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Control for a Class of Hybrid Systems

J.H. van Schuppen

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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Jan H. van Schuppen

CWI

P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

J.H.van.Schuppen@cwi.nl

ABSTRACT

A hybrid control system is a control theoretic model for a computer controlled engineering system. A definition of a hybrid control system is formulated that consists of a product of a finite state automaton and of a family of continuous control systems. An example of a transportation system consisting of a line of conveyor belts is used as a running example. The realization problem for this class of systems is discussed. Control synthesis of hybrid systems is in a first approach based on supervisory control of discrete event systems.

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1 Introduction

The purpose of this paper is to present problems, concepts, and theory for control of hybrid systems.

The motivation of the author for the study of hybrid systems is the use of computers for control of engineering systems. In the past control synthesis focused exclusively on control systems with a continuous state space. More recently, control of discrete event systems has been the subject of investigation. For control design it is in general no longer possible to separate the design at the discrete and at the continuous level. Hence the interest in hybrid systems in which these levels are combined. The motivation directs attention to a class of hybrid systems in which a computer science model is combined with a model for continuous control systems. The motivation is *not* the treatment of a continuous control system in which the switches between different modes are generated only by the continuous-time system.

The contribution of this paper is control synthesis for a class of hybrid systems. A definition of a hybrid control system is formulated that consists of a product of a finite-state automaton and of a family of continuous control systems. As example a model is treated of a transportation line that consists of a line of conveyor belts. For control synthesis of hybrid systems attention is restricted to a class of systems. A sufficient condition for the existence of a hybrid controller is presented that is based on supervisory control theory. The

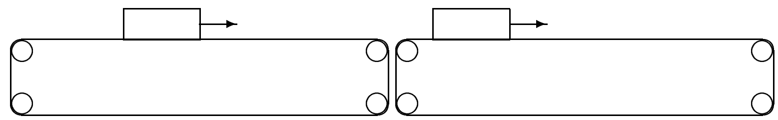

Figure 1: Two conveyors belts with belts of the transportation system.

sufficient conditions involve the checking of reachabilitiy and of controllability conditions. A controller for the hybrid control system of the conveyor belts is presented. The reachability problem for subclasses of hybrid control systems is discussed.

The novelty of this paper is in the approach to control synthesis, in the remarks on modeling and realization of hybrid systems, and in the example of the conveyor belts.

An overview of the contents follows. In Section 2 the example of a transportation system is described. A definition of a hybrid control system is presented in Section 3 together with the realization problem. In Section 4 an approach to control synthesis is formulated and illustrated for the example. Concluding remarks are stated in Section 5.

2 Example of conveyor belts

Example 2.1 *Conveyor system.* Consider a transportation system that consists of a line of conveyor belts. The purpose of the transportation system is to transport trays with products. The model was developed in the research group of J.E. Rooda at the Department of Mechanical Engineering of the Eindhoven University of Technology for a transportation system in a bike factory. A description may be found in [8].

Figure 1 shows a model of two conveyor belts. Each belt is driven by an electromotor of which the torque can be controlled. It is assumed that there is no friction in the belt system. The inertia of the mass of the conveyors and motors will be neglected. A conveyor has length l. A tray is assumed to have length a which is approximately $l/4$. This is a modeling assumption that may be relaxed. The trays carry batches of various products or parts. It is assumed that the belt which carries the major part of a tray determines the speed of the tray. The power supply to the motor determines the torque and hence the velocity of the belt and the tray. As a simplification, it is assumed that the controller sets the torque of the motor directly. It is assumed that the torque can be varied continuously from the value $-T_{max}$ to the value T_{max}. The resulting acceleration or deceleration depends on the torque applied and on the mass of the trays present on the belt.

A description of the model in the form of a hybrid control system follows. The concept of a hybrid system is formally defined in the next section. Partition every belt midway in a front end and a back end, thus both ends are of equal length. A model with different lengths for the front and back end is left for a future extension. Consider a discrete state

Q	i	j	k	
q_1	0	0	1	No tray present, down stream environment ready to accept a tray
q_2	1	0	1	Tray at front end, down stream is ready to accept
q_3	0	1	1	Tray at back end, down stream is ready to accept
q_4	1	1	1	Tray at front and at back end, down stream is ready to accept
q_5	0	0	0	No tray present, down stream not ready to accept a tray
q_6	1	0	0	Tray at front end, down stream is not ready to accept
q_7	0	1	0	Tray at back end, down stream is not ready to accept
q_8	1	1	0	Tray at front and back end, down stream not ready to accept

Table 1: Description of discrete states.

σ_{ar}	Arrival of tray at front end of the belt
σ_{mid}	Arrival of tray at mid point of the belt
σ_{dep}	Arrival of tray at end of belt
σ_{d1}	Message arrives that down stream environment can accept a tray
σ_{d0}	Message arrives that down stream environment cannot accept a tray
σ_{up0}	Message is sent to up stream environment that belt cannot accept a tray
σ_{up1}	Message is sent to up stream environment that belt is ready to accept a tray

Table 2: List of events.

set

$$Q = \{q_1, q_2, \dots, q_8\}.$$

Here each discrete state is represented by three variables, $q = (i, j, k) \in \{0, 1\}^3$, where $i = 1$ represents that a tray is present at the front end of the belt and $i = 0$ that it is not, where j represents in a similar way whether or not a tray is present at the back end of the belt, and $k = 1$ represents that, according to the latest information, the down stream environment can accept a tray and $k = 0$ that it cannot. The meaning of the discrete states is summarized in the Table 1.

Let the initial state be q_1. The possible events are listed in the Table 2. Let

$$\Sigma_{in} = \emptyset, \quad \Sigma_{env} = \{\sigma_{ar}, \sigma_{d0}, \sigma_{d1}\}, \quad \Sigma_{cd} = \{\sigma_{mid}, \sigma_{dep}\},$$
$$\Sigma_{int} = \{\sigma_{up0}, \sigma_{up1}\}, \quad \Sigma_{out} = \Sigma = \Sigma_{env} \cup \Sigma_{cd} \cup \Sigma_{int}.$$

The discrete transitions are described in Table 3. The first entry in Table 3 should be read as: If at time t the system is at discrete state q_1 and at continuous state x_{q_1}, if at that time event σ_{ar} occurs, then the system moves to discrete state q_2 and continuous state x_{q_2} where the first component of x_{q_2}, denoted by $x_{q_2,1}$, is set to the value 0 and the second component is set to the value $x_{q_1,2}$. In Table 3 the transitions from the discrete states q_5, q_6, q_7, q_8 have been omitted because they correspond in an obvious way to those displayed above. The environmental events from the set Σ_{env} can occur at all states. Thus the event σ_{ar}, that signals the arrival of a new tray at the belt, is allowed at all states. Control of the belt will prevent these events from happening at the discrete states q_2 and q_4. Therefore these transitions have been omitted from Table 3. The discrete event system associated with the hybrid control system is partly displayed in Figure 2.

$$
\begin{array}{ll}
q_2 = \delta(q_1, \sigma_{ar}), & x_{q_2} = r(q_1, q_2, x_{q_1}, \sigma_{ar}) = (0, x_{q_1,2}), \\
q_5 = \delta(q_1, \sigma_{d0}), & x_{q_5} = x_{q_1}, \\
q_1 = \delta(q_1, \sigma_{d1}), & \bar{x}_{q_1} = x_{q_1}, \\
q_3 = \delta(q_2, \sigma_{mid}), & x_{q_3} = (l/2, x_{q_2,2}), \\
q_6 = \delta(q_2, \sigma_{d0}), & x_{q_6} = x_{q_2}, \\
q_2 = \delta(q_2, \sigma_{d1}), & \bar{x}_{q_2} = x_{q_2}, \\
q_4 = \delta(q_3, \sigma_{ar}), & x_{q_4} = (x_{q_3,1}, x_{q_3,2}, 0), \\
q_1 = \delta(q_3, \sigma_{dep}), & x_{q_1} = (0, x_{q_3,2}), \\
q_7 = \delta(q_3, \sigma_{d0}), & x_{q_7} = x_{q_3}, \\
q_3 = \delta(q_3, \sigma_{d1}), & \bar{x}_{q_3} = x_{q_3}, \\
q_2 = \delta(q_4, \sigma_{dep}), & x_{q_2} = (x_{q_4,3}, x_{q_4,2}), \\
q_8 = \delta(q_4, \sigma_{d0}), & x_{q_8} = x_{q_4}, \\
q_4 = \delta(q_4, \sigma_{d1}), & \bar{x}_{q_4} = x_{q_4}.
\end{array}
$$

Table 3: Discrete transitions and resets.

