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Probability, Networks and Algorithms (PNA)

PNA-R9718 November 30, 1997

Report PNA-R9718
ISSN 1386-3711

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Abstract

Consider a fluid queue fed by N on/off sources. It is assumed that the silence periods of the sources are exponentially distributed, whereas the activity periods are generally distributed. The inflow rate of each source, when active, is at least as large as the outflow rate of the buffer.

We make two contributions to the performance analysis of this model. Firstly, we determine the Laplace-Stieltjes transforms of the distributions of the busy periods that start with an active period of source i, $i = 1, \dots, N$, as the unique solution in $[0, 1]^N$ of a set of N equations. Thus we also find the Laplace-Stieltjes transform of the distribution of an arbitrary busy period.

Secondly, we relate the tail behaviour of the busy period distributions to the tail behaviour of the activity period distributions. We show that the tails of all busy period distributions are regularly varying of index $-\nu$ iff the heaviest of the tails of the activity period distributions are regularly varying of index $-\nu$. We provide explicit equivalents of the former in terms of the latter, which show that the contribution of the sources with lighter associated tails is equivalent to a simple reduction of the outflow rate. These results have implications for the performance analysis of networks of fluid queues.

1991 Mathematics Subject Classification: 60K25, 68M20, 90B22.
Keywords & Phrases: Fluid queue, on/off sources, long-range dependence, regular variation, busy period.
Note: Work carried out under project PNA2.1 (LRD)

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1 Introduction: fluid queues and heavy tails

Fluid queueing models are queueing models in which work enters and leaves a buffer non-instantaneously, i.e., like a fluid. The basic fluid model is that of a buffer that is fed by a number of on/off sources, viz., sources that alternate between active (on) and silent (off) periods. This model has in the last 15 years become firmly established as a key model for capturing the behaviour of a wide range of, in particular ATM-based, communication networks.

Recently, there has been a rapidly growing interest in such fluid models, in which one or more of the probability distributions of the on- and/or off-periods have a heavy, non-exponential, tail. The main reason for this is the following. Plots of traffic measurements for traffic in Ethernet Local Area Networks [21], Wide Area Networks [18] and VBR video [2] have shown a striking similarity when one considers a time period of hours, minutes or milliseconds: bursty subperiods are alternated by less bursty subperiods on each scale. This scale-invariant or *self-similar* feature of traffic, and the related phenomenon of *long-range dependence* (i.e., the integral over time of the covariance of the input rate diverges), was also convincingly demonstrated in [16] using a careful statistical analysis. A natural possibility to introduce long-range dependence (LRD) in a traffic process is to take a fluid queue fed by one or more on/off sources, and to assume that either the on-period or the off-period of a source has the following 'heavy-tail' behaviour:

$$\mathbb{P}[A > t] \overset{t\to\infty}{\sim} ht^{-a}, \tag{1}$$

with h a positive constant and $1 < a < 2$ ($f(t) \overset{t\to\infty}{\sim} g(t)$, or simply $f(t) \sim g(t)$, denotes that $f(t)/g(t) \to 1$ as $t \to \infty$). As soon as one of the sources exhibits such behaviour, the cumulative input process is LRD [6]. As observed in [21], in many cases on- and/or off-periods of actual traffic sources do indeed exhibit such a heavy-tail behaviour.

The occurrence of heavy-tailed on- and/or off-periods of sources seems to provide the most natural explanation of LRD and self-similarity in aggregated packet traffic. It is therefore of considerable importance to study the performance of a fluid queue fed by on/off sources, with special attention for the effect of heavy-tailed on- and/or off-periods on key performance measures like buffer content and busy period. Recent studies have been devoted to the latter issue, focusing on buffer content [4, 5, 7, 13, 19]; see also the survey [6]. The typical result in those papers is that non-exponential tail behaviour of at least one of the on-period distributions gives rise to 'worse' non-exponential tail behaviour of the buffer content distribution. More precisely, if the heaviest of the tails of the on-periods are regularly varying at infinity of index $-\nu$ (see Subsection 2.1 below), then the tail of the buffer content is regularly varying of index $1 - \nu$; i.e., the latter tail is heavier.

The present paper is devoted to the analysis of *the busy period distribution* in fluid queues fed by on/off sources. Motivation for this study is provided by the following observation. The output process of the buffer (that may feed into another buffer) is an on/off process with on-periods the busy periods; therefore it is important to investigate whether the busy period distribution is heavy-tailed when at least one of the sources has heavy-tailed on-period distributions. Recent work in [11] (cf. its Section IV.C) suggests that heavy tails, and hence long-range dependence, propagate through a network. See also Anantharam [1], who considers a discrete-time network of quasi-reversible queues with Bernoulli routing; in his model, sessions arrive according to Poisson processes and are active during periods that obey a regularly varying distribution of index $-\nu \in (-2, -1)$. He shows that all internal traffic processes are LRD.

We consider a fluid queue fed by N on/off sources. It is assumed that the silence periods of the sources are exponentially distributed, whereas the activity periods are generally distributed. The inflow rate r_i of each source i, when active, is assumed to be at least equal to the outflow rate of the buffer – which is assumed to be 1. Our contribution is two-fold. Firstly, we determine the Laplace-Stieltjes transforms (LSTs) of the distributions of the busy periods that start with an active period of source i, $i = 1, \ldots, N$, as the unique solution in $[0, 1]^N$ of a set of N equations. This generalizes results of Rubinovitch [20] (N identical sources with inflow rate 1), Kaspi & Rubinovitch [15] (N nonidentical sources with inflow rate 1) and Cohen [8] (an infinite number of identical sources with inflow rate 1). Secondly, we relate the tail behaviour of the busy period distributions to the tail behaviour of the activity period distributions. Using the above-mentioned result on the distributions of the busy periods, and applying techniques developed in [17], we prove the following result: The tails of all busy period distributions are regularly varying of index $-\nu$ iff the heaviest of the tails of the activity period distributions is regularly varying of index $-\nu$. More precisely, we provide explicit equivalents of the former in terms of the latter. This shows that the sources whose associated tails are negligible with respect to those of the other sources, can equivalently be ignored if the outflow rate is subsequently reduced by their traffic intensities.

The paper is organized as follows. In Section 2 we first briefly discuss the concept of regular variation, and we present a lemma that plays a key role: It relates the regularly varying tail-behaviour of a distribution and that of its LST in the neighbourhood of 0 (Lemma 2.2). Then we describe our model and introduce the various busy periods under consideration. In Sections 3 and 4, we study the case of only *two* sources. Section 3 is devoted to the derivation of expressions for the LST's of the busy period distributions (Theorem 3.4). The regular variation results for busy periods are derived in Section 4 (Theorem 4.2). In Section 5 we provide the extension of the results in Sections 3 and 4 to an arbitrary number of sources (Theorems 5.2 and 5.3). Section 6 contains concluding remarks, with a brief discussion of the implications of our study for networks of fluid queues.

