)$ may be a biased estimator of (4.1), failing to account for edge effects (cf. Section 5).

In the remainder of this section we will extend (7.1) to continuous random set models observed in a compact window W of positive volume. Since the cardinality of the set X is no longer finite, we have to proceed by specifying a density $p(\cdot)$ with respect to some reference process, eg a Boolean model on W. Details can be found in Van Lieshout [17]. In analogy with (7.1), set

$$p(X) = \frac{1}{Z} \exp\left[- \int f(s) d\hat{P}_X(s) \right] \tag{7.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded (measurable) function and $\hat{P}_X(\cdot)$ is an estimator of the size distribution function (4.2) based on X.

Some care has to be taken to ensure that (7.2) is well-defined. In particular, the lack of monotonicity of the minus sampling estimator causes problems in defining the integral in the exponent of (7.2). However, by Lemma 3, for every X, the Hanisch-type estimator $\hat{P}_X^H(\cdot)$ can be normalised into a probability distribution function and hence $\int f(s) d\hat{P}_X^H(s)$ is well-defined for measurable functions $f(\cdot)$. Similarly, the naive estimator based on (5.1) ensures that the exponent in (7.2) is well-defined. Moreover, if $|f(s)| \leq F$, its integral with respect to the normalised Hanisch-type estimator is bounded as well and hence (7.2) is well-defined. The unnormalised Hanisch-type estimator (5.8)–(5.9) gives rise to a signed measure, bounded in absolute value by $|T|$, the number of sampling points in the grid T. Hence, under the same condition $|f(s)| \leq F$, (7.2) is well-defined.

Statistical inference for complex random sets usually relies on iterative procedures making 'local' changes to the random set. First, consider the case were both the germs and the grains of which the random set is composed are fully observable. In that case, $Y = \{(x_i, K_i)\}$ is a germ-grain process and (7.2) with $X = \cup_i (x_i \oplus K_i)$ is its density with respect to a Poisson marked point process with intensity measure $\lambda \, \text{leb}(\cdot) \times \mu(\cdot)$ (where leb denotes Lebesgue measure, λ is the point intensity and $\mu(\cdot)$ the mark distribution of the grains). Suppose that addition of a grain K at u is considered. Then the likelihood ratio

$$\frac{p(X \cup K_u)}{p(X)} = \exp\left[- \int f(s) d\hat{P}_{X \cup K_u}(s) + \int f(s) d\hat{P}_X(s) \right] \tag{7.3}$$

writing $K_u = u \oplus K$. If moreover $f(\cdot)$ is supported on $[-G, G]$, the log likelihood ratio reduces to

$$- \int_{-G}^{G} f(s) d\hat{P}_{X \cup K_u}(s) + \int_{-G}^{G} f(s) d\hat{P}_X(s). \tag{7.4}$$

Next, define a *neighbourhood relation* by

$$(u, K) \sim (v, L) \Leftrightarrow L_v \oplus (GB \oplus G\check{B}) \cap K_u \oplus (GB \oplus G\check{B}) \neq \emptyset \tag{7.5}$$

where $u, v \in W$ and K, L are non-empty compact sets.

In the following Theorem we will show that (7.4) only depends on $(x_i, K_i) \in Y$ with $(x_i, K_i) \sim (u, K)$. In other words, seen as a grain-marked point process, Y is Markov [23, 2] with respect to the neighbourhood relation $\sim$.

Theorem 4 *Let Y be a germ-grain model defined by its density (7.2) for some bounded function $f(\cdot)$ that is supported on $[-G, G]$ $(G > 0)$. Then Y is Markov with respect to $\sim$ for $\hat{P}_X(\cdot)$ either the naive estimator (5.1) or the Hanisch-type estimator (5.8)–(5.9).*

Proof : We start with proving that if $x \notin K_u \oplus (GB \oplus G\check{B})$, then for all $s \leq G$, $x \in (X \cup K_u) \circ sB \Leftrightarrow x \in X \circ sB$ and $x \in (X \cup K_u) \bullet sB \Leftrightarrow x \in X \bullet sB$.

To see this, let $x \in (X \cup K_u) \circ sB$. Then $\exists h$ such that $x \in (sB)_h \subseteq X \cup K_u$, and in particular we can write $x = h + sb'$ for some $b' \in B$. Now if $(sB)_h \cap K_u \neq \emptyset$, then $h + sb \in K_u$ for some $b \in B$ and hence $x = h + sb' = h + sb + (sb' - sb) \in K_u \oplus (sB \oplus s\check{B}) \subseteq K_u \oplus (GB \oplus G\check{B})$ using the convexity of the structuring element B. This contradicts the assumption that $x \notin K_u \oplus (GB \oplus G\check{B})$ and hence $x \in (sB)_h \subseteq X$, that is $x \in X \circ sB$.

