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ABSTRACT
A transient one-dimensional model of the vertical movement of water and salt in the mangrove root zone is investigated. This is an extension of a previous steady state model which assumed that the ability of the mangrove roots to take up water is uniformly distributed throughout the soil and that the root water uptake is reduced if there is nonzero salt concentration around the roots.

We show how both the time dependent and steady state salinity profiles in the soil depend on the strength of the root water uptake, the depth of the root zone and the porosity of the soil. The withdrawal of fresh water by the mangrove roots leads to salinization of the soil in the root zone, and significant reduction of the transpiration of the mangroves. When a steady state is reached, the salt that is excluded by the mangrove roots must diffuse back to the surface against the flow of soil water towards the roots. For a root zone of finite depth, a finite difference numerical scheme is used to investigate the rate of salinization of the root zone, and the diffusion of salt into the region below the root zone.

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## 1. Introduction

Mangroves grow on saturated soils or muds which are subject to regular inundation by tidal water with salt concentration $C_0$ close to that of sea water (see, for example, Hutchings and Saenger [13]). The mangrove roots take up fresh water from the saturated saline soil and leave behind most of the salt (Ball [4]). This results in a net flow of water downward from the soil surface, which carries salt with it. As pointed out by Passioura *et al.* [17], in the absence of lateral flow the steady state salinity profile in the root zone must be such that the salinity around the roots is higher than $C_0$, and that the concentration gradient is large enough so that the advective downward flow of salt is balanced by the diffusive flow of salt back up to the surface. Passioura *et al.* [17] neglected daily variations of evapotranspiration, and presented steady state equations governing the flow of salt and flow of uptake of water in the root zone, assuming that there is an upper limit $C_1$ to the salt concentration at which roots can take up water, and that the rate of uptake of water is proportional to the difference between the local concentration $C$ and the assumed upper limit $C_1$. They assumed that the root zone was unbounded, and that the constant of proportionality for root water uptake was independent of depth through the soil. They gave numerical results for variation of salt concentration with depth, obtained from a perturbation solution (unpublished) of the steady state water and salt flow equations.

Their results showed salt concentration increasing with depth from the value $C_0$ at the surface, and approaching the limiting value $C_1$ at large depth, with a corresponding decrease in water uptake from a maximum at the surface to the limiting value of zero at large depth. The assumption of a infinitely deep root zone requires that the concentration at depth approach $C_1$, otherwise the total uptake would be infinite. For real mangroves the root distribution is not uniform and does not extend to an infinite depth. For one mangrove species Lin and Sternberg [16] measured the root distribution and found that the root density decreased with depth, with more than half of the fine roots being contained in the top 50 cm of the soil. The depth distribution of root water uptake is expected to be related to the distribution of fine roots in the soil.

In this paper our aim is to extend the steady state model of Passioura *et al.* [17] in several important ways. Firstly, we will consider more general root water uptake functions, which vary with depth and which depend on a general power $p$ of the concentration difference $C_1 - C$. Denoting by $S$ the volume of water taken up by the roots per unit volume of porous material per unit time, we use

$$S := \begin{cases} K(Z)\left(1 - \dfrac{C}{C_1}\right)^p & \text{for } 0 \le C \le C_1, \\ 0 & \text{for } C > C_1, \end{cases} \tag{1.1}$$

where $K(Z)$ is determined by the root distribution as a function of the depth $Z$ below the soil surface and $p > 0$. This root distribution function will be non-negative, and we assume that it is non-increasing with $Z$. Passioura *et al.* [17] used the value $p = 1$ corresponding to a linear dependence of uptake on concentration difference, which is consistent with the assumption that uptake is governed by osmotic pressure difference. We will investigate more general values of $p$, and show that the behavior of the salinity profile differs between the two cases $p < 1$ and $p \ge 1$. For crops, Dirksen *et al.* [9] assumed that the root water uptake function $S$ could be expressed in a product form, with the effect of soil salinity embodied in one of the factors. However, the form of their osmotic reduction term differs from that in (1.1), and their exponent $p$ has a different meaning than ours.

Secondly, we will study time-dependent behavior of salt concentration and flow, and investigate how the evolution to steady state profiles is governed by the values of the parameters. Passioura *et al.* [17] asserted that where there was a root zone of finite depth with porous soil or mud below it, the increased salt concentration around the roots would diffuse down to the lower region, eventually filling it with salt water of the same high concentration as in the lower part of the root zone. We will prove the validity of this assertion.

In section 2 we formulate a mathematical model for arbitrary root distribution $K$. This model involves two coupled differential equations: a convection-diffusion equation for the transport of salt and an ordinary differential equation describing the fluid balance.

Further, in section 3 we consider the time-independent state. We present a detailed analysis for two simple but realistic functional forms of the root distribution $K(Z)$. Both choices allow us to use a phase plane argument to investigate the steady state.

For the first distribution we assume that the function $K$ is constant above certain depth $Z_m$, and zero below that depth, i.e.

$$K(Z) := \begin{cases} \dfrac{K_0}{Z_m} & 0 < Z < Z_m, \\ 0 & Z_m < Z < \infty. \end{cases} \tag{1.2}$$

For the second distribution we assume that the strength of the uptake decreases with depth according to

$$K(Z) := \frac{K_0}{Z_m \left(1 + \frac{Z}{Z_m}\right)^2}, \tag{1.3}$$

where $Z_m$ is a reference depth. Both distributions have the same weight $K_0$ since both of (1.2) and (1.3) satisfy

$$\int_0^\infty K(Z)dZ = K_0. \tag{1.4}$$

Therefore for both distributions the quantity $K_0$ is the total amount of root water uptake in the profile with no salt present, in volume per unit surface per unit time, i.e. the transpiration rate of the mangrove plants in the absence of salinity. For the first root distribution (1.2) the depth $Z_m$ is at the bottom of the root zone. For the second distribution (1.3)

$$\int_0^\infty K(Z)dZ = \frac{1}{2}K_0, \tag{1.5}$$

so $Z_m$ corresponds to the median depth in that case. Note that for the distribution defined by (1.3) a mean depth cannot be defined because the relevant integral diverges.

In section 4 we discuss some results concerning the evolution problem. Unlike the steady state problem, which can be reduced to the study of a nonlinear second order ordinary differential equation, the evolution problem consists of a system of nonlinear partial differential equations. The existence of solutions (in the sense of Definition 4.1) is proved by means of a standard fixed point argument meanwhile the uniqueness and comparison of solutions is a more subtle problem which we shall comment later on. Finally, in this section we also show, under suitable conditions on the solutions, the asymptotic stability of the problem.

In section 5 we study the formation of *dead cores*, i.e., the conditions under which the salt concentration reaches its threshold value $C = C_1$. We first state a result for the steady state problem for which a detailed analysis is possible thanks to the existence of a comparison principle. Later, we proceed with the evolution problem for which we need to use a more general technique based in energy methods.

## 2. The mathematical model

In this section we formulate the mathematical model which describes the salt movement below the surface where the mangroves are growing and the uptake of fresh water by the root system of the mangroves. We consider the case where the mangroves are present in the horizontal $X - Y$ plane, with an homogeneous porous medium located below this plane. This porous medium is characterized by a constant porosity $\theta$, indicating that we are assuming the mangroves roots to be homogenized throughout the porous medium, without affecting its properties. As discussed in the introduction, they are accounted for by the distribution function $K$. Assuming further that the hydrodynamic dispersion tensor $\mathbf{D}$ is constant and isotropic, i.e. neglecting the velocity dependence in the mechanical dispersion, we find for the salt concentration the equation (see Bear [5])

$$\theta \frac{\partial C}{\partial T} + div\,(C\mathbf{Q} - \theta \mathbf{D} \nabla C) = 0, \tag{2.1}$$

where the vector $\mathbf{Q}$ denotes the specific discharge of the fluid and $T$ denotes time. We also have a fluid balance. Disregarding density variations in the mass balance equation of the fluid, we obtain a fluid volume balance expressed by

$$div \mathbf{Q} + S = 0, \tag{2.2}$$

where $S$ is given by (1.1).

If the mangroves are uniformly distributed throughout the $X-Y$ plane and there is no lateral fluid flow, we may consider the problem as one-dimensional in the vertical $Z-$direction. If the $Z$ axis is positive when pointing downwards, the flow domain is characterized by the interval $0 < Z < L \leq \infty$, where a finite $L$ indicates a flow domain of finite depth.

In the one-dimensional setting equations (2.1) and (2.2) combined with (1.1) become

$$\theta \frac{\partial C}{\partial T} + div\,(C\mathbf{Q}) - \theta \mathbf{D} \frac{\partial^2 C}{\partial Z^2} = 0, \tag{2.3}$$

and

$$\frac{\partial \mathbf{Q}}{\partial Z} + K(Z)\left(1 - \frac{C}{C_1}\right)^p = 0, \tag{2.4}$$

which we want to solve for $0 < Z < L$ and $T > 0$, say. Note that in writing (2.4) we implicitly assume that $C \leq C_1$. Whenever $L < \infty$, we prescribe along the bottom of the domain a no-flow condition for the water and the salt:

$$\mathbf{Q}(L,T) = \frac{\partial C}{\partial Z}(L,T) = 0 \quad \text{for all } T > 0. \tag{2.5}$$

Along the top boundary we assume a constant salt concentration which is given by the seawater salt concentration

$$C(0,T) = C_0 \quad \text{for all } T > 0. \tag{2.6}$$

Initially, at $T = 0$, we start from the uniform concentration

$$C(Z,0) = C_0 \quad \text{for all } 0 < Z < L. \tag{2.7}$$

Finally, we recast the equations in an appropriate dimensionless form. Introducing the dimensionless variables and constants

$$\begin{array}{llll} u := \dfrac{C}{C_1}, & u_0 := \dfrac{C_0}{C_1}, & x := \dfrac{Z}{Z_m}, & d := \dfrac{L}{Z_m}, \\ t := \dfrac{DT}{Z_m^2}, & q := \dfrac{Z_m Q}{\theta D}, & k := \dfrac{Z_m^2 K}{\theta D}, & k_0 := \dfrac{Z_m K_0}{\theta D}, \end{array} \tag{2.8}$$

we arrive at the problem: find $u$ and $q$ such that

$$(\mathrm{P}) \begin{cases} u_t + (uq)_x - u_{xx} = 0 & \text{in } Q_T, \\ q_x + f(x,u) = 0 & \text{in } Q_T, \\ u(0,t) = u_D(t) & \text{in } (0,T), \\ u_x(d,t) = q(d,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.9}$$

with $Q_T := \Omega \times (0,T)$ and $\Omega := (0,d)$, $0 < d < \infty$ and $T$ arbitrarily fixed. We have written $f(x,s) := k(x)\,(1-s)_+^p$ with the root distribution, $k$, given by either

$$k(x) := \begin{cases} k_0 & 0 < x < 1, \\ 0 & 1 < x < d, \end{cases} \tag{2.10}$$

or

$$k(x) := \frac{k_0}{(1+x)^2}. \tag{2.11}$$

Although thorough all this article we will have on mind a function $f$ of this form, for most of the results we will only need to consider a function $f$ satisfying

$$\text{(H)}\begin{cases} f(x,s)\geq 0 \text{ in } \Omega\times(0,1)\,,\ f(x,\cdot) \text{ is non increasing in } (0,1) \text{ and } f(x,1)=0 \text{ a.e. in } \Omega,\\ f(x,\cdot)\in C^{0,\alpha}([0,1])\cap C^{\infty}([0,1)) \text{ for some } \alpha>0,\\ f(\cdot,s)\in L^{\infty}(\Omega). \end{cases} \tag{2.12}$$

## 3. The stationary problem

In general, it is difficult to construct explicit or semi-explicit solutions of a nonlinear time-dependent problem such as (2.9). However, when dropping the time dependence, i.e. when looking for the asymptotic state as $t\to\infty$, much can be done. We will first briefly comment on the well posedness of the stationary problem associated to (2.9) for a general function $f$ satisfying (2.12), namely

$$\text{(S)}\begin{cases} (uq)_x - u_{xx} = 0 & \text{in } \Omega,\\ q_x + f(x,u) = 0 & \text{in } \Omega,\\ u(0)=\tilde{u},\\ u_x(d)=q(d)=0, \end{cases} \tag{3.1}$$

with $\tilde{u} := \lim_{t\to\infty} u_D(t)$, and later analyze the special cases in which $f(x,s) := k(x)(1-s)_+^p$ and $k$ is given either by (2.10) or by (2.11). Integrating the first equation of (3.1) in $(x,d)$ we obtain

$$u_x(x) = u(x)q(x). \tag{3.2}$$

We assume $\tilde{u}>0$, which implies by the maximum principle that $u(x)\geq\tilde{u}>0$ in $\Omega$. We introduce the new unknown

$$w(x) := -\log u(x) \tag{3.3}$$

which satisfies $w_x = -q(x)$ and therefore from (3.1) we have that $(u,q)$ is a solution of (3.1) if and only if $w$ is a solution of

$$\begin{cases} -w_{xx} + f(x,e^{-w}) = 0,\\ w(0) = -\log\tilde{u}, \quad w_x(d)=0. \end{cases} \tag{3.4}$$

Notice that $f(x,e^{-s})$ is a non-decreasing function in $s$ and therefore we may apply well known results to prove the existence and uniqueness of solutions of (3.4) in the class $W^{2,\infty}(\Omega)$. We also point out that the eventual non Lipschitz continuity of $f(x,s)$ with respect to the second variable is inherited by $f(x,e^{-s})$.

For special choices of function $f$ and the root distribution $k$, as above mentioned, qualitative statements about the behavior of the solution can be made relatively easily. The reason is that in these cases the non-trivial part of the solution of (3.4) is determined by an autonomous type equation (see Hirsch and Smale [12] for background and examples). Next we sketch the phase plane analysis for the root distributions given by (2.10) and (2.11).

We first consider the piecewise constant distribution given by (2.10). Let $d>1$. In the interval $1\leq x\leq d$, where no uptake of water takes place, we deduce from (3.1) that

$$q(x)=0 \quad \text{for} \quad 1\leq x\leq d,$$

and, consequently, from (3.2) and $u_x(d)=0$ we get, $u_x(x)=0$ for $1\leq x\leq d$. Thus

$$u(x)=u(1) \quad \text{for} \quad 1\leq x\leq d.$$

In the interval $0 < x < 1$, we consider the system

$$\begin{cases} w_x = q, \\ q_x = -k_0(1-e^w)^p. \end{cases} \tag{3.5}$$

Without giving the details of the phase plane analysis, we show in Figure 3.1, where $\tilde{u} = 0.25$, $k_0 = 10$ and $p = 1$, the result of the shooting procedure. The behavior of the orbits is typical for all values of $p \geq 1$. The right hand side of the second equation of (3.5) is smooth for the range of $p$, which means that the singular point $(w, q) = (0, 0)$ (implying $u = 1$, $q = 0$) can never be reached at finite distance of the origin (see, for instance, Amann [1] or van Duijn and Knabner [10]). Consequently the threshold concentration $u = 1$ can never be attained.

This, however, may change as soon as $0 < p < 1$. Then the right hand side of the second equation of (3.5) loses its smoothness near $w = 0$ implying that now the singular point $(0, 0)$, or the threshold concentration, can be attained at finite depth. We notice that the situation in the $w, q$ plane is quite different from the $p \geq 1$ case. This is shown in Figure 3.2, where $p = 0.5$. For appropriate choice of the parameters, the orbit will enter the origin at a distance less than or equal to $z = 1$. That is to say the threshold concentration $u = 1$ occurs in or just below the mangrove root zone.

For other choices of the parameters a situation as in Figure 3.1 may occur. Then the $q = 0$ axis is reached at $z = 1$ with $w(1) < 0$, leading to a salt distribution that again can never attain the threshold value.

Following, we comment the case of the root distribution given by (2.11). A direct phase plane analysis for system (3.5), which is now non-autonomous, or otherwise a reduction of the boundary value problem (3.4) to first integrals, seems not so transparent. However, because of the special form of $k$ we can transform (3.5) into an autonomous system which we can analyze as before. Set $v(x) := q(x)(1 + x)$. Rewriting (3.5) in terms of $w$ and $v$ yields

$$\begin{cases} (1+x)\, w_x = v, \\ (1+x)\, v_x = v - k_0(1-e^w)^p. \end{cases} \tag{3.6}$$

Changing the independent variable into

$$s := \log(1 + x), \quad \text{when} \quad 0 \leq s \leq \log(1 + d),$$

we obtain

$$\begin{cases} \dfrac{dw}{ds} = v, \\ \dfrac{dv}{ds} = v - k_0(1-e^w)^p, \end{cases} \tag{3.7}$$

with $w(0) = \log \tilde{u} < 1$ and $v(0) = u_x(0)/\tilde{u}$. Again the shooting procedure is: find a value of $v(0)$, or equivalently of $u_x(0)$, so that the corresponding orbit intersects the $v = 0$ axis, precisely when $s = \log(1+d)$. This implies $u_x(d) = 0$, the desired boundary condition at $x = d$. A similar smoothness argument as mentioned previously gives that, for all $p \geq 1$, an orbit cannot enter the origin $w = 0$, $v = 0$ at finite distance. Hence for all $p \geq 1$ we have $w(\log(1 + d)) < 0$, implying that $u(d) < 1$. Since $u$ is monotone in $x$, which follows from the positivity of $v$ along the appropriate orbit, we conclude that

$$u(x) < u(d) < 1 \quad \text{for} \quad 0 \leq x \leq d \quad \text{and for all} \quad p \geq 1. \tag{3.8}$$

Again this changes for $0 < p < 1$. The corresponding orbits in the shooting procedure for the case $\tilde{u} = 0.25$, $k_0 = 10$ and $p = 1/2$ are shown in Figure 3.7. As before, for certain parameter combinations we may find an orbit that reaches the origin at a transformed distance $s^* < \log(1 + d)$. This implies that

$$u(x) = 1 \text{ and } q(x) = 0 \quad \text{for} \quad x^* = e^{s^*} - 1 \leq x \leq d. \tag{3.9}$$

For other parameter combinations the desired orbits intersects the $v = 0$ axis as $s = \log(1 + d)$ for negative $w$, giving for $u$ the inequalities from (3.8).To investigate the condition for which (3.9) holds, we first need to determine the orbit that ends up in the origin. For this we divide in (3.6) the first equation by the second and consider the initial value problem

$$\begin{cases} \dfrac{dv}{dw} = 1 - \dfrac{k_0(1-e^w)^p}{v} & \text{if } \log \tilde{u} < w < 0, \\ v(0) = 0. \end{cases}$$

Suppose that the solution is given by $V_p(w)$. Then from (3.6) and (3.9) we obtain

$$s^* = \int_{\log u_0}^{0} \frac{1}{V_p(\sigma)} d\sigma.$$

Remark 3.1 In the case of an arbitrary root distribution function $k(x) \geq 0$, a simple reduction to a phase plane analysis as in the previous examples is not possible. Nevertheless, it is possible to analyze solutions of (3.5) qualitatively. The results are:

1. If $p \geq 1$, then $u(z) < 1$ for all $0 \leq x \leq d$. Hence no maximal concentration can occur in or below the root zone.