A model of a line consisting of two conveyor belts is described next. Each conveyor belt is described by a hybrid control system as specified above. It will be assumed that

$$\sigma_{ar}(i) = \sigma_{up0}(i) = \sigma_{d0}(i-1), \quad \sigma_{mid}(i) = \sigma_{up1}(i) = \sigma_{d1}(i-1).$$

Thus when a tray has arrived at the front end of belt i, event $\sigma_{ar}(i)$ occurs, then that belt informs the upstream environment that it temporarily cannot accept new trays and the events $\sigma_{up0}(i) = \sigma_{d0}(i-1)$ occur simultaneously. When a tray moves from the front end to the back end of the belt then the event $\sigma_{mid}(i)$ occurs and at the same the upstream belt is notified that the belt can again accept trays, so the events $\sigma_{up1}(i) = \sigma_{d1}(i-1)$ occur simultaneously.

The control systems at the continuous level are rather elementary. Let in discrete state $q_2 \in Q$, in which only one tray is present, the first component of x_{q_2} denote the position of the front end of the tray with respect to the starting point of the belt and the second component denote the speed of the tray which equals the speed of the belt. Let u, the input signal, represent the torque of the motor and y, the output signal, represent the measurement of the velocity of the tray which is assumed to equal the velocity of the belt.

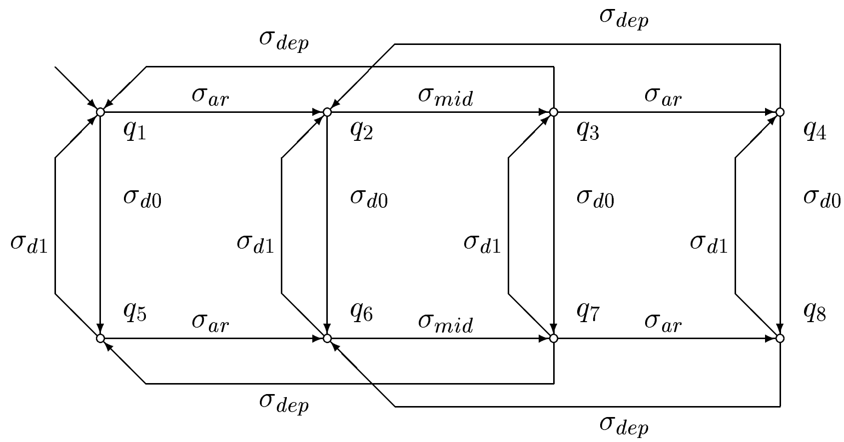

Figure 2: Discrete event system associated with hybrid control system of conveyor.

Then the movement of the tray is described by the control system

$$\begin{aligned}
\dot{x}_{q_2}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_{q_2}(t) + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t), \quad x_{q_2}(t_0) = x^+_{q_2}, \\
y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_{q_2}(t), \\
\dot{x}_{q_4}(t) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x_{q_4}(t) + \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix} u(t), \quad x_{q_4}(t_0) = x^+_{q_4}, \\
y(t) &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x_{q_4}(t),
\end{aligned}$$

with, $X_{q_2} = R^2$, $TX_{q_2} = R^2$ represents the tangent space, $U = R$, $Y = R$, $X_{q_4} = R^3$, and $TX_{q_4} = R^3$. The continuous dynamics in the discrete states $q_1, q_2, q_3, q_5, q_6, q_7$ are identical and in the discrete states q_4, q_8 are identical. If at the discrete state $q \in Q$ the trajectory x_q hits the subset $G_q(\sigma) \subset X_q$, say $x_q(t-) \in G_q(\sigma)$, then the event $\sigma \in \Sigma_{cd}$ occurs immediately. This differs from the computer science convention where an event can occur at any time after the state trajectory has entered a guard. The guards at the discrete states are

$$\begin{aligned}
&G_{q_2}(\sigma_{mid}) = [l/2, \infty) \times R,\ G_{q_3}(\sigma_{dep}) = [l, \infty) \times R,\ G_{q_4}(\sigma_{dep}) = [l, \infty) \times R^2, \\
&G_{q_6}(\sigma_{mid}) = G_{q_2}(\sigma_{mid}),\ G_{q_7}(\sigma_{dep}) = [l, \infty) \times R,\ G_{q_8}(\sigma_{dep}) = G_{q_4}(\sigma_{dep}).
\end{aligned}$$

At $t = t_0$ the initial condition at the discrete state q_0 is taken to be $x_{q_0,0}$. Let $U_c = \{u : T \to U\}$ be the class of admissable input functions.

In Section 4 a control problem for this belt will be described and solved.

The model of the conveyor belts defined so far is sufficient for several control problems. However, optimal control of the conveyor belt system requires a model with additional features. This extension of the model is left for future work.

3 Modeling of hybrid systems

3.1 Definitions

In this subsection a definition of a hybrid control system is formulated and discussed.

Definition 3.1 *A* continuous-time hybrid control system *is a tuple*

$$\left\{ \begin{array}{l} T, Q, \Sigma_{in}, \Sigma_{env}, \Sigma_{int}, \Sigma_{cd}, \Sigma_{out}, U, Y, \mathbf{U}_c, \mathbf{U}_{ex}, \\ \delta, r, \{X_q, TX_q, G_q, f_q, h_q, \forall q \in Q\}, (q_0, x_{q_0,0}) \end{array} \right\}, \tag{1}$$

where

$T = R_+$, *said to be the* time index set,
Q *is a finite set, the* discrete state set,
Σ_{in} *is a finite set, the* set of input events,
Σ_{env} *is a finite set, the set of* environmental events,
Σ_{int} *is a finite set, the set of* internal events,
Σ_{cd} *is a finite set, the set of* events generated by the continuous dynamics,
$\Sigma = \Sigma_{in} \cup \Sigma_{int} \cup \Sigma_{env} \cup \Sigma_{cd}$,
$\Sigma_{out} \subset \Sigma$ *the* set of output events,
$U \subseteq R^m$, *the* continuous input space,
$Y \subseteq R^p$ *the* continuous output space,
$\mathbf{U}_c \subset \{u : T \to U\}$, *set of* continuous input functions,
$\mathbf{U}_{ex} \subset (T \times \Sigma)^* \cup (T \times \Sigma)^\omega$ *the set of external timed-event sequences,*
$\delta : T \times Q \times X \times \Sigma \to Q$, *the discrete* transition function,
a, possibly partial, function,
$r : T \times Q \times Q \times X \times \Sigma \to X$, *the* reset map, *a, possibly partial, function,*
for all $q \in Q$,
$X_q \subseteq R^{n_q}$, *the* continuous state space at discrete state $q \in Q$, $X = \cup_{q \in Q} X_q$,
$TX_q(x) \subseteq R^{n_q}$ *the tangent space at* $x \in X_q$,
$G_q : \Sigma_{cd} \to P_{closed}(X_q)$, *the* guard *at* $q \in Q$, *a, possibly partial, function,*
$P_{closed}(X_q)$ *denotes the closed subsets of* X_q,
$f_q : T \times X_q \times U \to TX_q$, $h_q : T \times X_q \times U \to Y$,
are functions that determine a differential equation and a read-out map,
$(q_0, x_{q_0,0}) \in Q \times X_{q_0}$ *the* initial state.