Similarly for the closing, let $x \in (X \cup K_u) \bullet sB$. Then by duality $x \notin (X \cup K_u)^c \circ sB$ and hence for any h such that $x \in (sB)_h$ the intersection $(sB)_h \cap (X \cup K_u)$ must be non-empty. By the previous argument, K_u cannot be intersected and hence $(sB)_h \cap X \neq \emptyset$. Thus $x \in X \bullet sB$.

Secondly, for $x \notin K_u \oplus (GB \oplus G\check{B})$, suppose that $\eta(x, X \cup K_u) \leq G$. Then by Lemma 4 $x \in (X \cup K_u) \bullet GB$, hence by the above $x \in X \bullet GB$ or equivalently $\eta(x, X) \leq G$. Also $\eta(x, X \cup K_u) = \inf\{s \geq 0 : x \in (X \cup K_u) \bullet sB\} = \inf\{0 \leq s \leq G : x \in (X \cup K_u) \bullet sB\} = \inf\{0 \leq s \leq G : x \in X \bullet sB\} = \eta(x, X)$. Dually, supposely that $\rho(x, X) < G$. Then by Lemma 3, $x \notin X \circ GB$ hence by the above $x \notin (X \cup K_u) \circ GB$ or $\rho(x, X) < G$. Also $\rho(x, X) = \sup\{s \geq 0 : x \in X \circ sB\} = \sup\{0 \leq s \leq G : x \in X \circ sB\} = \sup\{0 \leq s \leq G : x \in (X \cup K_u) \circ sB\} = \rho(x, X \cup K_u)$.

Hence for $s \in [-G, G]$, $\hat{P}_{X \cup K_u}(s) - \hat{P}_X(s)$ depends only on $t_i \in K_u \oplus (GB \oplus G\check{B})$. By the local knowledge principle

$$|(Y \bullet GB) \cap (A \ominus (GB \oplus G\check{B}))| = |((Y \cap A) \bullet GB) \cap (A \ominus (GB \oplus G\check{B}))|$$

(similarly for openings) with $A = K_u \oplus (GB \oplus G\check{B}) \oplus (GB \oplus G\check{B})$ and noting that $A \ominus (GB \oplus G\check{B}) \supseteq K_u \oplus (GB \oplus G\check{B})$ only knowledge of Y in A is needed. The result follows. □

The function $f(\cdot)$ may favour some sizes and penalise others. For instance, if $f(\cdot)$ is the indicator function of $[-G, G]$, it will encourage the fore- and background to exceed size G. If $f(\cdot)$ is non-zero only for $r \geq 0$, it will influence only the size of the background, and similarly $f(r) \equiv 0$ for $r \geq 0$ influences the particle size only.

If grains are not individually observable, note that if

$$X = X_1 \cup \cdots \cup X_k$$

is partitioned into its connected components $X_1, \ldots, X_k$, the opening

$$X \circ B = \cup\{B_h \subseteq X\}$$

also partitions, as the convexity of B implies that B_h must fall entirely in one of the X_i. Thus, the naive (5.1) and Hanisch-type (5.9) estimators satisfy $\hat{P}_X(s) = \sum_{i=1}^{k}[\hat{P}_{X_i}(s)]$ for $s < 0$. Similarly, $X \bullet B$ factorises over the connected components of $W \setminus X$ and hence

$$p(X) = \prod_{i=1}^{k} \phi(X_i) \prod_{i=1}^{l} \phi(X_i^c).$$

Thus, altering X will only affect the connected components that are modified, a state-dependent Markov property as introduced by Baddeley and Møller [2]. See also [17, 21].

8. Examples

In this Section some realisations of size-biased Boolean models are given. More specifically, consider a reference Boolean model with intensity parameter $\lambda > 0$ and square primary grains with radius r distributed according to distribution $\mu(\cdot)$ on $\mathbb{R}^+$ and set

$$p(X) = \frac{1}{Z} \exp[-\int_{-G}^{G} \gamma d\hat{P}_X^H(s)] \tag{8.1}$$

For $\gamma > 0$, particle and pore sizes exceeding G will be favoured, while for $\gamma < 0$ the sizes tend to be smaller than G. As in Section 6, the structuring element will be a square.