2. If $p < 1$, then $u(x) = 1$, for some $0 < x \leq d$ is possible. As in the two examples, this depends on the value of the parameters $u_0$, $k_0$ and $d$.

3. When comparing solutions corresponding to different root distribution functions, we have the following ordering: if $k_i(x)$ implies the salt concentration $u_i(x)$, for $i = 1, 2$, and if $k_1(x) \geq k_2(x)$ then $u_1(x) \geq u_2(x)$ for all $0 \leq x \leq d$.

A more detailed analysis of the conditions under which the maximum concentration $u(x) = 1$ is reached is presented in section 5.

## 4. The evolution problem

In this section we study in some detail the mathematical settlement of the evolution problem (2.9). In this case, the integration and change of unknown performed in (3.3) are no longer useful and a direct treatment of the whole system is required, introducing some complications in the analysis of the problem. It is worth mentioning already that one of the main difficulties in studying problem (2.9) is that, in general, the comparison principle does not hold. As a consequence, the techniques involving comparison with sub and supersolutions are not available and more general techniques must be considered. Firstly, we present a result on existence of solutions by applying a fixed point argument. Secondly, we study by means of a duality technique the question of uniqueness and comparison of solutions, obtaining some sufficient conditions for them to hold. Finally, we show by using a general technique for nonlinear scalar problems the asymptotic stability of (2.9).

### *4.1 Existence of solutions*

We shall consider a notion of strong solution. Notice that the non homogeneity of the hyperbolic equation may cause a lack of regularity of solutions as it impose a restriction in the sense in which equalities in (2.9) take place. We introduce the functional space $\mathcal{V} := \left\{ v \in H^1(\Omega) : v(0) = 0 \right\}$.

Definition 4.1 *Let*

$$\begin{aligned} u &\in u_D + L^2(0,T;\mathcal{V}) \cap L^\infty(Q_T) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \quad \textit{and} \\ q &\in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T). \end{aligned}$$

*The pair* $(u,q)$ *is said to be a* strong solution *of* (2.9) *if it satisfies the equations and boundary conditions of* (2.9) *in a.e.* $(x,t) \in Q_T$ *and if*

$$\lim_{t\downarrow 0} \|u(x,t) - u_0(x)\|_{L^2(\Omega)} = 0. \tag{4.1}$$

We shall assume

$$u_D \in H^1(0,T), \quad u_0 \in H^2(\Omega) \quad \text{and} \quad 1 > u_D, u_0 > c \quad \text{for some } c \in (0,1). \tag{4.2}$$

In the following theorem we assert the existence of strong solutions:

THEOREM 4.1 *Assume* (2.12) *and* (4.2)*. Then there exists a strong solution* $(u,v)$ *of* (2.9) *with the following additional regularity:*

$$\begin{array}{l} u \in W^{1,\infty}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{2,\infty}(\Omega)) \quad \textit{and} \\ q \in L^\infty(0,T;W^{1,\infty}(\Omega)). \end{array} \tag{4.3}$$

*Besides,*

$$\min\left\{ \inf_{(0,T)} u_D, \inf_{\Omega} u_0 \right\} \le u \le 1 \quad \textit{in} \quad Q_T. \tag{4.4}$$

*Proof.* To prove the existence of a strong solution we only need to assume a bound on $f$ of the type

$$f(x,s) \le (1+b(x))\, g(s) \quad \text{with} \quad b \in L^2(\Omega), \quad g \in \mathcal{C}^\alpha([0,1]) \tag{4.5}$$

and both $b$ and $f$ non negative. It is clear that (2.12) implies (4.5). We proceed by a fixed point argument. Consider the set

$$K := \left\{ (F,G) \in L^2(Q_{T^*}) \times L^2(Q_{T^*}) : \|F\|_{L^2(Q_{T^*})} < R, \quad \|G\|_{L^2(Q_{T^*})} < \rho \right\},$$

for certain positive numbers $T^*$, $R$ and $\rho$ to be fixed. Clearly, $K$ is a convex weakly compact subset of $L^2(Q_{T^*}) \times L^2(Q_{T^*})$. We define the mapping $Q : K \to L^2(Q_{T^*}) \times L^2(Q_{T^*})$ by

$$Q(F,G) := (-(uq)_x, -f(x,u)),$$

where $u, q$ are solutions of

$$\left\{ \begin{array}{ll} u_t - u_{xx} = F & \text{in } Q_{T^*}, \\ u(0,t) = u_D(t) & \text{in } (0,T^*), \\ u_x(1,t) = 0 & \text{in } (0,T^*), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{array} \right. \tag{4.6}$$

and

$$\left\{ \begin{array}{ll} q_x = G & \text{in } Q_{T^*}, \\ q(1,t) = 0 & \text{in } (0,T^*). \end{array} \right. \tag{4.7}$$

Notice that a fixed point of $Q$ is a (local) solution of (2.9). The regularity of $F$ and $G$ implies (see, e.g., [14]) that $u, q$ are unique in the classes

$$\begin{array}{l} u \in H^1(0,T^*;L^2(\Omega)) \cap L^2(0,T^*;H^2(\Omega)) \cap L^\infty(0,T^*;H^1(\Omega)), \\ q \in L^2(0,T^*;H^1(\Omega)), \end{array} \tag{4.8}$$

and that the following estimates hold:

$$\int_{Q_{T^*}} \left(|u_t|^2 + |u_{xx}|^2\right) + \int_\Omega |u_x(T^*)|^2 \leq \int_\Omega |u_{0x}|^2 + \int_{Q_{T^*}} |F|^2 \tag{4.9}$$

and

$$\|q_x\|_{L^2(Q_{T^*})} \leq \|G\|_{L^2(Q_{T^*})} \,. \tag{4.10}$$

Since the spatial dimension of $\Omega$ is one, we have that the continuous imbedding $H^1(\Omega) \subset L^\infty(\Omega)$ holds and therefore, from (4.9) and (4.10) we also have

$$\|u\|_{L^\infty(Q_{T^*})} \leq c\left(\|u_{0x}\|_{L^2(\Omega)} + \|F\|_{L^2(Q_{T^*})}\right) \tag{4.11}$$

and

$$\|q\|_{L^2(0,T^*;L^\infty(\Omega))} \leq c\,\|G\|_{L^2(Q_{T^*})} \,.$$

Furthermore, the regularity $u \in H^1(0,T^*;L^2(\Omega))$ implies that $u = \hat{u}$ a.e. in $Q_T$, with $\hat{u} \in \mathcal{C}([0,T^*]\,;L^2(\Omega))$ and therefore the initial data is satisfied in the sense of (4.1). We shall apply the version of the fixed point theorem in [3] for which we need to show that: (i) $Q(K) \subset K$ and (ii) $Q$ is weakly-weakly sequentially continuous in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$. First point is equivalent to

$$\|(uq)_x\|_{L^2(Q_{T^*})} < R \quad \text{and} \quad \|f(x,u)\|_{L^2(Q_{T^*})} < \rho. \tag{4.12}$$

From (4.9), (4.10) and the theorem of Sobolev we get

$$\begin{aligned}\|(uq)_x\|_{L^2(Q_{T^*})} &\leq \|u_x\|_{L^\infty(0,T^*;L^2(\Omega))}\,\|q\|_{L^2(0,T^*;L^\infty(\Omega))} + \|u\|_{L^\infty(Q_{T^*})}\,\|q_x\|_{L^2(Q_{T^*})} \leq \\ &\leq c\left(\|u_{0x}\|_{L^2(\Omega)} + \|F\|_{L^2(Q_{T^*})}\right)\|G\|_{L^2(Q_{T^*})} < c\left(\|u_{0x}\|_{L^2(\Omega)} + R\right)\rho\end{aligned}$$

and from (4.5) and (4.11):

$$\|f(x,u)\|_{L^2(Q_{T^*})} \leq T^*\,\|1+b(x)\|_{L^2(\Omega)}\,\|g(u)\|_{L^\infty(Q_{T^*})} \leq cT^*\left(\|u_{0x}\|_{L^2(\Omega)} + R\right).$$

Therefore, choosing $R, \rho$ and $T^*$ small enough we get condition (4.12) satisfied.

Second point means that for every sequence $(F_n, G_n) \in K$ such that $(F_n, G_n) \rightharpoonup (F, G) \in K$ weakly-weakly in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$ it holds $Q(F_n, G_n) \rightharpoonup Q(F, G)$ weakly-weakly in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$ (at least in a subsequence). Since $(F_n, G_n)$ is bounded in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$ it follows that the solutions $(u_n, q_n)$ of (4.6) and (4.7) associated to these data are bounded in the spaces of (4.8). Therefore, there exist subsequences (still denoted by the subindex $n$) and functions $u, q$ such that

$$\begin{array}{ll} u_n \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(Q_{T^*}), \\ u_n \to u & \text{strongly in } L^2(0,T^*;H^1(\Omega)), \\ u_n \to u & \text{a.e. in } Q_{T^*}, \\ u_n \to u & \text{in } C([0,T^*];L^2(\Omega)) \\ q_n \rightharpoonup q & \text{weakly in } L^2(0,T^*;H^1(\Omega)). \end{array} \tag{4.13}$$

From (4.13) we deduce

$$\begin{array}{ll} u_{nx}q_n \to u_x q & \text{strongly in } L^2(Q_{T^*}), \\ u_n q_{nx} \rightharpoonup u q_x & \text{weakly in } L^2(Q_{T^*}), \\ f(x,u_n) \to f(x,u) & \text{strongly in } L^2(Q_{T^*}). \end{array}$$

Finally, because of the linearity of problems (4.6) and (4.7) it is clear that $(u,q)$ is a solution of (4.6) and (4.7) associated to $(F,G)$, respectively, concluding $Q(F_n,G_n) \rightharpoonup Q(F,G)$ in $L^2(Q_{T^*}) \times L^2(Q_{T^*})$ . Hence, there exists a fixed point $(u,q)$ of $Q$ that is a local (in time) solution of (2.9). But, since the estimates (4.9) and (4.10) are continuous with respect to $T^*$ we deduce that the solution is, in fact, global in time.

The additional regularity is deduced as follows: since $f(\cdot,s) \in L^\infty(\Omega)$ then from the second equation of (2.9) we get $q \in L^\infty(0,T;W^{1,\infty}(\Omega))$, and therefore from the first equation we get $u_t - u_{xx} \in L^\infty(Q_T)$. Then (4.3) follows from well known results (see, e.g., [14]).

Finally, (4.4) is just a result of the non negativity of $f$ and the maximum principle. Indeed, once that $u \geq 0$ is proved (by using the test function $\min\{u,0\}$, for instance) we have from (2.9)

$$u_t + qu_x - u_{xx} \geq u_t + qu_x - u_{xx} - uf(x,u) = 0,$$

and the maximum principle implies the assertion. □

REMARK 4.1 Under suitable assumptions on the regularity of the initial and boundary data as well as of $f(\cdot,s)$ it is possible to show the existence of classical solutions of (2.9). Indeed, assume $u_D \in \mathcal{C}^{1+\alpha}([0,T])$, $u_0 \in \mathcal{C}^{2+\beta}(\bar{\Omega})$, $f(\cdot,s) \in \mathcal{C}^{0,\gamma}(\bar{\Omega})$ and compatibility conditions between $u_D$ and $u_0$. Then, from the first equation of (2.9) we deduce that $u_t - u_{xx}$ is Hölder continuous and therefore $u \in \mathcal{C}^{1,2}_{t,x}([0,T] \times \bar{\Omega})$. Then, from the second equation of (2.9), we conclude that $q \in \mathcal{C}^{0,1}_{t,x}([0,T] \times \bar{\Omega})$.

*4.2 Uniqueness and comparison of solutions*

In this section we establish a result on uniqueness and comparison of solutions for problem (2.9). Very often, in parabolic scalar problems involving monotone or Lipschitz continuous nonlinearities, the uniqueness of solutions arises as a particular case of the comparison property. However, this situation, in general, changes for systems of equations, where uniqueness may hold regardless the existence of a comparison property. And indeed, this is the case of problem (2.9). We start by showing a counter-example to the comparison property:

let $\phi : [0,1] \to [0,1]$ be a smooth function satisfying $\phi(0) = 1$, $\phi(1) = 0$, and $\phi'(0) = \phi''(0) = \phi'(1) = \phi''(1) = 0$. Set $d = 3$ and define the initial data $u_{10}, u_{20} : [0,3] \to [0,1]$,

$$u_{10}(x) := \begin{cases} \phi(x) & x \in [0,1], \\ 0 & x \in (1,3], \end{cases} \qquad u_{20}(x) := \begin{cases} \phi(x) & x \in [0,1], \\ 0 & x \in (1,2), \\ \phi(3-x) & x \in [2,3], \end{cases}$$

the function $f(x,u) = 1-u$, and the boundary datum $u_D(t) \equiv 1$. By Theorem 4.1 there exist solutions $u_1, u_2$ with initial data $u_{10}, u_{20}$. Since $u_{10}$ and $u_{20}$ satisfy appropriate compatibility conditions at $(t,x) = (0,0),(0,3)$ the solutions have the regularity $C^{1,2}_{t,x}([0,T] \times \bar{\Omega})$ (see Remark 4.1). We can now derive a contradiction by considering the sign of $(u_1 - u_2)_t$ at $t = 0$. Since $u_{10} \leq u_{20}$, the comparison principle would imply $u_1 - u_2 \leq 0$ for all $(t,x) \in (0,T) \times \Omega$. At $t = 0$ we have

$$q_1(x) - q_2(x) = -\int_x^3 (u_{10}(s) - u_{20}(s))\, ds,$$

and for $x \leq 2$ this expression is equal to the constant $c = \int_0^1 \phi(x)dx > 0$. By subtracting the equations we find

$$(u_1 - u_2)_t = (u_1 - u_2)_{xx} - ((u_1 - u_2)q_1)_x - (u_2(q_1 - q_2))_x,$$

and if we consider this equation for $0 < x < 2$ and $t = 0$ we find

$$(u_1 - u_2)_t = -cu_{20x}.$$

Using the definition of $u_{20}$ it follows that $(u_1-u_2)_t$ can not be non-positive everywhere on $[0,2]$ at $t=0$, and that therefore we can find $(t,x)$ such that $u_1(t,x)>u_2(t,x)$. This contradicts the comparison principle.

In the following theorem we give sufficient conditions for uniqueness and comparison properties to hold. The proof relies on the study of a dual problem associated to (2.9).

THEOREM 4.2 *Assume* (2.12). *Let* $(u_1,q_1)$ *and* $(u_2,q_2)$ *be two strong solutions of* (2.9) *associated to the auxiliary data* $(u_{1D},u_{10})$, *and* $(u_{2D},u_{20})$, *respectively. Then:*

*(i) if* $(u_{1D},u_{10})=(u_{2D},u_{20})$ *and*

$$f(x,\cdot) \quad \text{is Lipschitz continuous in } [0,1] \tag{4.14}$$

*then* $(u_1,q_1)=(u_2,q_2)$ *in* $Q_T$,

*(ii) if* $u_{1D}\le u_{2D}$ *in* $(0,T)$, $u_{10}\le u_{20}$ *in* $\Omega$ *and one of the solutions satisfies*

$$u(x,t)>\int_0^x |u_x(y,t)|\,dy, \tag{4.15}$$

*then* $u_1\le u_2$ *and* $q_1\ge q_2$ *in* $Q_T$.

