Centrum voor Wiskunde en Informatica
REPORT *RAPPORT*

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Modelling, Analysis and Simulation (MAS)

MAS-R9730 October 31, 1997

Report MAS-R9730
ISSN 1386-3703

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ABSTRACT
In this paper, explicit formulas for the coefficients of the normal forms for all codim 2 equilibrium bifurcations of equilibria in autonomous ODEs are derived. They include second-order coefficients for the Bogdanov-Takens bifurcation, third-order coefficients for the cusp and fold-Hopf bifurcations, and coefficients of the fifth-order terms for the generalized Hopf (Bautin) and double Hopf bifurcations. The formulas are independent on the dimension of the phase space and involve only critical eigenvectors of the Jacobian matrix of the right-hand sides and its transpose, as well as multilinear functions from the Taylor expansion of the right-hand sides at the critical equilibrium.

1991 Mathematics Subject Classification: 58F36, 58F14
Keywords and Phrases: Normal forms, codim 2 bifurcations
Note: This work was supported by the NWO and has been carried out under the CWI/RIACA-project "Dynamical Systems Laboratory"

1. INTRODUCTION

Studying smooth differential equations

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m, \tag{1.1}$$

usually starts with one-parameter analysis, i.e. constructing the bifurcation diagram of (1.1) with respect to a selected control parameter, say α_1. In general, there might exist critical parameter values at which the system exhibits codim 1 bifurcations, for example, fold or Hopf bifurcations of equilibrium points. It has been clear since early 70s that such analysis is insufficient in many applications because thus obtained one-parameter diagrams revealed strong dependence on other (fixed) parameters $(\alpha_2, \alpha_3, \ldots, \alpha_m)$. If we allow only one more parameter, say α_2, to vary, such behavior of one-parameter diagrams can be explained by taking into account so called codim 2 bifurcation points. These are the points in the (α_1, α_2)-plane, where several curves corresponding to codim 1 bifurcations meet tangentially or intersect. A codim 2 points is of particular interest if it is an origin of some equilibrium bifurcation curves and some curves corresponding to bifurcations of periodic orbits (cycles). Such points can be detected by purely local analysis of equilibria and then be used to establish the existence of limit cycle bifurcations and other global phenomena that could hardly be proved otherwise.

The theory of codim 2 bifurcations of equilibria in generic systems (1.1) is well-developed (see, for example, Arnold [1983], Guckenheimer & Holmes [1983], Kuznetsov [1995]). A codim 2 bifurcation can be detected along the locus of codim 1 bifurcation points as the change of the dimension of the critical eigenspace (number n_c of the eigenvalues of $A = f_x$ with Re $\lambda = 0$) or as vanishing of a coefficient of the corresponding normal form of the reduced to the center manifold n_c-dimensional system. Then, generically, i.e. under conditions that exclude some relationships between the coefficients of the reduced normal forms at the codim 2 point and include the transversality to the codim 2 bifurcation manifold in the parameter space, the system (1.1) restricted to the center manifold is smoothly orbitally equivalent to a normal form plus higher order terms. The theory specifies which terms should be kept in the normal form; we call them *resonant terms.* In some of the codim 2 cases, truncating higher-order terms produces a topologically equivalent system, while in the others, it changes topology of the bifurcation diagram. In all generic cases, however, the truncated normal forms allow to predict important features of the system dynamics.

To apply the theory of codim 2 bifurcations to particular models, one needs to verify the non-degeneracy conditions at the bifurcation point, in other words, to compute the coefficients of the normal form up to certain order. There are powerful normalization algorithms (by Sanders [1994] and others) for symbolic computation of the normal form coefficients applicable when the system (1.1) is reduced to the center manifold. However, in most cases the critical parameter values and the equilibrium coordinates are known only approximately, from a numerical analysis, and not suitable for using any symbolic software. Therefore, numerical normalization techniques have to be developed and implemented into the standard software for the analysis of dynamical systems. For such implementation, one should have as explicit as possible formulas for the critical normal form coefficients. These formulas should satisfy the following requirements:

(a) be independent of the dimension n of the phase space of (1.1);
(b) involve only critical eigenvalues and eigenvectors of the Jacobian matrix and its transpose;
(c) be suitable for both numerical and symbolic evaluation.

Notice that there are such explicit formulas for all codim 1 bifurcations of equilibria in generic systems (1.1), i.e. fold and Hopf bifurcations (see Section 2).

The aim of this paper is to derive explicit formulas for the coefficients of the normal forms for all codim 2 equilibrium bifurcations in (1.1), namely: cusp, Bogdanov-Takens, generalized Hopf (Bautin), fold-Hopf, and double Hopf bifurcations. Some of these coefficients have been obtained earlier by Kurakin & Judovich [1986], who gave explicit criteria for stability of equilibria in n-dimensional ODEs in some critical cases. Other cubic and fifth-order coefficients of the normal forms for the generalized Hopf, fold-Hopf, and double Hopf bifurcations are irrelevant for the study of stability of the critical equilibrium but determine the topology of the bifurcation diagrams for nearby parameter values. In principle, these coefficients could be found by first computing the Taylor expansion of the center manifold and then evaluating the corresponding normal form coefficients. Beyn & Kless [1996] developed a numerical method to compute reduced equations on the center manifold. However, the resulting algorithms are complicated (see, for example, [Hassard, Kazarinoff & Wan 1981] in the generalized Hopf case). Moreover, for the double Hopf case, the fifth-order normal form coefficients seem to be never derived and published even in the four-dimensional case.

In this paper we derive all the coefficients using a reduction/normalization technique by Coullet & Spiegel [1983] and MAPLE V.R4 symbolic manipulation software. In this approach, the center-manifold reduction and normalization are performed simultaneously. Surprisingly enough, most formulas are rather compact and allow for straightforward implementation.

2. CODIM 1 NORMAL FORMS

Suppose (1.1) has an equilibrium $x=0$ at $\alpha=0$ and represent $F(x)=f(x,0)$ as

$$F(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \frac{1}{24}D(x,x,x,x) + \frac{1}{120}E(x,x,x,x,x) + \dots, \tag{2.1}$$

where $A = f_x(0,0)$ and

$$\begin{aligned}
B_i(x,y) &= \sum_{j,k=1}^{n} \left.\frac{\partial^2 F_i(\xi)}{\partial\xi_j\partial\xi_k}\right|_{\xi=0} x_j y_k, \\
C_i(x,y,z) &= \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 F_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0} x_j y_k z_l, \\
D_i(x,y,z,v) &= \sum_{j,k,l,m=1}^{n} \left.\frac{\partial^4 F_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l\partial\xi_m}\right|_{\xi=0} x_j y_k z_l v_m,
\end{aligned}$$

$$E_i(x,y,z,v,w) = \sum_{j,k,l,m,s=1}^{n} \left. \frac{\partial^5 F_i(\xi)}{\partial\xi_j \partial\xi_k \partial\xi_l \partial\xi_m \partial\xi_s} \right|_{\xi=0} x_j y_k z_l v_m w_s,$$

for $i = 1, 2, \ldots, n$.

If the equilibrium $x = 0$ of (1.1) exhibits a *fold bifurcation* at $\alpha = 0$, the Jacobian matrix A has a simple zero eigenvalue $\lambda_1 = 0$ and no other critical eigenvalues. Introduce two null-vectors:

$$Aq = 0, \quad A^T p = 0,$$

and normalized them according to

$$p^T q \equiv \langle p, q \rangle = 1.$$

The restriction of (1.1) at $\alpha = 0$ to the one-dimensional center manifold has the form

$$\dot{w} = aw^2 + O(|w|^3), \quad w \in \mathbb{R}^1, \tag{2.2}$$

where the coefficient a can be computed by the formula

$$a = \frac{1}{2}\langle p, B(q,q) \rangle, \tag{2.3}$$

If $a \neq 0$ and (1.1) depends generically on the parameter α_1, its restriction to the center manifold is locally topologically equivalent to the normal form

$$\dot{w} = \beta_1 + aw^2,$$

where β_1 is the unfolding parameter. This normal form predicts the collision of two equilibria when the parameter β_1 passes zero.

If the equilibrium $x = 0$ of (1.1) exhibits a *Hopf bifurcation* at $\alpha = 0$, the Jacobian matrix $A = f_x(0,0)$ has a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, and no other critical eigenvalues. Introduce two complex vectors:

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p,$$

and normalize them according to

$$\overline{p}^T q \equiv \langle p, q \rangle = 1.$$

The restriction of (1.1) at $\alpha = 0$ to the two-dimensional center manifold is locally smoothly orbitally equivalent to the complex normal form

$$\dot{w} = i\omega_0 w + l_1 w|w|^2 + O(|w|^4), \quad w \in \mathbb{C}^1, \tag{2.4}$$

where the normal form coefficient l_1 can be computed by the formula

$$l_1 = \frac{1}{2}\,\mathrm{Re}\,\langle p, C(q,q,\overline{q}) + B(\overline{q}, (2i\omega_0 I_n - A)^{-1}B(q,q)) - 2B(q, A^{-1}B(q,\overline{q})\rangle, \tag{2.5}$$

where I_n is the unit $n \times n$ matrix. If the *first Lyapunov coefficient* $l_1 \neq 0$ and (1.1) depends generically on the parameter α_1, its restriction to the center manifold is locally topologically equivalent to the normal form

$$\dot{w} = (\beta_1 + i\omega_0)w + l_1 w|w|^2.$$

This normal form describes a bifurcation of the unique limit cycle from the equilibrium $w = 0$, when the parameter β_1 passes the bifurcation value $\beta_1 = 0$.