2 Preliminaries

2.1 Regular variation and Laplace-Stieltjes transforms

Regular variation is an important concept in probability theory and various other fields. The main reference text is the book [3]. A measurable positive function f is called regularly varying of index ζ if, for all $x > 0$,

$$f(xt)/f(t) \to x^{\zeta}, \quad t \to \infty,$$

(cf. [3], p. 18). When $\zeta = 0$, one speaks of a *slowly varying function*; this could for instance be a constant, or a logarithmic function. In this paper, a slowly varying function is denoted by $l(\cdot)$. A basic result that we will often use without mention is:

$$\forall \epsilon > 0 : \quad \frac{l(t)}{t^{\epsilon}} \to 0, \quad t \to \infty. \tag{2}$$

We shall say that a stochastic variable $X \geq 0$ has a regularly varying tail when $\mathbb{P}[X > t]$ is a regularly varying function; an example is provided by the Pareto distribution. If the tail is regularly varying of index $-\nu$, $\nu \in (n, n+1)$ (for some $n \in \mathbb{N}$), then it is easy to see by (2) that $\mathbb{E}[X^n] < \infty$ and $\mathbb{E}[X^{n+1}] = \infty$. Of particular interest is the case that the activity period distribution of an/off source has a regularly varying tail of index $\zeta \in (-2, -1)$. In that case the first moment of the distribution exists, but the variance is infinite. This case is known to give rise to long-range dependence, see below (1).

A crucial property of variables with regularly varying tails is that they may be characterized in terms of their LST. First we need the following lemma (cf. Lemma 1 of [17] and the lines following it).

Lemma 2.1 *Let X be a non-negative random variable with LST $\phi[\omega]$.*
(i) If X has finite moments μ_k of order k, $k = 0, 1, \ldots, n$, then

$$\phi_n[\omega] \;:=\; (-1)^{n+1}\{\phi[\omega] - \sum_{j=0}^{n} \mu_j \frac{(-\omega)^j}{j!}\} \;=\; \mathrm{o}\,(\omega^n), \quad \omega \downarrow 0. \tag{3}$$

(ii) If there exist constants f_j, $j = 0, \ldots, n$, such that

$$\phi[\omega] - \sum_{j=0}^{n} f_j \omega^j = \mathrm{o}\,(\omega^n), \quad \omega \downarrow 0,$$

then $\mu_j < \infty$ and $f_j = (-1)^j \mu_j / j!$.

The next lemma links the behaviour of $\mathbb{P}[X > t]$ for $t \to \infty$ to the behaviour of its LST $\phi[\omega]$ for $\omega \to 0$. It plays a key role in Section 4.

Lemma 2.2 *Let X a non-negative random variable of LST $\phi[\omega]$, $l(t)$ a slowly varying function, $\nu \in (n, n+1)$ ($n \in \mathbb{N}$) and $C \geq 0$. Then the following are equivalent:*

(i) $\mathbb{P}[X > t] = (C + \mathrm{o}\,(1)) l(t)/t^\nu$, $t \to \infty$.

(ii) $\mathbb{E}[X^n] < \infty$ *and* $\phi_n[\omega] = (-1)^n \Gamma(1-\nu)(C + \mathrm{o}\,(1)) l(1/\omega)\omega^\nu$, $\omega \to 0$.

Proof:
Case $C > 0$ is part of Theorem 8.1.6 on p. 333/334 of [3], originally due to Bingham and Doney. (In Theorem 8.1.6 of [3] the somewhat more complicated case $\nu = n$ is also discussed.) Case $C = 0$ is treated in our Appendix. ♠

2.2 Model description

Consider a fluid queueing system with an infinite storage capacity and constant, unit, outflow rate. This system receives input from N independent on/off sources. Source i has mutually independent alternating silence periods S_{in} and activity periods A_{in}, $i = 1, \dots, N$, $n = 1, 2, \dots$. Source i constantly transmits at rate $r_i \geq 1$ when active, so it feeds $r_i A_{in}$ traffic into the buffer during its nth activity period. The silence periods S_{in} are negative exponentially distributed with mean $1/\lambda_i$, and the activity periods A_{in} are i.i.d. with mean $\alpha_i > 0$ (case $\alpha_i = 0$ is trivial) and LST $\alpha_i[\cdot]$.

We shall often use auxiliary parameters that naturally appear in the formulas: the *stationary probability of silence* of source i is

$$p_i := \frac{1}{1 + \lambda_i \alpha_i},$$

and its *traffic intensity* (that is the long-run average amount of fluid it sends per time unit) is

$$\rho_i := r_i(1 - p_i) = r_i \frac{\lambda_i \alpha_i}{1 + \lambda_i \alpha_i}.$$

Similarly, the stationary probability of *total* silence is: $p := \prod_{i=1}^N p_i$, and the *total* traffic intensity is: $\rho := \sum_{i=1}^N \rho_i$.

A busy period of the fluid queue is an uninterrupted period during which there is output leaving the buffer; it is followed by an idle period. Denote by P_n (resp. I_n) the nth busy period (resp. the nth idle period), assuming that they are all almost surely finite. Since silence periods are exponentially distributed, the idle periods are i.i.d. exponential variables, with rate $\lambda := \sum_{i=1}^N \lambda_i$; moreover, they are independent from the busy periods, which are i.i.d. too. Finally, during P_n (resp. during I_n) the queue generates an output at rate 1 (resp. no output), due to our assumption that the active sources send input at rate ≥ 1. Hence the traffic on the output line is as generated by an on/off source of input rate 1 whose active periods would be the busy periods of the fluid queue.

Our goal is to study the effect of one or several sources with active periods of regularly varying tails on the tail behaviour of the busy periods. In what follows, $(P_n)_{n\geq 1}$ (resp. $(P_{in})_{n\geq 1}$) will denote an i.i.d. sequence of busy periods (resp. an i.i.d. sequence of busy periods starting with an activity period of source i). To restrict the use of indices, we shall use the notation $A_{i\cdot}$ for a typical activity period of source i, and similarly $P_{\cdot}$ and $P_{i\cdot}$ for typical busy periods. We introduce their associated LSTs:

$$\pi[\omega] := \mathbb{E}[e^{-\omega P_{\cdot}}], \quad \pi_i[\omega] := \mathbb{E}[e^{-\omega P_{i\cdot}}],\ 1 \le i \le N, \quad \omega \ge 0.$$

Since silence periods are exponentially distributed:

$$\pi[\omega] = \sum_{i=1}^{N} \frac{\lambda_i}{\lambda} \pi_i[\omega]. \tag{4}$$

We first restrict ourselves to the case of *two* sources. This restriction is partly done for the sake of clarity of argument and notation; but the extension from two to an arbitrary number of sources also gives insight into the influence of individual sources on the busy period behaviour. This extension is presented in Section 5. In the next section we characterize the LST's $\pi_1[\omega]$ and $\pi_2[\omega]$.