REMARK 4.2 Condition (4.15) is satisfied, in particular, when $u$ is a non decreasing function in $\Omega$. Indeed, one finds that

$$\int_0^x |u_x(y,t)|\,dy=u(x,t)-u_D,$$

and since $u_D>0$ the conclusion follows. Other way to look at this property is the following: Let $u_D=u_0=$const. By the maximum principle we know that $u_x(0,t)>0$. Suppose that $x_1$ is a point of maximum for $u(x,t_1)$ and that, for simplicity, it is the unique local maximum attained in $t=t_1$. Then for $x>x_1$,

$$\int_0^x |u_x(y,t_1)|\,dy=u(x_1,t_1)-u_D+u(x_1,t_1)-u(x,t_1)$$

and therefore (4.15) holds if

$$u(x,t_1)>u(x_1,t_1)-\frac{u_D}{2} \quad \textit{for} \quad x>x_1, \quad t>0.$$

Since $u_D\le u(x,t)\le 1$ for all $(x,t)\in Q_T$ we find that if $u_D>\frac{2}{3}$ then (4.15) holds.

*Proof.* The first part of the proof is common to both assertions in the theorem. Let $(u_1,q_1)$ and $(u_2,q_2)$ be solutions associated to the auxiliary data $(u_{1D},u_{10})$ and $(u_{2D},u_{20})$, respectively, with $u_{1D}\le u_{2D}$ and $u_{10}\le u_{20}$, and set $(u,q):=(u_1-u_2,q_1-q_2)$. Then $(u,q)$ satisfies

$$\begin{cases} u_t+(q_1u+u_2q)_x-u_{xx}=0 & \text{in } Q_T,\\ q_x+f(x,u_1)-f(x,u_2)=0 & \text{in } Q_T,\\ u_D(0,t)\le 0 & \text{in } (0,T),\\ u_x(d,t)=q(d,t)=0 & \text{in } (0,T),\\ u_0(x)\le 0 & \text{in } \Omega. \end{cases} \tag{4.16}$$

Multiplying the equations of (4.16) by smooth functions $\varphi,\psi$ such that

$$\varphi(0,t)=\varphi_x(d,t)=\psi(0,t)=0, \tag{4.17}$$

integrating by parts and adding the resulting integral identities we get

$$\begin{aligned}\int_\Omega u(T)\varphi(T) &= \int_\Omega u_0\varphi(0) + \int_{Q_T} u\left[\varphi_t + q_1\varphi_x + \varphi_{xx}\right] - \int_0^T u_D(t)\varphi_x(0,t)dt + \\ &\quad + \int_{Q_T} \left(f(x,u_1) - f(x,u_2)\right)\left[u_2\varphi + \psi\right] - \int_{Q_T} q\left[\psi_x + u_{2x}\varphi\right]. \qquad (4.18)\end{aligned}$$

We define the function

$$h(x,t) := \begin{cases} \dfrac{f(x,u_1) - f(x,u_2)}{u} & \text{if} \quad u \neq 0, \\ 0 & \text{if} \quad u = 0, \end{cases}$$

which is non positive because $f(x,\cdot)$ is non increasing. We introduce sequences of $\mathcal{C}^\infty(Q_T)$ functions $q_1^n$, $u_2^n$, $h^m$ such that

$$q_1^n \to q_1, \quad u_2^n \to u_2 \quad \text{strongly in } L^\infty(0,T;W^{1,\infty}(\Omega)) \quad \text{as} \quad n \to \infty, \qquad (4.19)$$

$$h^m \to h \quad \text{a.e. in } Q_T \quad \text{as } m \to \infty, \quad \text{with } 0 \geq h^m(x,t) \geq \max\{-m, h(x,t)\}, \qquad (4.20)$$

and we choose $h^m$ as a monotone decreasing sequence. We can rewrite (4.18) as

$$\begin{aligned}\int_\Omega u(T)\varphi(T) &= \int_\Omega u_0\varphi(0) + \int_{Q_T} u\left[\varphi_t + q_1^n\varphi_x + \varphi_{xx} + h^m\left(u_2^n\varphi + \psi\right)\right] - \\ &\quad - \int_0^T u_D(t)\varphi_x(0,t)dt - \int_{Q_T} q\left[\psi_x + u_{2x}^n\varphi\right] + \\ &\quad + \int_{Q_T} u\left[\left(q_1 - q_1^n\right)\varphi_x + \left(h - h^m\right)\left(u_2\varphi + \psi\right)\right] + \\ &\quad + \int_{Q_T} uh^m\left(u_2 - u_2^n\right)\varphi - \int_{Q_T} q\left(u_{2x} - u_{2x}^n\right)\varphi. \qquad (4.21)\end{aligned}$$

We perform the change of variable $\tau := T - t$ and set the following problem to choose the functions $\varphi, \psi$ :

$$\begin{cases} \varphi_t - q_1^n\varphi_x - \varphi_{xx} - h^m\left(u_2^n\varphi + \psi\right) = 0 & \text{in } Q_T, \\ \psi_x + u_{2x}^n\varphi = 0 & \text{in } Q_T, \\ \varphi(0) = \chi_\delta(u(T)) & \text{in } \Omega, \end{cases} \qquad (4.22)$$

with $\chi_\delta \in \mathcal{C}^\infty(I\!R)$ such that $0 \leq \chi_\delta \leq 1$ and $\chi_\delta(s) \to sign_+(s)$ when $\delta \to 0$ and with the boundary conditions given in (4.17).

LEMMA 4.1 *For each $n, m, \delta$ there exists a unique solution of (4.22) with the regularity $\varphi, \psi \in \mathcal{C}^\infty(Q_T)$. Moreover,*

**(i)** *if $f(x,\cdot)$ if Lipschitz continuous in $[0,1]$ then $\|\varphi\|_{L^\infty(Q_T)}$ and $\|\psi\|_{L^\infty(Q_T)}$ are uniformly bounded with respect to $n, m, \delta$, and*

**(ii)** *if (4.15) holds then $0 \leq \varphi \leq 1$ in $Q_T$ and $\|\psi\|_{L^\infty(Q_T)}$ is uniformly bounded with respect to $n, m, \delta$ and $\varphi_x(0,t) \geq 0$ in $(0,T)$.*

*Continuation of the proof of Theorem* 4.2. Using the functions provided by Lemma 4.1 we obtain from (4.21)

$$\begin{aligned}\int_\Omega u(T)\chi_\delta(u(T)) &= \int_\Omega u_0\varphi(0) - \int_0^T u_D(t)\varphi_x(0,t)dt + \\ &+ \int_{Q_T} u\left[(q_1 - q_1^n)\varphi_x + (h - h^m)(u_2\varphi + \psi)\right] + \\ &+ \int_{Q_T} u h^m (u_2 - u_2^n)\varphi - \int_{Q_T} q\,(u_{2x} - u_{2x}^n)\varphi. \end{aligned} \tag{4.23}$$

Thanks to the uniform estimates in $n$ ensured by the lemma and to the convergences (4.19) we can pass to the limit $n \to \infty$ to obtain

$$\int_\Omega u(T)\chi_\delta(u(T)) = \int_\Omega u_0\varphi(0) - \int_0^T u_D(t)\varphi_x(0,t)dt + \int_{Q_T} u\,(h - h^m)(u_2\varphi + \psi)\,. \tag{4.24}$$

Since $\varphi, \psi$ have $L^\infty(Q_T)$ bounds independent of $m$ and since $h^m(x,t) \to h(x,t)$ a.e. $(x,t) \in Q_T$ as $m \to \infty$ we get

$$\int_{Q_T} u\,(h - h^m)(u_2\varphi + \psi) \to 0 \quad \text{as} \quad m \to \infty, \tag{4.25}$$

and therefore, from (4.24) we deduce

$$\int_\Omega u(T)\chi_\delta(u(T)) = \int_\Omega u_0\varphi(0) - \int_0^T u_D(t)\varphi_x(0,t)dt. \tag{4.26}$$

We distinguish now the two cases in the statement of the theorem. Firstly, from (4.26) and the assumption in (i) $u_{1D} = u_{2D}$ and $u_{10} = u_{20}$ we get when $\delta \to 0$

$$\int_\Omega |u(T)|_+ = 0,$$

from where we deduce $u_1 = u_2$ a.e. in $Q_T$ and then, from the second equation and the boundary condition for $q$ of (4.16) we also deduce $q_1 = q_2$ a.e. in $Q_T$. Secondly, since by Lemma 4.1 we have $0 \le \varphi \le 1$ in $Q_T$ and $\varphi_x(0,t) \ge 0$ in $(0,T)$ we obtain from (4.26) and (4.25) when letting $\delta \to 0$

$$\int_\Omega |u(T)|_+ \le \int_\Omega |u_0|_+ = 0,$$

from where we deduce the assertion on the comparison of solutions. □

*Proof of Lemma* 4.1. Since problem (4.22) is linear and the coefficients are $\mathcal{C}^\infty$ functions, the result on existence, uniqueness and regularity of the solution is well known. First notice that in the case in which $f(\cdot, s)$ is a Lipschitz continuous function we have that function $h$ is bounded and therefore $h^m$ and $h$ coincides for $m$ large enough. Clearly, the $L^\infty$ bounds of $\varphi$ and $\psi$ will not depend on $m$. A similar argument applies to the dependence on $n$ and $\delta$. In the case in which we assume (4.15) we proceed by reduction to the absurd. We assert that the global maximum and/or minimum of $\varphi$ are attained in the parabolic boundary of $Q_T$. Suppose this is false and let $(x_0, \tau_0)$ and $(x_1, \tau_1)$ be interior points of $Q_T$ where the global maximum and/or minimum, respectively, are attained. From the boundary conditions (4.17) we have that, in particular, $\varphi(0,0) = 0$ so necessarily $\varphi(x_0,\tau_0) > 0 > \varphi(x_1,\tau_1)$. Due to the regularity of $\varphi$ both $(x_0,\tau_0)$ and $(x_1,\tau_1)$ are critical points for $\varphi$ satisfying $\varphi_{xx}(x_0,\tau_0) \le 0$ and $\varphi_{xx}(x_1,\tau_1) \ge 0$. By using first equation of (4.22) we find

$$u_2(x_0,\tau_0)\varphi(x_0,\tau_0) + \psi(x_0,\tau_0) \le 0, \tag{4.27}$$

where we used that $f(x,\cdot)$ is non increasing (and therefore $h^m \leq 0$) . Integrating the second equation of (4.22) in $(0,x)$ we get

$$\psi(x,t) = \int_0^x \left(-u_{2x}(y,t)\right)\varphi(y,t)dy. \tag{4.28}$$

Therefore, from (4.27), (4.28) and assumption (4.15) we obtain

$$\begin{aligned} u_2(x_0,\tau_0)\varphi(x_0,\tau_0) &\leq \int_0^{x_0} u_{2x}(y,\tau_0)\varphi(y,\tau_0)dy \leq \sup_{y\in(0,x_0)} \varphi(y,\tau_0)\int_0^{x_0} |u_{2x}(y,\tau_0)|\, dy < \\ &< u_2(x_0,\tau_0)\varphi(x_0,\tau_0), \end{aligned}$$

a contradiction. So the maximum of $\varphi$ must be attained in the parabolic boundary. A similar argument shows that also the minimum of $\varphi$ must be attained in the parabolic boundary. In fact, by the strong maximum principle, we know that if the maximum (respectively, minimum) is attained in $(d,t)$, for some $t \in (0,T]$, then $\varphi_x(d,t) > 0$ (resp. $\varphi_x(d,t) < 0$) that contradicts the boundary condition (4.17) satisfied by $\varphi$. As a consequence, we get $0 \leq \varphi \leq 1$ in $Q_T$ and $\varphi_x(0,t) \geq 0$ in $(0,T)$. Finally from (4.28) we deduce

$$\|\psi\|_{L^\infty(Q_T)} \leq \|\varphi\|_{L^\infty(Q_T)} \|u_2\|_{L^\infty(0,T;W^{1,1}(\Omega))},$$

which is also independent of $m, n$ and $\delta$ (see (4.19)). □

*4.3 Asymptotic stability*

In this section we study the stabilization of the solutions of the evolution problem (2.9) to the solution of the stationary problem (3.4) by applying well known techniques developed for nonlinear scalar equations (see, e.g., [15] or [7]). To do this we first introduce some definitions:

Definition 4.2 *We say that* $(u,q)$ *is a* weak solution *of* (3.1) *if* $u \in \tilde{u} + \mathcal{V}$, $q \in W^{1,r}(\Omega)$ *for any* $r \in [1,\infty)$, $q_x(d) = 0$,

$$\int_\Omega (uq)_x\, \xi + \int_\Omega u_x \xi_x = 0, \quad \textit{for any } \xi \in \mathcal{V}, \quad \textit{and} \tag{4.29}$$

$$q_x = f(\cdot,u) \quad \textit{a.e. in } \Omega. \tag{4.30}$$

*We say that* $(u_\infty, q_\infty)$ belongs to the $\omega$-limit set $\omega(u_0,u_D)$ *if* $u_\infty \in \tilde{u} + \mathcal{V}$, $q_\infty \in W^{1,r}(\Omega)$ *for any* $r \in [1,\infty)$ *and there exists a sequence* $t_n \to \infty$ *such that*

$$\left.\begin{array}{lll} u(\cdot,t_n) \to u_\infty & \textit{strongly in } L^2(\Omega) & \\ q(\cdot,t_n) \to q_\infty & \textit{strongly in } L^r(\Omega) & \textit{for any } r \in [1,\infty) \end{array}\right. \quad \textit{as } n \to \infty.$$

Theorem 4.3 *Assume* (2.12) *and* $\lim_{t\to\infty} u_D(t) = \tilde{u}$. *Suppose that* $(u,q)$ *is a strong solution of* (2.9) *such that for some* $t_0 > 0$

$$u \in L^\infty(t_0,\infty;\mathcal{V}) \cap H^1(t_0,\infty;L^2(\Omega)). \tag{4.31}$$

*Then* $\omega(u_0,u_D)$ *is non empty and if* $(u_\infty,q_\infty) \in \omega(u_0,u_D)$ *then* $(u_\infty,q_\infty)$ *is a weak solution of* (3.1). *Besides, there exists a sequence* $\tilde{t}_n$ *with* $\tilde{t}_n \to \infty$ *as* $n \to \infty$ *such that*

$$u(\cdot,\tilde{t}_n) \to u_\infty \quad \textit{strongly in } \mathcal{V} \quad \textit{as } n \to \infty.$$

*Proof.* Let $t_n$ be a sequence such that $t_n \to \infty$ as $n \to \infty$. Due to (4.31) the sequence $u(\cdot, t_n)$ is bounded in $\mathcal{V}$ and therefore converges strongly in $L^2(\Omega)$ as $n \to \infty$ to an element that we denote by $u_\infty$. Since $\mathcal{V} \subset L^\infty(Q_T)$ due to the space dimension, we deduce $u(\cdot, t_n) \to u_\infty$ strongly in any $L^r(\Omega)$ with $r \in [1, \infty)$. Besides, from the second equation of (2.9) and the continuity of $f(x, \cdot)$ we obtain $q_x(\cdot, t_n) = -f(\cdot, u(\cdot, t_n)) \to -f(\cdot, u_\infty)$ strongly in $L^r(\Omega)$ for any $r \in [1, \infty)$, and therefore $q(\cdot, t_n)$ converges strongly in $W^{1,r}(\Omega)$, for any $r \in [1, \infty)$ to some function in this space which we denote by $q_\infty$ and which satisfies

$$q_{\infty x} = -f(\cdot, u_\infty) \quad \text{a.e.in } \Omega. \tag{4.32}$$

So we showed that $\omega(u_0, u_D)$ is non empty.