The formulas (2.3) and (2.5) are derived using the central manifold reduction and subsequent normalization on the central manifold in [Kuznetsov 1995]. Originally, the expression (2.5) has been obtained using the Lyapunov-Schmidt reduction and asymptotic expansions by Kopell and Howard in [Marsden & McCracken 1976] and by van Gils [1982]. These formulas will be rederived below as a part of codim 2 analysis. The formulas (2.3) and (2.5) are implemented in CONTENT [Kuznetsov & Levitin 1997].

3. LIST OF CODIM 2 NORMAL FORMS

In generic systems (1.1), there possible only five codim 2 bifurcations of equilibria (Arnold [1983], Guckenheimer & Holmes [1983], Kuznetsov [1995]):

- *cusp* ($\lambda_1 = 0,\ a = 0$)

  The restricted to the center manifold at the critical parameter values equation reads

$$\dot{w} = cw^3 + O(w^4), \quad w \in \mathbb{R}^1. \tag{3.1}$$

  If $c \neq 0$ and the system (1.1) depends generically on two parameters (α_1, α_2), its restriction to the center manifold is locally topologically equivalent near the bifurcation to the normal form

$$\dot{w} = \beta_1 + \beta_2 w + cw^3,$$

  where β_1 and β_2 are the unfolding parameters. This normal form predicts a hysteresis phenomenon near the bifurcation.

- *Bogdanov-Takens* ($\lambda_{1,2} = 0$)

  The restriction of (1.1) to the center manifold at the critical parameter values is locally smoothly equivalent to the normal form

$$\left\{ \begin{array}{lcl} \dot{w}_0 & = & w_1 \\ \dot{w}_1 & = & aw_0^2 + bw_0 w_1 + O(\|w\|^3), \end{array} \right. \tag{3.2}$$

  where $w = (w_0, w_1)^T \in \mathbb{R}^2$. If $ab \neq 0$ and the parameters (α_1, α_2) enter (1.1) generically, the restricted system is locally topologically equivalent to the normal form

$$\left\{ \begin{array}{lcl} \dot{w}_0 & = & w_1 \\ \dot{w}_1 & = & \beta_1 + \beta_2 w_0 + aw_0^2 + bw_0 w_1. \end{array} \right.$$

  Analysis of this normal form reveals a curve in the parameter plane emanating from the codim 2 point and corresponding to a saddle homoclinic bifurcation: The unique limit cycle born in the Hopf bifurcation approaches the homoclinic orbit and disappears while its period $T \to \infty$.

- *generalized Hopf* ($\lambda_{1,2} = \pm i\omega_0,\ l_1 = 0$)

  The restriction of (1.1) to the center manifold at the critical parameter values is locally smoothly orbitally equivalent to the normal form

$$\dot{w} = i\omega_0 w + l_2 w|w|^4 + O(|w|^6), \quad w \in \mathbb{C}^1, \tag{3.3}$$

  where the *second Lyapunov coefficient* l_2 is real. If $l_2 \neq 0$ then, generically, the restricted system (1.1) is locally topologically equivalent to the normal form

$$\dot{w} = (\beta_1 + i\omega_0)w + \beta_2 w|w|^2 + l_2 w|w|^4.$$

  This normal form predicts the existence of a curve originating at the codim 2 point in the parameter plane, where two limit cycles collide and disappear through a nonhyperbolic cycle with a nontrivial multiplier $\mu_1 = 1$.

- *fold-Hopf* ($\lambda_1 = 0,\ \lambda_{2,3} = \pm i\omega_0$)

  The normalized restriction of (1.1) to the center manifold at the critical parameter values has the form

$$\begin{cases} \dot{w}_0 &= \frac{1}{2}G_{200}w_0^2 + G_{011}|w_1|^2 + \frac{1}{6}G_{300}w_0^3 \\ & \quad + G_{111}w_0|w_1|^2 + O(\|(w_0,w_1,\overline{w}_1)\|^4) \\ \dot{w}_1 &= i\omega_0 w_1 + G_{110}w_0w_1 + \frac{1}{2}G_{210}w_0^2w_1 + \frac{1}{2}G_{021}w_1|w_1|^2 \\ & \quad + O(\|(w_0,w_1,\overline{w}_1)\|^4), \end{cases} \tag{3.4}$$

Here $w_0 \in \mathbb{R}^1, w_1 \in \mathbb{C}^1$, the coefficients G_{klm} in the first equation are real, while those in the second equation are complex. If $G_{200}G_{011} \neq 0$, the restricted to the center manifold system (1.1) is locally smoothly orbitally equivalent to the system

$$\begin{cases} \dot{u} &= \beta_1 + bu^2 + c|z|^2 + O(\|(u,z,\overline{z})\|^4) \\ \dot{z} &= (\beta_2 + i\omega_0)z + duz + eu^2z + O(\|(u,z,\overline{z})\|^4), \end{cases}$$

where b, c, e are real, while d is complex:

$$b = \frac{1}{2}G_{200}, \quad c = G_{011}, \quad d = G_{110} - i\omega_0 \frac{G_{300}}{3G_{200}} ,$$

and

$$e = \frac{1}{2} \operatorname{Re}\left[G_{210} + G_{110}\left(\frac{\operatorname{Re} G_{021}}{G_{011}} - \frac{G_{300}}{G_{200}} + \frac{G_{111}}{G_{011}}\right) - \frac{G_{021}G_{200}}{2G_{011}}\right].$$

In general, the O-terms can not be truncated, since they affect the topology of the bifurcation diagram of the system near the bifurcation. Depending upon the coefficients b, c, d, e, the system can have two-dimensional invariant tori and chaotic motions and exhibits Neimark-Sacker and Shil'nikov homoclinic bifurcations.

- *double Hopf* ($\lambda_{1,2} = \pm i\omega_1, \ \lambda_{3,4} = \pm i\omega_2$)

  Assume that

$$k\omega_1 \neq l\omega_2, \ k,l > 0, k+l \leq 5. \tag{3.5}$$

Then, the normalized restricted to the center manifold system (1.1) has the form

$$\begin{cases} \dot{w}_1 &= i\omega_1 w_1 + \frac{1}{2}G_{2100}w_1|w_1|^2 + G_{1011}w_1|w_2|^2 \\ & \quad + \frac{1}{12}G_{3200}w_1|w_1|^4 + \frac{1}{2}G_{2111}w_1|w_1|^2|w_2|^2 + \frac{1}{4}G_{1022}w_1|w_2|^4 \\ & \quad + O(\|(w_1,\overline{w}_1,w_2,\overline{w}_2)\|^6) \\ \dot{w}_2 &= i\omega_2 w_2 + G_{1110}w_2|w_1|^2 + \frac{1}{2}G_{0021}w_2|w_2|^2 \\ & \quad + \frac{1}{4}G_{2210}w_2|w_1|^4 + \frac{1}{2}G_{1121}w_2|w_1|^2|w_2|^2 + \frac{1}{12}G_{0032}w_2|w_2|^4 \\ & \quad + O(\|(w_1,\overline{w}_1,w_2,\overline{w}_2)\|^6), \end{cases} \tag{3.6}$$

where $G_{jklm} \in \mathbb{C}^1$. Moreover, if

$$(\operatorname{Re} G_{2100})(\operatorname{Re} G_{1011})(\operatorname{Re} G_{1110})(\operatorname{Re} G_{0021}) \neq 0,$$

the system (1.1) is locally smoothly orbitally equivalent near the bifurcation to the system

$$\begin{cases} \dot{v}_1 &= (\beta_1 + i\omega_1)v_1 + \frac{1}{2}P_{11}v_1|v_1|^2 + P_{12}v_1|v_2|^2 \\ & \quad + iR_1v_1|v_1|^4 + \frac{1}{4}S_1v_1|v_2|^4 + O(\|(v_1,\overline{v}_1,v_2,\overline{v}_2)\|^6) \\ \dot{v}_2 &= (\beta_2 + i\omega_2)v_2 + P_{21}v_2|v_1|^2 + \frac{1}{2}P_{22}v_2|v_2|^2 \\ & \quad + \frac{1}{4}S_2v_2|v_1|^4 + iR_2v_2|v_2|^4 + O(\|(v_1,\overline{v}_1,v_2,\overline{v}_2)\|^6), \end{cases}$$

where $(v_1, v_1)^T \in \mathbb{C}^2$, P_{jk} and S_k are complex, while R_k are real:

$$\text{Re } P_{11} = \text{Re } G_{2100}, \quad \text{Re } P_{12} = \text{Re } G_{1011}, \quad \text{Re } P_{21} = \text{Re } G_{1110}, \quad \text{Re } P_{22} = \text{Re } G_{0021},$$

and

$$\begin{aligned}
\text{Re } S_1 &= \text{Re } G_{1022} + \frac{1}{3}\text{Re } G_{1011}\left[6\frac{\text{Re } G_{1121}}{\text{Re } G_{1110}} - 4\frac{\text{Re } G_{0032}}{\text{Re } G_{0021}} - \frac{(\text{Re } G_{3200})(\text{Re } G_{0021})}{(\text{Re } G_{2100})(\text{Re } G_{1110})}\right], \\
\text{Re } S_2 &= \text{Re } G_{2210} + \frac{1}{3}\text{Re } G_{1110}\left[6\frac{\text{Re } G_{2111}}{\text{Re } G_{1011}} - 4\frac{\text{Re } G_{3200}}{\text{Re } G_{2100}} - \frac{(\text{Re } G_{2100})(\text{Re } G_{0032})}{(\text{Re } G_{1011})(\text{Re } G_{0021})}\right].
\end{aligned}$$

As in the fold-Hopf case, the O-terms can not be truncated, since they affect the topology of the bifurcation diagram of the system. Depending on the values of the normal form coefficients, the system can exhibit invariant tori and chaotic motions, as well as Neimark-Sacker and Shil'nikov homoclinic bifurcations.