3 Two sources: The busy period distributions

In this section we do not assume that $\alpha_i < \infty$ nor even $\mathbb{P}[A_{i.} < \infty] = 1$, $i = 1, 2$. The main result of the section is Theorem 3.4, which characterizes the distributions of the busy periods $P_{i.}$, $i = 1, 2$, and $P_{\cdot}$. A big step towards that result is made by the following statement (where notation $X \simeq Y$ denotes two random variables X and Y that follow the same law, and by convention $\sum_{n=1}^{0} = 0$).

Theorem 3.1

$$\begin{cases} P_{1.} \simeq r_1 A_{1.} + \displaystyle\sum_{n=1}^{K_1((r_1-1)A_{1.})} P_{1n} + \sum_{n=1}^{K_2(r_1 A_{1.})} P_{2n}, \\ \\ P_{2.} \simeq r_2 A_{2.} + \displaystyle\sum_{n=1}^{K_2((r_2-1)A_{2.})} P_{2n} + \sum_{n=1}^{K_1(r_2 A_{2.})} P_{1n}. \end{cases} \tag{5}$$

In the above expressions, the sequences $(P_{in})_{n\ge1}$ and the processes $(K_i(t))_{t\ge0}$, $i = 1, 2$, are independent of each other and of $A_{i.}$, $i = 1, 2$; and $(K_i(t))_{t\ge0}$ is the counting function of a Poisson process of intensity λ_i, $i = 1, 2$.

For the ordinary M/G/1 queue, the typical busy period $\tilde{P}_.$ satisfies:

$$\tilde{P}_. \simeq B_. + \sum_{j=1}^{K(B_.)} \tilde{P}_j, \tag{6}$$

with $B_.$ denoting service time and $(K(t))_{t\geq 0}$ being the counting function of the Poisson arrival process. The usual proof of (6), cf. [9], p. 250, is based on the observation that the order of service does not affect the busy period length, as long as the service discipline is work conserving. A suitable service discipline, like last-come-first-served preemptive resume, then yields (6).

In proving Theorem 3.1 we'd like to use a similar change-of-service-order discipline, cf. also [15] for the case $r_i \equiv 1$. The intuition behind (5) is that $P_{i.}$ contains at least a period $r_i A_{i.}$, and that source i can become active again during its last part $(r_i - 1)A_{i.}$, whereas the other source can become active during the whole period $r_i A_{i.}$; and such new activity periods of a source j give rise to independent busy periods $P_{j.}$. However, the proof of the latter part of the above statement is more delicate than in the ordinary $M/G/1$ queue.

Below we provide a proof of Theorem 3.1 based on two essential lemmas. Following the notations of [15], we denote by $Z(t)$ the content of the buffer at time $t \geq 0$, and by $Y_i(t)$ the residual activity time of source i at time t (with $Y_i(t) = 0$ if the source is silent), $i = 1, 2$. Let $T_{12}(z, y_1, y_2)$ be the residual busy time of the queue when $Z(0) = z$, $Y_1(0) = y_1$, $Y_2(0) = y_2$. Moreover set: $T_1(z, y) = T_{12}(z, y, 0)$, $T_2(z, y) = T_{12}(z, 0, y)$, and $T(z) = T_{12}(z, 0, 0)$. They are related to the busy periods by: $P_{i.} \simeq T_i(0, A_{i.})$, $i = 1, 2$. As a preliminary to the proof of Theorem 3.1, we need to prove the following two lemmas.

Lemma 3.2

$$T(z) \simeq z + \sum_{n=1}^{K_1(z)} P_{1n} + \sum_{n=1}^{K_2(z)} P_{2n}, \tag{7}$$

with the same conventions as in Theorem 3.1.

Proof:
A similar result has already been proved by Rubinovitch in [20]. If no activity period starts before time z, then $T(z) = z$. Otherwise, a source starts sending input at some time $z_1 < z$. Then we may assume that the processing of the

residual volume $z - z_1$ is interrupted, and the queue treats the new input until the first time $t > z_1$ such that $Z(t) = z - z_1$ and the two sources are silent again. Clearly, the time we have to wait until this event occurs is distributed as $P_.$. After that, the processing of the residual input $z - z_1$ can be restarted, possible interruptions being handled as the first one. Hence the successive interruptions form an i.i.d. sequence $(P_n)_{n\geq 1}$, and the number of interruptions is of the form $K(z)$, where $(K(t))_{t\geq 0}$ is the counting function of an independent Poisson process of intensity λ. We thus obtain:

$$T(z) \simeq z + \sum_{n=1}^{K(z)} P_n,$$

which yields Equation (7) in view of relation (4). ♠

Now we give a new formulation and a rigorous proof of an argument that was first introduced in [20] and used again in [15].

Lemma 3.3

$$T_{12}(z_1 + z_2, y_1, y_2) \simeq T_1(z_1, y_1) + T_2(z_2, y_2),$$

where $T_1(z_1, y_1)$ and $T_2(z_2, y_2)$ are independent.

Proof:
Denote by $r_i(t)$ the input rate at time t from source i, $i = 1, 2$, and $r(t) = r_1(t) + r_2(t)$. Now modify source 2 by skipping the residual activity time y_2 and starting with the next silence period; equivalently we replace $r_2(t)$ by $\tilde{r}_2(t) = r_2(y_2 + t)$, $t \geq 0$. Then wait until the first time $\tau_1 \geq 0$ such that $Z(\tau_1) = z_2$ and the two sources are silent again. Now modify source 2 again by re-inserting the residual activity period y_2 after time τ_1; equivalently replace $\tilde{r}_2(t)$ by $\approx{r}_2(t)$, with

$$\begin{cases} \stackrel{\approx}{r}_2(t) = \tilde{r}_2(t) = r_2(y_2 + t), \quad 0 \leq t < \tau_1, \\ \stackrel{\approx}{r}_2(t) = r_2, \quad \tau_1 \leq t < \tau_1 + y_2, \\ \stackrel{\approx}{r}_2(t) = \tilde{r}_2(t - y_2) = r_2(t), \quad t \geq \tau_1 + y_2. \end{cases} \tag{8}$$

Finally denote by τ_2 the residual busy period from time τ_1 for the system with the modified input $\stackrel{\approx}{r}(t) = r_1(t) + \stackrel{\approx}{r}_2(t)$, $t \geq 0$. Because all the silence periods are exponentially distributed, it is easy to check that $(\stackrel{\approx}{r}(t))_{0\leq t<\tau_1}$ and $(\stackrel{\approx}{r}(\tau_1+t))_{0\leq t<\tau_2}$ are independent; the distribution of the first process corresponds to the case when source 2 is initially silent, and $Y_1(0)=y_1$; the second term corresponds to the case when source 1 is initially silent, and $Y_2(0)=y_2$. Proceeding, we find that τ_1 and τ_2 are independent, with

$$\tau_1 \simeq T_1(z_1,y_1), \quad \tau_2 \simeq T_2(z_2,y_2).$$

What remains to be proved is that $T_{12}(z_1+z_2,y_1,y_2)=\tau_1+\tau_2$, or equivalently:

$$\inf\{t\geq 0 | z+\int_0^t (r(u)-1)du<0\} = \inf\{t\geq 0 | z+\int_0^t (\stackrel{\approx}{r}(u)-1)du<0\}.$$

