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ABSTRACT
Necessary and sufficient conditions for minimality of descriptor representations of impulsive-smooth behaviors are derived. We obtain a complete set of transformations by which minimal descriptor representations that give rise to the same behavior can be transformed into each other. In particular this leads to a jump-behavioral interpretation of the notion of strong equivalence of descriptor representations.

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1. Introduction and preliminaries

Our purpose in this paper is to investigate the use of descriptor representations for systems with possibly inconsistent initial conditions. Situations in which a set of differential and algebraic equations is inconsistently initialized may occur in particular in a context in which switches occur between several operating regimes or ‘modes’. When a switch occurs from one mode to another, the values of the state variables that are inherited from the preceding mode may not be consistent for the new mode, and in this case a state jump will occur. This type of modeling can be quite acceptable as a way of describing phenomena that are very fast with respect to the time scale of interest, as will be shown below in a simple example. We note that ‘multimodal’ or ‘hybrid’ systems have attracted considerable attention recently, see for instance [10].

When dealing with a multimodal system, one has to work with representations of the dynamics in each of the various modes; often it will be convenient to let these representations be similar to each other as much as possible. As illustrated below, the descriptor representation (or another representation in a generalized form) is in many cases more suitable for this purpose than the standard state space representation. Moreover, the descriptor representation also provides information about jumps that may occur, whereas the standard state space representation is by definition unable to provide such information.

To illustrate the above remarks, let us consider a very simple mechanical example (see Fig. 0.1). Two carts are connected to each other and to a fixed wall by springs. The motion of the left cart is restricted by a purely non-elastic stop. A control force can be exerted on the right cart; as an output, we consider the position of the right cart. For simplicity, we shall normalize all constants to 1 and let the springs be linear, and we shall assume that the stop is placed at the equilibrium position of the left cart. An ‘event’ takes place when the left cart hits the stop or when it is pulled away from a position at the stop. Such an event marks a change of mode: the system may switch from ‘constrained mode’

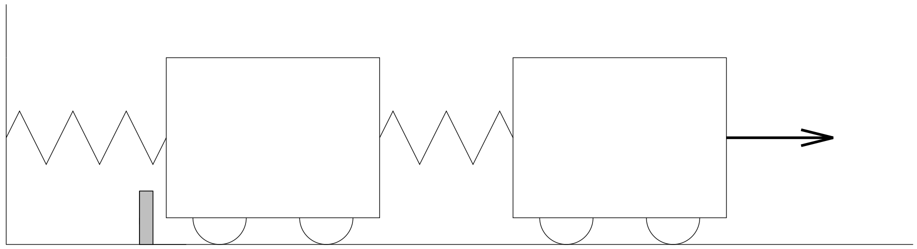

Figure 0.1: Multimodal system with continuous input

to 'unconstrained mode' or vice versa.

It is not difficult to write equations of motion for each of the two modes. Let $x_1(t)$ and $x_2(t)$ represent the deviations of the left and the right cart respectively from their equilibrium positions, and let $x_3(t)$ and $x_4(t)$ denote the corresponding velocities. In the unconstrained mode the equations are the ones that would hold if there were no block:

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_3(t) &= -2x_1(t) + x_2(t) \\ \dot{x}_4(t) &= x_1(t) - x_2(t) + u(t) \\ y(t) &= x_2(t). \end{aligned} \tag{1.1}$$

The equations of motion in the constrained mode are the ones that would hold if the first cart were nailed to the block:

$$\begin{aligned} x_1(t) &= 0 \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_4(t) &= -x_2(t) + u(t) \\ y(t) &= x_2(t). \end{aligned} \tag{1.2}$$

Note that the unconstrained mode has a minimal state space description of order 4, whereas the minimal state space for the constrained mode is 2; obviously this is the effect of the loss of one degree of freedom when a transition from the unconstrained mode to the constrained mode occurs. The difference in order already implies that the standard state space representations for the two modes are not very much alike. However, it is possible to write down a descriptor representation of order 5 with one discrete parameter that encapsulates both modes. Such a representation may be derived from the so-called complementarity formalism that is discussed in detail in [12, 6]. For the example considered above, one finds the representation

$$\frac{d}{dt}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \lambda \end{bmatrix}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 & 1-\epsilon \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \lambda \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = x_2(t)$$

where $\epsilon = 0$ in the unconstrained mode and $\epsilon = 1$ in the unconstrained mode. One can easily verify that a reduction to standard state space form for each of the two modes reproduces the state space equations that we wrote down before. In the descriptor representation the equations of motion of the two modes are related in a simple way. Even more importantly, the descriptor equations allow for jump solutions that, at the level of idealization that was aimed for in the model, correctly describe the physics of the system. For example, one can verify that the solution in the constrained mode of the above system with input identically zero and initial condition $(x_1, x_2, x_3, x_4, \lambda) = (0, -1, 0, 0, 0)$ is a jump to the origin. This corresponds to a situation in which the left cart hits the stop with velocity -1 at a moment when the right cart has zero velocity at its equilibrium position; the velocity of the left cart is instantaneously reduced to zero and the system remains at rest. The minimal state-space descriptions of the two separate modes do not contain any information about jump solutions.

To describe the jump behavior mathematically, we shall use a simple fragment of the calculus of distributions (following the framework laid out in [5]). Throughout the paper the first derivative of the Dirac distribution δ is denoted by p, and its k-th derivative is denoted by p^k. $\mathbb{R}[s]$ denotes, as usual, the ring of polynomials in s with real coefficients. We denote by $\mathcal{C}(t_0, t_1)$ the set of restrictions of $\mathcal{C}^\infty(\mathbb{R})$ -functions to (t_0, t_1) with $-\infty < t_0 < t_1 \leq +\infty$. The product space $(\mathbb{R}[p] \times \mathcal{C}(t_0, t_1))^k$ is denoted by $\mathcal{C}^k_{\text{imp}}(t_0, t_1)$; so elements v of this space consist of a polynomial part, which we shall refer to as the 'purely impulsive part' $v_{\text{p-imp}}$ (representing a pulse at time t_0) and a function part, which is called the 'smooth part' v_{sm}. We write $v = v_{\text{p-imp}} + v_{\text{sm}}$; the summation is motivated by the fact that elements of $\mathcal{C}^k_{\text{imp}}(t_0, t_1)$ may be identified with certain distributions (for further detail see [1] and [5]). Note that the convolution action of the operator p may be described by

$$pv = pv_{\text{p-imp}} + v_{\text{sm}}(t_0^+) + \dot{v}_{\text{sm}}$$

and for the purposes of the present paper this might actually be taken as the definition of p. It is easily verified that for instance the scalar differential equation $\dot{v} = av$ with initial condition $v(0) = v_0$ can be expressed in the present framework by the formula $pv = av + v_0$. To alleviate the notation, explicit mention of the interval (t_0, t_1) will be suppressed in what follows.

We shall consider systems with external variables w, which will sometimes be distinguished into inputs u and outputs y. The variable w takes values in a finite-dimensional real vector space $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. The *behavior* $\mathcal{B}$ of a given system of differential and algebraic equations in internal and external variables is defined, as in [15, 16], as the set of time trajectories of the external variables that are admitted by the system equations. In order to incorporate solutions that exhibit an initial jump we shall consider solutions in the space $\mathcal{C}^k_{\text{imp}}$ of impulsive-smooth distributions; in this we deviate from the setting of [15, 16]. Behaviors defined in the space of impulsive-smooth distributions will be referred to as *impulsive-smooth behaviors* or *jump behaviors*.