Proofs of the formulated above results could be found in Kuznetsov [1995] with all relevant bifurcation diagrams and bibliographical references.

4. THE METHOD

The following normalization technique is essentially due to Coullet & Spiegel [1983] (see also [Elphick, Tirapegui, Brachet, Coullet & Iooss 1987]). Suppose, the system (1.1) has at $\alpha = 0$ the equilibrium $x = 0$ such that the Jacobian matrix $A = f_x(0,0)$ has n_c eigenvalues counting multiplicities with zero real part and denote by T^c the corresponding generalized critical eigenspace of A. Write the system at $\alpha = 0$ as

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n, \tag{4.1}$$

where F is given by (2.1), and restrict it to its n_c-dimensional invariant center manifold parametrized by $w \in \mathbb{R}^{n_c}$:

$$x = H(w), \quad H : \mathbb{R}^{n_c} \to \mathbb{R}^n, \tag{4.2}$$

The restricted equation can be written as

$$\dot{w} = G(w), \quad G : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}. \tag{4.3}$$

Substitution of (4.2) and (4.3) into (4.1) gives the following *homological equation*:

$$H_w(w)G(w) = F(H(w)). \tag{4.4}$$

Now expand the functions G, H in (4.4) into multivariant Taylor series,

$$G(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu w^\nu, \quad H(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

and assume that the restricted equation (4.3) is put into the normal form up to a certain order. The coefficients g_ν of the normal form (4.3) and the coefficients h_ν of the Taylor expansion for $H(w)$ are unknown but can be found from (4.4) by a recurrent procedure, from lower to higher order terms.[1] Collecting the coefficients of the w^ν-terms in (4.4) gives a linear system for the coefficient h_ν

$$Lh_\nu = R_\nu. \tag{4.5}$$

Here the matrix L is determined by the Jacobian matrix A and its critical eigenvalues, while the right-hand side R_ν depends on the coefficients of G and H of order less or equal than $|\nu|$, as well on the terms of order less or equal to $|\nu|$ of the Taylor expansion (2.1) for F. When R_ν involves only

[1] Obviously, one has $\sum_{|\nu|=1} h_\nu w^\nu \in T^c$.

known quantities, the system (4.5) has a solution because either L is nonsingular, or R_ν satisfies the Fredholm's solvability condition

$$\langle p, R_\nu \rangle = 0,$$

where p is a null-vector of the adjoint matrix $\overline{L}^T$. When R_ν depends on the unknown coefficient g_ν of the normal form, L is singular and the above solvability condition gives the expression for g_ν.

For all codim 2 bifurcations except Bogdanov-Takens, the invariant subspace of $L(\overline{L}^T)$ corresponding to zero eigenvalue is one-dimensional in $\mathbb{C}^n$, i.e. there are unique (up to scaling) null-vectors q and p,

$$Lq = 0, \quad \overline{L}^T p = 0, \quad \langle p, q \rangle = 1,$$

and no generalized null-vectors. Then, the unique solution h_ν to (4.5) satisfying $\langle p, h_\nu \rangle = 0$ can be obtained by solving the following nonsingular $(n+1)$-dimensional *bordered system*:

$$\begin{pmatrix} L & q \\ \overline{p}^T & 0 \end{pmatrix} \begin{pmatrix} h_\nu \\ s \end{pmatrix} = \begin{pmatrix} R_\nu \\ 0 \end{pmatrix} \tag{4.6}$$

([Keller 1977], see [Govaerts & Pryce 1993] for generalizations). We write $h_\nu = L^{INV} R_\nu$.

The Taylor expansion of $H(w)$ simultaneously defines the expansions of the center manifold, the normalizing transformation on it, and the normal form itself. Since we know which terms are present in the normal form *a priori*, the described procedure is a powerful tool to compute their coefficients at the bifurcation parameter values. In the following sections, this method will be applied to all codim 2 cases.

5. CUSP BIFURCATION

At the cusp bifurcation, the system (1.1) has an equilibrium with a simple zero eigenvalue $\lambda_1 = 0$ and no other critical eigenvalues. Let $q, p \in \mathbb{R}^n$ satisfy

$$Aq = 0, \quad A^T p = 0, \quad \langle p, q \rangle = 1.$$

Any point $y \in T^c$ can be represented as

$$y = wq, \quad w \in \mathbb{R}^1,$$

where $w = \langle p, y \rangle$. The homological equation (4.4) has the form

$$H_w \dot{w} = F(H(w)),$$

where

$$F(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4)$$

(see (2.1)),

$$H(w) = wq + \frac{1}{2}h_2 w^2 + \frac{1}{6}h_3 w^3 + O(w^4),$$

$h_i \in \mathbb{R}^n$, and

$$\dot{w} = bw^2 + cw^3 + O(w^4)$$

with unknown coefficients b and c. Substituting these expressions into the homological equation gives

$$bw^2 q + (cq + bh_2)w^3 = \frac{1}{2}w^2[Ah_2 + B(q,q)] + \frac{1}{6}w^3[Ah_3 + 3B(q,h_2) + C(q,q,q)] + O(w^4). \tag{5.1}$$

The w^2-terms in (5.1) give the equation for h_2

$$Ah_2 = -B(q,q) + 2bq, \tag{5.2}$$

where the matrix A is obviously singular. The solvability of this system implies

$$\langle p, -B(q,q) + 2bq\rangle = -\langle p, B(q,q)\rangle + 2b\langle p, q\rangle = 0$$

and allows one to find b, namely:

$$b = \frac{1}{2}\langle p, B(q,q)\rangle$$

as specified by the formula (2.3) for the fold bifurcation. With this value of b, the linear system (5.2) becomes

$$Ah_2 = -B(q,q) + \langle p, B(q,q)\rangle q$$

and its unique solution $h_2 = -A^{INV}[B(q,q) - \langle p, B(q,q)\rangle q]$ satisfying $\langle p, h_2\rangle = 0$ can be computed by solving the nonsingular $(n+1)$-dimensional bordered system

$$\begin{pmatrix} A & q \\ p^T & 0 \end{pmatrix} \begin{pmatrix} h_2 \\ s \end{pmatrix} = \begin{pmatrix} -B(q,q) + \langle p, B(q,q)\rangle q \\ 0 \end{pmatrix}.$$

Collecting the w^3-terms in (5.1) yields

$$cq + bh_2 = \frac{1}{6}Ah_3 + \frac{1}{2}B(q,h_2) + \frac{1}{6}C(q,q,q)$$

which is equivalent to another singular system

$$Ah_3 = cq + bh_2 - \frac{1}{6}[C(q,q,q) + 3B(q,h_2)].$$

Its solvability implies

$$c\langle p, q\rangle + b\langle p, h_2\rangle - \frac{1}{6}\langle p, C(q,q,q) + 3B(q,h_2)\rangle = 0.$$

Since $\langle p, h_2\rangle = 0$, we obtain the following expression for the coefficient c:

$$c = \frac{1}{6}\langle p, C(q,q,q) + 3B(q,h_2)\rangle.$$

Now recall that $b = 0$ at the cusp bifurcation. Under this condition, the coefficient c in the normal form (3.1) can be expressed shortly as

$$c = \frac{1}{6}\langle p, C(q,q,q) - 3B(q, A^{INV}B(q,q))\rangle \tag{5.3}$$

6. BOGDANOV-TAKENS BIFURCATION

At the Bogdanov-Takens bifurcation, the system (1.1) has double zero eigenvalue $\lambda_{1,2} = 0$ and there exist two real linearly independent (generalized) eigenvectors, $q_{0,1} \in \mathbb{R}^n$, such that

$$Aq_0 = 0, \quad Aq_1 = q_0.$$

Moreover, there exist similar vectors $p_{1,0} \in \mathbb{R}^n$ of the transposed matrix A^T:

$$A^T p_1 = 0, \quad Ap_0 = p_1.$$

One can select these vectors to satisfy

$$\langle q_0, p_0\rangle = \langle q_1, p_1\rangle = 1, \quad \langle q_1, p_0\rangle = \langle q_0, p_1\rangle = 0.$$