But from the definition of τ_1 and relations (8), we see that:

$$\begin{cases} \forall t\leq \tau_1+y_2: \; z_1+\int_0^t r(u)du \geq z_1+\int_0^t \stackrel{\approx}{r}(u)du \geq t, \\ \forall t\geq \tau_1+y_2: \; \int_0^t r(u)du = \int_0^t \stackrel{\approx}{r}(u)du, \end{cases}$$

which completes the proof. ♠

Proof of Theorem 3.1:
Since $P_{1.}\simeq T_1(0,A_{1.})$ and the two sources play symmetric roles, it is sufficient to show that:

$$T_1(z,y)\simeq z+r_1y+\sum_{n=1}^{K_1(z+(r_1-1)y)} P_{1n}+\sum_{n=1}^{K_2(z+r_1y)} P_{2n} \tag{9}$$

(with the notations of (5)). So assume that $Z(0)=z$, $Y_1(0)=y$ and $Y_2(0)=0$. Source 2 remains silent until time τ_1, with $\tau_1\simeq S_{21}$; its first activity period is of length A_{21}.

If $\tau_1>y$, then clearly:

$$T_1(z,y)\simeq y+T(z+(r_1-1)y).$$

On the contrary, if $\tau_1\leq y$:

$$\begin{aligned} T_1(z,y) &\simeq \tau_1+T_{12}(z+(r_1-1)\tau_1, y-\tau_1, A_{21}) \\ &\simeq \tau_1+T_1(z+(r_1-1)\tau_1, y-\tau_1)+T_2(0,A_{21}), \end{aligned}$$

by Lemma 3.3. Moreover: $T_2(0, A_{21}) \simeq P_{21}$.

If we replicate τ_1 to form an i.i.d. sequence $(\tau_n)_{n\geq 1}$, then by iterating the above argument we finally obtain:

$$T_1(z,y) \simeq y + \sum_{n\geq 1} P_{2n} \mathbb{1}_{\{\tau_1 + \ldots + \tau_n \leq y\}} + T(z + (r_1 - 1)y),$$

or equivalently:

$$T_1(z,y) \simeq y + \sum_{n=1}^{K_2(y)} P_{2n} + T(z + (r_1 - 1)y),$$

(with the notations of Theorem 3.1). This yields (9) in view of Lemma 3.2. ♠

The next theorem shows that the obtained relations are sufficient to characterize the laws of $P_{i.}$, $i = 1, 2$; it also provides the conditions of finiteness and the formulas of the first moments.

Theorem 3.4 *In terms of Laplace-Stieltjes transforms, $(\pi_1[\omega], \pi_2[\omega])$ is for $\omega > 0$ the unique solution in $[0,1] \times [0,1]$ of the system of equations:*

$$\left\{ \begin{array}{l} \pi_1[\omega] = \alpha_1[r_1\omega + \lambda_1(r_1 - 1)(1 - \pi_1[\omega]) + \lambda_2 r_1(1 - \pi_2[\omega])], \\ \pi_2[\omega] = \alpha_2[r_2\omega + \lambda_2(r_2 - 1)(1 - \pi_2[\omega]) + \lambda_1 r_2(1 - \pi_1[\omega])]. \end{array} \right. \tag{10}$$

Furthermore, for $i = 1$ or 2, $P_{i.}$ is a.s. finite if and only if $A_{1.}$ and $A_{2.}$ are and $\rho \leq 1$; if this is the case:

$$\mathbb{E}[P_{i.}] = \frac{\rho_i}{\lambda_i(1-\rho)} \quad (= \infty \text{ if } \rho = 1), \;\; i = 1, 2, \quad \mathbb{E}[P_{.}] = \frac{\rho}{\lambda(1-\rho)}.$$

Remark 3.5

- *In [15] uniqueness had only been proved among solutions that are Laplace-Stieltjes transforms.*
- *Condition $\rho < 1$ is the ergodicity condition of the buffer content process, cf. [10].*

Proof of Theorem 3.4:
The right-hand side of (10) is obtained by integrating formula (9) (with $z = 0$) with respect to y according to the distribution of A_{11} (and similarly for source 2).

To prove that the system of equations (10) admits a unique solution, we first construct a *minimal* solution. Copying the argument of Feller for the M/G/1 queue (see [12], Section XIII.4), we set: $\xi_{i0}[\omega] = 0$, $i = 1, 2$, and for $n \geq 0$:

$$\begin{cases} \xi_{1(n+1)}[\omega] = \alpha_1[r_1\omega + \lambda_1(r_1 - 1)(1 - \xi_{1n}[\omega]) + \lambda_2 r_1(1 - \xi_{2n}[\omega])], \\ \xi_{2(n+1)}[\omega] = \alpha_2[r_2\omega + \lambda_2(r_2 - 1)(1 - \xi_{2n}[\omega]) + \lambda_1 r_2(1 - \xi_{1n}[\omega])]. \end{cases}$$

Then by an immediate induction we find: $\xi_{in}[\omega] \leq \xi_{i(n+1)}[\omega] \leq 1$ for all n, hence $(\xi_{in}[\omega])_{n\in\mathbb{N}}$ tends to a limit $\xi_i[\omega]$ for $i = 1, 2$, and $(\xi_1[\omega], \xi_2[\omega])$ is a solution of (10) in $[0, 1] \times [0, 1]$. (It can also be proved that the $\xi_i[\omega]$ are Laplace transforms as limits of sequences of completely monotone functions, see [12], but we don't need this result here.)

Moreover, the same inductive argument shows that *any* solution $(\tilde{\xi}_1[\omega], \tilde{\xi}_2[\omega])$ of (10) in $[0, 1] \times [0, 1]$ must satisfy: $\tilde{\xi}_i[\omega] \geq \xi_i[\omega]$, $i = 1, 2$. Now assume that $\omega > 0$ and, say, $\tilde{\xi}_1[\omega] > \xi_1[\omega]$, and w.l.o.g.:

$$T := (1 - \xi_1[\omega])/(\tilde{\xi}_1[\omega] - \xi_1[\omega]) \leq (1 - \xi_2[\omega])/(\tilde{\xi}_2[\omega] - \xi_2[\omega]).$$

Notice that $T > 1$ (because $\omega > 0$ and $\tilde{\xi}_1[\omega] \leq \alpha_1[r_1\omega] < 1$) and set:

$$\begin{cases} f_1(\omega, x_1, x_2) = \alpha_1[r_1\omega + \lambda_1(r_1 - 1)(1 - x_1) + \lambda_2 r_1(1 - x_2)] - x_1, \\ F(t) = f_1(\omega, (1 - t)\xi_1[\omega] + t\tilde{\xi}_1[\omega], (1 - t)\xi_2[\omega] + t\tilde{\xi}_2[\omega]). \end{cases}$$

Then it is easy to check that $F(t)$ is a convex function, which is defined on $[0, T]$. But $F(0) = F(1) = 0$, and $F(T) \leq \alpha_1[r_1\omega] - 1 < 0$, hence a contradiction. Thus for all $\omega > 0$, there is only one solution of (10), namely $(\pi_1[\omega], \pi_2[\omega])$.