The dynamics of the hybrid control system is described by the discrete transition function, the reset map, the differential equation, and the output map, according to

$$q^+ = \delta(t, q^-, x^-_{q^-}, \sigma), \; q_0, \tag{2}$$

$$x^+_{q^+} = r(t, q^-, q^+, x^-_{q^-}, \sigma), \tag{3}$$

$$\dot{x}_q(t) = f_q(t, x_q(t), u(t)), \quad x_q(0) = x^+_{q,0}, \tag{4}$$

$$y(t) = h_q(t, x_q(t), u(t)). \tag{5}$$

The operation of the hybrid control system is described below. At $t = 0$ the initial state is $(q_0, x_{q_0,0}) \in Q \times X_{q_0}$. Assume no immediate transition takes place at $t = 0$; see below on what to do if an event does occur at this time. At the discrete state $q = q_0$ the continuous dynamics proceeds according to the differential equation (4). It is assumed that for all $u \in \mathbf{U}_c$ this differential equation has an unique solution on R_+. The solution will be followed till the next event. The time interval till the next event will be denoted by $[t_0, t_1)$ for $t_1 \in R_+$ and for subsequent intervals by $[t_n, t_{n+1})$ for $n \in Z_+$.

At any time $t \in T$ an event may occur that results in a change of the discrete state. The possible events at state $q \in Q$ and at time $t \in T$ are:

- an *input event* $\sigma \in \Sigma_{in}$ occurs if such an event is supplied on the input channel;
- an *environmental event* $\sigma \in \Sigma_{env}$ occurs if such an event is supplied by the environment;
- an *event generated by the continuous dynamics* $\sigma \in \Sigma_{cd}$ occurs immediately when $x_q(t-) \in G_q(\sigma)$, thus if the state of the system hits a guard. (Here the notation $x_q(t-) = \lim_{s \uparrow t} x_q(s)$ is used.)

If the timed-event (t, σ_1) occurs then the transition is described by the discrete transition function and the reset map (2,3). The transition should be read as that the timed-event (t, σ_1) transfers the system from the state $(q^-, x^-_{q^-})$ to the state $(q^+, x^+_{q^+})$. It may be the case that the new state $(q^+, x^+_{q^+}) \in Q \times X_{q^+}$ is such that $x^+_{q^+} \in G_{q^+}(\sigma_2)$. In this case the event $\sigma_2 \in \Sigma_{cd}$ takes place at the same time. It will be assumed that only a finite number of events can occur at any time. After the last event of the sequence of events occuring at moment t, the new state is $(q_f, x^+_{q_f})$ where $x^+_{q_f}$ is the initial condition of the differential equation in the discrete state q_f. A further extension is to make the guards time-varying.

The *internal behavior* of the hybrid control system consists of a sequence of states and transitions and of a sequence of trajectories as in

$$\begin{aligned} b_{in,d} &= ((q_0, x^+_{q_0,0}), (t_1, \sigma_1), (q_1, x^+_{q_1}), (t_2, \sigma_2), \ldots) \\ &\in ((Q \times X) \cup (T \times \Sigma))^* \cup ((Q \times X) \cup (T \times \Sigma))^\omega, \quad (6) \\ b_{in,c} &= ((q_0, x_{q_0}(t), u(t), y(t), t \in [t_0, t_1)), (q_1, x_{q_1}(t), u(t), y(t), t \in [t_1, t_2)), \ldots), \\ &(Q \times X_q^I \times U^I \times Y^I)^* \cup (Q \times X_q^I \times U^I \times Y^I)^\omega, \quad (7) \\ &\text{where } I \subset R_+ \text{ is a left closed and right open interval} \\ &\text{that varies over the sequence.} \end{aligned}$$

For a projection operator $p : \Sigma \to \Sigma_{out} \cup \{\epsilon\}$, define the *behavior* of the hybrid control system as

$$\begin{aligned} b_d &= ((t_1, \bar{\sigma}_1), (t_2, \bar{\sigma}_2), \ldots) \in (T \times \Sigma_{out})^* \cup (T \times \Sigma_{out})^\omega, \quad (8) \\ b_c &= ((u(t), y(t), t \in [t_0, t_1)), (u(t), y(t), t \in [t_1, t_2)), \ldots), \\ &\in (U^I \times Y^I)^* \cup (U^I \times Y^I)^\omega, \quad (9) \\ b_{HCS} &= (b_d, b_c) \in B(T, \Sigma, U, Y). \quad (10) \end{aligned}$$

where $\bar{\sigma}_i = p(\sigma_j)$ only if $p(\sigma_j) \neq \epsilon$.

Comments on the definition follow. The concept of a hybrid control system is a product of an input-output automaton and of a family of control systems with continuous state spaces. The coupling between the discrete level, the input-output automaton, and the continuous level, the family of control systems, is in the dependence of the control system on the discrete state and in the events generated by the continuous dynamics. As in system theory, a hybrid control system exists in relation to its environment with which it is connected by both discrete and continuous inputs and outputs.

The definition of a hybrid control system involves a choice for the input/output automaton at the discrete level. An option *not* taken is to have a model with as class of input functions both continuous and discrete input trajectories. Then questions of existence and uniqueness of a solution to the differential equation and of properties of the solution can easily be studied. The choice for a computer science model at the discrete level is motivated by the use of the definition for modeling of computer controlled engineering systems.

The definition of a hybrid control system involves the forcing of events by an input event or by an environmental event. This differs from the model introduced by W.M. Wonham and P.J. Ramadge, see [19], for control of discrete event systems in which the events just happen. That is so because in that model the continuous systems are not explicitly modelled.

In the definition an input event and an environmental event can arrive at any time. This allows a rich behavior. In practice not every input event can be executed immediately, its arrival often leads to a short procedure consisting of several events and time trajectories. For example, in case an operator pushes the button to shut down a chemical plant this is followed by an elaborate procedure that stretches out over time.

For a hybrid control system to be well defined it must be proven that (1) at any time only a finite number of events can occur; and (2) on any finite interval only a finite number of events can occur (Non-Zeno behavior). Condition (1) can be checked from the definitions by an analysis of the discrete part of the system. Condition (2) requires analysis of switched differential equations. Possibly the solution concept of a differential equation introduced by A.P. Filippov is useful in this regard.

The definition of a hybrid control system is fairly general. Experience with examples will have to establish how useful it is for modeling of computer controlled engineering systems.

Because of space limitations, a discussion of synchronization and of interconnections of hybrid control systems is omitted. The synchronization may take rather complex forms.

Other definitions of hybrid systems or hybrid control systems have been proposed. For an overview see the thesis of M. Branicky, [6]. Definitions similar in character to the above are [6, Ch. 5] and [9, 17, 21, 23].