Realisations are obtained using the Metropolis-Hastings sampler of Geyer and Møller [7], a special case of Green's reversible jump technique (cf.[8]). The basic idea is that although Z in (8.1) is not available in closed form, the ratios (7.3) do not depend on the normalisation constant and are 'local' (see Theorem 4).

Briefly, given an initial configuration $Y_0 = \{(x_i, K_i)\}$ of germs and grains with associated set $X_0 = \bigcup_i (x_i \oplus K_i)$, with probability 1/2 propose adding a grain ('birth'); with probability 1/2 propose deleting one of the grains in Y_0 if any ('death'). Births are proposed uniformly with respect to $\text{leb}(\cdot) \times \mu(\cdot)$ and (u, K) is accepted with probability

$$\min\left\{1, \frac{p(X_0 \cup K_u)}{p(X_0)} \frac{\lambda |W|}{n(Y_0) + 1}\right\}$$

where $n(Y_0)$ denotes the cardinality of Y_0. If the new grain K at u is accepted, set $Y_1 = Y_0 \cup \{(u, K)\}$, $X_1 = X_0 \cup K_u$; otherwise $Y_1 = Y_0$ and $X_1 = X_0$. Similarly, select grain K_i at x_i for deletion from Y_0 with probability $1/n(Y_0)$. The proposal is accepted with probability

$$\min\left\{1, \frac{p(X_0 \setminus (K_i)_{x_i})}{p(X_0)} \frac{n(Y_0)}{\lambda |W|}\right\}$$

and if so, $Y_1 = Y_0 \setminus \{(x_i, K_i)\}$, $X_1 = X_0 \setminus (K_i)_{x_i}$. Otherwise $Y_1 = Y_0$ and $X_1 = X_0$. Continuing in this fashion, we obtain a sequence $X_k, k \in \mathbb{N}_0$, which converges to $p(\cdot)$ as $k \to \infty$.

In the samples depicted in Figure 9, we took a geometric distribution for μ with parameter $\delta = .2$, $\lambda = 1000$, $G = 5$ and $|\gamma| = 2500$.

In order to check whether the Metropolis-Hastings sampler has converged, time series of the sufficient statistic $\hat{P}_X^H(G) - \hat{P}_X^H(-G)$ are plotted in Figure 10 over 200000 iterations. There does not seem to be any reason to doubt convergence.

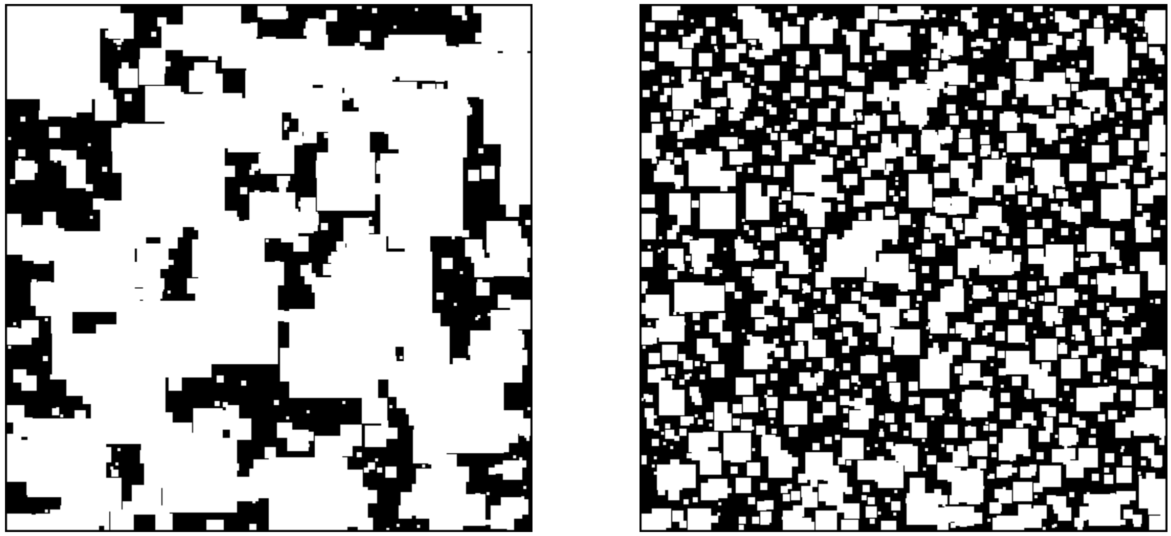

Figure 9: Samples after 200000 Metropolis-Hastings steps for (8.1) with $G = 5$ in a 512x512 image for $\gamma = 2500.0$ (left) and $\gamma = -2500.0$ (right).