Let us show now that if $(u_\infty, q_\infty) \in \omega(u_0, u_D)$ then $(u_\infty, q_\infty)$ is a weak solution of (3.1). To this end it is enough to show that $(u_\infty, q_\infty)$ satisfies (4.29) because then (4.30) follows directly from (4.32). Consider the function $v(x,t) := \xi(x)\varphi(t)$, with $\xi \in \mathcal{V}$ and $\varphi \in \mathcal{D}(-1,1)$ satisfying $\varphi \geq 0$ and $\int_{-1}^1 \varphi(s)ds = 1$. Choosing $t_n > 1$ and $T \geq t_n + 1$ and multiplying the first equation of (2.9) by $v$ we get

$$-\int_{t_n-1}^{t_n+1} \int_\Omega u\xi\varphi' + \int_{t_n-1}^{t_n+1} \int_\Omega (uq)_x \xi\varphi + \int_{t_n-1}^{t_n+1} \int_\Omega u_x \xi_x \varphi = 0, \tag{4.33}$$

where we used that $\varphi(0) = \varphi(T) = 0$. We introduce the change of variable $s := t - t_n$ and define $U_n(x,s) := u(x, t_n + s)$ and $Q_n(x,s) := q(x, t_n + s)$. On one hand, for each $s \in (-1,1)$ it holds

$$\int_\Omega |U_n(x,s) - u(x,t_n)|^2 \, dx \leq \int_\Omega \int_{t_n-1}^{t_n+1} |u_t(x,t)|^2 \, dt dx.$$

Integrating in $(-1,1)$ and using (4.31) we obtain

$$\int_{-1}^1 \int_\Omega |U_n(x,s) - u(x,t_n)|^2 \, dx \leq 2 \int_\Omega \int_{t_n-1}^{t_n+1} |u_t(x,t)|^2 \, dt dx \to 0$$

as $n \to \infty$. Since $u(\cdot, t_n) \to u_\infty$ strongly in $L^2(\Omega)$ we deduce $U_n \to u_\infty$ strongly in $L^2((-1,1) \times \Omega)$. On the other hand, since $f(x, \cdot)$ is continuous, we have

$$Q_{nx}(\cdot, s) = -f(\cdot, U_n(\cdot, s)) \to -f(\cdot, u_\infty) \quad \text{strongly in } L^2((-1,1) \times \Omega).$$

But, by definition, $q_{\infty x} := -f(\cdot, u_\infty)$ and therefore $Q_{nx} \to q_{\infty x}$ strongly in $L^2((-1,1) \times \Omega)$. Next we identify the limits as a weak solution of (3.1). From the introduced change of variable and unknowns and (4.33) we get

$$-\int_{-1}^1 \int_\Omega U_n \xi\varphi' + \int_{-1}^1 \int_\Omega (U_n Q_n)_x \xi\varphi + \int_{-1}^1 \int_\Omega U_{nx} \xi_x \varphi = 0. \tag{4.34}$$

Since, by assumption, $U_n$ is bounded in $L^\infty(-1,1;\mathcal{V})$ there exists a subsequence (still denoted by $n$) such that $U_n \rightharpoonup u_\infty$ weakly$*$ - weakly in $L^\infty(-1,1;\mathcal{V})$ and since by Sobolev's theorem $\mathcal{V} \subset L^\infty(\Omega)$ we also have that $U_n \rightharpoonup u_\infty$ weakly$*$ in $L^\infty((-1,1) \times \Omega)$. We conclude that $U_n \to u_\infty$ strongly in $L^r((-1,1) \times \Omega)$ for any $r \in [1, \infty)$. This implies, as before, that $Q_{nx} \to q_{\infty x}$ strongly in $L^r((-1,1) \times \Omega)$ for any $r \in [1, \infty)$ and therefore $(U_n Q_n)_x \rightharpoonup (u_\infty q_\infty)_x$ weakly in $L^2((-1,1) \times \Omega)$. Letting $n \to \infty$ in (4.34) we obtain

$$-\int_{-1}^1 \varphi' ds \int_\Omega u_\infty \xi dx + \int_{-1}^1 \varphi ds \int_\Omega (u_\infty q_\infty)_x \xi dx + \int_{-1}^1 \varphi ds \int_\Omega u_{\infty x} \xi_x = 0 \tag{4.35}$$

and since $\int_{-1}^{1} \varphi' ds = 0$, we deduce

$$\int_\Omega (u_\infty q_\infty)_x \, \xi + \int_\Omega u_{\infty x} \xi_x = 0. \tag{4.36}$$

Finally, we shall show that there exists a sequence $\tilde{t}_n$ with $\tilde{t}_n \to \infty$ as $n \to \infty$ such that

$$u(\cdot, \tilde{t}_n) \to u_\infty \quad \text{strongly in } \mathcal{V} \quad \text{as } n \to \infty.$$

With this purpose we shall prove that

$$\int_{-1}^{1} \int_\Omega |U_{nx} - u_{\infty x}|^2 \, \varphi \to 0 \tag{4.37}$$

because since this is impossible if for some $\varepsilon \geq 0$

$$\int_\Omega |U_{nx}(s,x) - u_{\infty x}(x)|^2 \, dx \geq \varepsilon \quad \text{for a.e. } s \in (-1,1)\,,$$

we conclude that there exists a sequence $s_n$ in $(-1,1)$ such that

$$\lim_{n\to\infty} \int_\Omega |u_x(t_n + s_n, x) - u_{\infty x}(x)|^2 \, dx = \lim_{n\to\infty} \int_\Omega |U_{nx}(s_n, x) - u_{\infty x}(x)|^2 \, dx = 0.$$

So let us prove (4.37). Multiplying the first equation of (2.9) by $v(x,t) := (u(x,t) - u_D(t))\, \varphi(t - t_n)$ and integrating we obtain

$$-\frac{1}{2} \int_{t_n-1}^{t_n+1} \int_\Omega \left( \left(u^2\right)_t - u_t u_D \right) \varphi + \int_{t_n-1}^{t_n+1} \int_\Omega (uq)_x \, (u - u_D)\, \varphi + \int_{t_n-1}^{t_n+1} \int_\Omega |u_x|^2 \, \varphi = 0.$$

We have

$$\int_{t_n-1}^{t_n+1} \int_\Omega \left(u^2\right)_t \varphi = - \int_{t_n-1}^{t_n+1} \int_\Omega u^2 \varphi' = \int_{-1}^{1} \varphi' \int_\Omega U_n^2 dx ds \to \int_{-1}^{1} \varphi' ds \int_\Omega u_\infty^2 dx = 0$$

and

$$\int_{t_n-1}^{t_n+1} u_t u_D \varphi = - \int_{t_n-1}^{t_n+1} u \left(u_{Dt} \varphi + u_D \varphi'\right) \to -\tilde{u} \int_{-1}^{1} \varphi' ds \int_\Omega u_\infty dx = 0.$$

Therefore,

$$\lim_{n\to\infty} \int_{-1}^{1} \int_\Omega (U_n Q_n)_x \, (U_n - u_D(t_n + s))\, \varphi + \int_{-1}^{1} \int_\Omega |U_{nx}|^2 \, \varphi = 0.$$

By the previous result on convergence we have

$$\lim_{n\to\infty} \int_{-1}^{1} \int_\Omega (U_n Q_n)_x \, (U_n - u_D(t_n + s))\, \varphi = \int_\Omega (u_\infty q_\infty)_x \, (u_\infty - \tilde{u})$$

and therefore

$$\lim_{n\to\infty} \int_{-1}^{1} \int_\Omega |U_{nx}|^2 \, \varphi = - \int_\Omega (u_\infty q_\infty)_x \, (u_\infty - \tilde{u}) = \int_\Omega |u_{\infty x}|^2 \,.$$

Then

$$\int_{-1}^{1} \int_\Omega |U_{nx} - u_{\infty x}|^2 \, \varphi = \int_{-1}^{1} \int_\Omega U_{nx} \left(U_{nx} - u_{\infty x}\right) \varphi - \int_{-1}^{1} \int_\Omega u_{\infty x} \left(U_{nx} - u_{\infty x}\right) \varphi \to 0,$$

and we get (4.37).□

## 5. Formation of a dead core

In this section we present results concerning the existence and spatial localization of free boundaries (sets where $\{u = 1\}$) for problems (3.1) and (2.9). When and where the root uptaking of fresh water is *strong* we expect the formation of a region (*dead core*) where the threshold concentration value $u = 1$ is reached. It is already well known that this kind of situations are linked, in the mathematical sense, to some type of degeneracy in the problem, that may be caused, as it is our case, by a degenerated or strong absorption term in the problem, typically, a non Lipschitz continuous absorption term. For the stationary problem, due to its simplicity, we can make use of the comparison principle and the tools associated to it to give accurate conditions that imply the existence of such a dead core. However, for the evolution problem this is no longer possible and more sophisticated techniques have to be employed. In this case we use a local energy method for free boundary problems. The idea is to introduce an energy functional (given by the norm in the natural energy space associated to the problem) and to derive a differential inequality for the energy functional. From this inequality the desired qualitative property of solutions can be deduced.

The energy method we use (see [2], [8]) has two principal features. Firstly, it is a local method, i.e., it operates in subsets of the corresponding domain without need of global informations like boundary conditions or boundedness of the domain. Secondly, it has a very general setting, allowing to consider, for instance, systems of equations in any space dimension or with coefficients depending on the space and time variables. This method does not need any monotonicity assumption on the non linearities and it requires no comparison principle to hold. In return, however, it only provides a qualitative insight in the problem but not valuable quantitative estimates.

We first analyze the steady state problem and find sufficient conditions for a dead core to arise. This conditions are related both with the non Lipschitz continuity of function $f(\cdot, s)$ in $s = 1$ and with the size of the auxiliary data. Then we show that there exist situations in which the size of the auxiliary data prevents the formation of a dead core even when $f$ is non Lipschitz continuous. Next, we turn to the evolution problem and show the corresponding result on formation of a dead core. Finally, we prove a crucial inequality used in the energy method that becomes specially simple in the one dimensional case.

### *5.1 The steady state*

To study the conditions that imply the formation of a dead core in the stationary problem (3.1) we will work with the equivalent formulation of the problem given by (3.4). We recall it here:

$$\begin{cases} -w_{xx} = -f(x, e^{-w}) & \text{in } (0,d), \\ w(0) = -\log \tilde{u}, \\ w_x(d) = 0. \end{cases} \tag{5.1}$$

Since $f$ is monotone decreasing and non negative, it is well known that the comparison principle holds for this problem. In particular, the maximum of $w$ is attained in $x = 0$ and it holds $w \geq 0$, so

$$0 \leq w \leq -\log \tilde{u} \quad \text{in} \quad (0,d).$$

Notice that the dead core, if it does exists, is given by

$$\{x \in [0,d] : w(x) = 0\}.$$

We will construct a supersolution $\bar{w}$ of (5.1) such that $\bar{w} = 0$ in $(a,d)$ with $a \in (0,d)$ implying by the comparison principle that also $w = 0$ in $(a,d)$, and therefore, that a free boundary arises for the original steady state problem (3.1). Before doing this we will assume, in addition to (2.12), the following structural property on function $f$:

$$f(x,s) \geq k_0 (1-s)_+^p, \quad \text{for} \quad x \in (0,d) \quad \text{and} \quad s \geq 0, \tag{5.2}$$

with $k_0$ a positive constant and $p \in (0,1)$. We consider the auxiliary problem

$$\left\{\begin{array}{ll} -\bar{w}_{xx} = -k_0 c_*^p \bar{w}^p & \text{in } (0,d), \\ \bar{w}(0) = -\log \tilde{u}, & \\ \bar{w}_x(d) = 0, & \end{array}\right. \tag{5.3}$$

with $c_*$ to be fixed. After (5.2), a sufficient condition for a solution of (5.3) to be a supersolution of (5.1) is that

$$c_* s \leq 1 - e^{-s} \quad \text{for} \quad s \in (0, -\log \tilde{u}).$$

An easy computation shows that this is fulfilled for $c_* = \frac{1-\tilde{u}}{-\log \tilde{u}}$. Now we will give conditions on the data that will imply that the function

$$\bar{w}(x) = \left\{\begin{array}{ll} c(a-x)^{\frac{2}{1-p}} & \text{if } x \in (0,a), \\ 0 & \text{if } x \in (a,d), \end{array}\right.$$

with $a \in (0,d)$ is a solution of (5.3). The differential equation is satisfied if $2c\frac{1+p}{(1-p)^2} = k_0 c_*^p c^p$,so we choose

$$c := \left(k_0 c_*^p \frac{(1-p)^2}{2(1+p)}\right)^{\frac{1}{1-p}}.$$

The boundary conditions are fulfilled if $a < d$ and $ca^{\frac{2}{1-p}} = -\log \tilde{u}$, so we fix

$$a := \left(\frac{-\log \tilde{u}}{c}\right)^{\frac{1-p}{2}}, \tag{5.4}$$

and it only remains the condition $a \in (0,d)$ to be satisfied .It is clear that $a > 0$. On the other hand, $a < d$ if $-\log \tilde{u} < cd^{\frac{2}{1-p}}$. After substituting the expressions for $c$ (and $c_*$) in this latter inequality we arrive to the condition

$$\frac{-\log \tilde{u}}{(1-\tilde{u})^p} < \frac{k_0 d^2 (1-p)^2}{2(1+p)}. \tag{5.5}$$

Notice finally that the left hand side of (5.5) tends to zero when $\tilde{u} \to 1$, and therefore, for any fixed triple $k_0$, $d$, $p$ there always exist $\tilde{u} \in (0,1)$ such that (5.5) holds and then a free boundary arises. We have proved the following

THEOREM 5.1 *Assume* (2.12), (5.2) *and that the data of the problem satisfy* (5.5). *Then the first component of the solution of* (3.1) *has the property* $\{x : u(x) = 1\} \supset (a,d)$, *with a given by* (5.4).

*5.2 The evolution case*

For convenience, we perform the change of unknown $\hat{u} := 1 - u$ in (2.9) to remove the singularity from $u = 1$ to $\hat{u} = 0$. We get (dropping the hats)

$$\left\{\begin{array}{ll} u_t + (uq)_x - u_{xx} + f(x, 1-u) = 0 & \text{in } Q_T, \\ q_x + f(x,u) = 0 & \text{in } Q_T, \\ u(0,t) = 1 - u_D(t) & \text{in } (0,T), \\ u_x(d,t) = q(d,t) = 0 & \text{in } (0,T). \end{array}\right. \tag{5.6}$$

For any $t \in (0,T)$ we consider the paraboloid

$$\mathcal{P} \equiv \mathcal{P}(t) := \{(x,\tau) : X_-(\tau) \leq x \leq X_+(\tau), \quad \tau \in (t,T)\},$$

with $X_-(\tau) := x_0 - R(\tau;t)$, $X_+(\tau) := x_0 + R(\tau;t)$, $R(\tau;t) := (\tau - t)^\nu$, $0 < \nu < 1$ to be fixed and $x_0 \in (0,d)$ satisfying $R(T;t) < x_0 < d - R(T;t)$, so $\mathcal{P}(t) \subset (0,d)$ for all $t \in (0,T)$. We introduce the non negative functions (*energies*)

$$E(t) := \int_{\mathcal{P}(t)} |u_x|^2 \, dx \, d\tau, \quad C(t) := \int_{\mathcal{P}(t)} u^{p+1} \, dx \, d\tau. \tag{5.7}$$

THEOREM 5.2 *Assume that there exist constants $k_0$ and $k_1$ such that*

$$0 < k_0 \, (1-s)^{p+1} \le (1-s) \, f(x,s) \le k_1 \, (1-s)^{p+1} \quad in \quad x \in \mathcal{P}(t), \quad a.e \; t \in (0.T), \tag{5.8}$$

*for any $s \in (0,1)$ and with $p < 1$ and $k_0 > k_1/2$. Then, if $E(0) + C(0)$ is small enough, there exists a $t^* \in (0,T)$ such that any first component of a solution of (2.9) satisfies $u \equiv 0$ in $\mathcal{P}(t^*)$.*

*Proof.* We proceed in several steps:
Step **1.** Thanks to the regularity of $u$ and $q$ we may multiply the first equation of (5.6) by $u$ and integrate in $\mathcal{P}$, obtaining

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{2}(u^2)_t + \tfrac{1}{2} \left( \left(u^2 q\right)_x + u^2 q_x \right) + \left( |u_x|^2 - (u u_x)_x \right) + u f(x, 1-u) \right\} \, dx d\tau = 0.$$

Using the divergence theorem, the second equation of (5.6) and the bounds (5.8) we find

$$\begin{aligned} \tfrac{1}{2} \int_{X_-(T)}^{X_+(T)} u^2(T,x) \, dx + \int_{\mathcal{P}} |u_x|^2 \, dx d\tau \quad & + k_0 \int_{\mathcal{P}} u^{p+1} \, dx d\tau \le \\ & \le \tfrac{1}{2} \int_{\partial\mathcal{P}} u^2 n_\tau \, dx d\tau + \int_{\partial\mathcal{P}} u u_x n_x \, dx d\tau + \\ & + \tfrac{k_1}{2} \int_{\mathcal{P}} u^{p+2} \, dx d\tau - \frac{1}{2} \int_{\partial\mathcal{P}} u^2 q n_x \, dx d\tau, \end{aligned}$$

where $(n_x, n_\tau)$ is the unitary outward normal vector to $\mathcal{P}$, given by

$$(n_x, n_\tau) := \begin{cases} (0,1) & \text{if } \tau = T \text{ and } x \in (X_-(T), X_+(T)), \\ \text{non defined} & \text{if } \tau = T \text{ and } x = X_\pm(T), \\ \frac{\left((\tau-t)^{1-\nu}, -\nu\right)}{\left(\nu^2 + (\tau-t)^{2(1-\nu)}\right)^{1/2}} & \text{in the rest.} \end{cases}$$

Since $\|u\|_{L^\infty(Q_T)} \le 1$, $\|q\|_{L^\infty(\mathcal{P})} \le dk_1$and $(n_x, n_\tau)$ is unitary

$$\tfrac{1}{2} \int_{X_-(T)}^{X_+(T)} u^2(T,x) \, dx + E(t) + \tfrac{2k_0 - k_1}{2} C(t) \le \tfrac{1 + dk_1}{2} \int_t^T \left[u^2\right] d\tau + \int_{\partial\mathcal{P}} |u| \, |u_x| \, dx d\tau, \tag{5.9}$$

where we introduced the notation $[v] := |v(X_+(\tau))| + |v(X_-(\tau))|$.
Step **2.** Our aim is to estimate the right hand side of (5.9) by means of the functions in the left hand side and their derivatives. First notice that

$$\frac{dE}{dt}(t) = \int_t^T \frac{\partial R}{\partial t}(\tau, t) \left[ |u_x|^2 \right] d\tau$$

and therefore we can use Hölder's inequality to get

$$\begin{aligned} \int_{\partial\mathcal{P}} |u| \, |u_x| \, dx d\tau \quad & \le \quad \left( \int_t^T - \frac{\partial R}{\partial t} \left[ |u_x|^2 \right] \, d\tau \right)^{1/2} \left( \int_t^T \left( - \frac{\partial R}{\partial t} \right)^{-1} \left[u^2\right] \, d\tau \right)^{1/2} = \\ & = \quad I_1 \left( - \frac{dE}{dt}(t) \right)^{1/2} \le I_1 \left( - \frac{d\,(E+C)}{dt}(t) \right)^{1/2}, \end{aligned} \tag{5.10}$$

where $I_1 := \left( \int_t^T \left( -\frac{\partial R}{\partial t} \right)^{-1} [u^2] \, d\tau \right)^{1/2}$. To handle the term $I_1$ as well as the term $I_2 := \int_t^T [u^2] \, d\tau$ of (4.23) we shall apply a simple version of an interpolation-trace inequality deduced in a more general setting in [8]. It reads: for any $v \in H^1(x_0 - R, x_0 + R)$ it holds

$$|v(x_0 - R)| + |v(x_0 + R)| \leq L_0 \left( \|v_x\|_2 + (2R)^{-\delta} \|v\|_{p+1} \right)^{\gamma} \|v\|_r^{1-\gamma}, \tag{5.11}$$

with $\|\cdot\|_s := \|\cdot\|_{L^s(x_0 - R, x_0 + R)}$, $L_0$ an universal constant independent of $v$, $r \in [1, \infty]$,

$$\gamma := \frac{2}{2+r} \quad \text{and} \quad \delta := \frac{p+3}{2(p+1)}. \tag{5.12}$$