3.2 Subclasses of the class of hybrid control systems

An example of a hybrid control system is the conveyor belt model of Example 2.1.

E.D. Sontag has introduced a class of hybrid control systems in a paper that appeared in 1981, see [21]. The definition requires the concept of a polyhedral set, of a PL-set, and of a PL-map, see [22]. A closed polyhedron is the intersection of a finite number of closed half spaces of a vector space. A PL-set is the finite union of relatively open polyhedra. A PL-map is a map from a vector space to another such vector space such that its graph is a PL-set. Below a linear hybrid control system is defined that is a special case of that introduced in [21].

Definition 3.2 *A continuous-time time-invariant* linear hybrid control system*, see [21], is a hybrid control system with*

$$\begin{aligned} U &\subset R^m, Y \subset R^p, X_q \subset R^{n_q},\ \forall q \in Q,\ \textit{are polyhedral sets,} \\ G_q(\sigma) &\subset X_q, \forall q \in Q,\ \sigma \in \Sigma_{cd},\ \textit{are polyhedral sets,} \\ q^+ &= \delta(q^-, x^-_{q^-}, \sigma), q^- = q_0, \\ x^+_{q^+} &= A_d(q^-, q^+, \sigma)x^-_{q^-} + B_d(q^-, q^+, \sigma), \\ \dot{x}_q(t) &= A_q x_q(t) + B_q u(t),\quad x_q(0) = x^+_q, \\ y(t) &= C_q x_q(t) + D_q u(t). \end{aligned}$$

An overview of classes of hybrid control systems follows.

1. *Timed automata*. This is a subclass of the class of hybrid control systems in which the right hand side of the differential equation is constant. This class has been introduced by R. Alur and D. Dill in [4]. Other references on this class are [3, 2].

2. *Linear hybrid control systems.*

3. *Polyhedral hybrid systems* in which the input, output, and state spaces are polyhedral sets and the continuous dynamics is described by a differential inclusion in which the inclusion is also specified by a polyhedral set. This subclass has been explored by T.A. Henzinger, see [10].

4. Arbitrary nonlinear hybrid control systems. This class is considered in, for example, the thesis of J. Lygeros, [15].

The choice of a model class has to be based on a compromise between the expressive power and the complexity of the model class. At the discrete level of a hybrid control system complexity may relate to the number of discrete states and the discrete dynamics as described by events or logical formulas. At the continuous level of a hybrid control system, complexity may relate to the spaces: a finite-dimensional state space, and to geometric constraints on the input, output, and state spaces; and to the dynamics: linear dynamics or dynamics described by a differential inclusion described by a polyhedral set.

A major problem of control and system theory is to formulate subclasses of the class of hybrid control systems that are interesting for control and system theory and for which problems and questions are computationally tractable. More so than in classical realization theory, properties of decidability and of complexity should be used in the selection of the subclass.

3.3 Realization of hybrid systems

In this subsection the realization problem for hybrid control systems is formulated and discussed.

Problem 3.3 *Consider a time set $T \subset R_+$, an event set Σ, an input set $U \subset R^m$, and an output set $Y \subset R^p$. Consider a set of behaviors*

$$B_{given} = \{b_j \in B(T, \Sigma, U, Y), j \in J\},$$

for some index set J. Construct a hybrid control system HCS such that for suitable initial conditions the set of behaviors of the system equals the given set of behaviors,

$$B_{given} = B_{HCS}.$$

Such a hybrid control system is then called a realization *of the given set of behaviors.*

Questions of this problem include:

1. *Does a hybrid control system exists which does the job? Because a hybrid control system by definition has a finite number of discrete states and has at each discrete state a finite-dimensional state space, not every set of behaviors can be represented.*

2. *When is a realization minimal in a to be defined sense? Even the definition of the concept of minimality requires thought.*

3. *Which subclasses of hybrid control systems are equivalent in the sense that they represent the same behaviors?*

The problem defined above differs from the realization problem for linear finite-dimensional systems given the impulse response function in that here behaviors are given consisting of input-output trajectories.

The problem formulated above is likely to be undecidable in full generality. It is of interest to find subclasses of hybrid control systems for which a useful realization theory can be developed.

Recent developments in the computer science literature on hybrid systems are based on language equivalence of two subclasses of hybrid systems, see the result by D.L. Dill, [7], by R. Alur etal, [1], by T. Henzinger, [10], and by K. Inan [13]. The importance of the result is that for questions up to untimed language equivalence the hybrid control system can be reduced to a finite-state automaton for which the questions may be easier to solve. For other problems other equivalence relations may have to be used. For each equivalence relation the smallest subclass of hybrid control systems is of interest.

4 Control of hybrid systems

4.1 Problem formulation

Consider a hybrid control system. The problem is to construct a controller which in closed-loop with the hybrid control system meets the control objectives. An interconnection of a hybrid control system and of a hybrid controller is displayed in Figure 3. A hybrid controller is a hybrid control system as defined in Definition 3.1.

In the approach taken in this paper, the control objectives at the discrete level will receive relatively more attention than those at the continuous level. An argument for this approach is that control synthesis for the continuous part of a hybrid control systems is well developed. Control synthesis is therefore separated into that at the discrete and that at the continuous level. For other approaches to control synthesis see [12, 15, 16, 21]. Below the discrete event system of a hybrid control system is considered in a finite string framework. The extension to an infinite string framework is of interest and will be investigated later.

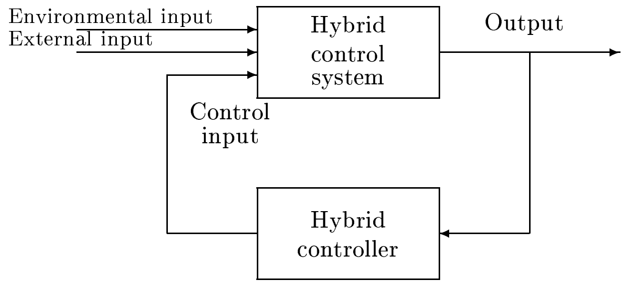

Figure 3: Interconnection of a hybrid control system and of a hybrid controller

There are two cases to be distinguished. In Case 1 $\Sigma_{in} = \emptyset$, $\Sigma_{env} \neq \emptyset$, and $\Sigma_{cd} \neq \emptyset$. For this case supervisory control of discrete event systems is useful. In Case 2 $\Sigma_{env} = \emptyset$, $\Sigma_{in} \neq \emptyset$, and $\Sigma_{cd} \neq \emptyset$. In this case an input-output automaton is obtained at the discrete level. Control synthesis for Case 2 will be described elsewhere.

4.2 Example

Example 4.1 Consider Example 2.1 in which a line of conveyor belts is modeled by a hybrid control system. Below the control objectives and a hybrid controller are specified.

Control objectives for the control of the conveyor belt follow. The first objective is that trays should not collide. As two trays on the same belt both travel at the same speed they cannot collide. Two trays can collide if a tray moves from one belt onto the next down stream belt where a tray stays put because the belt is not moving or moving slowly. This situation has to be prevented. This is a safety control objective.

A second control objective is that a tray which the up stream environment delivers to the belt is eventually transported and delivered to the down stream environment when the down stream environment is ready to accept it. This is called a control objective of acceptable behavior.

A third control objective is performance optimization. How to control the hybrid control system such that the throughput of trays is optimized while guaranteeing safety. The throughput of the conveyor belt system is control dependent.