A study of impulsive-smooth behaviors was based in [1] and [2] on the so-called *pencil* representations (which for brevity will sometimes be referred to as P representations below). Actually two types were considered, called *conventional* and *unconventional* pencil representations respectively. These are equally expressive but differ in the way in which the initial conditions are incorporated. Unconventional pencil representations are defined as follows.

Definition 1.1 For a matrix triple (F, G, H) $(F, G \in \mathbb{R}^{n \times (n+k)}, H \in \mathbb{R}^{q \times (n+k)})$, the *unconventionally associated impulsive-smooth behavior* $\mathcal{B}(F, G, H)$ is

$$\mathcal{B}(F, G, H) = \{w \in \mathcal{C}^q_{\text{imp}} \mid \exists z \in \mathcal{C}^{n+k}_{\text{imp}},\ x_0 \in \mathbb{R}^n \text{ s.t. } pGz = Fz + x_0,\ w = Hz\}.$$

The definition of conventional pencil representations is as follows.

Definition 1.2 For a matrix triple (F, G, H) $(F, G \in \mathbb{R}^{n \times (n+k)}, H \in \mathbb{R}^{q \times (n+k)})$, the *conventionally associated impulsive-smooth behavior* $\mathcal{B}_c(F, G, H)$ is

$$\mathcal{B}_c(F, G, H) = \{w \in \mathcal{C}^q_{\text{imp}} \mid \exists z \in \mathcal{C}^{n+k}_{\text{imp}}, z_0 \in \mathbb{R}^{n+k} \text{ s.t. } pGz = Fz + Gz_0,\ w = Hz\}.$$

Analogously we shall distinguish between conventional and unconventional *descriptor* representations (D representations) given by quintuples (E, A, B, C, D) with $E, A \in \mathbb{R}^{n_1 \times n_2}$, $B \in \mathbb{R}^{n_1 \times m}$, $C \in \mathbb{R}^{p \times n_2}$, and $D \in \mathbb{R}^{p \times m}$). The unconventional form is given in the following way.

Definition 1.3 For a matrix quintuple (E, A, B, C, D), the *unconventionally associated impulsive-smooth behavior* $\mathcal{B}(E, A, B, C, D)$ is

$$\begin{aligned} &\mathcal{B}(E, A, B, C, D) = \\ &= \{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{C}_{\rm imp}^{p+m} \mid \exists z \in \mathcal{C}_{\rm imp}^{n_2},\ x_0 \in \mathbb{R}^{n_1} \text{ s.t. } pEz = Az + Bu + x_0,\ y = Cz + Du \}. \end{aligned}$$

The conventional form is the following.

Definition 1.4 : For a matrix quintuple (E, A, B, C, D), the *conventionally associated impulsive-smooth behavior* $\mathcal{B}_c(E, A, B, C, D)$ is

$$\begin{aligned} &\mathcal{B}_c(E, A, B, C, D) = \\ &= \{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{C}_{\rm imp}^{p+m} \mid \exists z \in \mathcal{C}_{\rm imp}^{n_2},\ z_0 \in \mathbb{R}^{n_2} \text{ s.t. } pEz = Az + Bu + Ez_0,\ y = Cz + Du \}. \end{aligned}$$

As soon as we associate behaviors to matrix tuples, we obtain a notion of *equivalence* [15, 16]; we say that systems are *externally equivalent* if their associated behaviors are the same. Of course the notion depends on the behavior that is being associated, and so one must distinguish between external equivalence in the sense of smooth behaviors, external equivalence in the sense of conventionally associated impulsive-smooth behaviors, and external equivalence in the sense of unconventionally associated impulsive-smooth behaviors. In this paper, we shall consider the latter two equivalences for descriptor representations, as a follow-up to [1, 2] where a similar study was made for pencil representations. One of the main aims is to derive necessary and sufficient conditions for minimality of descriptor representations with respect to equivalence in the sense of impulsive-smooth behaviors. We will give a characterization of minimality in terms of the matrices E, A, B, C and D. Secondly, we will give the complete set of transformations by which minimal conventional (or unconventional) descriptor representations that give rise to the same behavior can be transformed into each other.

For the situation in which external equivalence is considered in the sense of *smooth* behaviors, the minimality conditions for both pencil and descriptor representations were given in [8] and [9]. Note that, in the case of smooth behaviors, the difference between conventional and unconventional representations becomes irrelevant because the equation $pGz = Fz + x_0$ does not give rise to smooth solutions unless $x_0 \in \operatorname{im} G$. In the case of of impulsive-smooth behaviors, the minimality conditions for both conventional and unconventional pencil representations were obtained in [2]. Of course it should be specified what is understood by minimality; a pencil representation (F, G, H) is said to be minimal if both the number of rows and the number of columns of the matrices F and G are minimal among the set of all triples equivalent to (F, G, H). For easy reference, we summarize the minimality and equivalence results of [2].

Theorem 1.5 [2, Thm. 4.2] *The following conditions are necessary and sufficient for a triple (F, G, H) to be a minimal representation of its unconventionally associated impulsive-smooth behavior $\mathcal{B}(F, G, H)$:*

(i) *$sG - F$ has full row rank as a rational matrix*

(ii) $\begin{bmatrix} G \\ H \end{bmatrix}$ *has full column rank*

(iii) $\begin{bmatrix} sG - F \\ H \end{bmatrix}$ *has full column rank for all $s \in \mathbb{C}$.*

Theorem 1.6 [2, Thm. 4.4] *A conventional representation (F, G, H) is minimal iff the matrices F, G and H satisfy the conditions (i)-(iii) of Theorem 1.5 and the additional condition*

(iv) $F[\ker G] \subset \operatorname{im} G$.

Theorem 1.7 [2, Thm. 4.1] *If the matrix triples* (F, G, H) *and* $(\tilde{F}, \tilde{G}, \tilde{H})$ *both satisfy the conditions (i)-(iii) of Theorem 1.5, then* $\mathcal{B}(F, G, H) = \mathcal{B}(\tilde{F}, \tilde{G}, \tilde{H})$ *if and only if there exist constant nonsingular matrices* S *and* T *such that* $F = S\tilde{F}T^{-1}$, $G = S\tilde{G}T^{-1}$ *and* $H = \tilde{H}T^{-1}$.

An equivalence result for conventional pencil representations was not given in [2]; this void is filled below (Thm. 4.6).

2. Representations of jump behaviors

In this section we present algorithms for the systems which have impulsive-smooth behaviors to obtain D representations from P representations and vice versa. These algorithms will be used in the next section where we derive results on the minimality of D representations by using the known results for P representations that were mentioned in the introduction.

In [8] it is shown that a close connection exists between a P representation and a D representation. An algorithm is given for rewriting a pencil representation in descriptor form in such a way that minimality is preserved. In [9] an algorithm with a similar property for rewriting a D representation in pencil form was presented and both algorithms were used for deriving minimality conditions for D representations of smooth behaviors.