We restrict $r$ to be $r < 2$ and find, by applying Hölder's inequality with exponent $Q := \frac{1-p}{2-r}$ to the norm $\|v\|_r$ that

$$\|v\|_r \leq \|v\|_2^{\frac{2}{rQ'}} \|v\|_{p+1}^{\frac{p+1}{rQ}}. \tag{5.13}$$

Step **3.** Combining (5.11) and (5.13) with $v(x) := u(x, \tau)$ we get

$$[u^2] \leq c\,[u]^2 \leq Km(R) \left( \|u_x\|_2^2 + \|v\|_{p+1}^{p+1} \right)^{\gamma} \|u\|_2^{\frac{4(1-\gamma)}{rQ'}} \|u\|_{p+1}^{\frac{2(1-\gamma)(p+1)}{rQ}}$$

with

$$K := 2L_0^2 \max \left\{ 1, 2^{-2\delta} \sup_{t<\tau<T} \|u(\tau)\|_{p+1}^{1-p} \right\}^{\gamma}, \quad m(R) := \max\left\{1, R^{-2\delta}\right\}.$$

Notice that since $\|u\|_{L^\infty(Q_T)} \leq 1$ and $R(T; t) \leq d/2$ we get $K \leq 2L_0^2 \max\left\{1, 2^{-\frac{4}{p+1}} d^{\frac{1-p}{p+1}}\right\}^{\gamma}$, where we have used the value of $\delta$ given in (5.12). From the non negativeness of the norms we deduce

$$[u^2] \leq Km(R) \left( \|u_x\|_2^2 + \|v\|_{p+1}^{p+1} \right)^{\gamma + \frac{2(1-\gamma)}{rQ}} \|u\|_2^{\frac{4(1-\gamma)}{rQ'}}.$$

We then obtain

$$I_1 \leq K^{1/2} b(t)^{\frac{1-\gamma}{rQ'}} \left( \int_t^T m(R) \left( -\frac{\partial R}{\partial t} \right)^{-1} \left( \|u_x\|_2^2 + \|v\|_{p+1}^2 \right)^{\gamma + \frac{2(1-\gamma)}{rQ}} \right)^{1/2}$$

with $b(t) := \sup_{t<\tau<T} \|u(\tau)\|_2^2$. Using Hölder's inequality with exponent $\mu$, given by

$$\mu^{-1} := \gamma + \frac{2(1-\gamma)}{rQ} \tag{5.14}$$

and substituting the explicit expression of $R$ we obtain

$$I_1 \leq K^{1/2} \Lambda(t) b(t)^{\frac{1-\gamma}{rQ'}} (E(t) + C(t))^{\frac{\gamma}{2} + \frac{1-\gamma}{rQ}} \leq K^{1/2} \Lambda(t) \left(b(t) + E(t) + C(t)\right)^{\frac{\gamma}{2} + \frac{1-\gamma}{r}}, \tag{5.15}$$

with

$$\Lambda(t) := \left( \int_t^T (\tau - t)^{\mu'(1-\nu-2\delta\nu\gamma)} \right)^{1/2\mu'}. \tag{5.16}$$

We have that $\mu$ is an admissible Holder's exponent if $\mu^{-1} \leq 1$, that turns to be equivalent to $r \geq \frac{4}{3-p}$. The function $\Lambda$ is finite whenever we choose $\nu < \frac{\mu+1}{\mu(1+2\delta\gamma)}$ that is always possible since the only restriction assumed on $\nu$ is that $0 < \nu < 1$. Gathering (5.10) and (5.15) we get

$$\int_{\partial\mathcal{P}} |u|\,|u_x|\; dx d\tau \leq K^{1/2}\Lambda(t)\left(-\frac{d\,(E+C)}{dt}(t)\right)^{1/2}(b(t)+E(t)+C(t))^{\frac{\gamma}{2}+\frac{1-\gamma}{r}}.$$

In a similar way (but choosing $r = 2$ in (5.11)) we get the following estimate

$$I_2 \leq K\Gamma(t)\,(b(t)+E(t)+C(t)),$$

with

$$\Gamma(t) := \int_t^T (\tau - t)^{-2\delta\nu\tilde{\gamma}\tilde{\mu}'}\,d\tau,$$

with $\tilde{\gamma} := 1/2$ and $\tilde{\mu} := 2$. $\Gamma(t)$ is finite if we choose $\nu < \frac{p+1}{p+3}$.
Step **4.** From (4.23) we deduce

$$\begin{aligned}\tfrac{1}{2}\int_{X_-(T)}^{X_+(T)} u^2(T,x)\,dx + E(t) \quad &+ \tfrac{2k_0-k_1}{2}C(t) \leq \tfrac{1+dk_1}{2}K\Gamma(t)\,(b(t)+E(t)+C(t)) + \\ &+K^{1/2}\Lambda(t)\left(-\frac{d\,(E+C)}{dt}(t)\right)^{1/2}(b(t)+E(t)+C(t))^{\frac{\gamma}{2}+\frac{1-\gamma}{r}}.\end{aligned}$$

Since the right hand side of this expression is increasing with $T$ we may replace $\int_{X_-(T)}^{X_+(T)} u^2(T,x)\,dx$ by $b(t)$, obtaining

$$c_0\,(b(t)+E(t)+C(t)) \leq K^{1/2}\Lambda(t)\left(-\frac{d\,(E+C)}{dt}(t)\right)^{1/2}(b(t)+E(t)+C(t))^{\frac{\gamma}{2}+\frac{1-\gamma}{r}},$$

with $c_0 := \min\left\{\frac{1}{2}, k_0 - \frac{k_1}{2}\right\} - \frac{1+dk_1}{2}K\Gamma(t)$. Notice that making $T-t$ small enough (say $T-t \leq \varepsilon$) we can ensure that $c_0 > 0$. We therefore arrive to the inequality

$$c_0^2\,(E(t)+C(t))^{2\left(1-\frac{\gamma}{2}-\frac{1-\gamma}{r}\right)} \leq -K\Lambda(t)^2\frac{d\,(E+C)}{dt}(t). \tag{5.17}$$

Due to the crucial assumption $p < 1$ we find that the exponent $\sigma := 2\left(1-\frac{\gamma}{2}-\frac{1-\gamma}{r}\right) < 1$. First we assume that $T := \varepsilon$ so the restriction $T - t \leq \varepsilon$ is fulfilled for all $t \geq 0$. Then we can integrate (5.17) in $t \in (0, t^*)$ with $t^* \leq \varepsilon$ to get

$$(E+C)^{1-\sigma}(t^*) \leq (E+C)^{1-\sigma}(0) - (1-\sigma)\frac{c_0^2}{K}\int_0^{t^*}\Lambda(t)^{-2}dt.$$

Therefore, if

$$(E+C)^{1-\sigma}(0) \leq (1-\sigma)\frac{c_0^2}{K}\int_0^{t^*}\Lambda(t)^{-2}dt$$

then $E(t^*) + C(t^*) = 0$ and therefore $u = 0$ a.e. in $\mathcal{P}(t^*)$. □

*Appendix: an interpolation trace inequality.* The following lemma is a particular case of a more general result obtained in [8] for any spatial dimension and a wider range of exponents. However, due to the space dimension, the proof is much simpler in our case, and all the constants appearing in it can be explicitly computed.

LEMMA 5.1 *Let $v \in H^1(x_0 - R, x_0 + R)$, for some $x_0 \in I\!R$ and some positive constant $R$. Then it holds*

$$|v(x_0 - R)| + |v(x_0 + R)| \leq L_0 \left( \|v_x\|_2 + R^{-\delta} \|v\|_{p+1} \right)^{\gamma} \|v\|_r^{1-\gamma}, \tag{5.18}$$

*with $L_0 \leq 16$ , $r \in [1, 2]$, $p \geq 0$, $\gamma := \frac{2}{2+r}$ and $\delta := \frac{p+3}{2(p+1)}$.*

*Proof.* We proceed in several steps:
Step **1.** We first consider $v \in H^1(0,1)$. From the identity

$$v^2(x) - v^2(0) = 2\int_0^x v(x)v_x(x)dx \tag{5.19}$$

and

$$v^2(0) = \int_0^1 v^2(x)dx - 2\int_0^1 \int_0^x v(x)v_x(x)dx \leq \|v\|_2^2 + 2\|v\|_2 \|v_x\|_2 ,$$

we obtain

$$\left[v^2\right] := v^2(1) + v^2(0) \leq 8\|v\|_2 (\|v\|_2 + \|v_x\|_2) = 8\|v\|_2 \|v\|_{H^1(0,1)} . \tag{5.20}$$

Step **2.** If $u \in H^1(0,1)$ with $u(0) = 0$ then from (5.19) and Hölder's inequality we get $\|u\|_2 \leq \sqrt{8}\|u_x\|_2$ . Taking $u(x) := v(x) - v(0)$ we find

$$\int_0^1 v^2 + v(0)^2 \leq 8\int_0^1 v_x^2 + 2v(0)\int_0^1 v \leq 8\int_0^1 v_x^2 + v(0)^2 + 2\left(\int_0^1 v\right)^2$$

and then $\|v\|_2 \leq \sqrt{8}(\|v_x\|_2 + \|v\|_1)$, from where we deduce

$$\|v\|_{H^1(0,1)} \leq \left(\sqrt{8} + 1\right)(\|v_x\|_2 + \|v\|_1) . \tag{5.21}$$

Step **3.** We use Hölder's interpolation inequality:

$$\|v\|_s \leq \|v\|_l^{\alpha} \|v\|_q^{1-\alpha} \quad \text{with} \quad \frac{1}{s} = \frac{\alpha}{l} + \frac{1-\alpha}{q} \quad \text{and} \quad 1 \leq l \leq s \leq 2 \leq \infty. \tag{5.22}$$

This inequality is true even if $l \in (0,1)$. Indeed, if we set $v := u^m$ with $m < \frac{1}{l}$ the above inequality reads $\|v\|_{sm} \leq \|v\|_{lm} \|v\|_{qm}$ , with $\frac{1}{s} = \frac{\alpha}{l} + \frac{1-\alpha}{q}$ and $lm < 1$. Applying (5.22) to the function $v^2$ with the parameters $s := 1$, $l = \alpha := r/2$, $q = \infty$ we get

$$\|v\|_1^2 \leq \|v\|_r^r \left\|v^2\right\|_{\infty}^{1-r/2} .$$

By Sobolev's theorem

$$\left\|v^2\right\|_{\infty} \leq 2\left\|v^2\right\|_{W^{1,1}(0,1)} = 2\left(\|v\|_2^2 + \left\|\left(v^2\right)_x\right\|_1\right),$$

but $\left\|\left(v^2\right)_x\right\|_1 \leq 2\|v\|_2 \|v_x\|_2$ and therefore

$$\left\|v^2\right\|_{\infty} \leq 4\|v\|_2 (\|v\|_2 + \|v_x\|_2) = 4\|v\|_2 \|v\|_{H^1(0,1)} .$$

We then obtain

$$\|v\|_2^2 \le \|v\|_r^r \left(4\,\|v\|_2\,\|v\|_{H^1(0,1)}\right)^{1-r/2},$$

implying

$$\|v\|_2 \le 4^{\frac{2-r}{2+r}}\,\|v\|_r^{\frac{2r}{2+r}}\,\|v\|_{H^1(0,1)}^{\frac{2-r}{2+r}}. \tag{5.23}$$

Step **4.** From (5.20) and (5.23) we get

$$[v] \le \left[v^2\right]^{1/2} \le \left(4^{2+\frac{2-r}{2+r}}\,\|v\|_r^{\frac{2r}{2+r}}\,\|v\|_{H^1(0,1)}^{\frac{2-r}{2+r}+1}\right)^{1/2} = 2^{\frac{6+r}{2+r}}\,\|v\|_r^{\frac{r}{2+r}}\,\|v\|_{H^1(0,1)}^{\frac{2}{2+r}},$$

and using (5.21) we obtain

$$[v] \le 2^{\frac{10+r}{2+r}}\left(\|v_x\|_2 + \|v\|_1\right)^{\frac{2}{2+r}}\,\|v\|_r^{\frac{r}{2+r}}.$$

Finally, since $\|v\|_1 \le \|v\|_{p+1}$ (remind that the measure of the domain is 1) we find

$$[v] \le 2^{\frac{10+r}{2+r}}\left(\|v_x\|_2 + \|v\|_{p+1}\right)^{\frac{2}{2+r}}\,\|v\|_r^{\frac{r}{2+r}}. \tag{5.24}$$

Notice that since $r \in [1,2]$ we have $2^{\frac{10+r}{2+r}} \le 16$.
Step **5.** Finally, we consider the change of unknown $y := x_0 - R + 2xR$ that maps the interval $(0,1)$ onto $(x_0 - R, x_0 + R)$. We obtain that for any $q \ge 1$ it holds

$$\|u\|_{L^q(0,1)} = \frac{1}{(2R)^q}\,\|u\|_{L^q(x_0-R,x_0+R)} \quad \text{and} \quad \|u\|_{H^1(0,1)} = \sqrt{2R}\,\|u\|_{H^1(x_0-R,x_0+R)}.$$

Therefore, from (5.24) we deduce

$$[v] \le 2^{\frac{10+r}{2+r}}\left(\|v_x\|_2 + (2R)^{-\frac{3+p}{2(p+1)}}\,\|v\|_{p+1}\right)^{\frac{2}{2+r}}\,\|v\|_r^{\frac{r}{2+r}}$$

for any $v \in H^1(x_0 - R, x_0 + R)$. □

## 6. Conclusions

Under the hypothesis that water and salt movement in the mangrove root zone is in the vertical direction and that there is an impermeable base at some distance below the soil surface, we have shown that the salt concentration builds up around the mangrove roots until a steady gradient is established. In the steady state, the convection of salt downwards from the surface in the water flowing to the roots balances the upward diffusion of salt due to the concentration gradient. The buildup of salt in the lower part of the root zone reduces the effectiveness of the roots there, and lowers the transpiration rate of the mangroves. We have shown that the steady state salinity profile is governed by the dimensionless parameter $k_0$ defined in (2.8) as

$$k_0 := \frac{Z_m K_0}{\theta D}. \tag{6.1}$$

The salt concentration at the bottom of the root zone is governed by $k_0$, and is higher for higher values of $k_0$. In our model the root zone has a finite total water uptake rate $K_0$ in the absence of salinity, and $Z_m$ is a characteristic depth of the root zone. It is evident from the form of (6.1) that not just the value of $K_0$ but also the distribution of the roots over depth is important, and a deeper root zone will lead to higher salinity levels. The occurrence of the porosity $\theta$ in the denominator of (6.1) means that in more porous soils or those with macropores the effective root zone can extend to a greater depth before salt build-up becomes limiting.

If the soil is assumed to start with the salt concentration $C_0$ of the surface water, as the mangroves take up water the salt concentration $C$ builds up in the root zone until the steady state is reached, and the actual transpiration rate $Q_0$ falls substantially. At early times, the increase in the salinity in the root zone has the approximate characteristic time $T_1$,

$$T_1 := \frac{\theta Z_m}{K_0},$$

which means that the rise is more rapid for higher values of $K_0$. For sufficiently high $K_0$ the concentration at the bottom of the root zone will be at or close to threshold concentration $C_1$ above which the roots no longer take up water. In this situation the mangrove plant will cope best the salinity build-up by having most of its roots close to the surface. These theoretical results are in accord with the observations of Lin and Sternberg [16], who found that the mangroves they studied had most of their roots close to the surface and used mainly surface water.

If there is a region of porous soil between the bottom of the root zone at depth $Z_m$ and an impermeable base at depth $L$, salt will diffuse downwards from the root zone until the soil water below the root zone has the same elevated salt concentration, and the approximate characteristic time $T_2$ for this is

$$T_2 := \frac{(L - Z_m)^2}{D}, \tag{6.2}$$

which does not depend on $K_0$. This estimate will give a reasonably accurate upper bound on the time for the soil below the root zone to become salinized. When $T_2 \gg T_1$, i.e., when the diffusional time is much greater than the time for the root zone to salinize, (6.2) will give a reasonably accurate estimate of the upper bound on the time for the soil below the root zone to become salinized. In this study we have ignored the effect of density differences caused by concentration differences. These could give rise to instabilities of the kind studied by Wooding et al. [18], which would cause convective fingering and which would lead to mixing on a much shorter time scale.

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## Figure captions

Fig 3.1 Construction of the orbit for $p = 1$, $u_0 = 0.25$ and $k_0 = 10.0$, and root distribution $k$ given by (2.10). Numbers on curves are values of $q(0)$ used for shooting, with $q_0 = 3.5369$. The curve $z = 1.0$ was obtained numerically by shooting, with initial values $w(0)$ and $q(0)$ satisfying $w(0) = w_0 = log(u_0)$, and $3.4 \leq q(0) \leq 3.8$, and is evidently monotone.