For the safety control objective the condition is imposed that the following inequality holds

$$\frac{m_{trays} \times v_{max}^2}{2b_2 T_{max}} < l/2. \tag{11}$$

If the inequality does not hold then the conveyor system should have motors with a higher maximum torque, less mass on the trays, or a lower maximum speed of the belt. The control objective of acceptable behavior can be met if the following conditions hold on the parameters of the system

$$b_2 > 0, \quad T_{max} > 0. \tag{12}$$

Control design for the conveyor belt system follows. Safety. If the down stream environment is not ready to accept a tray, which happens in the discrete states q_5, q_6, q_7, q_8, then no tray should be transferred from the belt to the down stream environment, or the event σ_{dep} should not occur or, if it cannot be prevented, then it should not lead to a collision. At the discrete states mentioned, the event σ_{dep} is possible only in the states q_7, q_8. The event σ_{dep} in the discrete states q_7, q_8 is generated by the continuous dynamics, the moving of the belt. Thus in those states the belt has to be decelerated till it becomes to a stand still and be kept at this zero speed. It has to be proven that in no circumstance this action leads to a collision of the trays.

Acceptable behavior. A tray that is delivered from the up stream environment to the belt must be passed on to the down stream environment if the latter environment is ready to accept it. This can be achieved at the discrete states in which a tray is present and in which the down stream environment is ready to accept a tray, q_2, q_3, q_4, by forcing the event σ_{mid} or the event σ_{dep}. The latter events can be forced by the continuous input, thus by accelerating the belt to the maximal speed v_{max} and keeping the speed at that level. In discrete state q_6 in which a tray is present at the front end of the belt but not at the back end, and the down stream belt cannot accept a tray, the belt may be brought to maximum speed so as to force the event σ_{mid}.

A controller is described such that the interconnection of the hybrid control system and of the hybrid controller achieves the control objectives. The hybrid controller consists of a hybrid control system as defined in Definition 3.1 with the same discrete states as those of the control system but with the following details. Let

$$\begin{aligned} &Q_2 = Q_1, \quad \Sigma_{1,in} = \Sigma = \Sigma_{env} \cup \Sigma_{cd} \cup \Sigma_{int}, \quad \Sigma_{2,env} = \emptyset, \quad \Sigma_{2,int} = \emptyset, \quad \Sigma_{2,cd} = \emptyset, \\ &U_2 = Y_1, \quad Y_2 = U_1 = R, \quad u_2 = y_1, \quad u_1 = y_2. \end{aligned}$$

The hybrid controller is specified in Table 4. The controller has no continuous dynamics. In the table $y_1(t)$ denotes the speed of the belt which equals the speed of any tray present and v_{max} denotes the maximal speed of the belt.

It will be argued that the controller designed above meets the control objectives. Consider the safety control objective and distinguish two cases. In Case 1 a tray passes the mid point of the belt and hence moves from the front end to the back end of the belt while the down stream environment cannot accept a tray. Thus the transition

$$q_7 = \delta(q_6, \sigma_{mid}), \quad x_{q_7} = r(q_6, q_7, x_{q_6}, \sigma_{mid}) = (l/2, x_{q_6,2}),$$

occurs. Because the tray and the belt are moving, the speed of the belt $y(t)$ is strictly positive. According to the controller defined above, in discrete state q_7 the input signal is $u_1(t) = y_2(t) = -T_{max}$. Thus the belt is decelerated till it stops. The distance travelled by the tray from the mid point of the belt till the time the belt stops may be calculated to be

$$\frac{m_{trays} v_{max}^2}{2 b_2 T_{max}}.$$

Discrete-state	Output signal of controller
q_1	$y_2(t) = \begin{cases} -T_{max}, & \text{if } y_1(t) > 0, \\ 0, & \text{if } y_1(t) = 0, \end{cases}$
q_2	$y_2(t) = \begin{cases} T_{max}, & \text{if } y_1(t) < v_{max}, \\ 0, & \text{if } y_1(t) = v_{max}, \end{cases}$
q_3	$y_2(t) = \begin{cases} T_{max}, & \text{if } y_1(t) < v_{max}, \\ 0, & \text{if } y_1(t) = v_{max}, \end{cases}$
q_4	$y_2(t) = \begin{cases} T_{max}, & \text{if } y_1(t) < v_{max}, \\ 0, & \text{if } y_1(t) = v_{max}, \end{cases}$
q_5	$y_2(t) = \begin{cases} -T_{max}, & \text{if } y_1(t) > 0, \\ 0, & \text{if } y_1(t) = 0, \end{cases}$
q_6	$y_2(t) = \begin{cases} T_{max}, & \text{if } y_1(t) < v_{max}, \\ 0, & \text{if } y_1(t) = v_{max}, \end{cases}$
q_7	$y_2(t) = \begin{cases} -T_{max}, & \text{if } y_1(t) > 0, \\ 0, & \text{if } y_1(t) = 0, \end{cases}$
q_8	$y_2(t) = \begin{cases} -T_{max}, & \text{if } y_1(t) > 0, \\ 0, & \text{if } y_1(t) = 0. \end{cases}$

Table 4: Description of hybrid controller.

If Condition (11) holds then the tray will come to a stand still before reaching the next belt.

In Case 2 a tray is present on the back of the belt, it is moving at nonzero speed, and the down stream environment can accept a tray. The discrete state is thus either q_3 or q_4. Because the down stream environment can accept a new tray, there is no tray present on the front end of the down stream belt. Suppose that the belt considered receives the event σ_{d0} signalling that the down stream belt from then on cannot receive a tray. The reason for this may be a communication that the down stream belt receives from the environment further down stream. The hybrid control system moves to state q_7 or q_8 depending on whether it was in respectively q_3 or q_4. According to the controller defined above the belt will then be decelerated at maximal torque. Because of Condition (11) the tray will come to a stand still before having travelled distance $l/2$ on the belt. Depending on when the event σ_{d0} occured, the tray may be shifted onto the next belt. Because the front end of the down stream belt was free of trays, no collision of trays occurs.

The optimization control objective requires additional work. The model used so far can be used to satisfy the safety control objective but it may not be good for the optimization of throughput. Optimal control theory requires another model. Let the belt not be partitioned in a front end and a back end of equal lengths but in a front end of length l_f. When a tray arrives at a belt then the upstream environment is notified that no new tray can be accepted. When a tray leaves the shortened front end then the up stream environment is notified that the belt can accept trays again. How small can the length of the front end be so as to satisfy the safety objective? This question requires a model in which the back end can carry two or more trays. Although the total number of trays is bounded in this case, the hybrid control system must keep track of a countable number of trays in general. Hence another model is required. Such a model would also have to include the time delay between the time of sending a message with event σ_{up0} in belt i and the time of event σ_{d0} in belt $i-1$. Such an extended model is left for future work.

4.3 Control synthesis for a special class of systems

In this subsection a control synthesis approach is developed based on an extension of control synthesis for discrete event systems.

It will be assumed in this subsection that the hybrid control system is time-invariant and that there are no input events ($\Sigma_{in} = \emptyset$). The only possible events are environmental events and events generated by the continuous dynamics.