To derive the algorithm for systems which have impulsive-smooth behavior, first we decompose the external variable space $\mathcal{W}$ into input space $\mathcal{U}$ and output space $\mathcal{Y}$ and denote P representation with behavior $\mathcal{B}(F, G, H)$ as follows:

$$\begin{array}{rcl} pGz &=& Fz + x_0 \\ y &=& H_y z \\ u &=& H_u z. \end{array} \tag{2.1}$$

If we have a D representation (E, A, B, C, D) with behavior $\mathcal{B}(E, A, B, C, D)$ or $\mathcal{B}_c(E, A, B, C, D)$, it is always possible to obtain a P representation via the simple transformations

$$G = \begin{bmatrix} E & 0 \end{bmatrix},\ F = \begin{bmatrix} A & B \end{bmatrix},\ H = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \tag{2.2}$$

In case (E, A, B, C, D) is minimal, P representation with behavior $\mathcal{B}(F, G, H)$ is minimal but P representation with behavior $\mathcal{B}_c(F, G, H)$ is not minimal. In the second case, we need to make reduction on the dimensions of both descriptor and equation space to obtain a minimal P representation.

Now, we state the following lemma which is essential for the proof of external equivalence of the systems which have impulsive-smooth behavior.

Lemma 2.1 *Let* $P(s) \in \mathbb{R}^{n\times m}[s]$ *and* $Q(s) \in \mathbb{R}^{q\times m}[s]$. *Consider*

$$\mathcal{B}(P, Q) := \{w \in \mathcal{C}^q_{\text{imp}} \mid \exists z \in \mathcal{C}^m_{\text{imp}},\ x_0 \in \mathbb{R}^n \text{ s.t. } x_0 = P(p)z,\ w = Q(p)z\}.$$

Moreover, assume that $P(s)$ *and* $Q(s)$ *have the following form (with respect to conformable partitionings):*

$$\begin{array}{rcl} P(s) &=& \begin{bmatrix} P_1(s) & P_2(s) \\ 0 & P_3(s) \end{bmatrix} \\ Q(s) &=& \begin{bmatrix} 0 & Q_2(s) \end{bmatrix} \end{array}$$

where $P_1(s)$ *has full row rank. Then,* $\mathcal{B}(P, Q) = \mathcal{B}(P_3, Q_2)$.

Proof It is immediately seen from the definitions that $\mathcal{B}(P, Q) \subset \mathcal{B}(P_3, Q_2)$. To show the converse, let $w \in \mathcal{B}(P_3, Q_2)$ so that there exists an impulsive-smooth z_2 and a constant x_{20} such that $w = Q_2(p)z_2$ and $P_3(p)z_2 = x_{20}$. Since $P_1(s)$ has full row rank, we can partition $P_1(s)$ as

$$P_1(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \end{bmatrix}$$

where $P_{11}(s)$ is nonsingular. Using the fact that the operator $P_{11}(p)$ (as a mapping between spaces of vector-valued impulsive-smooth functions) is invertible (cf. [5, 3, 1]), we can define

$$z_{11} = -{P_{11}}^{-1}(p)P_2(p)z_2, \quad z_1 = \begin{bmatrix} z_{11} \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ x_{20} \end{bmatrix}.$$

With these definitions, we have $P_1(p)z_1 + P_2(p)z_2 = x_0$ and it follows that $w \in \mathcal{B}(P,Q)$. □

In the following we consider the transformation from pencil to descriptor representations, both in the conventional and in the unconventional case. First, let us consider the P representation given by the equation (2.1). Decompose the internal variable space $\mathcal{Z}$ (the space on which F and G act) as $\mathcal{Z} = \mathcal{Z}_1 \bigoplus \mathcal{Z}_2 \bigoplus \mathcal{Z}_3$, where $\mathcal{Z}_2 = \ker G \cap \ker H_u$ and $\mathcal{Z}_2 \bigoplus \mathcal{Z}_3 = \ker G$. Accordingly, write

$$\begin{aligned} G &= \begin{bmatrix} G_1 & 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}, \\ H_y &= \begin{bmatrix} H_{y1} & H_{y2} & H_{y3} \end{bmatrix}, \; H_u = \begin{bmatrix} H_{u1} & 0 & H_{u3} \end{bmatrix} \end{aligned} \tag{2.3}$$

The matrices G_1 and H_{u3} both have full column rank.

Algorithm 2.2 Consider the behavior $\mathcal{B}(F, G, H_y, H_u)$. Assume that $H_u[\ker G] = \mathcal{U}$ and that the matrices F, G, H_y and H_u are of the form as in (2.3). The matrix H_{u3} is invertible (see the proof of Lemma 2.3 below). Define descriptor matrices by

$$E = \begin{bmatrix} G_1 & 0 \end{bmatrix}, \; A = \begin{bmatrix} \hat{F}_1 & F_2 \end{bmatrix}, \; B = \hat{F}_3, \; C = \begin{bmatrix} \hat{H}_{y1} & H_{y2} \end{bmatrix}, \; D = \hat{H}_{y3} \tag{2.4}$$

where

$$\hat{F}_1 = F_1 - F_3 H_{u3}^{-1} H_{u1}, \; \hat{F}_3 = F_3 H_{u3}^{-1}, \; \hat{H}_{y1} = H_{y1} - H_{y3} H_{u3}^{-1} H_{u1}, \; \hat{H}_{y3} = H_{y3} H_{u3}^{-1}. \tag{2.5}$$

Lemma 2.3 *Let (E, A, B, C, D) be a* D *representation with behavior $\mathcal{B}(E, A, B, C, D)$ that results from applying Algorithm 2.2 to a* P *representation with behavior $\mathcal{B}(F, G, H_y, H_u)$, where $H_u[\ker G] = \mathcal{U}$. Then these two representations are externally equivalent.*

Proof We note that the assumption $H_u[\ker G] = \mathcal{U}$ and the decomposition on the internal variable space imply $H_u[\ker G] = \operatorname{im}[0 \;\; H_{u3}] = \mathcal{U}$. Therefore, H_{u3} is nonsingular. Now, multiply F, G, H_y and H_u on the right by

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -H_{u3}^{-1} H_{u1} & 0 & H_{u3}^{-1} \end{bmatrix}. \tag{2.6}$$

Since the only operation that is involved in this algorithm is to choose another basis for the internal variable space, according to Theorem 1.7 we will obtain the following equivalent representation to the representation given in (2.1):

$$\begin{aligned} x_0 &= \begin{bmatrix} pG_1 - \hat{F}_1 & -F_2 & -\hat{F}_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} \hat{H}_{y1} & H_{y2} & \hat{H}_{y3} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \end{aligned} \tag{2.7}$$