Any vector $y \in T^c$ can be uniquely represented as

$$y = w_0 q_0 + w_1 q_1,$$

where $w_0 = \langle p_0, y\rangle, \; w_1 = \langle p_1, y\rangle$. The homological equation (4.4) has the form

$$H_{w_0}\dot{w}_0 + H_{w_1}\dot{w}_1 = F(H(w_0, w_1)), \tag{6.1}$$

where

$$F(H) = AH + \frac{1}{2}B(H, H) + O(\|H\|^3),$$

$$H(w_0, w_1) = w_0 q_0 + w_1 q_1 + \frac{1}{2}h_{20}w_0^2 + h_{11}w_0 w_1 + \frac{1}{2}h_{02}w_1^2 + O(\|(w_0, w_1)\|^3)$$

with $h_{jk} \in \mathbb{R}^n$, and $\dot{w}_0, \dot{w}_1$ are defined by the normal form (3.2) with unknown coefficients a and b. Substituting these expressions into (6.1) and collecting the w_0^2-terms, give the singular linear system for h_{20}:

$$Ah_{20} = 2aq_1 - B(q_0, q_0). \tag{6.2}$$

The solvability condition for this system is

$$\langle p_1, 2aq_1 - B(q_0, q_0)\rangle = 2a\langle p_1, q_1\rangle - \langle p_1, B(q_0, q_0)\rangle = 0,$$

which gives

$$a = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle. \tag{6.3}$$

Taking the scalar product of both sides of (6.2) with p_0 yields $\langle p_0, Ah_{20}\rangle = 2a\langle p_0, q_1\rangle - \langle p_0, B(q_0, q_0)\rangle$, which implies

$$\langle p_1, h_{20}\rangle = -\langle p_0, B(q_0, q_0)\rangle, \tag{6.4}$$

The $w_0 w_1$-terms in (6.1) give the linear system

$$Ah_{11} = h_{20} + bq_1 - B(q_0, q_1).$$

Its solvability means

$$\langle p_1, h_{20} + bq_1 - B(q_0, q_1)\rangle = \langle p_1, h_{20}\rangle + b\langle p_1, q_1\rangle - \langle p_1, B(q_0, q_1)\rangle = 0.$$

Taking into account (6.4), we get

$$b = \langle p_0, B(q_0, q_0)\rangle + \langle p_1, B(q_0, q_1)\rangle \tag{6.5}$$

The coefficients a, b of the normal form (3.2) are computed.

7. BAUTIN (GENERALIZED HOPF) BIFURCATION

At the Bautin bifurcation, the system (1.1) has an equilibrium with a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, and no other critical eigenvalues. As in the simple Hopf case, introduce two complex eigenvectors:

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p,$$

and normalize them according to

$$\langle p, q\rangle = 1.$$

Any vector $y \in T^c \subset \mathbb{R}^n$ can be represented as

$$y = wq + \overline{w}\,\overline{q},$$

where $w = \langle p, y\rangle \in \mathbb{C}^1$. The homological equation (4.4) now takes the form

$$H_w\dot{w} + H_{\overline{w}}\dot{\overline{w}} = F(H(w, \overline{w})), \tag{7.1}$$

where F is given by (2.1),

$$H(w,\overline{w}) = wq + \overline{w}\,\overline{q} + \sum_{1\le j+k\le 5} \frac{1}{j!k!} h_{jk} w^j \overline{w}^k + O(|w|^6),$$

with $h_{jk} \in \mathbb{C}^n,\ h_{kj} = \overline{h}_{jk}$, and

$$\dot{w} = i\omega_0 w + \frac{1}{2} G_{21} w|w|^2 + \frac{1}{12} G_{32} w|w|^4 + O(|w|^6),$$

where $G_{jk} \in \mathbb{C}^1$.

Collecting the coefficients of the quadratic terms in (7.1) and solving appearing linear systems, we get

$$h_{20} = (2i\omega_0 I_n - A)^{-1} B(q,q), \tag{7.2}$$
$$h_{11} = -A^{-1} B(q,\overline{q}). \tag{7.3}$$

The coefficient in front of the w^3-term in (7.1) leads to the following expression for h_{30}:

$$h_{30} = (3i\omega_0 I_n - A)^{-1}[C(q,q,q) + 3B(q,h_{20})], \tag{7.4}$$

while the $w^2\overline{w}$-terms give the singular system for h_{21}:

$$(i\omega_0 I_n - A)h_{21} = C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q. \tag{7.5}$$

The solvability of this system is equivalent to

$$\langle p, C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q\rangle = 0,$$

so the cubic normal form coefficient can be expressed as

$$G_{21} = \langle p, C(q,q,\overline{q}) + B(\overline{q},(2i\omega_0 I_n - A)^{-1}B(q,q)) - 2B(q,A^{-1}B(q,\overline{q}))\rangle \tag{7.6}$$

and $l_1 = \frac{1}{2}\mathrm{Re}\ G_{21}$ coincides with (2.5). Then,

$$h_{21} = (i\omega_0 I_n - A)^{INV}[C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q]. \tag{7.7}$$

Here the complex vector h_{21} satisfying $\langle p, h_{21}\rangle = 0$ can be found by solving the nonsingular $(n+1)$-dimensional complex system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \overline{p}^T & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q \\ 0 \end{pmatrix}.$$

For the fourth-order coefficients, we get

$$h_{40} = (4i\omega_0 I_n - A)^{-1}[D(q,q,q,q) + 6C(q,q,h_{20}) + 4B(q,h_{30}) + 3B(h_{20},h_{20})], \tag{7.8}$$
$$\begin{aligned} h_{31} = {} & (2i\omega_0 I_n - A)^{-1}[D(q,q,q,\overline{q}) + 3C(q,q,h_{11}) + 3C(q,\overline{q},h_{20}) + 3B(h_{20},h_{11}) \\ & + B(\overline{q},h_{30}) + 3B(q,h_{21}) - 3G_{21}h_{20}], \end{aligned} \tag{7.9}$$
$$\begin{aligned} h_{22} = {} & -A^{-1}[D(q,q,\overline{q},\overline{q}) + 4C(q,\overline{q},h_{11}) + C(\overline{q},\overline{q},h_{20}) + C(q,q,\overline{h}_{20}) \\ & + 2B(h_{11},h_{11}) + 2B(q,\overline{h}_{21}) + 2B(\overline{q},h_{21}) + B(\overline{h}_{20},h_{20}) - 2h_{11}(G_{21} + \overline{G}_{21})]. \end{aligned} \tag{7.10}$$

Recall now that at the Bautin bifurcation $l_1 = 0$ or $G_{21} + \overline{G}_{21} = 0$. Taking this into account and using the equality $\langle p, h_{21}\rangle = 0$, one can check that the solvability condition of the linear system for h_{32} provides the following formula for $l_2 = \frac{1}{12}\mathrm{Re}\ G_{32}$ in (3.3):

$$\begin{aligned} l_2 = {} & \frac{1}{12}\mathrm{Re}\ \langle p, E(q,q,q,\overline{q},\overline{q}) \\ & + D(q,q,q,\overline{h}_{20}) + 3D(q,\overline{q},\overline{q},h_{20}) + 6D(q,q,\overline{q},h_{11}) \\ & + C(\overline{q},\overline{q},h_{30}) + 3C(q,q,\overline{h}_{21}) + 6C(q,\overline{q},h_{21}) + 3C(q,\overline{h}_{20},h_{20}) \\ & + 6C(q,h_{11},h_{11}) + 6C(\overline{q},h_{20},h_{11}) \\ & + 2B(\overline{q},h_{31}) + 3B(q,h_{22}) + B(\overline{h}_{20},h_{30}) + 3B(\overline{h}_{21},h_{20}) + 6B(h_{11},h_{21})\rangle, \end{aligned} \tag{7.11}$$

where all h_{jk} are defined earlier. Notice that h_{40} does not enter (7.11).