Notation needs to be introduced. For $q_1, q_2 \in Q$ and $\sigma \in \Sigma$ define the *arrival set*

$$A(q_1,\sigma,q_2) = \left\{ x_{q_2} \in X_{q_2} | \begin{array}{l} \exists t \in T,\ x_{q_1} \in X_{q_1},\ \text{such that} \\ q_2 = \delta(q_1, x_{q_1}, \sigma),\quad x_{q_2} = r(q_1, q_2, x_{q_1}, \sigma) \end{array} \right\}, \tag{13}$$

and denote by AR the set of all such arrival sets. The set $A(q_1,\sigma,q_2)$ consists of those continuous states in X_{q_2} at which one arrives after a transition from the discrete state q_1 with event σ. For $\sigma \in \Sigma_{cd}$, $q \in Q$, and guard $G_{q_2}(\sigma)$ denote the *controllability set* of this guard by

$$Con(G_{q_2}(\sigma)) = G_{q_2}(\sigma) \cup \left\{ x_{q_2} \in X_{q_2} | \begin{array}{c} \exists t_0, t_1 \in R_+, t_0 < t_1, \exists u \in \mathbf{U}[t_0,t_1], \\ \text{such that } x_{q_2}(t_0) = x_{q_2},\quad \forall t \in (t_0,t_1), \\ x_{q_2}(t) \in X_{q_2},\quad x_{q_2}(t_1-) \in G_{q_2}(\sigma) \end{array} \right\}. \tag{14}$$

In words, the controllability set consists of all states of the state space X_{q_2} that are either in $G_{q_2}(\sigma)$ or, if not, for which there exists an input trajectory that transfers the state of the system to the set $G_{q_2}(\sigma)$.

For $q \in Q$ and $x_{q,0} \in X_q$ an initial state let the *reachable set* be defined as

$$Reach_q(\{x_{q,0}\}) = \{x_{q,0}\} \cup \left\{ x_{q,1} \in X_q | \begin{array}{l} \exists t_0, t_1 \in R_+, t_0 < t_1,\quad \exists u \in \mathbf{U}[t_0,t_1) \\ \text{such that } x_q(t_0) = x_{q,0}, \\ \forall t \in (t_0,t_1), x_q(t) \in X_q,\ x_q(t_1-) = x_{q,1} \end{array} \right\}. \tag{15}$$

Consider then a discrete event system with as event set $\Sigma = \Sigma_{env} \cup \Sigma_{cd}$. The restriction will be imposed that the control objective can be formulated at the discrete event level only. The extension to the control objective of unsafe sets of states at the continuous level is under study.

Theorem 4.2 *Consider a hybrid control system. Denote the discrete event system associated with the hybrid control system by $G = (Q, \Sigma, \delta_1, q_0)$. Only finite-string languages are considered. Assume that control objectives of legal and acceptable behavior are given at the discrete level, thus let $L(A), L(E) \subset \Sigma^*$ be languages with*

$$L(A) \subset L(E) \subset L(G),$$

where $L(G)$ is the language generated by the discrete event G. Denote for $q^- \in Q$ the set of possible events generated by the continuous dynamics at this discrete state by

$$\Sigma_{cd}(q^-) = \left\{ \sigma \in \Sigma_{cd} | \exists \; x^-_{q^-} \in X_{q^-}, q^+ \in Q, \text{ such that } q^+ = \delta_1(q^-, x^-_{q^-}, \sigma) \right\}$$

a *If there exist a non-empty language $K \subset \Sigma^*$ and a subset $\Sigma_{cdc} \subset \Sigma_{cd}$, called the* set of controllable continuous-dynamics events*, such that:*

1.

$$L(A) \subset K \subset L(E); \tag{16}$$

2. *K is prefix-closed;*

3.

$$K(\Sigma_{env} \cup \Sigma_{cduc}) \cap L(G) \subset K, \tag{17}$$

where $\Sigma_{cduc} = \Sigma_{cd} \cap (\Sigma_{cdc})^c$; this is a controllability condition, see [19];

4. the events of Σ_{cdc} can be forced to occur in due time*: for all $q \in Q$, all $\sigma_1 \in \Sigma_{cdc} \cap \Sigma_{cd}(q)$ that occur at this discrete state, and all arrival sets $AR(q_i, \sigma_0, q) \in AR$ there holds*

$$AR(q_i, \sigma_0, q) \subset Con(G_q(\sigma_1)); \tag{18}$$

in words, for all discrete states and for all controllable continuous-dynamics events $\sigma_1 \in \Sigma_{cdc}$ that are possible at that state, and for all arrival sets $AR(q_i, \sigma_0, q)$, it is possible to find a continuous input that steers the system from any initial state in $AR(q_i, \sigma_0, q)$ to the guard $G_q(\sigma_1)$ at which the timed-event (t, σ_1) occurs;

5. the events of Σ_{cdc} can be disabled during any time interval*: for all $q \in Q$, all $\sigma_1 \in \Sigma_{cdc} \cap \Sigma_{cd}(q)$, all $AR(q_i, \sigma_0, q) \in AR$, and all $t_1 \in R_+$ there exists a continuous input $u \in \mathbf{U}_c$ such that for all initial states in $AR(q_i, \sigma_0, q)$ the state trajectory is defined on $[t_0, t_1]$ and for all $t \in [t_0, t_1]$, $x_q(t) \notin G_q(\sigma_1)$;*

then there exists a hybrid controller such that the closed-loop system G/S at the discrete level satisfies

$$L(A) \subset L(G/S) \subset L(E). \tag{19}$$

b *If there exists a non-empty hybrid controller such that the closed-loop system satisfies (19) then the conditions 1, 2, and 3 of (a) hold. The Conditions 4 and 5 will not hold in general at every discrete state, at every continuous state, and at every event.*

Proof a. The Conditions 4 and 5 state that at any discrete state and for any initial condition outside the guards of the continuous control system, the event $\sigma \in \Sigma_{cdc}(q)$ can be made to occur or can be prevented from occuring. Therefore, this event can be enabled or disabled in the sense of control of discrete event systems and hence $\Sigma_{in} \cup \Sigma_{cdc}(q)$ is the set of

controllable events and $\Sigma_{env} \cup \Sigma_{cduc}(q)$ the set of uncontrollable events at the discrete state q. The Conditions 1, 2, and 3 imply by the results of P. Ramadge and W.M. Wonham, see [19], that there exists a supervisor such that (19) holds. This supervisor, in combination with a controller at the continuous level, guaranteed to exist by the Conditions 4 and 5, forms the hybrid controller.
b. The existence of a hybrid controller implies, by results for control of discrete event systems, that there exists a language $K \subset \Sigma^*$ such that the Conditions 1, 2, and 3 hold with $\Sigma_{cdc} \subset \Sigma_{cd}$. There are simple examples in which either Condition 4 or Condition 5 does not hold. This is a consequence of the fact that, because the hybrid control system as defined in this paper has a fixed initial condition, the trajectories will never pass through all initial conditions at every discrete state. □

The result above does not say anything about the time behavior of the controlled system. Thus the belt can be moved at extremely low speed. Therefore optimal control requires an extended control synthesis and another model.

It is straigthforward to formulate a control design algorithm based on the previous theorem such that, if a controller exists, it will be computed.

Example 4.3 Consider the conveyor belt system. Note that $\Sigma_{env} = \{\sigma_{ar}, \sigma_{d0}, \sigma_{d1}\}$ and $\Sigma_{cd} = \{\sigma_{mid}, \sigma_{dep}\}$. As mentioned in Example 4.1, the safety control objective implies that in the states q_7 and q_8 the event σ_{dep} should not occur. The acceptable behavior control objective implies that in the states q_3, q_4 the event σ_{dep} must occur and in the states q_2, q_6 the event σ_{mid} must occur.