Thus, $\mathcal{B}(E, A, B, C, D) = \mathcal{B}(F, G, H_y, H_u)$. □

ALGORITHM 2.4 Consider the behavior $\mathcal{B}_c(F,G,H_y,H_u)$. Assume that the matrices F, G, H_y and H_u are of the form as in (2.3). Then, by renumbering the u variables, we can write

$$H_{u1} = \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}, \quad H_{u3} = \begin{bmatrix} H_{13} \\ H_{23} \end{bmatrix} \tag{2.8}$$

where H_{23} is invertible (or empty, if $\ker G \subset \ker H_u$). Now, define descriptor matrices by

$$E = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \bar{F}_1 & F_2 \\ -\bar{H}_{11} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \bar{F}_3 & 0 \\ -\bar{H}_{13} & I \end{bmatrix}$$

$$C = \begin{bmatrix} \bar{H}_{y1} & H_{y2} \end{bmatrix}, \quad D = \begin{bmatrix} \bar{H}_{y3} & 0 \end{bmatrix} \tag{2.9}$$

where

$$\begin{aligned} \bar{F}_1 &= F_1 - F_3 H_{23}^{-1} H_{21} \\ \bar{F}_3 &= F_3 H_{23}^{-1} \\ \bar{H}_{y1} &= H_{y1} - H_{y3} H_{23}^{-1} H_{21} \\ \bar{H}_{11} &= H_{11} - H_{13} H_{23}^{-1} H_{21} \\ \bar{H}_{y3} &= H_{y3} H_{23}^{-1} \\ \bar{H}_{13} &= H_{13} H_{23}^{-1}. \end{aligned} \tag{2.10}$$

LEMMA 2.5 *Let (E,A,B,C,D) be a* D *representation with behavior $\mathcal{B}_c(E,A,B,C,D)$ that results from applying Algorithm 2.4 to a* P *representation with behavior $\mathcal{B}_c(F,G,H_y,H_u)$. Then, these two representations are externally equivalent.*

PROOF Let us consider the representation given in (2.1) with behavior $\mathcal{B}_c(F,G,H_y,H_u)$ and assume that F, G, H_y and H_u are given of the form as in equation (2.3). Then, there exists an initial condition z_0 such that $x_0 = Gz_0$. By taking $H_{u1} = H_{21}$ and $H_{u3} = H_{23}$ in T and multiplying F, G, H_y and H_u on the right by T we will obtain a conventional representation which is equivalent to the representation in 2.1 with initial condition Gz_0 and by Lemma 2.1 it is also equivalent to the representation given below:

$$\begin{bmatrix} Gz_0 \\ 0 \end{bmatrix} = \begin{bmatrix} pG_1 - \bar{F}_1 & -F_2 & -\bar{F}_3 & 0 \\ \bar{H}_{11} & 0 & \bar{H}_{13} & -I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ u_1 \end{bmatrix}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \bar{H}_{y1} & H_{y2} & \bar{H}_{y3} & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ u_1 \end{bmatrix}.$$

Thus, $\mathcal{B}_c(E,A,B,C,D) = \mathcal{B}_c(F,G,H_y,H_u)$. □

In the following we will present two algorithms for obtaining a P representation from a D representation for an impulsive-smooth behavior.

First, let us consider D representation. Decompose the descriptor space $\mathcal{X}_d$ (the space on which E and A act) as $\mathcal{X}_{d1} \bigoplus \mathcal{X}_{d2}$, where $\mathcal{X}_{d2} = \ker E$. Decompose the equation space $\mathcal{X}_e$ (the space that E and A map into) as $\mathcal{X}_{e1} \bigoplus \mathcal{X}_{e2} \bigoplus \mathcal{X}_{e3}$, where $\mathcal{X}_{e1} = \operatorname{im} E$ and $\mathcal{X}_{e1} \bigoplus \mathcal{X}_{e2} = \operatorname{im}[E \;\; B]$. Accordingly, write

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{2.11}$$

Since the matrix B_2 is surjective, by renumbering the u variables we can write

$$\begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix} \tag{2.12}$$

where B_{22} is invertible.

ALGORITHM 2.6 Consider the behavior $\mathcal{B}(E,A,B,C,D)$ and the matrices in (2.11) and (2.12). Define pencil matrices as follows:

$$\begin{aligned} F &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} & 0 \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} & -I \\ -A_{31} & -A_{32} & 0 & 0 \end{bmatrix}, \\ G &= \begin{bmatrix} E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ H = \begin{bmatrix} C_1 & C_2 & D_1 & D_2 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \bar{A}_{11} &= A_{11} - B_{12}B_{22}^{-1}A_{21} \\ \bar{A}_{12} &= A_{12} - B_{12}B_{22}^{-1}A_{22} \\ \bar{B}_{11} &= B_{11} - B_{12}B_{22}^{-1}B_{21} \\ \bar{A}_{21} &= B_{22}^{-1}A_{21} \\ \bar{A}_{22} &= B_{22}^{-1}A_{22} \\ \bar{B}_{21} &= B_{22}^{-1}B_{21}. \end{aligned} \tag{2.14}$$

LEMMA 2.7 *Let (F,G,H) be a* P *representation with behavior $\mathcal{B}(F,G,H)$ that results from applying Algorithm 2.6 to a* D *representation with behavior $\mathcal{B}(E,A,B,C,D)$. Then, these two representations are externally equivalent.*

PROOF Let us consider a D representation with behavior $\mathcal{B}(E,A,B,C,D)$ determined by the equations

$$pEz = Az + Bu + x_0 \tag{2.15}$$

$$y = Cz + Du. \tag{2.16}$$

Assume that descriptor matrices are given in the form as in the equations (2.11) and (2.12).Then, if we multiply the equation (2.15) on the left by

$$S = \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 \\ 0 & B_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \tag{2.17}$$

we will obtain an equivalent representation. If we define F, G and H as in (2.2) then the D representation can be regarded as a P representation (F,G,H) with behavior $\mathcal{B}(F,G,H)$, where F, G, and H are of the form given in (2.13). Thus, the D representation is externally equivalent to the P representation obtained by Algorithm 2.6. □

The following algorithm is similar to Algorithm 3.28 in [7].

ALGORITHM 2.8 Consider the behavior $\mathcal{B}_c(E,A,B,C,D)$ and the matrices in equations (2.11) and (2.12). Define pencil matrices as follows:

$$\begin{aligned} G &= \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} \\ A_{31} & A_{32} & 0 \end{bmatrix} \\ H_y &= \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \bar{D}_1 \end{bmatrix}, \quad H_u = \begin{bmatrix} 0 & 0 & I \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} \end{bmatrix} \end{aligned} \tag{2.18}$$

where

$$\bar{C}_1 = C_1 - D_2\bar{A}_{21}, \quad \bar{C}_2 = C_2 - D_2\bar{A}_{22}, \quad \bar{D}_1 = D_1 - D_2\bar{B}_{21} \tag{2.19}$$

and the other matrices are the same as in equations (2.15).

LEMMA 2.9 *Let (F, G, H) be a* P *representation with behavior $\mathcal{B}_c(F, G, H)$ that results from applying Algorithm 2.8 to a* D *representation with behavior $\mathcal{B}_c(E, A, B, C, D)$. Then, these two representations are externally equivalent.*

PROOF Consider a conventional D representation $\mathcal{B}_c(E, A, B, C, D)$. Then, for any $x_0 \in \mathrm{im}E$ there exists z_0 such that $x_0 = Ez_0$. Thus, if we follow the procedure given in the proof of the previous lemma we will obtain the representation below, which is externally equivalent to P representation with conventional behavior:

$$\begin{bmatrix} E_{11}z_{10} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} pE_{11} - \bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_{11} & 0 \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} & -I \\ -A_{31} & -A_{32} & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \\ u_2 \end{bmatrix} \tag{2.20}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & D_1 & D_2 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \\ u_2 \end{bmatrix}. \tag{2.21}$$

Here, z_{10}, z_1, z_2 and u_1, u_2 are obtained by a suitable partitioning of z_0, z, and u respectively. By Lemma 2.1 this representation is equivalent to the following representation:

$$\begin{bmatrix} E_{11}z_{10} \\ 0 \end{bmatrix} = \begin{bmatrix} pE_{11} - \bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_{11} \\ -A_{31} & -A_{32} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \end{bmatrix} \tag{2.22}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \bar{D}_1 \\ 0 & 0 & I \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \end{bmatrix} \tag{2.23}$$

Thus, $\mathcal{B}_c(E, A, B, C, D) = \mathcal{B}_c(F, G, H)$. □

Finally we consider descriptor representations with a zero feedthrough term. The following algorithm is similar to Algorithm 3.36 in [7].