To obtain the conditions of finiteness and the first moments, we use the approach of Kaspi and Rubinovitch in [15]. First denote $t[\omega] = (t_1[\omega], t_2[\omega])$, where:

$$\begin{cases} t_1[\omega] = r_1\omega + \lambda_1(r_1 - 1)(1 - \pi_1[\omega]) + \lambda_2 r_1(1 - \pi_2[\omega]), \\ t_2[\omega] = r_2\omega + \lambda_2(r_2 - 1)(1 - \pi_2[\omega]) + \lambda_1 r_2(1 - \pi_1[\omega]). \end{cases} \quad (11)$$

By differentiation of (10) for $\omega > 0$, we obtain:

$$\begin{cases} \pi_1'[\omega] = \alpha_1'[t_1](r_1 - \lambda_1(r_1-1)\pi_1'[\omega] - \lambda_2 r_1 \pi_2'[\omega]), \\ \pi_2'[\omega] = \alpha_2'[t_2](r_2 - \lambda_2(r_2-1)\pi_2'[\omega] - \lambda_1 r_2 \pi_1'[\omega]). \end{cases}$$

The solution of these two linear equations is:

$$-(1-\rho[t])\pi_i'[\omega] = \frac{\rho_i[t_i]}{\lambda_i}, \tag{12}$$

with $\rho_i[x_i] = -r_i\lambda_i\alpha_i'[x_i]/(1-\lambda_i\alpha_i'[x_i])$, $i = 1,2$, $\rho[x] = \rho_1[x_1] + \rho_2[x_2]$, $x = (x_1, x_2)$.

If $P_{1.}$ is a.s. finite, then $\pi_1[0+] = 1$, which by (10) implies that $\pi_2[0+] = 1$ and $\alpha_1[0+] = 1$: thus $P_{2.}$ and $A_{1.}$ are a.s. finite, and for the same reason $A_{2.}$ is a.s. finite too. Hence $t_i[\omega] \to 0$ and $\rho_i[t_i] \to \rho_i$ when $\omega \to 0$, and consequently:

$$\mathbb{E}[P_{i.}] = \lim_{\omega\to 0}(-\pi_i'[\omega]) = \rho_i/\lambda_i(1-\rho), \quad i = 1,2.$$

This shows that ρ must be smaller than or equal to one (because $\rho_i > 0$ and $\mathbb{E}[P_{i.}] \geq 0$) and provides the formulas of the first moments ($\mathbb{E}[P_.]$ comes from (4)).

If $P_{1.}$ or $P_{2.}$ is not a.s. finite, set:

$$F_i(t) = f_i(0, (1-t)\pi_1[0+] + t, (1-t)\pi_2[0+] + t), \quad i = 1,2,$$

(with f_2 defined like f_1 above). Since $F_i(0) = F_i(1) = 0$, then $F_i'(\theta_i) = 0$ for some $\theta_i \in (0,1)$, which yields:

$$\begin{cases} 1-\pi_1[0+] = \alpha_1'[\psi_1](-\lambda_1(r_1-1)(1-\pi_1[0+]) - \lambda_2 r_1(1-\pi_2[0+])), \\ 1-\pi_2[0+] = \alpha_2'[\psi_2](-\lambda_2(r_2-1)(1-\pi_2[0+]) - \lambda_1 r_2(1-\pi_1[0+])), \end{cases}$$

with

$$\begin{cases} \psi_1/(1-\theta_1) = \lambda_1(r_1-1)(1-\pi_1[0+]) + \lambda_2 r_1(1-\pi_2[0+]), \\ \psi_2/(1-\theta_2) = \lambda_2(r_2-1)(1-\pi_2[0+]) + \lambda_1 r_2(1-\pi_1[0+]). \end{cases}$$

Similarly as (12) was derived, we find:

$$-(1-\rho[\psi])(1-\pi_i[0+]) = 0, \quad i = 1,2,$$

where $\psi = (\psi_1, \psi_2)$. Since $\pi_i[0+] < 1$ for $i = 1$ or 2, we obtain on the one hand: $\rho[\psi] = 1$; on the other hand, if $A_{1.}$ and $A_{2.}$ are a.s. finite, then $\rho[0+] = \rho$, hence: $\rho[\psi] < \rho$ (since $\psi \neq 0$), which completes the proof. ♠

4 Regularly varying tails

In this section we relate the tail behaviour of the busy period distributions to the tail behaviour of the activity period distributions in the case when one or more activity period distributions have regularly varying tails. Throughout this section we aim to extend arguments of [17] for the tail behaviour of the busy period distribution of the ordinary $M/G/1$ queue to the fluid queue with two sources. From now on *we assume that* $A_{1.}$ *and* $A_{2.}$ *are a.s. finite and* $\rho < 1$.

Starting-point for relating the tail behaviour of the busy period distributions to the tail behaviour of the activity period distributions is Theorem 3.4. We rewrite the relations (10) into: for $\omega \geq 0$,

$$\begin{cases} \pi_1[\omega] = \alpha_1[t_1], \\ \pi_2[\omega] = \alpha_2[t_2], \end{cases} \tag{13}$$

with $t_i[\omega]$, $i = 1, 2$, being defined in (11). Since it is assumed that $\rho < 1$, it follows that

$$\pi_i := \mathbb{E}[P_{i.}] = \frac{\rho_i}{\lambda_i(1-\rho)} < \infty, \quad i = 1, 2,$$

so Lemma 2.1 (i) with $n = 1$ implies that, for $i = 1, 2$:

$$t_i[\omega] = \frac{r_i p_i}{1-\rho}\omega + \mathrm{o}(\omega) = \frac{\pi_i}{\alpha_i}\omega + \mathrm{o}(\omega), \quad \omega \downarrow 0. \tag{14}$$

Use the notation of Lemma 2.1 to introduce $\alpha_{1n}[t_1]$, $\alpha_{2n}[t_2]$, $\pi_{1n}[\omega]$ and $\pi_{2n}[\omega]$, $n = 1, 2, \ldots$ if their corresponding first n moments are finite; in particular, $\pi_{i1}[\omega] := \pi_i[\omega] - 1 + \pi_i\omega$, $i = 1, 2$. An easy calculation, based on (13) and (11), shows that

$$\begin{cases} \pi_{11}[\omega] = c_{11}\alpha_{11}[t_1] + c_{12}\alpha_{21}[t_2], \\ \\ \pi_{21}[\omega] = c_{21}\alpha_{11}[t_1] + c_{22}\alpha_{21}[t_2]. \end{cases} \tag{15}$$

Here $c_{11} := \frac{p_1(1-\rho_2)}{1-\rho}$, $c_{12} := p_2\frac{\lambda_2}{\lambda_1}\frac{\rho_1}{1-\rho}$, $c_{21} := p_1\frac{\lambda_1}{\lambda_2}\frac{\rho_2}{1-\rho}$, and $c_{22} := \frac{p_2(1-\rho_1)}{1-\rho}$. Inverting these relations (note that the determinant equals $p_1p_2/(1-\rho)$) yields:

$$\begin{cases} \alpha_{11}[t_1] = d_{11}\pi_{11}[\omega] + d_{12}\pi_{21}[\omega], \\ \\ \alpha_{21}[t_2] = d_{21}\pi_{11}[\omega] + d_{22}\pi_{21}[\omega]. \end{cases} \tag{16}$$

Here $d_{11} := \frac{1-\rho_1}{p_1}$, $d_{12} := -\frac{\lambda_2}{\lambda_1}\frac{\rho_1}{p_1}$, $d_{21} := -\frac{\lambda_1}{\lambda_2}\frac{\rho_2}{p_2}$, and $d_{22} := \frac{1-\rho_2}{p_2}$.