Fig 3.2 Construction of the orbit for $p = 0.5$, $u_0 = 0.25$ and $k_0 = 10$. Numbers on curves are values of $q(0)$ used for shooting, with $q_0 = 4.2471$.

Fig 3.3 Similarity profiles of (a) reduced flux $qk_0^{-1/2}$ and (b) dimensionless salt concentration $u$ as functions of reduced depth $zk_0^{1/2}$, for $p$ values of 0, 0.5, 1 and 2. Numbers on curves are values of $p$. These profiles are for an infinitely deep root zone, with uniform $k(z) := k_0$.

Fig 3.4 Graphs of $k_0(u_0, u_1, p)$ as a function of $u_1$, for value $u_0 = 0.25$, and for $p$ values of 0, 0.5, 1 and 2. Numbers on curves are values of $p$. For $p < 1$, $k_0(u_0, 1, p)$ has the finite value $k_0^*(u_0, p)$ which is 2.77 for $p = 0$ and 14.7 for $p = 0.5$, as indicated by the symbol *. It is evident that the curves are monotonically increasing functions of $u_1$.

Fig 3.5 Profiles of (a) dimensionless fluid discharge $q$ and (b) dimensionless salt concentration $u$ as a function of dimensionless depth $z$, for $p = 0.5$ and root uptake distribution given by (2.10). Numbers on curves are values of the transpiration parameter $k_0$, with the critical value $k_0^* = 14.705$.

Fig 3.6 Profiles of (a) dimensionless fluid discharge $q$ and (b) dimensionless salt concentration $u$ as a function of dimensionless depth $z$, for $p = 2$ and root uptake distribution given by (2.10). Numbers on curves are values of the transpiration parameter $k_0$.

Fig 3.7 Construction of the orbit entering the origin at $s = s^* < log(1 + d)$. Root distribution is given by (2.11) and parameter values are $p = 0.5$, $u_0 = 0.25$ and $k_0 = 10.$. Shooting with $v_0 = 3.4843$ gives $s^* = 1.2456$.

Fig 4.1 Evolution of dimensionless salinity $u$ for root distribution given by (2.10), and for parameter values $p = 0.5$, $u_0 = 0.25$, $d = 4.$ and $k_0 = 10.$. Contours in the $z - t$ plane are shown. Numbers on curves are values of $u$.

Fig 4.2 As for Fig. 4.1, with root distribution decreasing with $z$ according to (2.11).

Fig 4.3 Evolution of dimensionless salinity $u$ for root distribution uniform in $0 \leq z \leq 1$, and zero in $1 < z \leq d$, for parameters values $p = 1$, $u_0 = 0.25$, $d = 2.$ and $k_0 = 10$. Profiles of $u(z)$ at dimensionless times $t$ are marked on curves.

Fig 4.4 As for Fig. 4.3 with $d = 4.0$.

Fig 4.5 As for Fig. 4.3 with $k_0 = 20.0$ and $d = 4.0$.

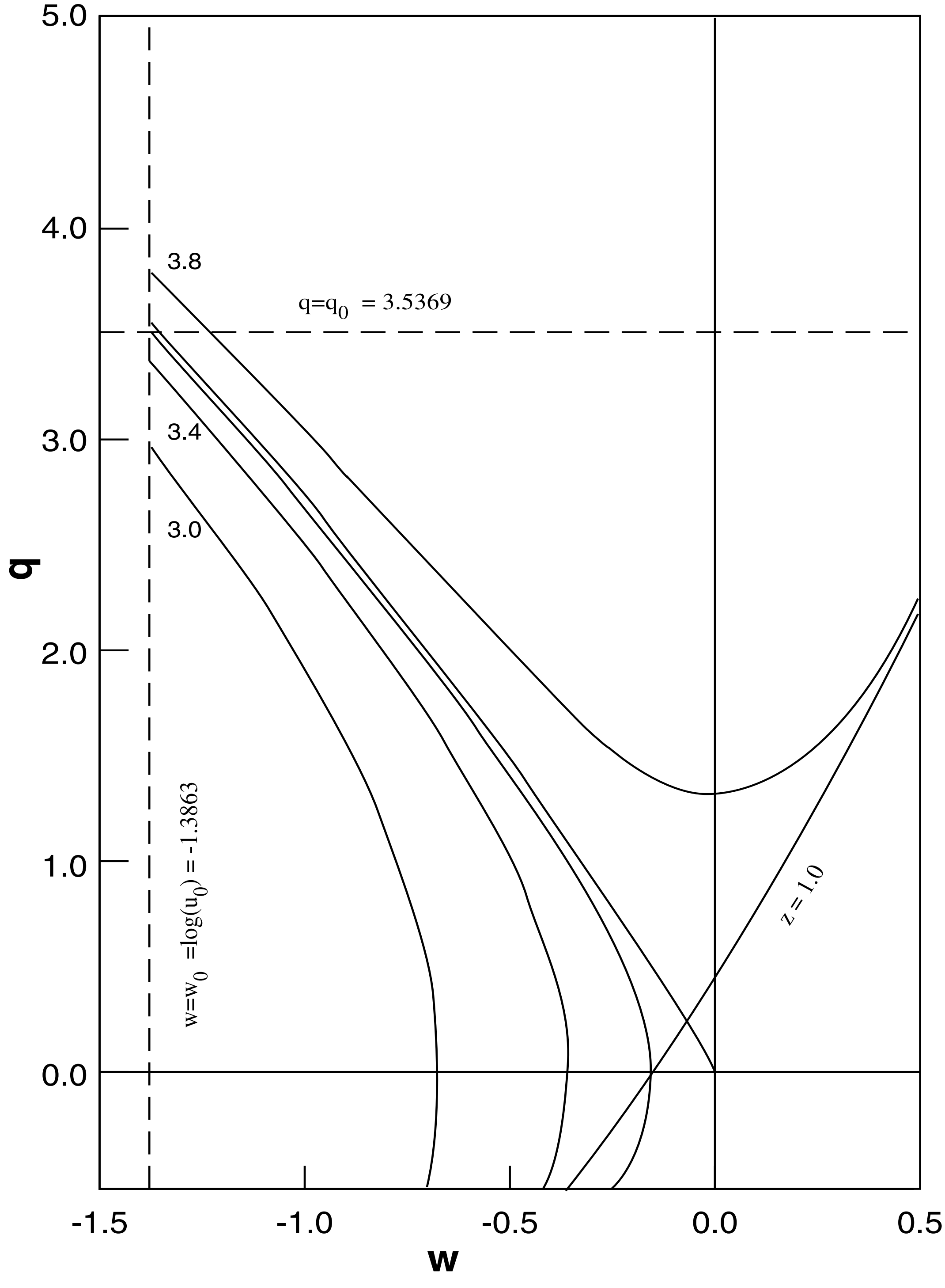


FIGURE 3.1. Construction of the orbit for $p = 1$, $u_0 = 0.25$ and $k_0 = 10.0$, and root distribution $k(z)$ given by (2.15). Numbers on curves are values of $q(0)$ used for shooting, with $q_0 = 3.5369$. The curve $z = 1.0$ was obtained numerically by shooting, with initial values $w(0)$ and $q(0)$ satisfying $w(0) = w_0 = \log(u_0)$, and $3 \leq q(0) \leq 3.8$, and is evidently monotone.

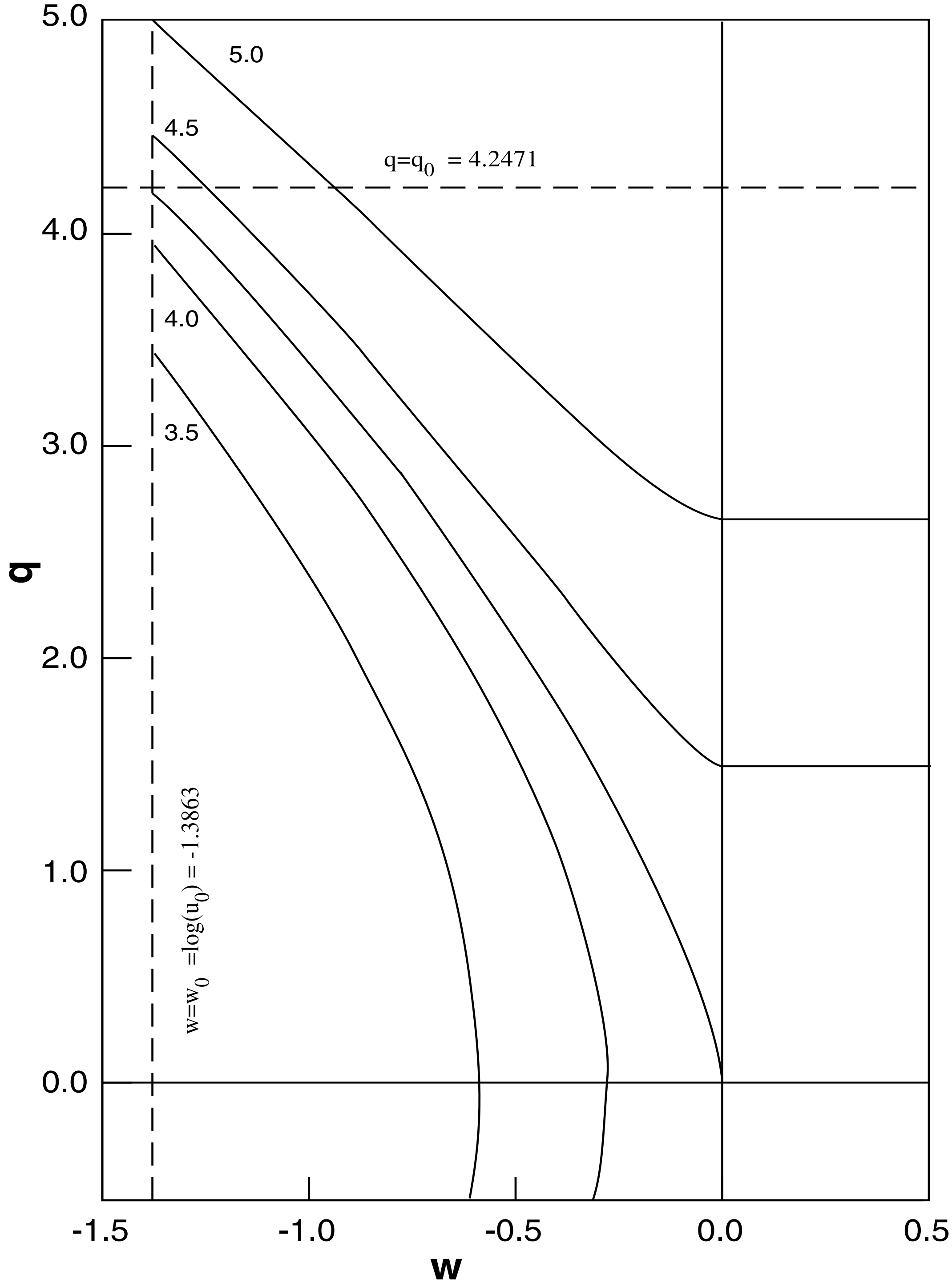


FIGURE 3.2. Construction of the orbit for $p = 0.5$, $u_0 = 0.25$ and $k_0 = 10.0$. Numbers on curves are values of $q(0)$ used for shooting, with $q_0 = 4.2471$.

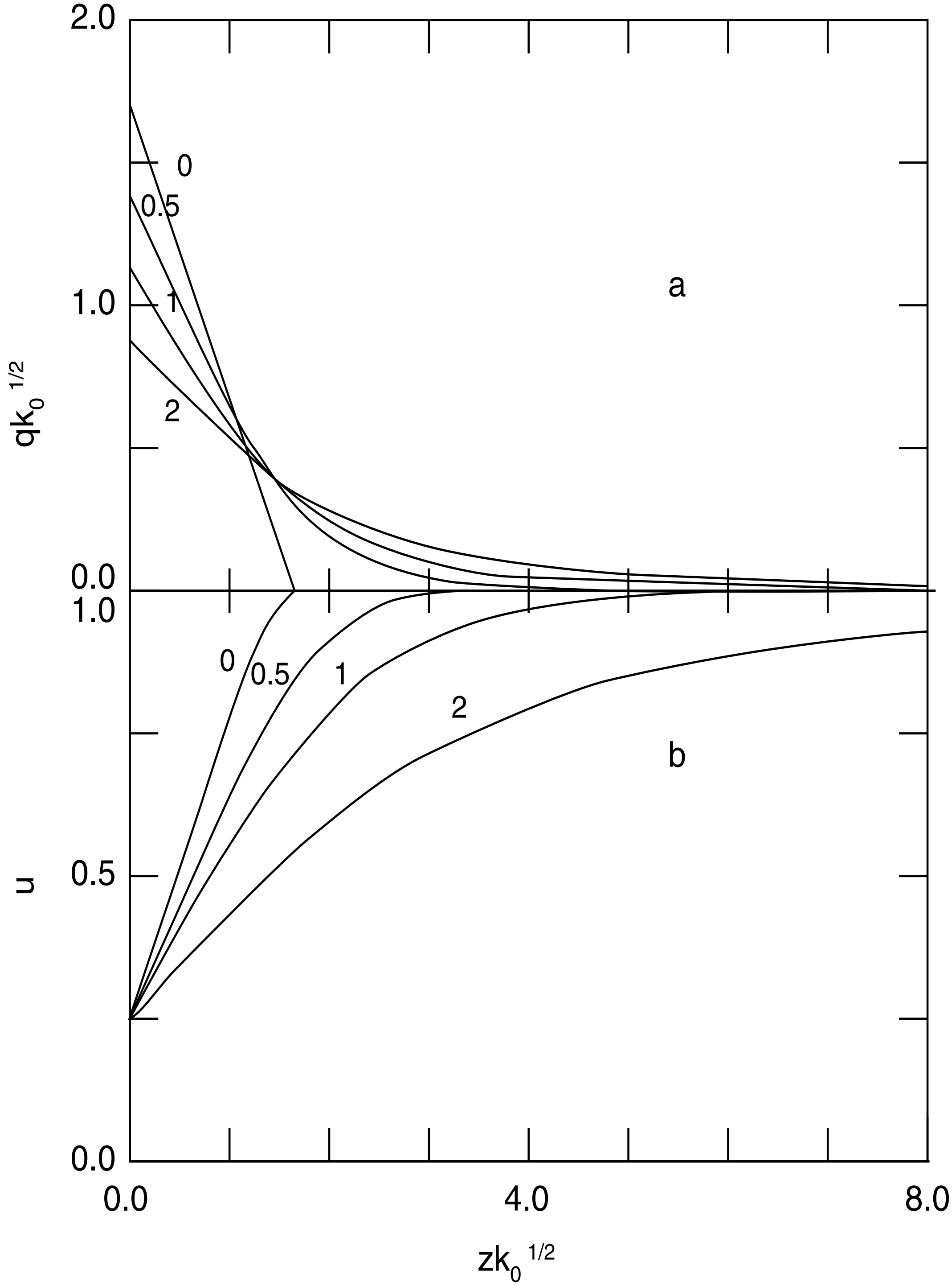


FIGURE 3.3. Similarity profiles of (a) reduced flux $qk_0^{-1/2}$ and (b) dimensionless salt concentration $u$ as functions of reduced depth $zk_0^{1/2}$, for $p$ values of 0, 0.5, 1 and 2. Numbers on curves are values of $p$. These profiles are for an infinitely deep root zone, with uniform $k(z) = k_0$.