ALGORITHM 2.10 Let a D representation be given by (E, A, B, C) (i.e. $D = 0$). Decompose the descriptor space X_d as $X_{d1} \bigoplus X_{d2} \bigoplus X_{d3}$, where $X_{d3} = A^{-1}[\mathrm{im}\, E] \cap \ker E$ and $X_{d2} \bigoplus X_{d3} = \ker E$. Decompose the equation space X_e as $X_{e1} \bigoplus X_{e2} \bigoplus X_{e3}$, where $X_{e1} = \mathrm{im}\, E$ and $X_{e2} = AX_{d2}$. Accordingly, write

$$\begin{bmatrix} E_{11}z_{01} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} pE_{11} - A_{11} & -A_{12} & -A_{13} \\ -A_{21} & -A_{22} & 0 \\ -A_{31} & 0 & 0 \end{bmatrix} z - \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u \tag{2.24}$$

$$y = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} z \tag{2.25}$$

where E_{11} and A_{22} are nonsingular. Since

$$C_2A_{22}^{-1}([\, -A_{21} \quad -A_{22} \quad 0\,]z - B_2u) = 0$$

we can write

$$y = \begin{bmatrix} C_1 - C_2A_{22}^{-1}A_{21} & 0 & C_3 \end{bmatrix} z - C_2A_{22}^{-1}B_2u.$$

Now, define D representation $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ with conventional behavior by

$$\tilde{E} = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} A_{11} & A_{13} \\ A_{31} & 0 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_1 \\ B_3 \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}, \ \tilde{D} = -C_2 A_{22}^{-1} B_2 \tag{2.26}$$

where $\tilde{C}_1 = C_1 - C_2 A_{22}^{-1} A_{21}$, $\tilde{C}_2 = C_3$. Since A_{22} is nonsingular and the rows of z corresponding to the columns of A_{22} do not affect the behavior, then by Lemma 2.1 it is clear that

$$\mathcal{B}_c(E,A,B,C) = \mathcal{B}_c(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$$

and

$$\tilde{A}[\ker \tilde{E}] \subset \operatorname{im} \tilde{E}.$$

3. Minimality of descriptor representations

In this section, we discuss the minimality of both conventional and unconventional descriptor representations. In descriptor systems there are three indices which play a role to determine the minimality: the rank of E, the *column defect* of E ($\dim \ker E$) and the *row defect* of E ($\operatorname{codim} \operatorname{im} E =$ the number of rows of E minus the rank of E). A *minimal* descriptor representation is by definition one in which each of these three indices is minimal within the set of descriptor representations for a given behavior.

In Lemma 2.3 above, we carried out the transition from unconventional P to unconventional D representation under the condition $H_u[\ker G] = \mathcal{U}$. It will be important below that this property is preserved under a certain transformation as shown in the next lemma.

Lemma 3.1 *Let $G : \mathcal{Z} \to \mathcal{X}$ be of the form*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$$

with G_2 full column rank and consider $H_u = [H_{u1} \quad H_{u2}] : \mathcal{Z} \to \mathcal{U}$. If $H_u[\ker G] = \mathcal{U}$, then also $H_{u1}[\ker G_1] = \mathcal{U}$.

Proof Take $u \in \mathcal{U}$, then we may write

$$u = [H_{u1} \ H_{u2}] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \text{ with } \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.$$

From $G_2 z_2 = 0$ it follows that $z_2 = 0$, so actually $u = H_{u1} z_1$ with $G_1 z_1 = 0$, i.e. $u \in H_{u1}[\ker G_1]$. □

Lemma 3.2 *The following condition is necessary for minimality of a descriptor representation (E,A,B,C,D) with behavior $\mathcal{B}(E,A,B,C,D)$:*

(i) $[\ sE - A \quad -B\ ]$ has full row rank as a rational matrix.

Proof Define a pencil representation as in (2.2). Note that $H_u[\ker G] = \mathcal{U}$. Let E and A have size $n_1 \times n_2$, $\dim \mathcal{Y} = p$ and $\dim \mathcal{U} = m$. Then, G has size $n_1 \times (n_2 + m)$. If condition (i) does not hold, then $sG - F$ will not have full row rank as a rational matrix and hence a reduction is possible as in [2] (proof of Thm. 2.3) to a representation of size $\tilde{n}_1 \times (\tilde{n}_2 + m)$ with $\tilde{n}_1 < n_1$ and $\tilde{n}_2 \le n_2$. By Lemma 3.1, we still have $\tilde{H}_u[\ker \tilde{G}] = \mathcal{U}$ in the reduced representation. By Lemma 2.3, we can therefore find a descriptor representation of size $\tilde{n}_1 \times \tilde{n}_2$. Because $\tilde{n}_1 < n_1$ and $\tilde{n}_2 \le n_2$, the representation that we started with is not minimal. □

We now obtain the minimality conditions for unconventional descriptor representations.

Theorem 3.3 *A D representation (E,A,B,C,D) is a minimal representation of its unconventionally associated behavior $\mathcal{B}(E,A,B,C,D)$ if and only if the following conditions hold:*

(i) $\begin{bmatrix} sE-A & -B \end{bmatrix}$ *has full row rank as a rational matrix*

(ii) $\begin{bmatrix} E \\ C \end{bmatrix}$ *has full column rank*

(iii) $\begin{bmatrix} sE-A \\ C \end{bmatrix}$ *has full column rank for all* $s \in \mathbb{C}$.

Proof The necessity of condition (i) has already been shown in Lemma 3.2. The other conditions are shown to be necessary exactly as in the case of smooth behaviors (see [8] (proof of Lemma 4.7) for (ii) and [7] (proof of Thm. 4.12) for (iii)), by using the property given in Lemma 2.1. To prove the sufficiency, suppose that (E,A,B,C,D) satisfies (i)-(iii). Then, it is readily verified on the basis of Thm. 1.5 that the associated pencil representation defined by the equations in (2.2) is minimal. Hence, there can be no smaller descriptor representation of the same behavior. □

The analogous result for conventional representations is the following.

Theorem 3.4 *A D representation (E,A,B,C,D) is a minimal representation of its conventionally associated behavior $\mathcal{B}_c(E,A,B,C,D)$ if and only if the conditions (i)-(iii) of Lemma 3.2 and the additional condition*

(iv) $A[\ker E] \subset \operatorname{im} E$

are satisfied.