We are now ready to prove the following theorem, which is an extension of Lemma 3 and Corollary 1 of [17].

Theorem 4.1 *For $n \geq 1$, the following two statements are equivalent:*

(i) $\mathbb{E}[A_{1.}^n] < \infty$ *and* $\mathbb{E}[A_{2.}^n] < \infty$; **(ii)** $\mathbb{E}[P_{1.}^n] < \infty$ *and* $\mathbb{E}[P_{2.}^n] < \infty$.

In addition, both **(i)** *and* **(ii)** *imply that, for $\omega \downarrow 0$:*

$$\begin{cases} \pi_{1n}[\omega] = c_{11}\alpha_{1n}[t_1] + c_{12}\alpha_{2n}[t_2] + \mathrm{O}\,(\omega^{n+1}), \\ \\ \pi_{2n}[\omega] = c_{21}\alpha_{1n}[t_1] + c_{22}\alpha_{2n}[t_2] + \mathrm{O}\,(\omega^{n+1}), \end{cases} \tag{17}$$

$$\begin{cases} \alpha_{1n}[t_1] = d_{11}\pi_{1n}[\omega] + d_{12}\pi_{2n}[\omega] + \mathrm{O}\,(t_1^{n+1}), \\ \\ \alpha_{2n}[t_2] = d_{21}\pi_{1n}[\omega] + d_{22}\pi_{2n}[\omega] + \mathrm{O}\,(t_2^{n+1}). \end{cases} \tag{18}$$

Proof:
(i) $\Rightarrow$ **(ii)**. Since $\rho < 1$, the result is valid for $n = 1$. Using induction, assume that the result has been shown for $k = 1, \ldots, n-1$ and that $\mathbb{E}[A_{1.}^n] < \infty$ and $\mathbb{E}[A_{2.}^n] < \infty$; hence (3) holds for $\pi_{1,n-1}[\omega]$ and $\pi_{2,n-1}[\omega]$, while (3) is also assumed to hold for $\alpha_{1n}[\omega]$ and $\alpha_{2n}[\omega]$. Hence, cf. (15) and (14), for $\omega \downarrow 0$,

$$\begin{aligned} \pi_1[\omega] = 1 - \mathbb{E}[P_{1.}]\omega + c_{11}\alpha_{11}[t_1] + c_{12}\alpha_{21}[t_2] \ &= \\ 1 - \mathbb{E}[P_{1.}]\omega + c_{11}\sum_{k=2}^{n}\mathbb{E}[A_{1.}^k]\frac{(-t_1)^k}{k!} + c_{12}\sum_{k=2}^{n}\mathbb{E}[A_{2.}^k]\frac{(-t_2)^k}{k!} + \mathrm{o}\,(\omega^n)&, \end{aligned} \tag{19}$$

and a similar relation holds for $\pi_2[\omega]$.

Now use (11) and the induction assumption about the finiteness of the first $n-1$ moments of $P_{1.}$ and $P_{2.}$, to write for $\omega \downarrow 0$:

$$t_i[\omega] \ = \ \sum_{j=1}^{n-1} z_{ij}\omega^j + \mathrm{o}\,(\omega^{n-1}), \quad i = 1, 2, \tag{20}$$

and hence for $k \geq 2$:

$$t_i^k[\omega] \ = \ \sum_{j=k}^{n} z_{ij}^{(k)}\omega^j + \mathrm{o}\,(\omega^n), \quad i = 1, 2. \tag{21}$$

Substitution in (19) shows that there exist constants g_{1j}, $j = 0, \ldots, n$ such that

$$\pi_1[\omega] - \sum_{j=0}^{n} g_{1j}\omega^j \; = \; \mathrm{o}\,(\omega^n), \quad \omega \downarrow 0.$$

Lemma 2.1 (ii) now implies that $\mathbb{E}[P_{1.}^n] < \infty$. In exactly the same way the finiteness of $\mathbb{E}[P_{2.}^n]$ is derived.

(ii) ⇒ (i). The proof proceeds similarly as the proof of the reverse statement. Since $\rho < 1$, the result is valid for $n = 1$. Using induction, assume that the result has been shown for $k = 1, \ldots, n-1$ and that $\mathbb{E}[P_{1.}^n] < \infty$ and $\mathbb{E}[P_{2.}^n] < \infty$; hence (3) holds for $\pi_{1n}[\omega]$ and $\pi_{2n}[\omega]$. Hence, cf. (16), for $\omega \downarrow 0$,

$$\begin{aligned} \alpha_1[t_1] - 1 + \alpha_1 t_1 = & \\ & d_{11}\left((-1)^{n+1}\pi_{1n}[\omega] + \sum_{k=2}^{n}\mathbb{E}[P_{1.}^k]\frac{(-\omega)^k}{k!}\right) \\ & +d_{12}\left((-1)^{n+1}\pi_{2n}[\omega] + \sum_{k=2}^{n}\mathbb{E}[P_{2.}^k]\frac{(-\omega)^k}{k!}\right). \end{aligned} \tag{22}$$

A similar relation holds for $\alpha_2[t_2]$.

Now use (11) and the induction assumption to express $t_1[\omega]$ into powers $\omega^1, \ldots, \omega^n$. Note that the assumption on the finiteness of the first n moments of the busy periods allows extending the sum in (20) with a term $z_{in}\omega^n$:

$$t_i[\omega] \;=\; \sum_{j=1}^{n} z_{ij}\omega^j + \mathrm{o}\,(\omega^n), \quad i = 1, 2.$$

Since $t_1[\omega]$ is increasing in ω, we can invert, expressing ω into powers of t_1: for $t_1 \downarrow 0$,

$$\omega(t_1) \;=\; \sum_{j=1}^{n} v_{1j} t_1^j + \mathrm{o}\,(t_1^n),$$

and hence, for $t_1 \downarrow 0$,

$$\omega^k(t_1) \;=\; \sum_{j=k}^{n+1} v_{1j}^{(k)} t_1^j + \mathrm{o}\,(t_1^{n+1}).$$