The conditions of Theorem 4.2 will be verified with $\Sigma_{cdc} = \Sigma_{cd}$. Condition 4. The controllable continuous-dynamics events can be forced to occur. Note that the events of $\Sigma_{cdc} = \Sigma_{cd}$ can occur only in the discrete states $q_2, q_3, q_4, q_6, q_7, q_8$. The inclusion relation of Condition 4 is satisfied as follows from

$$\begin{array}{ll} A(q_1, \sigma_{ar}, q_2) \subset Con(G_{q_2}(\sigma_{mid})), & A(q_4, \sigma_{dep}, q_2) \subset Con(G_{q_2}(\sigma_{mid})), \\ A(q_6, \sigma_{d1}, q_2) \subset Con(G_{q_2}(\sigma_{mid})), & A(q_2, \sigma_{mid}, q_3) \subset Con(G_{q_3}(\sigma_{dep})), \\ A(q_3, \sigma_{ar}, q_4) \subset Con(G_{q_4}(\sigma_{dep})), & A(q_2, \sigma_{d0}, q_6) \subset Con(G_{q_6}(\sigma_{mid})), \end{array}$$

and several more such relations for arrivals at the discrete states q_5, q_6, q_7, q_8. That the inclusion relation holds in each of these cases can be proven by explicitly evaluating the effect of moving the trays by applying an input to the motor.

Condition 5. The action to be taken to prevent the event σ_{dep} is to decelerate the belt at maximal torque and to keep the belt at the zero speed level. This event can occur only at the discrete states q_3, q_4, q_7, q_8. As argued in Subsection 4.2, it is not always possible to prevent the event σ_{dep} from occuring. But, if it occurs then this is in a case in which the front end of the down stream belt does not carry a tray.

It is possible to stop or to decelerate the belt at these discrete states for initial states of the continuous system in the arrival sets:

$$\begin{array}{llll} A(q_2, \sigma_{mid}, q_3), & A(q_7, \sigma_{d1}, q_3), & A(q_3, \sigma_{ar}, q_4), & A(q_8, \sigma_{d1}, q_4), \\ A(q_6, \sigma_{mid}, q_7), & A(q_3, \sigma_{d0}, q_7), & A(q_7, \sigma_{ar}, q_8), & A(q_4, \sigma_{d0}, q_8), \end{array}$$

etc. The conditions 1, 2, and 3, of Theorem 4.2 are easily checked by inspection of the state diagram. The conclusion obtained from the theorem is that there exists a hybrid controller which meets the discrete level control objectives.

4.4 Reachability problems

Theorem 4.2 leads to a reachability problem for the continuous part of a hybrid control system.

Problem 4.4 *Consider a hybrid control system. Let $q \in Q$ be a discrete state, $X_{q,0} \subset X_q$ be a set of possible initial conditions at this state, and $X_{q,f} \subset X_q$ be a set of terminal conditions.*

a *Determine whether*

$$X_{q,0} \subseteq Con(X_{q,f}). \tag{20}$$

In words, the inclusion holds if for every initial condition $x_{q,0} \in X_{q,0}$ there exists a continuous input such that the continuous state x_q is transferred from state $x_{q,0}$ to a state in the set $X_{q,f}$ while the state trajectory never leaves the space X_q.

b *Determine whether for every $t_0,\ t_1 \in R_+$ there exists a continouous input $u \in U_c$ on $[t_0, t_1]$ such that the state trajectory will not enter the set $X_{q,f}$ on the interval $[t_0, t_1]$, or*

$$\forall q_0 \in Q, \quad x_q(t_0) = x_{q,0} \notin X_{q,f}, \quad x_q(t) \notin X_{q,f}, \ \forall t \in [t_0, t_1]. \tag{21}$$

Below these problems are discussed for linear hybrid control systems. The reachability problem for a class of hybrid system without inputs was treated in [5, 18].

Consider a discrete-time time-invariant linear hybrid control system at discrete state $q \in Q$ represented by the equations

$$x_q(t+1) = A_q x_q(t) + B_q u(t), \quad x_q(t_0) = x_q^+. \tag{22}$$

Denote

$$\begin{aligned}
X_q^k(\{x_q^+\}) &= \left\{ x_q^+ + RM^k(A_q, B_q)v \in X_q \subset R^{n_q} | \forall v \in U^k \right\}, \\
&\quad \text{the set of states reachable at time } k \geq t_0, \\
Reach_q^k(\{x_q^+\}) &= \cup_{t_0 \leq r \leq k} X_q^r(\{x_q^+\}), \\
X_q^{-k}(\{x_q^f\}) &= \left\{ x_q^f - RM^k(A_q, B_q)v \in X_q \subset R^{n_q} | \forall v \in U^k \right\}, \\
&\quad \text{the set of states controllable at time } t_0 - k < t_0, \\
Con_q^k(\{x_q^f\}) &= \cup_{k \leq r \leq t_0} X_q^{-r}(\{x_q^f\}), \\
RM^k(A_q, B_q) &= \begin{pmatrix} B_q & A_q B_q & \ldots & A_q^{k-1} B_q \end{pmatrix} \in R^{n_q \times km}.
\end{aligned}$$

If $U = R^m$ and $k \geq n$, then

$$X_q^k(\{x_q^+\}) = \left\{ x_q^+ + RM^n(A_q, B_q)v \in X_q \subset R^{n_q} | \forall v \in U^n \right\}.$$

This reduction does not hold if $U \subset R^m$ is an arbitrary polyhedral set.

In the discrete-time case it is possible to propagate the set of states reachable at time $t \geq t_0$, $X^t_q(\{x^+_q\})$, in time. The reachability condition

$$X_{q,f} \subseteq Reach^t_q(\{x^+_q\}), \tag{23}$$

can then be checked for each time moment. From the formulas above it may be deduced that if U is a polyhedral set then $X^t_q(\{x^+_q\})$ is at any time t a polyhedral set but that the number of vertices that spans the set in general grows with time. The long run behavior of this set can exhibit chaotic dynamics, see [14]. The controllability condition is checked analogously. Note that the compliment of a polyhedral set is a finite union of polyhedra.

In continuous time the approach of checking (23) for every t cannot be followed and one must compute the reachable set $Reach^t_q(\{x^+_q\})$ for large values of $t \in T$. Even if the hybrid control system is linear then the reachable set will in general not be polyhedral. Because a hybrid control system is used as a model for computer control of engineering systems and computers use an internal clock, one can restrict attention to the class of piece-wise constant inputs with regular switching times. After a reduction one is then in the discrete time case covered above.

As proven in [21], if the reachable set contains the goal set then one can construct a control law $g_q : X_q \to U$ such that the required input trajectory is generated by $u(t) = g(x_q(t))$. The use of a control law is preferred over the open-loop approach in which only the input as a time function is supplied to the system.

What is of interest to control of hybrid systems is the determination of the types of reachable sets that admit computations. Consider a continuous-time hybrid control system. Let at time $t_0 \in T$ the initial discrete state be q_0 and the set of continuous initial states be $X_{q_0,0} \subset X_{q_0}$. One can then in principle compute the set of states that can be reached at any time, to be denoted by $X(t)$. Because of the way a hybrid control system is defined, the set of states at time t may consist of several pieces, each piece at a different discrete state to be denoted by $X_q(t) \subset X_q$. The state set at time t is thus represented by the Cartesian product

$$\prod_{q \in Q} X_q(t) \subset \prod_{q \in Q} X_q = Prod(X). \tag{24}$$

The dynamics of the state set of a hybrid control system may be described by

$$\begin{aligned} &\prod_{q \in Q} X_q(t+1) = F(t, \prod_{q \in Q} X_q(t), U(t)), \\ &F : T \times Pwr(Prod(X) \times U) \to Pwr(Prod(X)). \end{aligned}$$

Let

$$PL(Prod(X)) = \left\{ \prod_{q \in Q} A_q \subseteq \prod X_q \mid A_q \subseteq X_q \text{ is PL set} \right\}.$$

If then follows from the definition of a linear hybrid control system that

$$F(t, .) : PL(Prod(X)) \times PL(U) \to PL(Prod(X)).$$

This invariance property is what makes the class of linear hybrid control systems interesting.