Proof The proofs of the conditions (i)-(iii) are similar to the proofs of the same conditions in Lemma 3.3, with the initial condition being taken as $x_0 = Ez_0$ since we now consider the conventional behavior $\mathcal{B}_c(E,A,B,C,D)$. To prove the necessity of (iv), apply Algorithm 2.10 to (E,A,B,C,D). Then, we have $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ as in (2.26) except that $\tilde{D} = D - C_2A_{22}^{-1}B_2$. Because of the equality between $\mathcal{B}_c(E,A,B,C,D)$ and $\mathcal{B}_c(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ and the minimality of (E,A,B,C,D), the matrix A_{22}, which is given in the equation (2.25) in Algorithm 2.10, should be empty. Thus, (iv) holds. (Compare the argument in the proof of [7, Lemma 4.8].)

To prove sufficiency suppose that (E,A,B,C,D) satisfies (i)-(iv). It can be verified that, when Algorithm 2.8 is applied to (E,A,B,C,D), the resulting pencil representation (F,G,H_y,H_u) defined by the matrices in 2.18 satisfies the conditions (i)-(iv) of Theorem 1.6. Note in particular that the condition (iv) implies $A_{32}=0$ and the condition (i) implies that A_{31} in F (in (2.18)) has full row rank. So, by Theorem 1.6 (F,G,H_y,H_u) is minimal with respect to the behavior $\mathcal{B}_c(F,G,H_y,H_u)$. Hence, there can be no smaller conventional descriptor representation having the same behavior. □

The result above can be compared to the minimality conditions for descriptor representations of smooth behaviors as given in [9]. The two sets of minimality conditions are identical, except that condition (i) above is replaced by the stronger requirement that the matrix $[E \quad B]$ should have full row rank for minimality in the sense of smooth behaviors.

4. Equivalence of descriptor representations

Definition 4.1 The triples (F,G,H) and $(\tilde{F},\tilde{G},\tilde{H})$ are said to be *strongly similar* if there exist invertible matrices S and T such that

$$\begin{bmatrix} s\tilde{G}-\tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sG-F \\ H \end{bmatrix} T^{-1}. \tag{4.1}$$

Definition 4.2 The triples (F,G,H) and $(\tilde F,\tilde G,\tilde H)$ are said to be *weakly similar* if there exist constant invertible matrices S and T and a constant matrix X such that

$$\begin{bmatrix} s\tilde G-\tilde F \\ \tilde H \end{bmatrix} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG-F \\ H \end{bmatrix} T^{-1}. \tag{4.2}$$

The condition (4.2) is equivalent to the requirements $\tilde F = SFT^{-1}$, $\tilde G = SGT^{-1}$, $\tilde H = (H-XF)T^{-1}$ and $XG=0$. It is straightforward to prove the following lemma.

Lemma 4.3 *Among matrix triples of equal dimensions, weak similarity is an equivalence relation.*

It is also easy to verify that the minimality conditions of Thm. 1.5 and Thm. 1.6 are similarity invariants, i. e. if a triple (F,G,H) satisfies the conditions of these theorems than the same holds for any triple that is weakly similar to (F,G,H). The following lemma takes a little bit more effort.

Lemma 4.4 *Weakly similar triples generate the same conventional behavior.*

Proof Assume that the triples (F,G,H) and $(\tilde F,\tilde G,\tilde H)$ are weakly similar. Then, there exist constant invertible matrices S and T and a constant matrix X such that (4.2) holds. Take $w \in \mathcal{B}_c(F,G,H)$. Then by definition there exist $z \in \mathcal{C}_{\rm imp}^{n+k}$ and $z_0 \in \mathbb{R}^{n+k}$ such that

$$\begin{bmatrix} Gz_0 \\ w \end{bmatrix} = \begin{bmatrix} pG-F \\ H \end{bmatrix} z. \tag{4.3}$$

Define $\tilde z = Tz$ and $\tilde z_0 = Tz_0$. Since $XG=0$, we can then write

$$\begin{aligned} \begin{bmatrix} p\tilde G-\tilde F \\ \tilde H \end{bmatrix} \tilde z &= \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} pG-F \\ H \end{bmatrix} T^{-1}\tilde z = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} Gz_0 \\ w \end{bmatrix} = \\ &= \begin{bmatrix} SGT^{-1}\tilde z_0 \\ w \end{bmatrix} = \begin{bmatrix} \tilde G\tilde z_0 \\ w \end{bmatrix}. \end{aligned} \tag{4.4}$$

It follows that $w \in \mathcal{B}_c(\tilde F,\tilde G,\tilde H)$. So we have $\mathcal{B}_c(F,G,H) \subset \mathcal{B}_c(\tilde F,\tilde G,\tilde H)$; since weak similarity is an equivalence relation, it follows that actually equality must hold. □

The following lemma, which relates conventional pencil representations to unconventional ones, will be of use below.

Lemma 4.5 *If a triple (F,G,H) satisfies the minimality conditions (i)-(iv) mentioned in Thm. 1.6, then (F,G,H) is weakly similar to a triple $(\hat F,\hat G,\hat H)$, where*

$$\hat F = \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix}, \quad \hat G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}, \quad \hat H = [\, H_1 \;\; 0 \,] \tag{4.5}$$

in which $[G_{11} \;\; G_{12}]$ has full row rank and

$$\text{no. of columns of } G_{12} \;=\; \operatorname{codim} \operatorname{im} G_{11}. \tag{4.6}$$

Moreover, the triple (F_{11},G_{11},H_1) satisfies the minimality conditions (i)-(iii) of Thm. 1.5, and we have

$$\mathcal{B}_c(F,G,H) = \mathcal{B}(F_{11},G_{11},H_1). \tag{4.7}$$

Proof Let U be a constant nonsingular matrix such that

$$UG \;=\; \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \tag{4.8}$$

where G_1 has full row rank, and define F_1 and F_2 by the comformable partitioning

$$UF = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{4.9}$$

By minimality condition (i), the matrix F_2 must have full row rank. Then there exists a constant nonsingular matrix V such that

$$UFV = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \tag{4.10}$$

where F_{22} is nonsingular. Let

$$G_1V = [G_{11} \;\; G_{12}], \quad HV = [H_1 \;\; H_2] \tag{4.11}$$

with partitionings corresponding to those in (4.10). Since F_{22} is nonsingular, we can write down the following equation:

$$\begin{bmatrix} I & -F_{12}F_{22}^{-1} & 0 \\ 0 & F_{22}^{-1} & 0 \\ 0 & H_2F_{22}^{-1} & I \end{bmatrix} \begin{bmatrix} sG_{11} - F_{11} & sG_{12} - F_{12} \\ 0 & -F_{22} \\ H_1 & H_2 \end{bmatrix} = \begin{bmatrix} sG_{11} - F_{11} & sG_{12} \\ 0 & -I \\ H_1 & 0 \end{bmatrix}. \tag{4.12}$$

By defining

$$\begin{bmatrix} 0 & -F_{12}F_{22}^{-1} \\ 0 & F_{22}^{-1} \end{bmatrix} U =: S, \quad [\,0 \quad H_2F_{22}^{-1}]U =: X, \quad V^{-1} =: T \tag{4.13}$$

we will obtain

$$\begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1} = \begin{bmatrix} s\hat{G} - \hat{F} \\ \hat{H} \end{bmatrix}. \tag{4.14}$$

Thus, (F, G, H) and $(\hat{F}, \hat{G}, \hat{H})$ are weakly similar.