Substitution in (22) shows that there exist constants h_{1j}, $j = 0, \ldots, n$ such that

$$\alpha_1[t_1] - \sum_{j=0}^{n} h_{1j} t_1^j \;=\; d_{11}(-1)^{n+1}\pi_{1n}[\omega] + d_{12}(-1)^{n+1}\pi_{2n}[\omega] + \mathrm{O}\,(t_1^{n+1}), \quad t_1 \downarrow 0.$$

By Lemma 2.1 (i): $\pi_{in}[\omega] = \mathrm{o}\,(\omega^n)$, or equivalently: $\pi_{in}[\omega] = \mathrm{o}\,(t_1^n)$ in view of Formula (14), $i = 1, 2$. Lemma 2.1 (ii) then implies that $\mathbb{E}[A_{1.}^n] < \infty$, and moreover:

$$\alpha_{1n}[t_1] = d_{11}\pi_{1n}[\omega] + d_{12}\pi_{2n}[\omega] + \mathrm{O}\,(t_1^{n+1}), \quad \omega \downarrow 0.$$

In exactly the same way the finiteness of $\mathbb{E}[A_{2.}^n]$ is derived and Formula (18) follows. By inversion we obtain (17). ♠

We are now ready to study the effect of regularly varying tails of activity period distributions on the tail behaviour of the busy period distributions.

Theorem 4.2 *Assume that as $t \to \infty$:*

$$\mathbb{P}[A_{i.} > (\alpha_i/\pi_i)t] = (a_i + \mathrm{o}\,(1))l(t)/t^\nu, \tag{23}$$

with $a_i \geq 0$, $i = 1, 2$, $\nu \in (n, n+1)$ (for some $n \geq 1$) and $l(t)$ a slowly varying function. Then as $t \to \infty$:

$$\begin{cases} \mathbb{P}[P_{1.} > t] = \left(\dfrac{(1-\rho_2)p_1\lambda_1 a_1 + \rho_1 p_2 \lambda_2 a_2}{(1-\rho)\lambda_1} + \mathrm{o}\,(1) \right) \dfrac{l(t)}{t^\nu}, \\ \\ \mathbb{P}[P_{2.} > t] = \left(\dfrac{(1-\rho_1)p_2\lambda_2 a_2 + \rho_2 p_1 \lambda_1 a_1}{(1-\rho)\lambda_2} + \mathrm{o}\,(1) \right) \dfrac{l(t)}{t^\nu}. \end{cases}$$

Conversely, assume that

$$\mathbb{P}[P_{i.} > t] = (c_i + \mathrm{o}\,(1))l(t)/t^\nu, \quad t \to \infty,$$

with $c_i \geq 0$ ($i = 1, 2$), $\nu \in (n, n+1)$ (for some $n \geq 1$) and $l(t)$ a slowly varying function. Then necessarily:

$$(1-\rho_1)(\lambda_1 c_1) - \rho_1(\lambda_2 c_2) \geq 0, \;\; (1-\rho_2)(\lambda_2 c_2) - \rho_2(\lambda_1 c_1) \geq 0. \tag{24}$$

Moreover, either $c_1 = c_2 = 0$ or $c_1 > 0$ and $c_2 > 0$, and in the latter case at least one of the above expressions is positive. Finally, for $t \to \infty$:

$$\begin{cases} \mathbb{P}[A_{1.} > (\alpha_1/\pi_1)t] = \left(\dfrac{(1-\rho_1)(\lambda_1 c_1) - \rho_1(\lambda_2 c_2)}{\lambda_1 p_1} + \mathrm{o}\,(1)\right)\dfrac{l(t)}{t^\nu}, \\ \\ \mathbb{P}[A_{2.} > (\alpha_2/\pi_2)t] = \left(\dfrac{(1-\rho_2)(\lambda_2 c_2) - \rho_2(\lambda_1 c_1)}{\lambda_2 p_2} + \mathrm{o}\,(1)\right)\dfrac{l(t)}{t^\nu}. \end{cases}$$

Corollary 4.3 *Under assumption (23) of Theorem 4.2:*

$$\mathbb{P}[P_{.} > t] = \left(\frac{p_1\lambda_1 a_1 + p_2\lambda_2 a_2}{(1-\rho)\lambda} + \mathrm{o}\,(1)\right)\frac{l(t)}{t^\nu}.$$

Proof:
Immediate by Formula (4). ♠

Proof of Theorem 4.2:
Assumption (23) is equivalent to: $\mathbb{P}[A_{i.} > t] = (a_i + \mathrm{o}\,(1))l(t)(\pi_i t/\alpha_i)^{-\nu}$ (which implies that $\mathbb{E}[A_{i.}^n] < \infty$), $i = 1, 2$. So it follows from Lemma 2.2 and Formula (14) that as $\omega \to 0$:

$$\begin{aligned} \alpha_{in}[t_i] &= (-1)^n\Gamma(1-\nu)(a_i + \mathrm{o}\,(1))(\alpha_i t_i/\pi_i)^\nu l(1/t_i) \\ &= (-1)^n\Gamma(1-\nu)(a_i + \mathrm{o}\,(1))\omega^\nu l(1/\omega), \quad i = 1, 2. \end{aligned}$$

Using Formula (17) of Theorem 4.1 we finally obtain:

$$\pi_{in}[\omega] = (-1)^n\Gamma(1-\nu)(c_{i1}a_1 + c_{i2}a_2 + \mathrm{o}\,(1))l(1/\omega)\omega^\nu, \quad i = 1, 2.$$

In view of Lemma 2.2, the proof of the direct part is completed. For the converse part, similar arguments yield, for $\omega \to 0$:

$$\alpha_{in}[\omega] = (-1)^n\Gamma(1-\nu)(d_{i1}c_1 + d_{i2}c_2 + \mathrm{o}\,(1))l(1/\omega)(\pi_i\omega/\alpha_i)^\nu, \quad i = 1, 2.$$

Since $\alpha_{in}[\omega] \geq 0$ for all $\omega \geq 0$, $i = 1, 2$, the factors of $l(1/\omega)\omega^\nu$ are necessarily non-negative. It can easily be checked that they can be both null only if $c_1 = c_2 = 0$ (because $\rho < 1$); from conditions (24) is clear that $c_1 > 0$ implies $c_2 > 0$ and conversely. The conclusion finally comes from Lemma 2.2. ♠

Remark 4.4 *For the ordinary $M/G/1$ queue, De Meyer and Teugels [17] have proven the following result: The tail of the distribution of the busy period $P_{M/G/1}$ is regularly varying of index $-\nu$ iff the tail of the distribution of the service time $B_{M/G/1}$ is regularly varying of index $-\nu$, and then*

$$\mathbb{P}[P_{M/G/1} > t] \overset{t\to\infty}{\sim} \frac{1}{1-\rho}\mathbb{P}[B_{M/G/1} > (1-\rho)t].$$

Theorem 4.2 and its corollary show that a similar result holds for the fluid queue with 2 on/off sources. In particular, the regularly varying behaviour of the heaviest *of the activity period tails is related to the regularly varying behaviour of* all *the busy period tails, with the same index; and again a factor $(1-\rho)^{-(\nu+1)}$ appears in the quotient of the tails of the busy period and activity period distributions.*

Remark 4.5 *In Theorem 4.2 we have refrained from discussing the - mathematically intricate - case of ν being integer. We refer to De Meyer and Teugels [17] for a discussion of the tail behaviour of the $M/G/1$ busy period for integer ν.*

5 Extension to N sources ($N \geq 2$)

5.1 The distributions of the busy periods in the general case

In order to extend the results obtained for two on/off sources to the case of $N \geq 2$ sources, we are going to use an argument of work-conservation. Since the speed at which the buffer size increases at some time t is completely determined by which sources are active at t, some specific sources may be given priority for transmission by the queue without affecting the busy periods. So considering our model with N sources and isolating some source i, we may assume that sources j, $j \neq i$, are given preemptive priority over source i. This means that sources $j \neq i$ are treated as if source i were absent, hence they generate their own busy periods $P_n^{(i)}$, namely the busy periods of the model without source i. As for source i, it is served only during the corresponding idle periods $I_n^{(i)}$.