Exact simulation of the germ-marked point process is theoretically possible [15], since the likelihood ratio

$$\lambda\mu(K_u)\exp[-\gamma(\hat{P}^H_{X\cup K_u}(G) - P^H_{X\cup K_u}(-G) - \hat{P}^H_X(G) + P^H_X(-G))]$$

for model (8.1) is uniformly bounded. However, since efficient upper and lower bounds on the likelihood ratio based on the current state of the sampler would have to be computed at every iteration, we prefer to use the computationally easier Metropolis-Hastings method.

To conclude this section Figure 11 gives the estimated size distributions for the two realisations in Figure 9.

Acknowledgements

I would like to thank Adri Steenbeek for programming assistance and Henk Heijmans for valuable comments. This research was partially carried out while the author was at the University of Warwick.

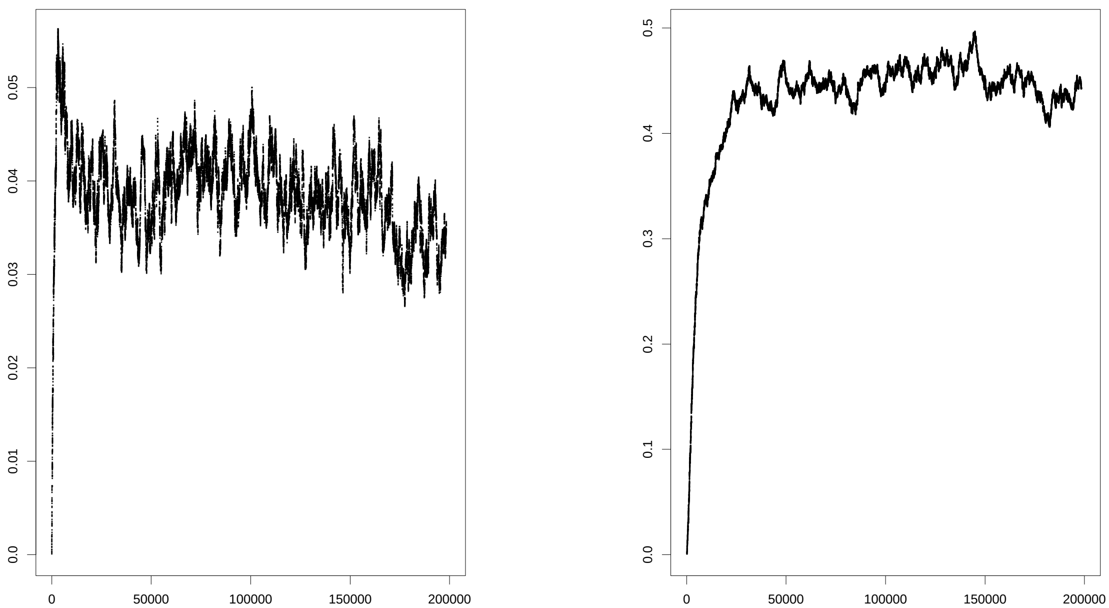

Figure 10: Time series of the sufficient statistic over 200000 Metropolis-Hastings steps of (8.1) with $G = 5$ and $\gamma = 2500.0$ (left) or $\gamma = -2500.0$ (right).

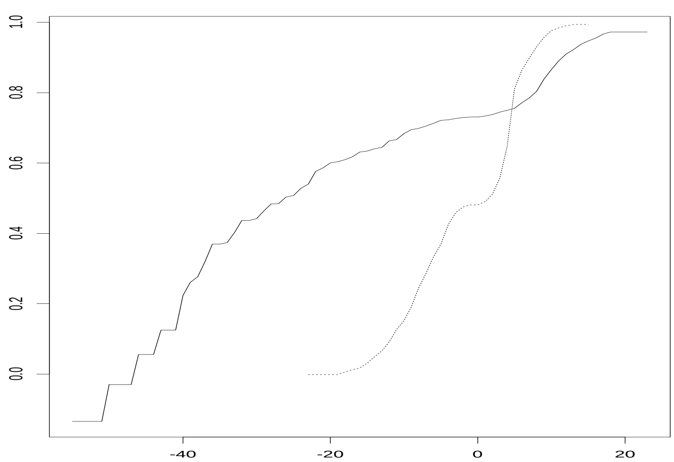

Figure 11: Estimated size distribution of the Markov samples in Figure 9 with $\gamma = 2500.0$ (solid line) or $\gamma = -2500.0$ (dotted line).

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