8. FOLD-HOPF BIFURCATION

At the fold-Hopf bifurcation the system (1.1) has an equilibrium with a simple zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary simple eigenvalues of the Jacobian matrix $A = f_x(0,0)$:

$$\lambda_1 = 0, \ \lambda_{2,3} = \pm i\omega_0,$$

with $\omega_0 > 0$, and no other critical eigenvalues. Introduce two eigenvectors, $q_0 \in \mathbb{R}^n$ and $q_1 \in \mathbb{C}^n$,

$$Aq_0 = 0, \quad Aq_1 = i\omega_0 q_1,$$

and two adjoint eigenvectors, $p_0 \in \mathbb{R}^n$ and $p_1 \in \mathbb{C}^n$,

$$A^T p_0 = 0, \quad A^T p_1 = -i\omega_0 p_1.$$

Normalize them such that

$$\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1.$$

The following orthogonality properties hold:

$$\langle p_1, q_0 \rangle = \langle p_0, q_1 \rangle = 0.$$

Now any vector $y \in T^c \in \mathbb{R}^n$ can be represented as

$$y = w_0 q_0 + w_1 q_1 + \overline{w}_1 \overline{q}_1,$$

where $w_0 = \langle p_0, y \rangle \in \mathbb{R}^1$ and $w_1 = \langle p_1, y \rangle \in \mathbb{C}^1$. The homological equation (4.4) can be written as

$$H_{w_0}\dot{w}_0 + H_{w_1}\dot{w}_1 + H_{\overline{w}_1}\dot{\overline{w}}_1 = F(H(w_0, w_1, \overline{w}_1)), \tag{8.1}$$

where

$$F(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4),$$

$$H(w_0, w_1, \overline{w}_1) = w_0 q_0 + w_1 q_1 + \overline{w}_1 \overline{q}_1 + \sum_{2 \le j+k+l \le 3} \frac{1}{j!k!l!} h_{jkl} w_0^j w_1^k \overline{w}_1^l + O(\|(w_0, w_1, \overline{w}_1)\|^4),$$

$h_{jkl} \in \mathbb{C}^n$, $h_{jlk} = \overline{h}_{jkl}$, and $(\dot{w}_0, \dot{w}_1)$ are defined by (3.4).

Collecting the $w_0^j w_1^k \overline{w}_1^l$-terms in (8.1) with $j + k + l = 2$, one gets from the solvability conditions the expressions for the quadratic coefficients in (3.4):

$$G_{200} = \langle p_0, B(q_0, q_0) \rangle, \quad G_{110} = \langle p_1, B(q_0, q_1) \rangle, \quad G_{011} = \langle p_0, B(q_1, \overline{q}_1) \rangle, \tag{8.2}$$

and the following formulas for the coefficients h_{jkl} with $j + k + l = 2$:

$$h_{200} = -A^{INV}[B(q_0, q_0) - \langle p_0, B(q_0, q_0) \rangle q_0], \tag{8.3}$$

$$h_{020} = (2i\omega_0 I_n - A)^{-1} B(q_1, q_1), \tag{8.4}$$

$$h_{110} = (i\omega_0 I_n - A)^{INV}[B(q_0, q_1) - \langle p_1, B(q_0, q_1) \rangle q_1], \tag{8.5}$$

$$h_{011} = -A^{INV}[B(q_1, \overline{q}_1) - \langle p_0, B(q_1, \overline{q}_1) \rangle q_0]. \tag{8.6}$$

Here the vectors h_{200} and h_{011} can be computed by solving the nonsingular $(n+1)$-dimensional real systems

$$\begin{pmatrix} A & q_0 \\ p_0^T & 0 \end{pmatrix} \begin{pmatrix} h_{200} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_0, q_0) + \langle p_0, B(q_0, q_0) \rangle q_0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} A & q_0 \\ p_0^T & 0 \end{pmatrix} \begin{pmatrix} h_{011} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_1, \overline{q}_1) + \langle p_0, B(q_1, \overline{q}_1) \rangle q_0 \\ 0 \end{pmatrix},$$

while the vector h_{110} can be found by solving the nonsingular $(n+1)$-dimensional complex system

$$\begin{pmatrix} i\omega_0 I_n - A & q_1 \\ \bar{p}_1^T & 0 \end{pmatrix} \begin{pmatrix} h_{110} \\ s \end{pmatrix} = \begin{pmatrix} B(q_0,q_1) - \langle p_1, B(q_0,q_1)\rangle q_1 \\ 0 \end{pmatrix}.$$

Finally, the solvability conditions applied to the systems coming out from the resonant $w_0^j w_1^k \overline{w}_1^l$-terms in (8.1) with $j+k+l=3$ yield

$$G_{300} = \langle p_0, C(q_0,q_0,q_0) + 3B(q_0,h_{200})\rangle, \tag{8.7}$$
$$G_{111} = \langle p_0, C(q_0,q_1,\overline{q}_1) + B(q_1,h_{101}) + B(\overline{q}_1,h_{110}) + B(q_0,h_{011})\rangle, \tag{8.8}$$
$$G_{210} = \langle p_1, C(q_0,q_0,q_1) + 2B(q_0,h_{110}) + B(q_1,h_{200})\rangle, \tag{8.9}$$
$$G_{021} = \langle p_1, C(q_1,q_1,\overline{q}_1) + 2B(q_1,h_{011}) + B(\overline{q}_1,h_{020})\rangle, \tag{8.10}$$

where the coefficients h_{jkl} are defined by (8.3)-(8.6).

9. DOUBLE HOPF BIFURCATION

At the double Hopf bifurcation the system (1.1) has an equilibrium with two pairs of purely imaginary simple eigenvalues of the Jacobian matrix $A = f_x(0,0)$:

$$\lambda_{1,4} = \pm i\omega_1,\ \ \lambda_{2,3} = \pm i\omega_2,$$

with $\omega_1 > \omega_2 > 0$, and no other critical eigenvalues. Assume that the conditions (3.5) hold. Since the eigenvalues are simple, there are two complex eigenvectors, $q_{1,2} \in \mathbb{C}^n$, corresponding to these eigenvalues:

$$Aq_1 = i\omega_1 q_1,\ \ Aq_2 = i\omega_2 q_2.$$

Introduce the adjoint eigenvectors $p_{1,2} \in \mathbb{C}^n$ by

$$A^T p_1 = -i\omega_1 p_1,\ \ A^T p_2 = -i\omega_2 p_2,$$

where T denotes transposition. These eigenvectors can be normalized using the standard scalar product in $\mathbb{C}^n$,

$$\langle p_1, q_1\rangle = \langle p_2, q_2\rangle = 1,$$

and satisfy the orthogonality conditions

$$\langle p_2, q_1\rangle = \langle p_1, q_2\rangle = 0.$$

Any vector $y \in T^c \subset \mathbb{R}^n$ from the critical eigenspace can be represented as

$$y = w_1 q_1 + \overline{w}_1 \overline{q}_1 + w_2 q_2 + \overline{w}_2 \overline{q}_2, \quad w_i \in \mathbb{C}^1,$$

where

$$w_1 = \langle p_1, y\rangle, \quad w_2 = \langle p_2, y\rangle.$$

Therefore, the homological equation (4.4) can be written as

$$H_{w_1}\dot{w}_1 + H_{\overline{w}_1}\dot{\overline{w}}_1 + H_{w_2}\dot{w}_2 + H_{\overline{w}_2}\dot{\overline{w}}_2 = F(H(w_1,\overline{w}_1,w_2,\overline{w}_2)), \tag{9.1}$$

where F is defined by (2.1),

$$H(w_1,\overline{w}_1,w_2,\overline{w}_2) = w_1 q_1 + \overline{w}_1\overline{q}_1 + w_2 q_2 + \overline{w}_2 \overline{q}_2 + \sum_{j+k+l+m\geq 2} \frac{1}{j!k!l!m!} h_{jklm} w_1^j \overline{w}_1^k w_2^l \overline{w}_2^m$$

$h_{jklm} \in \mathbb{C}^n$, $h_{kjml} = \overline{h}_{jklm}$, and $(\dot{w}_1, \dot{\overline{w}}_2)$ are specified by the normal form (3.6).

Collecting the coefficients of the $w_1^j \overline{w}_1^k w_2^l \overline{w}_2^m$-terms with $j+k+l+m=2$ in (9.1) gives the following expressions for h_{jklm}:

$$\begin{aligned}
h_{1100} &= -A^{-1}B(q_1,\overline{q}_1), && (9.2)\\
h_{2000} &= (2i\omega_1 I_n - A)^{-1}B(q_1,q_1), && (9.3)\\
h_{1010} &= [i(\omega_1+\omega_2)I_n - A]^{-1}B(q_1,q_2), && (9.4)\\
h_{1001} &= [i(\omega_1-\omega_2)I_n - A]^{-1}B(q_1,\overline{q}_2), && (9.5)\\
h_{0020} &= (2i\omega_2 I_n - A)^{-1}B(q_2,q_2), && (9.6)\\
h_{0011} &= -A^{-1}B(q_2,\overline{q}_2). && (9.7)
\end{aligned}$$

All matrices involved in (9.3)-(9.7) are invertible in ordinary sense due to the assumptions (3.5) on the critical eigenvalues.