Because G_1 has full row rank, the claim (4.6) is equivalent to saying that $\operatorname{im} G_1$ is the direct sum of $\operatorname{im} G_{11}$ and $\operatorname{im} G_{12}$, and that the columns of G_{12} are independent. This in turn is the same as saying that a vector z_2 satisfies $G_{12}z_2 \in \operatorname{im} G_{11}$ if and only if $z_2 = 0$. So, let us assume that z_2 is such that $G_{12}z_2 \in \operatorname{im} G_{11}$. Then there exists z_1 such that $G_{11}z_1 + G_{12}z_2 = 0$, i.e. $z := [z_1^T \;\; z_2^T]^T$ belongs to $\ker \hat{G}$. By condition (iv) and weak similarity, we have $\hat{F}[\ker \hat{G}] \subset \operatorname{im} \hat{G}$. By the conformable partitionings of $\hat{F}$ and $\hat{G}$, the relation

$$\hat{F} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} F_{11}z_1 \\ z_2 \end{bmatrix} \in \operatorname{im} \hat{G}$$

implies $z_2 = 0$.

Due to the special structure of the matrices in (4.5), it is straightforward to verify that the triple (F_{11}, G_{11}, H_1) satisfies the minimality conditions for unconventional pencil representations. To prove the final claim, let $w \in \mathcal{B}_c(F, G, H)$; then there exist $z_0 \in \mathbb{R}^{n+k}$ and $z \in \mathcal{C}_{\mathrm{imp}}^{n+k}$ such that $pGz = Fz + Gz_0$, $w = Hz$. From the equation (4.5) we obtain

$$\begin{bmatrix} G_{11}z_{10} + G_{12}z_{20} \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} pG_{11} - F_{11} & pG_{12} \\ 0 & -I \\ H_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{4.15}$$

where

$$V^{-1}z_0 =: \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}, \quad V^{-1}z =: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

It is clear from the equation above that $z_2 = 0$ and $w = H_1 z_1$. Then, if we let $x_0 = G_{11}z_{10} + G_{12}z_{20}$ by Lemma 2.1 we have $w \in \mathcal{B}(F_{11}, G_{11}, H_1)$. Conversely, since $[G_{11} \;\; G_{12}]$ has full row rank, for given x_0 it is always possible to find z_{10} and z_{20} such that $x_0 = G_{11}z_{10} + G_{12}z_{20}$ and (4.15) holds (setting $z_2 = 0$). Consequently, $w \in \mathcal{B}(F_{11}, G_{11}, H_1)$ implies $w \in \mathcal{B}_c(F, G, H)$. □

The following theorem completes the results in [2] on equivalence of minimal pencil representations of impulsive-smooth behaviors.

THEOREM 4.6 *Suppose (F,G,H) and $(\tilde{F},\tilde{G},\tilde{H})$ both satisfy the conditions (i)-(iv) of Theorem 1.6. Then, $\mathcal{B}_c(F,G,H) = \mathcal{B}_c(\tilde{F},\tilde{G},\tilde{H})$ iff (F,G,H) and $(\tilde{F},\tilde{G},\tilde{H})$ are weakly similar.*

PROOF The 'if' part has already been proved in Lemma 4.4. So, let us prove the 'only if' part. By the lemmas 4.4 and 4.5 and by Thm. 1.5, we may assume without loss of generality that

$$F = \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix} = \tilde{F}, \quad H = \begin{bmatrix} H_1 & 0 \end{bmatrix} = \tilde{H},$$

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G_{11} & \tilde{G}_{12} \\ 0 & 0 \end{bmatrix}. \tag{4.16}$$

Because $[G_{11} \;\; \tilde{G}_{12}]$ has full row rank, we can write $G_{12} = G_{11}T_{12} + \tilde{G}_{12}T_{22}$ for certain matrices T_{12} and T_{22}, where T_{22} must be square (by property (4.6)). Suppose $T_{22}z_2 = 0$; then $G_{12}z_2 = G_{11}T_{12}z_2$ and it follows from (4.6) that $z_2 = 0$. So T_{22} must be invertible. Now note that

$$\tilde{F} = \begin{bmatrix} I & F_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1} \tag{4.17}$$

$$\tilde{G} = \begin{bmatrix} I & F_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1} \tag{4.18}$$

$$\tilde{H} = (\begin{bmatrix} H_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & H_1T_{12} \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix}) \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1} \tag{4.19}$$

If we let

$$\begin{bmatrix} I & F_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} =: S \;, \; \begin{bmatrix} 0 & H_1T_{12} \end{bmatrix} =: X \;\; \text{and} \;\; \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix} =: T \tag{4.20}$$

then

$$\begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1}. \tag{4.21}$$

□

We can now characterize the relations between minimal unconventional descriptor representations.

THEOREM 4.7 *Let (E,A,B,C,D) and $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ be* D *representations. Assume that both of them satisfy the conditions (i)-(iii) of Thm. 3.3. Then*

$$\mathcal{B}(E,A,B,C,D) = \mathcal{B}(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \tag{4.22}$$

if and only if there exist constant nonsingular matrices M and N and a constant matrix Y such that

$$\begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} s\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \tag{4.23}$$

Proof To prove the 'if' part let $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}(E,A,B,C,D)$. Then, there exist $z \in \mathcal{C}^{n_2}_{\text{imp}}$ and $x_0 \in \mathbb{R}^{n_1}$ such that

$$\begin{bmatrix} x_0 \\ y \end{bmatrix} = \begin{bmatrix} pE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \tag{4.24}$$

It follows that

$$\begin{bmatrix} x_0 \\ y \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \tag{4.25}$$

and so $w \in \mathcal{B}(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. The reverse inclusion follows in the same way and so we have (4.22).

To prove the 'only if' part let us assume that $\mathcal{B}(E,A,B,C,D) = \mathcal{B}(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. By means of (2.2), let us define (F,G,H) from (E,A,B,C,D) and $(\tilde{F},\tilde{G},\tilde{H})$ from $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. Since both (E,A,B,C,D) and $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ satisfy the conditions (i)-(iii) of Thm. 3.3, both (F,G,H) and $\tilde{F},\tilde{G},\tilde{H})$ satisfy the conditions (i)-(iii) of Theorem 1.5. Thus, both (F,G,H) and $(\tilde{F},\tilde{G},\tilde{H})$ are minimal in the sense of unconventionally associated behaviors $\mathcal{B}(F,G,H)$ and $\mathcal{B}(\tilde{F},\tilde{G},\tilde{H})$, and also the following properties in behaviors hold:

$$\mathcal{B}(F,G,H) = \mathcal{B}(E,A,B,C,D) = \mathcal{B}(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D}) = \mathcal{B}(\tilde{F},\tilde{G},\tilde{H}). \tag{4.26}$$

So, by Theorem 1.7 there exist constant nonsingular matrices S and T such that

$$F = S\tilde{F}T^{-1}\ ,\ G = S\tilde{G}T^{-1} \ \text{ and } \ H = \tilde{H}T^{-1} \tag{4.27}$$

so that, by (2.2),

$$\begin{bmatrix} sE - A & -B \\ C & D \\ 0 & I \end{bmatrix} = \begin{bmatrix} S(s\tilde{E} - \tilde{A}) & -S\tilde{B} \\ \tilde{C} & \tilde{D} \\ 0 & I \end{bmatrix} T^{-1}. \tag{4.28}$$

Now, let

$$T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{4.29}$$

Then (4.28) and (4.29) imply

$$T_{21} = 0, \quad T_{22} = I. \tag{4.30}$$

Since T is nonsingular, T_{11} is nonsingular and we can define

$$T_{11} =: N, \quad T_{12} =: Y, \quad S =: M \tag{4.31}$$

to satisfy (4.23). □

The analogous result for conventional representations is the following.