Another class of geometrical figures that is of interest is that of the rectangles. Denote

$$Rec(Prod(X)) = \left\{ \prod_{q \in Q} \prod_{i=1}^{n_q} [u_{q,i}, v_{q,i}] \subseteq Prod(X) | u_{q,i} \leq v_{q,i} \right\}.$$

In [18] a class of linear hybrid control systems is considered in which the initial state set is rectangular and the matrices A_q for all $q \in Q$ and the matrix A_d are diagonal. For this particular class of hybrid systems it may be proven that

$$F(t,.) : Rec(Prod(X)) \times Rec(U) \to Rec(Prod(X)).$$

Conclusions for this subsection follow. A problem in control of hybrid systems is to characterize the reachable set at any time, the set of states that can be reached on or before the time specified. In full generality this problem in intractable. If the initial state set and the input space are polyhedral sets or PL-sets, then the set of states reachable at any time is a PL-set. However, the complexity of the set grows with time. Does there exist a class of discrete-time hybrid control systems for which the complexity of the set of states reachable at any time remains finite in some sense? So far only the class of systems with rectangular sets has been explored, see [18]. It is not clear whether there exist other classes with the finite complexity property for which interesting control problems can be solved. An alternative approach, which has been used in the conveyor belt system, is to solve explicitly for the state function and to check the controllability or the reachability conditions by inspection. More research is required here.

5 Concluding remarks

The study of control problems of hybrid systems is motivated by computer control of engineering systems. As a running example has been used the control of a transportation system consisting of several conveyor belts.

The concept of hybrid control system has been defined. It is based on the product of an input-output automaton and a family of continuous control systems. The realization problem for hybrid control systems has been formulated and several questions of this problem discussed.

A control synthesis approach for a class of hybrid systems has been described. It is based on supervisory control of discrete event systems. A sufficient condition for the existence of a controller has been formulated. The reachability problem for a class of hybrid control systems has been discussed.

What is needed in regard to modeling and control of hybrid control systems? It is clear that the control and computation requirements for control synthesis impose conditions on the subclass of hybrid control systems for which interesting results can be derived. A refinement of the classes considered in this paper is necessary for the development of useful theory. Questions of computation should receive more attention in this regard, see the article [11] and for the theory of computation the book [20]. Control problems with time constraints require attention, such problems arise in applications.

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References

[1] R. Alur, C. Courcoubetis, and D.L. Dill. Model checking for real-time systems. In *Proc. 5th Annual Symposium on Logic in Computer Science*, pages 414–425, New York, 1990. IEEE Computer Society Press.

[2] R. Alur, C. Courcoubetis, N. Halbwachs, T.A. Henzinger, P.-H. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138:3–34, 1995.

[3] R. Alur, C. Courcoubetis, T.A. Henzinger, and P.H. Ho. Hybrid automaton: An algorithmic approach to the specification and verification of hybrid systems. In R.L. Grossman, A. Nerode, A.P. Ravn, and H. Rischel, editors, *Hybrid systems*, pages 209–229, New York, 1993. Springer.

[4] R. Alur and D. Dill. The theory of timed automata. In J.W. de Bakker, C. Huizing, and W.P. de Roever, editors, *Real-time: Theory and practice, Proceedings of the REX Workshop, Mook, The Netherlands, June 3-7, 1991*, number 600 in Lecture Notes in Computer Science, pages 45–74, Berlin, 1992. Springer.

[5] A. Asarin, O. Maler, and A. Pnueli. Reachability analysis of dynamical systems having piecewise-constant derivaties. *Th. Computer Science*, 138:35–65, 1995.

[6] M.S. Branicky. *Studies in hybrid systems: Modeling, analysis, and control*. PhD thesis, M.I.T., Cambridge, MA, 1995.

[7] D.L. Dill. *Trace theory for automatic hierarchical verification of spped independent circuits*. ACM Distinguished Dissertations. M.I.T. Press, Cambridge, MA, U.S.A., 1989.

[8] J.J.H. Fey. Control and verification of industrial hybrid systems using models specified with the formalism χ. Note BS-N9601, CWI, Amsterdam, 1996.

[9] R.L. Grossman and R.G. Larson. Viewing hybrid systems as products of control systems and automata. In *Proceedings of the 31st IEEE Conference on Decision and Control*, pages 2953–2955, New York, 1992. IEEE Press.

[10] T.A. Henzinger. Hybrid automata with finite bisimulations. In Z. Fülöp and F. Gécseg, editor, *ICALP 95: Automata, languages, and programming*, volume 944 of *Lecture Notes in Computer Science*, pages 324–335, New York, 1994. Springer.

[11] T.A. Henzinger, P.W. Kopke, A. Puri, and P. Varaiya. What's decidable about hybrid automata. In X, editor, *Proceedings of the 27th Annual Symposium on Theory of Computing*, pages 373–382, X, 1995. ACM Press.

[12] M. Heymann, Feng Lin, and G. Meyer. Control synthesis for a class of hybrid systems subject to configuration-based safety constraints. In O. Maler, editor, *Hybrid and real-time systems - Proceedings of International Workshop HART97*, number 1201 in Lecture Notes in Computer Science, pages 376–390, Berlin, 1997. Springer.

[13] K. Inan. On a class of timer hybrid systems reducible to finite state automata. *J. Discrete Event Dynamics Systems*, 4:83–96, 1995.

[14] M. Kourjanski and P. Varaiya. Stability of hybrid systems. In R. Alur, T.A. Henzinger, and E.D. Sontag, editors, *Hybrid systems: Verification and control*, number 1066 in Lecture Notes in Computer Science, pages 413–423, Berlin, 1995. Springer.

[15] J. Lygeros. *Hierarchical, hybrid control of large scale systems*. PhD thesis, University of California, Berkeley, 1996.

[16] J. Lygeros, C. Tomlin, and S. Sastry. Multiobjective hybrid controller synthesis. In O. Maler, editor, *Hybrid and real-time systems - Proceedings of International Workshop HART97*, number 1201 in Lecture Notes in Computer Science, pages 109–123, Berlin, 1997. Springer.

[17] N. Lynch, R. Segala, F. Vaandrager, and H.B. Weinberg. Hybrid I/O automata. In R. Alur, T.A. Henzinger, and E.D. Sontag, editors, *Hybrid systems: Verification and control*, volume 1066 of *Lecture Notes in Computer Science*, pages x–y, Berlin, 1996. Springer.

[18] A. Puri and P. Varaiya. Decidable hybrid systems. *Mathl. Comput. Modelling*, 23:191–202, 1996.

[19] P.J.G. Ramadge and W.M. Wonham. The control of discrete event systems. *Proc. IEEE*, 77:81–98, 1989.

[20] M. Sipser. *Introduction to the theory of computation*. PWS Publishing Company, Boston, 1997.

[21] E.D. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Trans. Automatic Control*, 26:346–358, 1981.

[22] E.D. Sontag. Remarks on piecewise-linear algebra. *Pacific J. Math.*, 98:183–201, 1982.

[23] E.D. Sontag. Interconnected automata and linear systems: A theoretical framework in discrete-time. In R. Alur, B. Kurshan, and E.D. Sontag, editors, *Proceedings Workshop on Verification and Control of Hybrid Systems*, pages 436–448, Berlin, 1996. Springer.