Collecting the coefficients in front of the nonresonant $w_1^j \overline{w}_1^k w_2^l \overline{w}_2^m$-terms with $j+k+l+m=3$ in (9.1), one obtains the following expressions for h_{jklm}:

$$\begin{aligned}
h_{3000} &= (3i\omega_1 I_n - A)^{-1}[C(q_1,q_1,q_1)+3B(h_{2000},q_1)], && (9.8)\\
h_{2010} &= [i(2\omega_1+\omega_2)I_n - A]^{-1}[C(q_1,q_1,q_2)+B(h_{2000},q_2)+2B(h_{1010},q_1)], && (9.9)\\
h_{2001} &= [i(2\omega_1-\omega_2)I_n - A]^{-1}[C(q_1,q_1,\overline{q}_2)+B(h_{2000},\overline{q}_2)+2B(h_{1001},q_1)], && (9.10)\\
h_{1020} &= [i(\omega_1+2\omega_2)I_n - A]^{-1}[C(q_1,q_2,q_2)+B(h_{0020},q_1)+2B(h_{1010},q_2)], && (9.11)\\
h_{1002} &= [i(\omega_1-2\omega_2)I_n - A]^{-1}[C(q_1,\overline{q}_2,\overline{q}_2)+B(\overline{h}_{0020},q_1)+2B(h_{1001},\overline{q}_2)], && (9.12)\\
h_{0030} &= (3i\omega_2 I_n - A)^{-1}[C(q_2,q_2,q_2)+3B(h_{0020},q_2)]. && (9.13)
\end{aligned}$$

The matrices in (9.8)-(9.13) are invertible. Collecting the coefficients of the resonant cubic terms in (9.1), one obtains the resonant cubic coefficients in the normal form

$$\begin{aligned}
G_{2100} &= \langle p_1, C(q_1,q_1,\overline{q}_1)+B(h_{2000},\overline{q}_1)+2B(h_{1100},q_1)\rangle, && (9.14)\\
G_{1011} &= \langle p_1, C(q_1,q_2,\overline{q}_2)+B(h_{1010},\overline{q}_2)+B(h_{1001},q_2)+B(h_{0011},q_1)\rangle, && (9.15)\\
G_{1110} &= \langle p_2, C(q_1,\overline{q}_1,q_2)+B(h_{1100},q_2)+B(h_{1010},\overline{q}_1)+B(\overline{h}_{1001},q_1)\rangle, && (9.16)\\
G_{0021} &= \langle p_2, C(q_2,q_2,\overline{q}_2)+B(h_{0020},\overline{q}_2)+2B(h_{0011},q_2)\rangle, && (9.17)
\end{aligned}$$

and the corresponding cubic coefficients h_{jklm} satisfying the orthogonality conditions:

$$\begin{aligned}
h_{2100} &= (i\omega_1 I_n - A)^{INV}[C(q_1,q_1,\overline{q}_1)+B(h_{2000},\overline{q}_1)+2B(h_{1100},q_1)-G_{2100}\,q_1], && (9.18)\\
h_{1011} &= (i\omega_1 I_n - A)^{INV}[C(q_1,q_2,\overline{q}_2)+B(h_{1010},\overline{q}_2)+B(h_{1001},q_2)+B(h_{0011},q_1)\\
&\qquad -G_{1011}\,q_1], && (9.19)\\
h_{1110} &= (i\omega_2 I_n - A)^{INV}[C(q_1,\overline{q}_1,q_2)+B(h_{1100},q_2)+B(h_{1010},\overline{q}_1)+B(\overline{h}_{1001},q_1)\\
&\qquad -G_{1110}\,q_2], && (9.20)\\
h_{0021} &= (i\omega_2 I_n - A)^{INV}[C(q_2,q_2,\overline{q}_2)+B(h_{0020},\overline{q}_2)+2B(h_{0011},q_2)-G_{0021}\,q_2], && (9.21)
\end{aligned}$$

Here the vectors of the form $h=(i\omega_i I_n - A)^{INV}b$ can be found by solving the nonsingular $(n+1)$-dimensional complex systems

$$\begin{pmatrix} i\omega_i I_n - A & q_i \\ \overline{p}_i^T & 0 \end{pmatrix}\begin{pmatrix} h \\ s \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Collecting the coefficients of the $w_1^j \overline{w}_1^k w_2^l \overline{w}_2^m$-terms with $j+k+l+m=4$ in (9.1) gives the following expressions for h_{jklm}:

$$\begin{aligned}
h_{4000} &= (4i\omega_1 I_n - A)^{-1}[3B(h_{2000},h_{2000})+4B(h_{3000},q_1)+6C(h_{2000},q_1,q_1)\\
&\qquad +D(q_1,q_1,q_1,q_1)], && (9.22)\\
h_{3100} &= (2i\omega_1 I_n - A)^{-1}[3B(h_{2000},h_{1100})+3B(h_{2100},q_1)+3C(h_{1100},q_1,q_1)
\end{aligned}$$

$$
\begin{aligned}
& \quad + 3C(h_{2000}, q_1, \overline{q}_1) + D(q_1, q_1, q_1, \overline{q}_1) - 3G_{2100}h_{2000}], && (9.23)\\
h_{3010} &= [i(3\omega_1+\omega_2)I_n - A]^{-1}[3B(h_{2010}, q_1) + 3B(h_{2000}, h_{1010}) + B(h_{3000}, q_2)\\
& \quad + 3C(h_{2000}, q_1, q_2) + 3C(h_{1010}, q_1, q_1) + D(q_1, q_1, q_1, q_2)], && (9.24)\\
h_{3001} &= [i(3\omega_1-\omega_2)I_n - A]^{-1}[3B(h_{2000}, h_{1001}) + 3B(h_{2001}, q_1) + B(h_{3000}, \overline{q}_2)\\
& \quad + 3C(h_{2000}, q_1, \overline{q}_2) + 3C(h_{1001}, q_1, q_1) + D(q_1, q_1, q_1, \overline{q}_2)], && (9.25)\\
h_{2200} &= -A^{-1}[2B(h_{1100}, h_{1100}) + B(h_{2000}, \overline{h}_{2000}) + 2B(\overline{h}_{2100}, q_1) + 2B(h_{2100}, \overline{q}_1)\\
& \quad + C(\overline{h}_{2000}, q_1, q_1) + 4C(h_{1100}, q_1, \overline{q}_1) + C(h_{2000}, \overline{q}_1, \overline{q}_1)\\
& \quad + D(q_1, q_1, \overline{q}_1, \overline{q}_1) - 4\,\mathrm{Re}(G_{2100})h_{1100}], && (9.26)\\
h_{2110} &= [i(\omega_1+\omega_2)I_n - A]^{-1}[2B(h_{1100}, h_{1010}) + B(h_{2010}, \overline{q}_1) + 2B(h_{1110}, q_1)\\
& \quad + B(h_{2000}, \overline{h}_{1001}) + B(h_{2100}, q_2) + C(h_{2000}, \overline{q}_1, q_2)\\
& \quad + 2C(h_{1100}, q_1, q_2) + C(\overline{h}_{1001}, q_1, q_1) + 2C(h_{1010}, q_1, \overline{q}_1)\\
& \quad + D(q_1, q_1, \overline{q}_1, q_2) - (G_{2100} + 2G_{1110})h_{1010}] && (9.27)\\
h_{2101} &= [i(\omega_1-\omega_2)I_n - A]^{-1}[2B(h_{1100}, h_{1001}) + 2B(\overline{h}_{1110}, q_1) + B(h_{2000}, \overline{h}_{1010})\\
& \quad + B(h_{2100}, \overline{q}_2) + B(h_{2001}, \overline{q}_1) + 2C(h_{1100}, q_1, \overline{q}_2)\\
& \quad + C(\overline{h}_{1010}, q_1, q_1) + 2C(h_{1001}, q_1, \overline{q}_1) + C(h_{2000}, \overline{q}_1, \overline{q}_2)\\
& \quad + D(q_1, q_1, \overline{q}_1, \overline{q}_2) - (2\overline{G}_{1110} + G_{2100})h_{1001}], && (9.28)\\
h_{2020} &= [2i(\omega_1+\omega_2)I_n - A]^{-1}[2B(h_{1020}, q_1) + 2B(h_{2010}, q_2) + B(h_{2000}, h_{0020})\\
& \quad + 2B(h_{1010}, h_{1010}) + C(h_{2000}, q_2, q_2) + 4C(h_{1010}, q_1, q_2)\\
& \quad + C(h_{0020}, q_1, q_1) + D(q_1, q_1, q_2, q_2)], && (9.29)\\
h_{2011} &= (2i\omega_1 I_n - A)^{-1}[B(h_{2010}, \overline{q}_2) + B(h_{2000}, h_{0011}) + B(h_{2001}, q_2)\\
& \quad + 2B(h_{1010}, h_{1001}) + 2B(h_{1011}, q_1) + C(h_{2000}, q_2, \overline{q}_2)\\
& \quad + 2C(h_{1010}, q_1, \overline{q}_2) + 2C(h_{1001}, q_1, q_2) + C(h_{0011}, q_1, q_1)\\
& \quad + D(q_1, q_1, q_2, \overline{q}_2) - 2G_{1011}h_{2000}] && (9.30)\\
h_{2002} &= [2i(\omega_1-\omega_2)I_n - A]^{-1}[2B(h_{1002}, q_1) + B(h_{2000}, \overline{h}_{0020}) + 2B(h_{2001}, \overline{q}_2)\\
& \quad + 2B(h_{1001}, h_{1001}) + C(h_{2000}, \overline{q}_2, \overline{q}_2) + C(\overline{h}_{0020}, q_1, q_1)\\
& \quad + 4C(h_{1001}, q_1, \overline{q}_2) + D(q_1, q_1, \overline{q}_2, \overline{q}_2)] && (9.31)\\
h_{1120} &= (2i\omega_2 I_n - A)^{-1}[B(h_{1100}, h_{0020}) + B(h_{1020}, \overline{q}_1) + 2B(h_{1110}, q_2)\\
& \quad + B(\overline{h}_{1002}, q_1) + 2B(h_{1010}, \overline{h}_{1001}) + C(h_{1100}, q_2, q_2)\\
& \quad + 2C(\overline{h}_{1001}, q_1, q_2) + C(h_{0020}, q_1, \overline{q}_1) + 2C(h_{1010}, \overline{q}_1, q_2)\\
& \quad + D(q_1, \overline{q}_1, q_2, q_2) - 2G_{1110}h_{0020}], && (9.32)\\
h_{1111} &= -A^{-1}[B(h_{1100}, h_{0011}) + B(h_{1110}, \overline{q}_2) + B(\overline{h}_{1110}, q_2) + B(\overline{h}_{1011}, q_1)\\
& \quad + B(\overline{h}_{1001}, h_{1001}) + B(h_{1010}, \overline{h}_{1010}) + B(h_{1011}, \overline{q}_1)\\
& \quad + C(h_{1100}, q_2, \overline{q}_2) + C(\overline{h}_{1010}, q_1, q_2) + C(\overline{h}_{1001}, q_1, \overline{q}_2)\\
& \quad + C(h_{1010}, \overline{q}_1, \overline{q}_2) + C(h_{1001}, \overline{q}_1, q_2) + C(h_{0011}, q_1, \overline{q}_1)\\
& \quad + D(q_1, \overline{q}_1, q_2, \overline{q}_2) - 2\,\mathrm{Re}(G_{1011})h_{1100} - 2\,\mathrm{Re}\ (G_{1110})h_{0011}], && (9.33)\\
h_{1030} &= [3i(\omega_1+\omega_2)I_n - A]^{-1}[3B(h_{1020}, q_2) + 3B(h_{1010}, h_{0020}) + B(h_{0030}, q_1)\\
& \quad + 3C(h_{0020}, q_1, q_2) + 3C(h_{1010}, q_2, q_2) + D(q_1, q_2, q_2, q_2)], && (9.34)\\
h_{1021} &= [i(\omega_1+\omega_2)I_n - A]^{-1}[B(h_{1020}, \overline{q}_2) + B(h_{0020}, h_{1001}) + 2B(h_{1010}, h_{0011})\\
& \quad + B(h_{0021}, q_1) + 2B(h_{1011}, q_2) + C(h_{1001}, q_2, q_2)\\
& \quad + C(h_{0020}, q_1, \overline{q}_2) + 2C(h_{1010}, q_2, \overline{q}_2) + 2C(h_{0011}, q_1, q_2)\\
& \quad + D(q_1, q_2, q_2, \overline{q}_2) - (2G_{1011} + G_{0021})h_{1010}] && (9.35)\\
h_{1012} &= [i(\omega_1-\omega_2)I_n - A]^{-1}[B(h_{1002}, q_2) + B(h_{1010}, \overline{h}_{0020}) + 2B(h_{1001}, h_{0011})\\
& \quad + 2B(h_{1011}, \overline{q}_2) + B(\overline{h}_{0021}, q_1) + C(h_{1010}, \overline{q}_2, \overline{q}_2)
\end{aligned}
$$