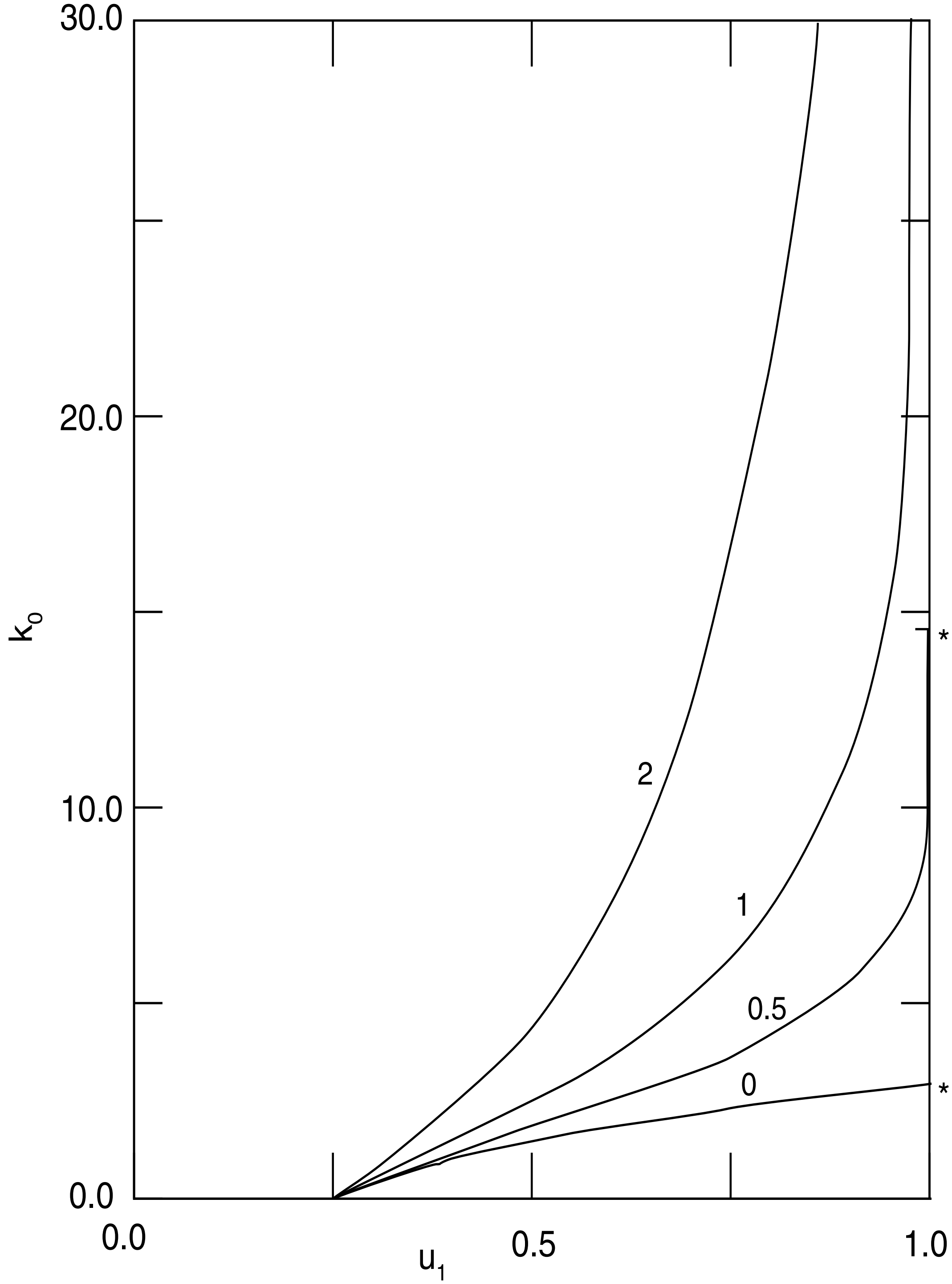


FIGURE 3.4. Graphs of $k_0(u_0, u_1, p)$ as a function of $u_1$, for the value $u_0 = 0.25$, and for $p$ values of 0, 0.5, 1 and 2. Numbers on curves are values of $p$. For $p < 1$, $k_0(u_0, 1, p)$ has the finite value $k_0^*(u_0, p)$, which is 2.77 for $p = 0$, and 14.7 for $p = 0.5$, as indicated by the symbol *. It is evident that the curves are monotonically increasing functions of $u_1$.

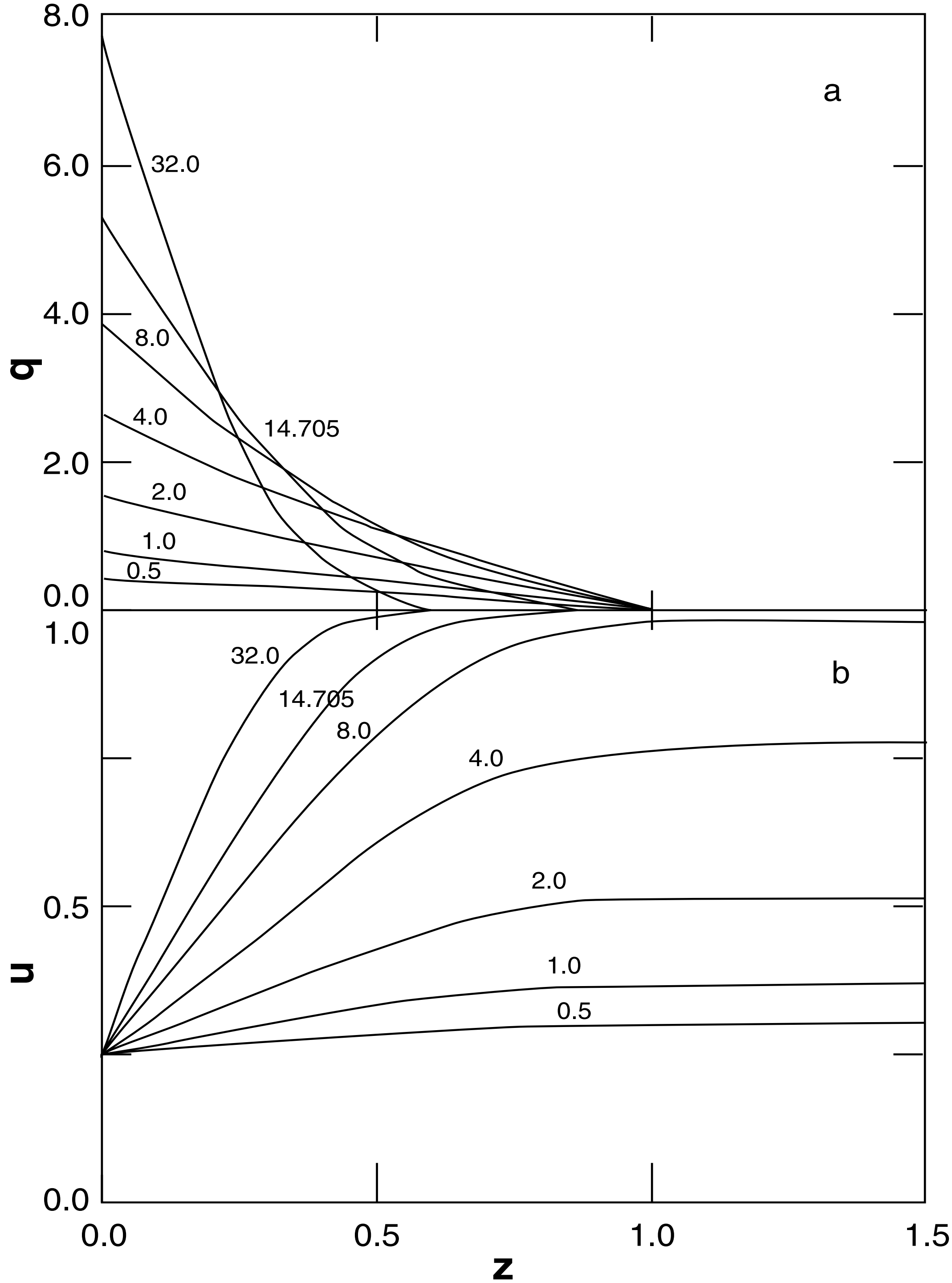


FIGURE 3.5. Profiles of (a) dimensionless fluid discharge $q$ and (b) dimensionless salt concentration $u$ as a function of dimensionless depth $z$, for $p = 0.5$ and root uptake distribution given by (2.15). Numbers on curves are values of the transpiration parameter $k_0$, with the critical value $k_0^* = 14.705$.

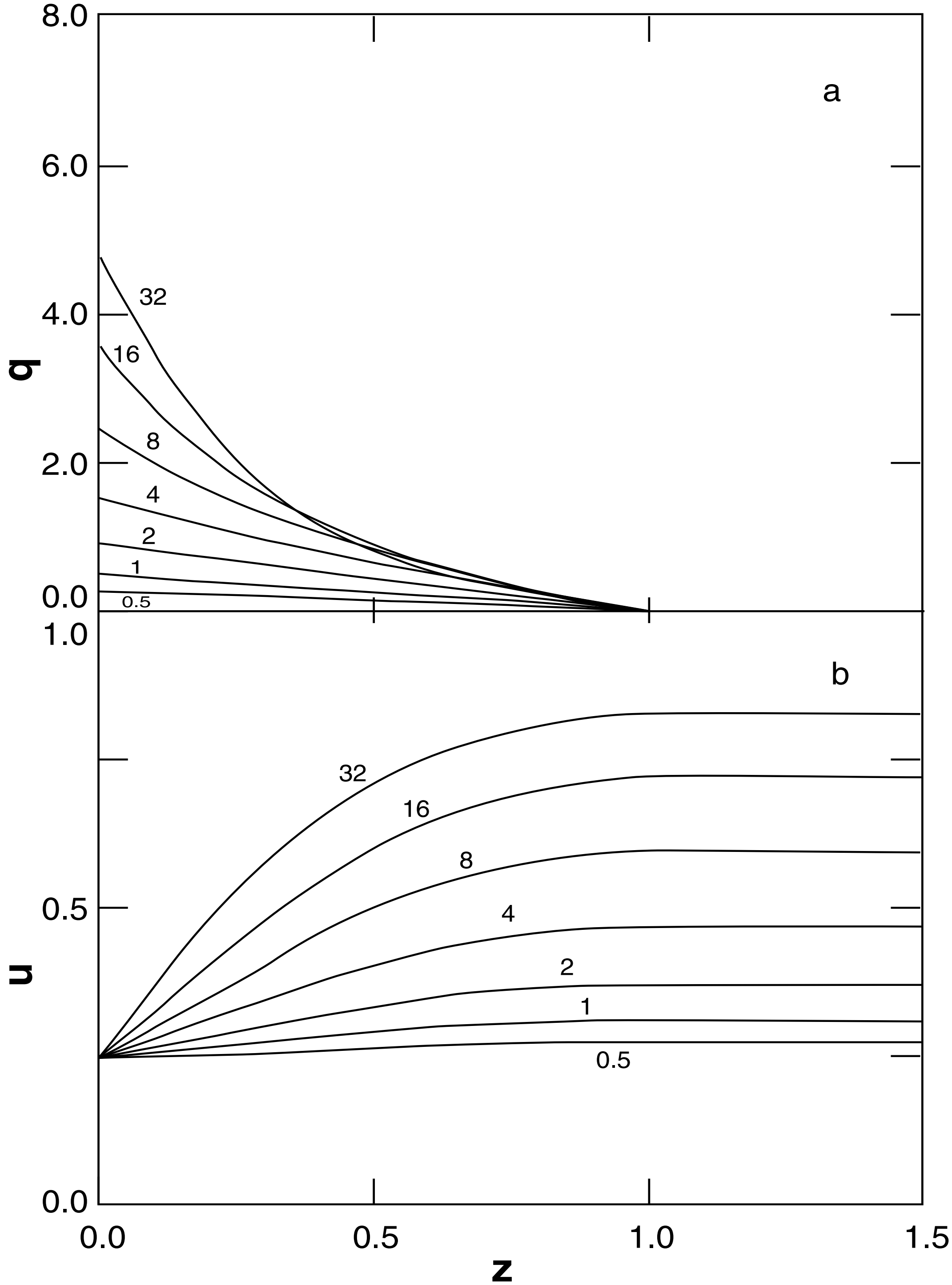


FIGURE 3.6. Profiles of (a) dimensionless fluid discharge $q$ and (b) dimensionless salt concentration $u$ as a function of dimensionless depth $z$, for $p = 2$ and root uptake distribution given by (2.15). Numbers on curves are values of the transpiration parameter $k_0$.

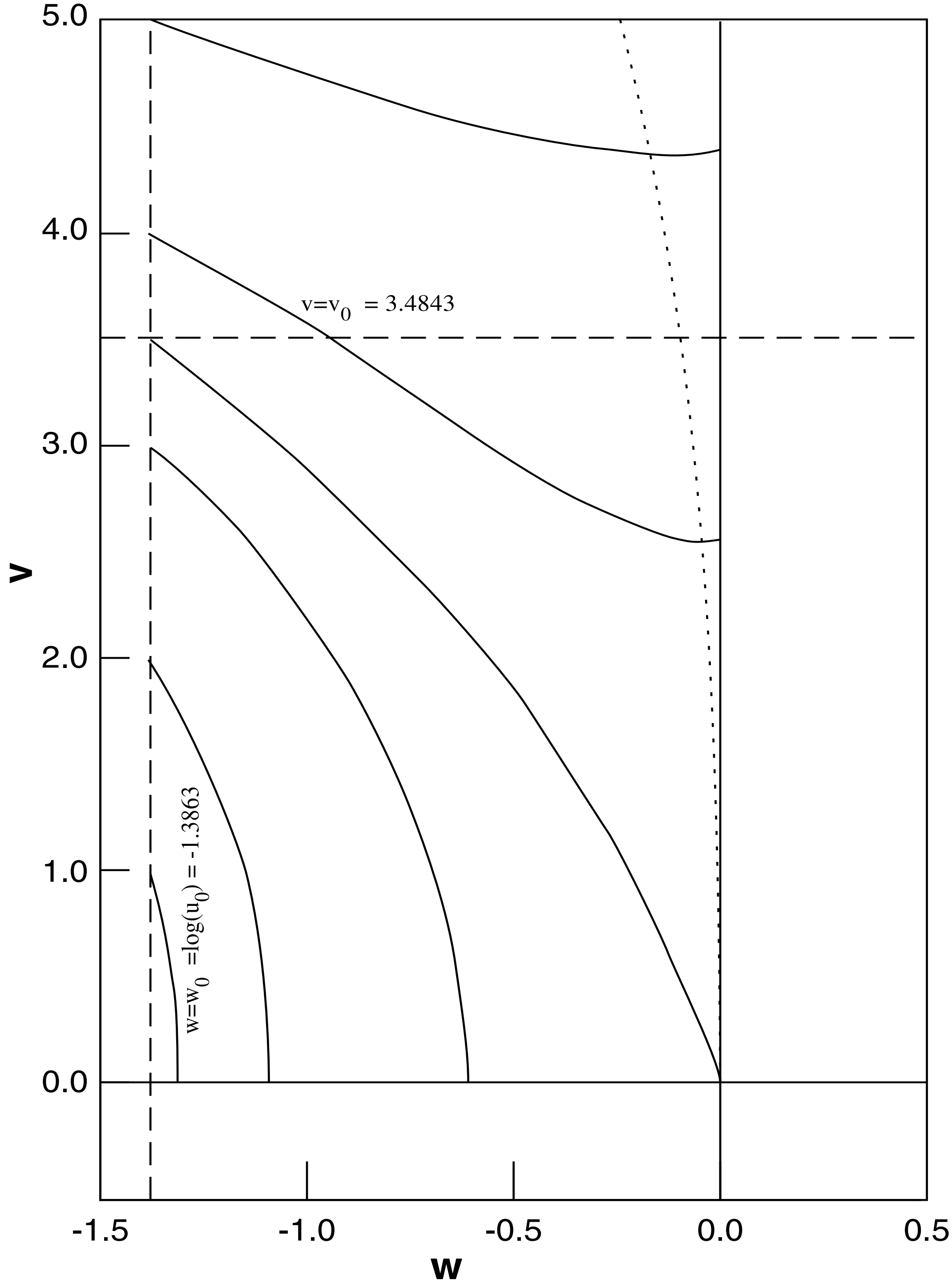


FIGURE 3.7. Construction of the orbit entering the origin at $s = s^* < \log(1+d)$. Root distribution $k(z)$ is given by (2.14) and parameter values are $p = 0.5$, $u_0 = 0.25$ and $k_0 = 10.0$; shooting with $v_0 = 3.4843$ gives $s^* = 1.2456$.

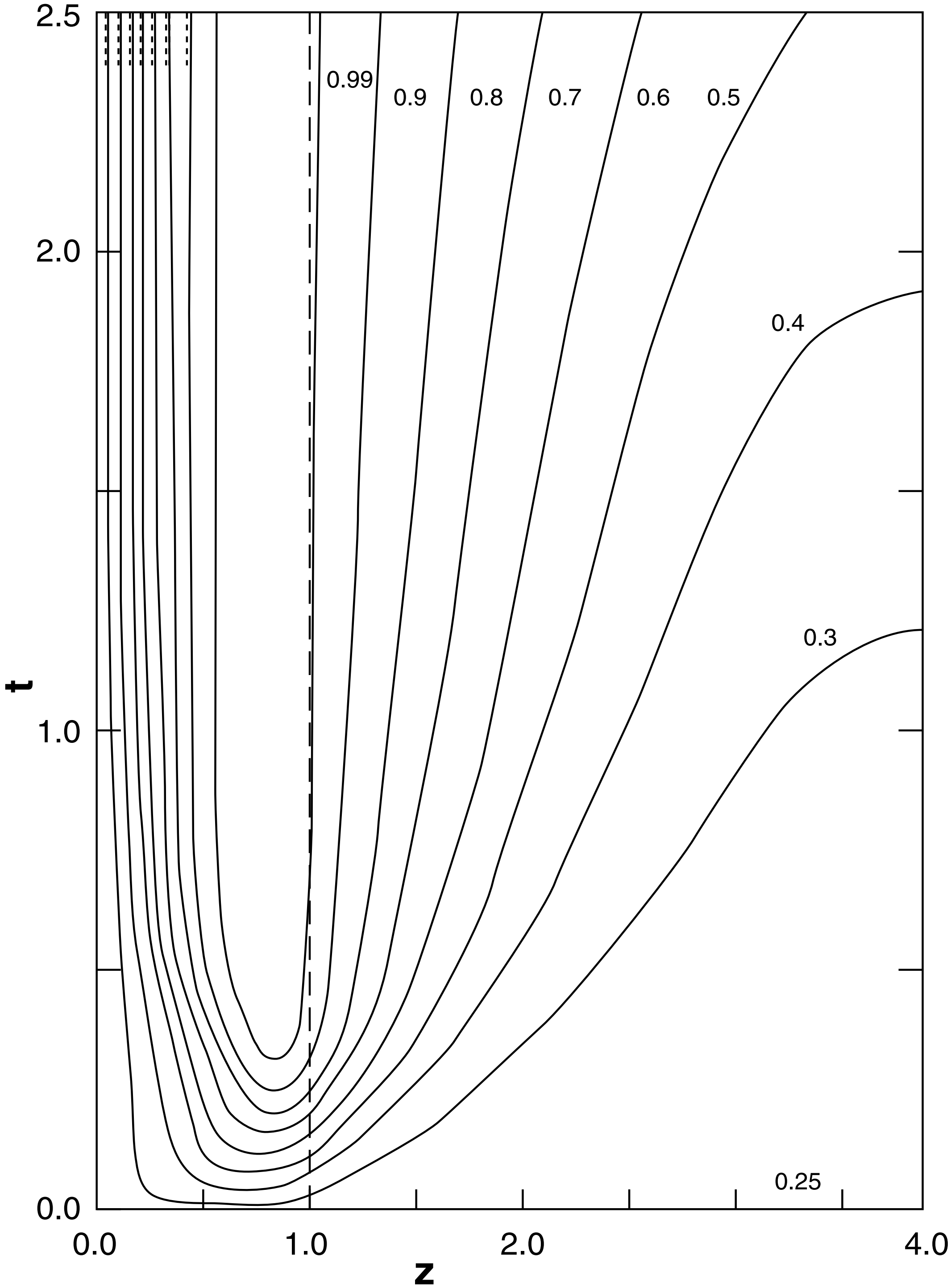


FIGURE 4.1. Evolution of dimensionless salinity $u$ for root distribution $k(z)$ given by (2.15), and for parameter values $p = 0$, $k_0 = 10.0$, $d = 4.0$ and $u_0 = 0.25$. Contours of $u$ in the $z - t$ plane are shown: numbers on curves are values of $u$.