Theorem 4.8 *Let (E,A,B,C,D) and $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ be* D *representations. Assume that both of them satify the conditions (i)-(iv) of Theorem 3.4. Then*

$$\mathcal{B}_c(E,A,B,C,D) = \mathcal{B}_c(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \tag{4.32}$$

if and only if there exist constant and nonsingular matrices S, T and constant matrices X and Y such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}. \tag{4.33}$$

PROOF For the 'if' part, take $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}_c(E,A,B,C,D)$. By definition, there exist a constant z_0 and an impulsive-smooth z such that

$$\begin{bmatrix} Ez_0 \\ y \end{bmatrix} = \begin{bmatrix} pE-A & -B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \tag{4.34}$$

Note that (4.33) implies that $ME = \tilde{E}N$ and $XE = 0$. Therefore, it follows from (4.33) that

$$\begin{bmatrix} \tilde{E}Nz_0 \\ y \end{bmatrix} = \begin{bmatrix} p\tilde{E}-\tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} Nz+Yu \\ u \end{bmatrix} \tag{4.35}$$

so that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}_c(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. The argument is completed as in the proof of the previous theorem.

In order to prove the 'only if' part, let us assume that (E,A,B,C,D) and $(\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ satisfy the conditions (i)-(iv) of Thm. 3.4 and their conventionally associated behaviors are the same. Next, apply Algorithm 2.8 to both of them; this yields conventionally externally equivalent pencil representations (F,G,H) and $(\tilde{F},\tilde{G},\tilde{H})$ which are minimal. Then by Theorem 4.6 they are weakly similar and so there exist constant invertible matrices S and T and a constant matrix X such that

$$\begin{bmatrix} s\tilde{G}-\tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG-F \\ H \end{bmatrix} T^{-1}. \tag{4.36}$$

We may assume that both descriptor representations are in the form (2.11)-(2.12) with $E_{11} = I$ and $B_{22} = -I$. Then (4.36) may be written in further detail as

$$\begin{bmatrix} S_1 & S_2 & 0 & 0 & 0 \\ S_3 & S_4 & 0 & 0 & 0 \\ X_1 & X_2 & I & 0 & 0 \\ X_3 & X_4 & 0 & I & 0 \\ X_5 & X_6 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} sI-A_{11} & -A_{12} & 0 \\ -A_{31} & 0 & 0 \\ C_1+D_2A_{21} & C_2 & D_1+D_2B_{21} \\ 0 & 0 & I \\ A_{21} & 0 & B_{21} \end{bmatrix} =$$

$$= \begin{bmatrix} sI-\tilde{A}_{11} & -\tilde{A}_{12} & 0 \\ -\tilde{A}_{31} & 0 & 0 \\ \tilde{C}_1+\tilde{D}_2\tilde{A}_{21} & \tilde{C}_2 & \tilde{D}_1+\tilde{D}_2\tilde{B}_{21} \\ 0 & 0 & I \\ \tilde{A}_{21} & 0 & \tilde{B}_{21} \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{bmatrix}.$$

It now follows immediately that the matrices S_3, X_1, X_3, X_5, T_2, T_3, T_8 must all be zero matrices, and $T_9 = I$. Then since S and T are nonsingular S_1, S_4, T_1, and T_5 are nonsingular. Also note that $T_7 = -X_4A_{31}$. After tedious but in principle straightforward calculations, it can be verified that (4.33) is satisfied with

$$M := \begin{bmatrix} S_1 & S_1B_{12}-\tilde{B}_{12} & S_2-\tilde{B}_{12}X_6+\tilde{B}_{11}X_4 \\ 0 & I & -X_6+\tilde{B}_{21}X_4 \\ 0 & 0 & S_4 \end{bmatrix}, \quad N := \begin{bmatrix} T_1 & 0 \\ T_4 & T_5 \end{bmatrix},$$

$$X := \begin{bmatrix} 0 & -D_2+\tilde{D}_2 & X_2-\tilde{D}_2X_6-\tilde{D}_1X_4 \end{bmatrix}, \quad Y := \begin{bmatrix} 0 & 0 \\ T_6 & 0 \end{bmatrix}. \tag{4.37}$$

□

The equivalence relation (4.33) is well-known in the literature; it was introduced by Verghese *et al.* [14] under the name of *strong equivalence operation*. The same transformation group was used earlier for descriptor representations with zero feedthrough term by Van der Weiden and Bosgra [13], who used the name *restricted system equivalence*. Compare also Rosenbrock's notion of *strict system equivalence*

[11, p. 52] which uses polynomial matrices in a format similar to (4.33). The above theorem provides a motivation in terms of impulsive-smooth behaviors for the notion of strong equivalence.

The first to find a motivation for strong equivalence from a more intrinsically defined notion of equivalence was Grimm [4]. The minimality conditions used by Grimm were the same as the ones mentioned in Thm. 3.4 above, except that the requirement (i) is replaced in his paper by the stronger condition that the matrix $[\,sE - tA \;\; -B\,]$ should have full row rank for *all* pairs of complex numbers $(s,t) \neq (0,0)$. This condition can be interpreted as a controllability condition. A weaker notion of minimality (so one that applies to a wider class of systems) was used by Kuijper and Schumacher [9]; they used external equivalence in the sense of smooth behaviors, which leads to the minimality conditions (i)-(iv) of Thm. 3.4 with condition (i) replaced by the requirement that $[\,E \;\; B\,]$ should have full row rank. This requirement can be interpreted as a condition of 'controllability at infinity'. The operations relating minimal representations (in the sense of smooth behaviors) to each other were again identified in [9] as the operations of strong equivalence. The result above gives an interpretation of strong equivalence that goes even further, since it applies to systems satisfying the conditions of Thm. 3.4 as such; note that the condition that $[sE - A \;\; -B]$ should have full row rank as a rational matrix is equivalent to requiring that the matrix $[sE - tA \;\; -B]$ should have full row rank for *some* pair of complex numbers (s,t). The condition (i) as given in Thm. 3.4 is no longer a controllability condition but rather a nonredundancy condition, as it requires that none of the equations given by the rows of $pEz = Az + Bu + Ez_0$ should be obtainable from the other equations by differentiating and taking linear combinations.

5. Conclusions

In this paper we have discussed minimality and equivalence of descriptor representations for impulsive-smooth behaviors. As can be expected, the minimality conditions are weaker than those for descriptor representations of smooth behaviors; in particular, no form of controllability is required for minimality in the sense of impulsive-smooth behaviors. In the case of conventional representations, minimal representations turn out to be related by operations of strong equivalence as defined in [14]. We have therefore given a motivation for strong equivalence that applies to a wider class of systems than the classes considered earlier in [4] and [9]. The operations that we found for minimal unconventional representations have to our knowledge not been considered before. We have also identified the transformation group that describes the relations between minimal conventional pencil representations of impulsive-smooth behaviors, thus completing the results in [2].

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