$$
\begin{aligned}
& + 2C(h_{1001},q_2,\overline{q}_2) + C(\overline{h}_{0020},q_1,q_2) + 2C(h_{0011},q_1,\overline{q}_2) \\
& + D(q_1,q_2,\overline{q}_2,\overline{q}_2) - (2G_{1011} + \overline{G}_{0021})h_{1001}], \qquad (9.36)
\end{aligned}
$$

$$
\begin{aligned}
h_{1003} &= [i(\omega_1 - 3\omega_2)I_n - A]^{-1}[3B(h_{1002},\overline{q}_2) + B(\overline{h}_{0030},q_1) + 3B(h_{1001},\overline{h}_{0020}) \\
& \quad + 3C(\overline{h}_{0020},q_1,\overline{q}_2) + 3C(h_{1001},\overline{q}_2,\overline{q}_2) \\
& \quad - \frac{1}{2}\overline{G}_{0032}h_{1001}], \qquad (9.37)
\end{aligned}
$$

$$
\begin{aligned}
h_{0040} &= (4i\omega_2 I_n - A)^{-1}[3B(h_{0020},h_{0020}) + 4B(h_{0030},q_2) + 6C(h_{0020},q_2,q_2) \\
& \quad + D(q_2,q_2,q_2,q_2)], \qquad (9.38)
\end{aligned}
$$

$$
\begin{aligned}
h_{0031} &= (2i\omega_2 I_n - A)^{-1}[3B(h_{0020},h_{0011}) + 3B(h_{0021},q_2) + B(h_{0030},\overline{q}_2) \\
& \quad + 3C(h_{0020},q_2,\overline{q}_2) + 3C(h_{0011},q_2,q_2) + D(q_2,q_2,q_2,\overline{q}_2) \\
& \quad - 3G_{0021}h_{0020}], \qquad (9.39)
\end{aligned}
$$

$$
\begin{aligned}
h_{0022} &= -A^{-1}[B(h_{0020},\overline{h}_{0020}) + 2B(h_{0011},h_{0011}) + 2B(h_{0021},\overline{q}_2) + 2B(\overline{h}_{0021},q_2) \\
& \quad + C(h_{0020},\overline{q}_2,\overline{q}_2) + 4C(h_{0011},q_2,\overline{q}_2) + C(\overline{h}_{0020},q_2,q_2) \\
& \quad + D(q_2,q_2,\overline{q}_2,\overline{q}_2) - 4\,\mathrm{Re}(G_{0021})h_{0011}]. \qquad (9.40)
\end{aligned}
$$

Finally, the solvability conditions applied to the systems coming out from the resonant $w_1^j\overline{w}_1^k w_2^l\overline{w}_2^m$-terms in (8.1) with $j+k+l+m=5$ yield

$$
\begin{aligned}
G_{3200} &= \langle p_1, 3B(h_{2200},q_1) + B(h_{3000},\overline{h}_{2000}) + 2B(h_{3100},\overline{q}_1) + 6B(h_{1100},h_{2100}) \\
& \quad + 3B(h_{2000},\overline{h}_{2100}) + 6C(h_{2100},q_1,\overline{q}_1) + 3C(\overline{h}_{2100},q_1,q_1) + 6C(h_{1100},h_{1100},q_1) \\
& \quad + C(h_{3000},\overline{q}_1,\overline{q}_1) + 3C(h_{2000},\overline{h}_{2000},q_1) + 6C(h_{2000},h_{1100},\overline{q}_1) \\
& \quad + 6D(h_{1100},q_1,q_1,\overline{q}_1) + 3D(h_{2000},q_1,\overline{q}_1,\overline{q}_1) + D(\overline{h}_{2000},q_1,q_1,q_1) \\
& \quad + E(q_1,q_1,q_1,\overline{q}_1,\overline{q}_1)\rangle, \qquad (9.41)
\end{aligned}
$$

$$
\begin{aligned}
G_{2111} &= \langle p_1, 2B(h_{1100},h_{1011}) + B(h_{2000},\overline{h}_{1011}) + B(\overline{h}_{1001},h_{2001}) + B(h_{2100},h_{0011}) \\
& \quad + B(h_{2110},\overline{q}_2) + 2B(h_{1110},h_{1001}) + B(h_{2101},q_2) + 2B(h_{1111},q_1) \\
& \quad + 2B(h_{1010},h_{1101}) + B(h_{2010},\overline{h}_{1010}) + B(h_{2011},\overline{q}_1) + 2C(h_{1100},h_{1010},\overline{q}_2) \\
& \quad + C(h_{2000},\overline{h}_{1010},q_2) + C(h_{2000},h_{0011},\overline{q}_1) + C(h_{2000},\overline{h}_{1001},\overline{q}_2) \\
& \quad + 2C(h_{1110},q_1,\overline{q}_2) + 2C(h_{1100},h_{1001},q_2) + C(h_{2100},q_2,\overline{q}_2) \\
& \quad + 2C(h_{1100},h_{0011},q_1) + 2C(h_{1010},\overline{h}_{1010},q_1) + 2C(h_{1010},h_{1001},\overline{q}_1) \\
& \quad + 2C(\overline{h}_{1001},h_{1001},q_1) + C(h_{2010},\overline{q}_1,\overline{q}_2) + C(\overline{h}_{1011},q_1,q_1) \\
& \quad + 2C(h_{1011},q_1,\overline{q}_1) + C(h_{2001},\overline{q}_1,q_2) + 2C(h_{1101},q_1,q_2) \\
& \quad + D(h_{0011},q_1,q_1,\overline{q}_1) + D(h_{2000},\overline{q}_1,q_2,\overline{q}_2) + 2D(h_{1001},q_1,\overline{q}_1,q_2) \\
& \quad + 2D(h_{1010},q_1,\overline{q}_1,\overline{q}_2) + 2D(h_{1100},q_1,q_2,\overline{q}_2) + D(\overline{h}_{1001},q_1,q_1,\overline{q}_2) \\
& \quad + D(\overline{h}_{1010},q_1,q_1,q_2) + E(q_1,q_1,\overline{q}_1,q_2,\overline{q}_2)\rangle, \qquad (9.42)
\end{aligned}
$$