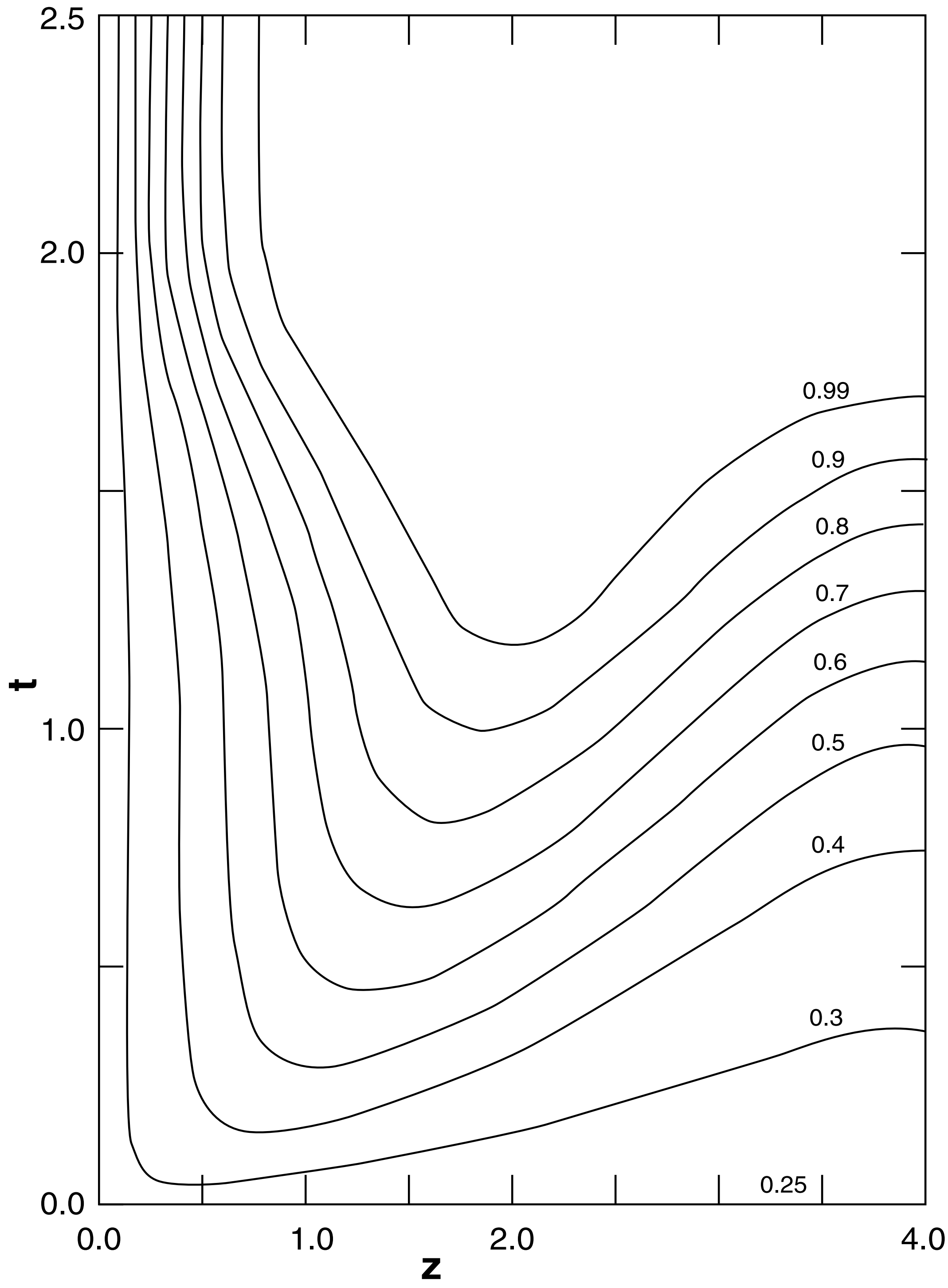


FIGURE 4.2. As for Fig. 4.1, with root distribution $k(z)$ decreasing with depth according to (2.14).

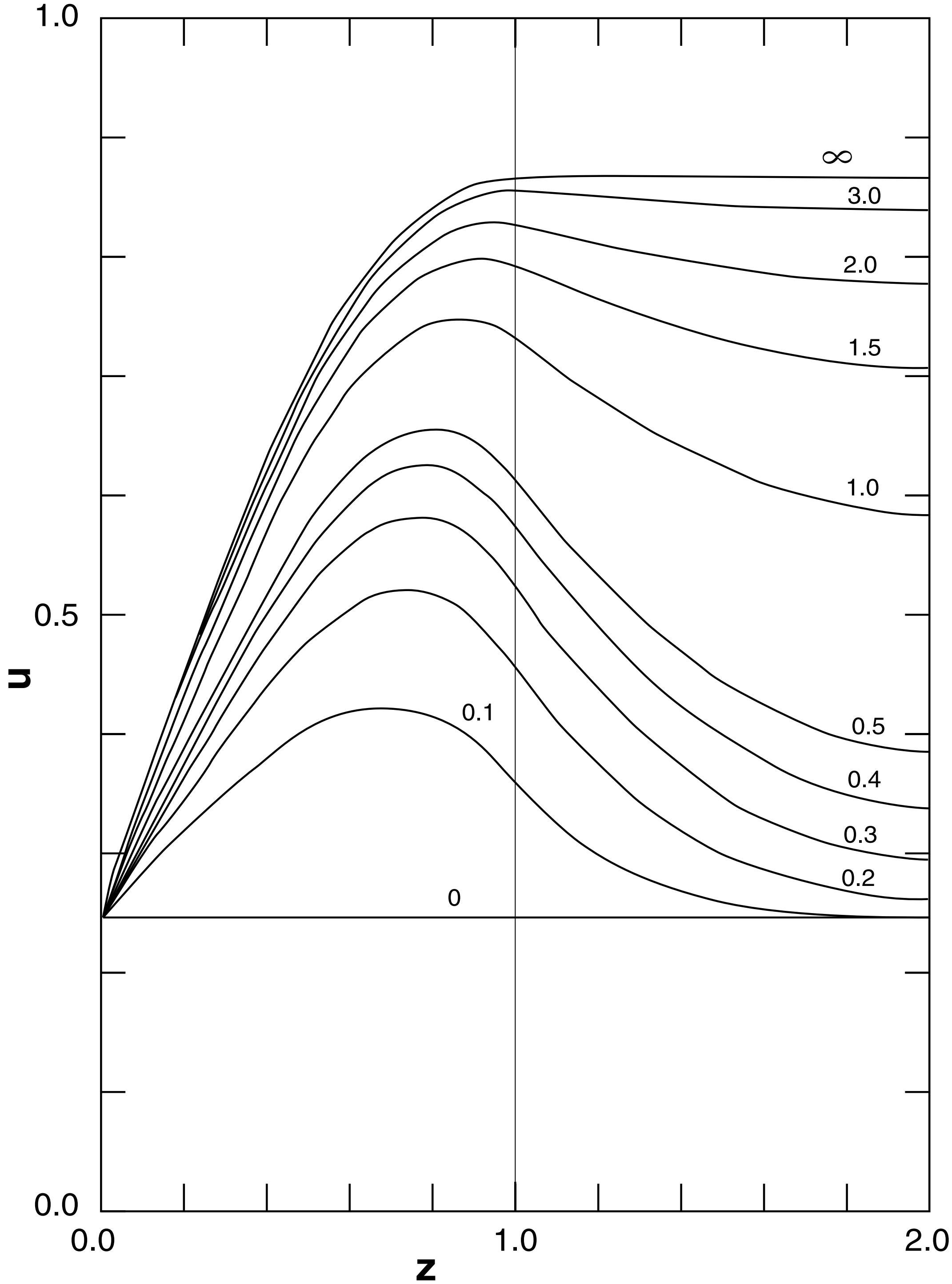


FIGURE 4.3. Evolution of dimensionless salinity $u$ for root distribution $k(z)$ uniform in $0 \leq z \leq 1$, and zero in $1 < z \leq d$, for parameter values $p = 1$, $k_0 = 10.0$, $d = 2.0$ and $u_0 = 0.25$. Profiles of $u(z)$ at dimensionless times $t$ as marked on curves.

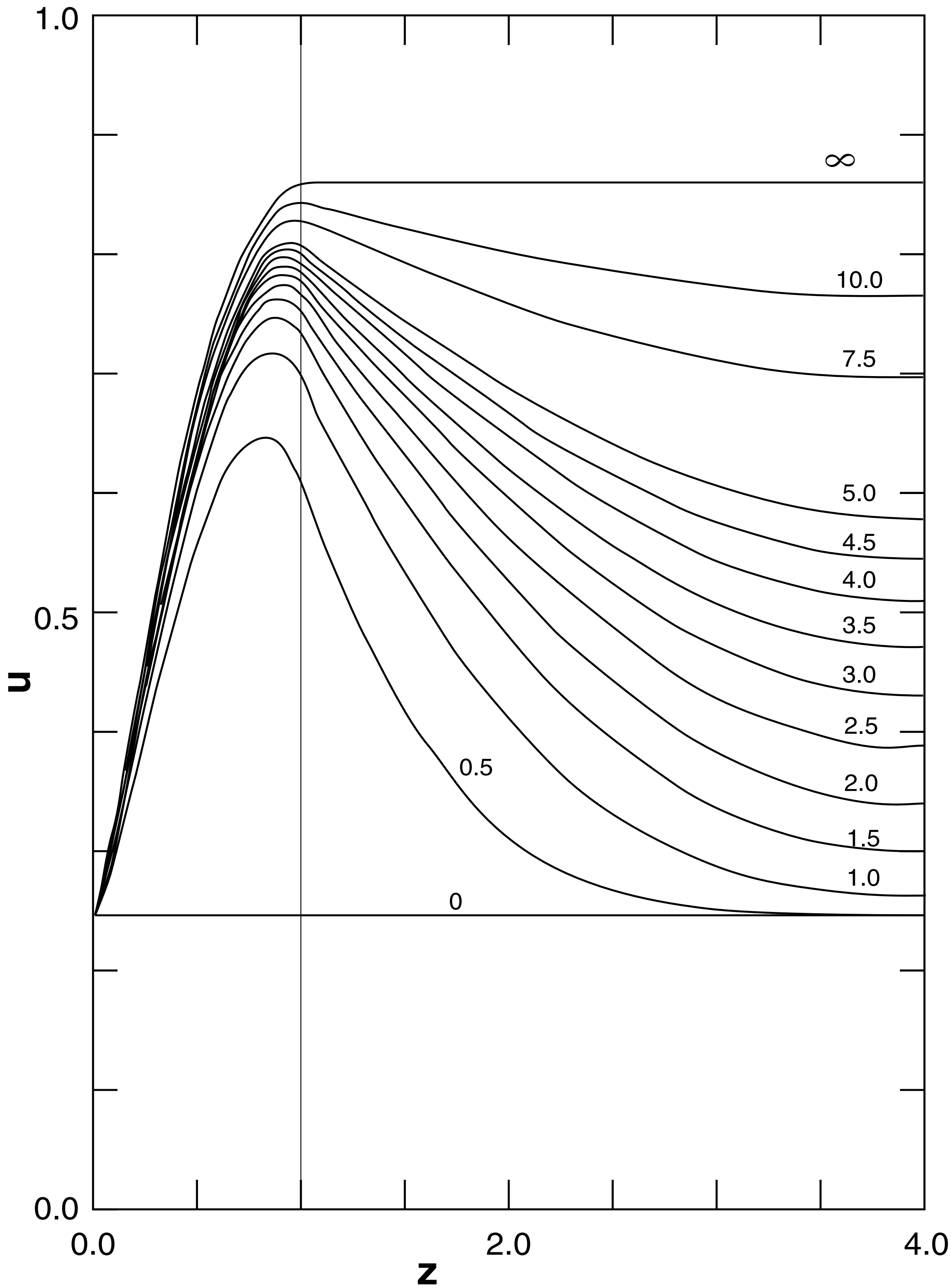


FIGURE 4.4. As for Fig. 4.3, with $d = 4.0$.

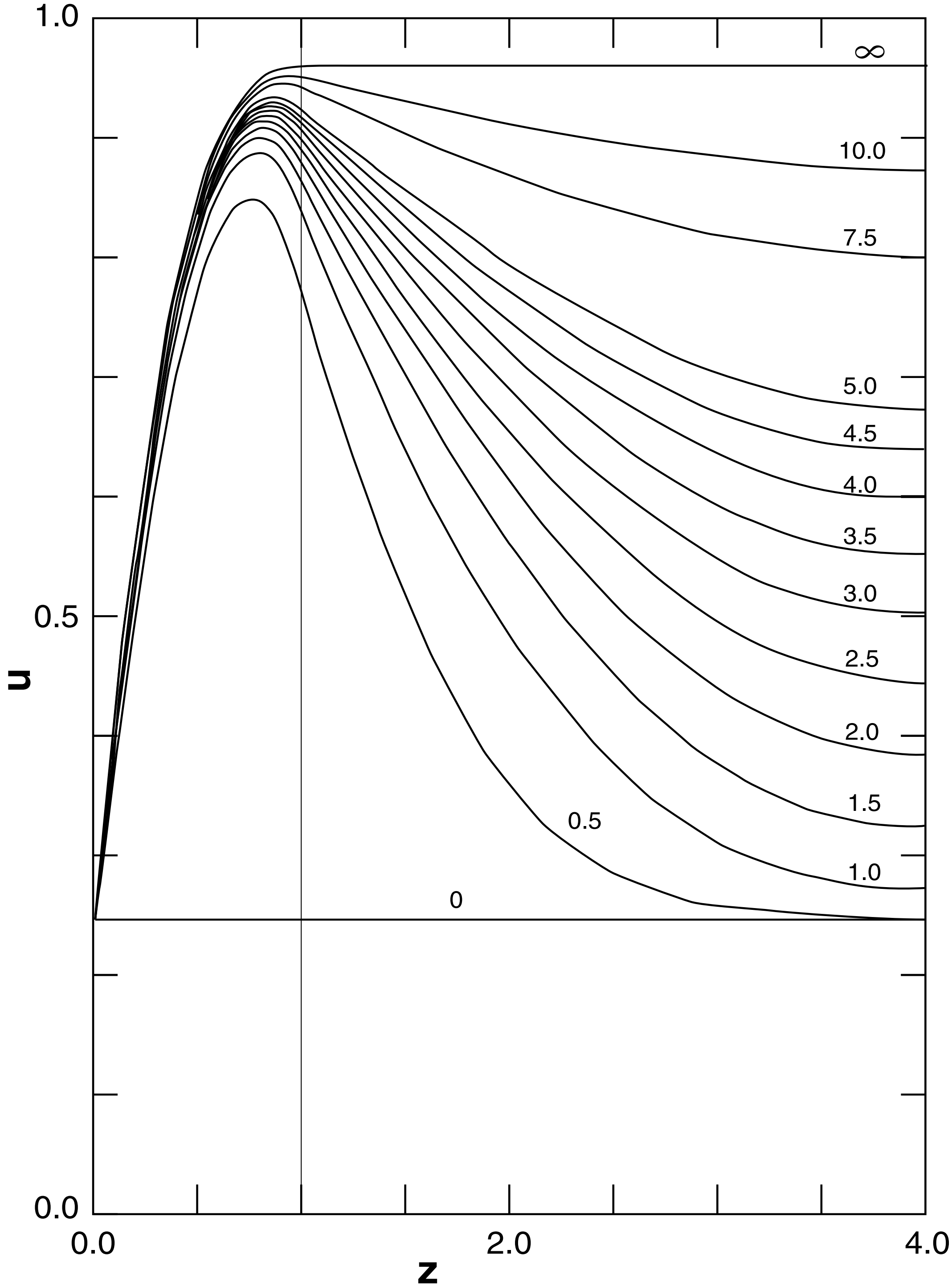


FIGURE 4.5. As for Fig. 4.3, with $k_0 = 20.0$ and $d = 4.0$.