$$
\begin{aligned}
G_{1022} &= \langle p_1, B(h_{0020},h_{1002}) + 4B(h_{0011},h_{1011}) + 2B(h_{1010},\overline{h}_{0021}) + 2B(h_{1012},q_2) \\
& \quad + B(h_{1020},\overline{h}_{0020}) + 2B(h_{1001},h_{0021}) + 2B(h_{1021},\overline{q}_2) + B(h_{0022},q_1) \\
& \quad + 4C(h_{1010},h_{0011},\overline{q}_2) + 2C(h_{0011},h_{0011},q_1) + 2C(h_{1010},\overline{h}_{0020},q_2) \\
& \quad + 4C(h_{1001},h_{0011},q_2) + 2C(h_{0021},q_1,\overline{q}_2) + 2C(h_{0020},h_{1001},\overline{q}_2) \\
& \quad + C(h_{0020},\overline{h}_{0020},q_1) + 2C(\overline{h}_{0021},q_1,q_2) + 4C(h_{1011},q_2,\overline{q}_2) \\
& \quad + C(h_{1020},\overline{q}_2,\overline{q}_2) + C(h_{1002},q_2,q_2) + 4D(h_{0011},q_1,q_2,\overline{q}_2) \\
& \quad + D(h_{0020},q_1,\overline{q}_2,\overline{q}_2) + 2D(h_{1010},q_2,\overline{q}_2,\overline{q}_2) + D(\overline{h}_{0020},q_1,q_2,q_2) \\
& \quad + 2D(h_{1001},q_2,q_2,\overline{q}_2) + E(q_1,q_2,q_2,\overline{q}_2,\overline{q}_2)\rangle \qquad (9.43)
\end{aligned}
$$

$$
\begin{aligned}
G_{2210} = \; & \langle p_2, 4B(h_{1100}, h_{1110}) + 2B(\overline{h}_{2100}, h_{1010}) + B(\overline{h}_{2000}, h_{2010}) + B(h_{2000}, \overline{h}_{2001}) \\
& + 2B(h_{2100}, h_{0110}) + B(h_{2200}, q_2) + 2B(h_{2110}, \overline{q}_1) + 2B(\overline{h}_{2101}, q_1) \\
& + 2C(h_{2000}, \overline{h}_{1001}, \overline{q}_1) + 4C(h_{1100}, h_{1010}, \overline{q}_1) + 2C(\overline{h}_{2000}, h_{1010}, q_1) \\
& + 2C(h_{2100}, \overline{q}_1, q_2) + 2C(\overline{h}_{2100}, q_1, q_2) + 4C(h_{1100}, \overline{h}_{1001}, q_1) \\
& + C(h_{2010}, \overline{q}_1, \overline{q}_1) + 2C(h_{1100}, h_{1100}, q_2) + 4C(h_{1110}, q_1, \overline{q}_1) \\
& + C(\overline{h}_{2001}, q_1, q_1) + C(h_{2000}, \overline{h}_{2000}, q_2) + 4D(h_{1100}, q_1, \overline{q}_1, q_2) \\
& + 2D(\overline{h}_{1001}, q_1, q_1, \overline{q}_1) + 2D(h_{1010}, q_1, \overline{q}_1, \overline{q}_1) + D(\overline{h}_{2000}, q_1, q_1, q_2) \\
& + D(h_{2000}, \overline{q}_1, \overline{q}_1, q_2) + E(q_1, q_1, \overline{q}_1, \overline{q}_1, q_2)\rangle
\end{aligned} \tag{9.44}
$$

$$
\begin{aligned}
G_{1121} = \; & \langle p_2, 2B(h_{1110}, h_{0011}) + B(\overline{h}_{1002}, h_{1001}) + B(h_{1100}, h_{0021}) + B(h_{0020}, h_{1101}) \\
& + 2B(\overline{h}_{1001}, h_{1011}) + 2B(\overline{h}_{1001}, h_{1011}) + B(h_{1020}, \overline{h}_{1010}) + B(h_{1120}, \overline{q}_2) \\
& + 2B(h_{1111}, q_2) + B(h_{1021}, \overline{q}_1) + 2B(h_{1010}, \overline{h}_{1011}) + B(\overline{h}_{1012}, q_1) \\
& + C(h_{1100}, h_{0020}, \overline{q}_2) + 2C(\overline{h}_{1001}, h_{0011}, q_1) + C(h_{0020}, h_{1001}, \overline{q}_1) \\
& + 2C(h_{1010}, h_{0011}, \overline{q}_1) + C(h_{0020}, \overline{h}_{1010}, q_1) + C(h_{1101}, q_2, q_2) \\
& + 2C(h_{1010}, \overline{h}_{1001}, \overline{q}_2) + 2C(h_{1010}, \overline{h}_{1010}, q_2) + C(\overline{h}_{1002}, q_1, \overline{q}_2) \\
& + 2C(h_{1100}, h_{0011}, q_2) + 2C(\overline{h}_{1001}, h_{1001}, q_2) + 2C(h_{1110}, q_2, \overline{q}_2) \\
& + C(h_{0021}, q_1, \overline{q}_1) + 2C(\overline{h}_{1011}, q_1, q_2) + C(h_{1020}, \overline{q}_1, \overline{q}_2) \\
& + 2C(h_{1011}, \overline{q}_1, q_2) + D(h_{0020}, q_1, \overline{q}_1, \overline{q}_2) + D(h_{1100}, q_2, q_2, \overline{q}_2) \\
& + 2D(h_{0011}, q_1, \overline{q}_1, q_2) + 2D(\overline{h}_{1001}, q_1, q_2, \overline{q}_2) + D(h_{1001}, \overline{q}_1, q_2, q_2) \\
& + D(\overline{h}_{1010}, q_1, q_2, q_2) + 2D(h_{1010}, \overline{q}_1, q_2, \overline{q}_2) + E(q_1, \overline{q}_1, q_2, q_2, \overline{q}_2)\rangle
\end{aligned} \tag{9.45}
$$

$$
\begin{aligned}
G_{0032} = \; & \langle p_2, 3B(h_{0022}, q_2) + B(h_{0030}, \overline{h}_{0020}) + 2B(h_{0031}, \overline{q}_2) + 6B(h_{0011}, h_{0021}) \\
& + 3B(h_{0020}, \overline{h}_{0021}) + 6C(h_{0011}, h_{0011}, q_2) + 3C(h_{0020}, \overline{h}_{0020}, q_2) + C(h_{0030}, \overline{q}_2, \overline{q}_2) \\
& + 6C(h_{0020}, h_{0011}, \overline{q}_2) + 3C(\overline{h}_{0021}, q_2, q_2) + 6C(h_{0021}, q_2, \overline{q}_2) \\
& + 3D(h_{0020}, q_2, \overline{q}_2, \overline{q}_2) + D(\overline{h}_{0020}, q_2, q_2, q_2) + 6D(h_{0011}, q_2, q_2, \overline{q}_2) \\
& + E(q_2, q_2, q_2, \overline{q}_2, \overline{q}_2)\rangle
\end{aligned} \tag{9.46}
$$

10. DISCUSSION

The above derived formulas for the normal form coefficients allow to verify the nondegeneracy conditions (see Section 3 and [Kuznetsov 1995]) for all codim 2 equilibrium bifurcations. In particular, computing the coefficients for the fold-Hopf and double Hopf bifurcations allows to distinguish between "simple" and "difficult" cases implying "chaotic motions".

The formulas are independent on the dimension n of the phase space and involve only critical eigenvectors of A and A^T. They are also valid when $n_c = n$ (the dimension of the center manifold is equal to the phase space dimension).

The formulas are easily programmable if algebraic operations with complex matrices are supported, like in MAPLE, Mathematica, or MATLAB. Finding the intermediate coefficients h_ν reduces via bordering technique to solving nonsingular (complex) linear systems. If symbolic derivatives of the right-hand side of (1.1) are available, they can be used directly to evaluate the multilinear functions B, C, D, E by the formulas given after the equation (2.1). If no symbolic derivatives are given, these functions (and scalar products involving them) can be approximated numerically using only directional derivatives of the right-hand side of (1.1) (see Kuznetsov [1995] for finite-difference approximations of $B(p, q)$ and $C(q, q, p)$).

The formulas for the double Hopf bifurcation are complicated. Publishing such formulas is dangerous due to possible errors. The author tried to reduce the possibility of misprints by automatic conversion of the MAPLE output into LaTeX with minimal postprocessing. The MAPLE V.R4 sessions to derive all the coefficients are available on request. In actual implementations of these formulas, the corresponding `C`-codes should be generated automatically.

Acknowledgement

The author is thankful to Dr. A. Heck (UvA) for his help in dealing with multilinear functions in MAPLE V.R4.

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