As far as busy periods are concerned, everything thus works as if all the sources $j \neq i$ were replaced by a single on/off source i' (with preemptive priority over i) of activity periods $A_{i'n} = P_n^{(i)}$, input rate $r_{i'} = 1$, and exponential silence periods $S_{i'n} = I_n^{(i)}$ of rate $\lambda_{i'} = \sum_{j \neq i} \lambda_j$. Besides, the busy periods $P_{i'n}$ starting with an activity period of source i' are just the busy periods starting with an activity period of some source $j \neq i$. So if we set $\pi_{i'}[\omega] = \mathbb{E}[e^{-\omega P_{i'}.}]$, we have:

$$\pi_{i'}[\omega] = \sum_{j \neq i} \frac{\lambda_j}{\lambda_{i'}} \pi_j[\omega]. \tag{25}$$

From Theorem 3.1, we may now write:

$$P_{i.} \simeq r_i A_{i.} + \sum_{n=1}^{K_i((r_i-1)A_{i.})} P_{in} + \sum_{n=1}^{K_{i'}(r_i A_{i.})} P_{i'n}.$$

Using decomposition (25), we obtain:

Theorem 5.1 *For all i:*

$$P_{i.} \simeq r_i A_{i.} + \sum_{n=1}^{K_i((r_i-1)A_{i.})} P_{in} + \sum_{j \neq i} \sum_{n=1}^{K_j(r_i A_{i.})} P_{jn}; \tag{26}$$

here the sequences $(P_{jn})_{n \geq 1}$ and the processes $(K_j(t))_{t \geq 0}$, $1 \leq j \leq N$, are independent of each other and of $A_{i.}$, and for $1 \leq j \leq N$, $(K_j(t))_{t \geq 0}$ is the counting function of a Poisson process of intensity λ_j.

Proceeding, we are going to show the following generalization of Theorem 3.4.

Theorem 5.2 *In terms of Laplace-Stieltjes transforms, $(\pi_i[\omega])_{1 \leq i \leq N}$ is for $\omega > 0$ the unique solution in $[0,1]^N$ of the system of equations:*

$$\pi_i[\omega] = \alpha_i[r_i\omega + \lambda_i(r_i - 1)(1 - \pi_i[\omega]) + \sum_{j \neq i} \lambda_j r_i (1 - \pi_j[\omega])], \quad 1 \leq i \leq N. \tag{27}$$

Furthermore, for all i, $P_{i.}$ is a.s. finite if and only if all the $A_{j.}$'s are and $\rho \leq 1$; if this is the case:

$$\mathbb{E}[P_{i.}] = \frac{\rho_i}{\lambda_i(1-\rho)} \quad (= \infty \text{ if } \rho = 1),\ 1 \leq i \leq N, \quad \mathbb{E}[P_.] = \frac{\rho}{\lambda(1-\rho)}.$$

Proof:
Equation (27) is just the LST form of (26). We shall obtain uniqueness, conditions of finiteness and means by induction on N. Everything has been proved for a system with $N = 2$ sources in Theorem 3.4. Assume that the results are valid for any system with $N - 1$ sources, and consider a solution $(\xi_j[\omega])_{1\leq j\leq N}$ of the system with N sources, for some $\omega > 0$. Then clearly $(\xi_j[\omega])_{j\neq i}$ is a solution of the system obtained by deleting source i, but for $\omega' = \omega + \lambda_i(1 - \xi_i[\omega])$ instead of ω. By the induction hypothesis, this system of $N - 1$ equations admits a unique solution in $[0, 1]^{N-1}$, namely given by the LST $\pi_j^{(i)}[\omega]$ of the busy periods $P_j^{(i)}$, $j \neq i$, for the model without source i. Therefore:

$$\xi_j[\omega] = \pi_j^{(i)}[\omega'] = \pi_j^{(i)}[\omega + \lambda_i(1 - \xi_i[\omega])],\ j \neq i.$$

With the notations introduced at the beginning of the section, set:

$$\alpha_{i'}[\omega] := \mathbb{E}[e^{-\omega A_{i'\cdot}}] = \mathbb{E}[e^{-\omega P_{\cdot}^{(i)}}] = \sum_{j\neq i} \frac{\lambda_j}{\lambda_{i'}} \pi_j^{(i)}[\omega],\ \xi_{i'}[\omega] = \sum_{j\neq i} \frac{\lambda_j}{\lambda_{i'}} \xi_j[\omega],$$

so that we finally obtain:

$$\begin{cases} \xi_{i'}[\omega] = \alpha_{i'}[\omega + \lambda_i(1 - \xi_i[\omega])], \\ \xi_i[\omega] = \alpha_i[r_i\omega + \lambda_i(r_i - 1)(1 - \xi_i[\omega]) + \lambda_{i'} r_i(1 - \xi_{i'}[\omega])] \end{cases}$$

By Theorem 3.4, $\xi_i[\omega]$ is uniquely determined by these equations, hence it is equal to $\pi_i[\omega]$, and so for all i. Moreover, $P_{i.}$ is a.s. finite if and only if $A_{i.}$ and $A_{i'.}$ are a.s. finite and $\rho_i + \rho_{i'} \leq 1$ (where $\rho_{i'}$ denotes the traffic intensity of source i'), and then:

$$\mathbb{E}[P_{i.}] = \frac{\rho_i}{\lambda_i(1 - \rho_i - \rho_{i'})}.$$

By the induction hypothesis, since $A_{i'.} \simeq P_{\cdot}^{(i)}$, we find that $A_{i'.}$ is a.s. finite if and only if all the $A_{j.}$'s with $j \neq i$ are and $\rho^{(i)} := \sum_{j\neq i} \rho_j \leq 1$; moreover:

$$\alpha_{i'} := \mathbb{E}[A_{i'}] = \rho^{(i)}/(\lambda_{i'}(1 - \rho^{(i)})),$$

which yields: $\rho_{i'} = \rho^{(i)}$. This completes the proof by induction. ♠

5.2 Regularly varying tails

Thanks to the fictitious source i', we can easily extend the tail analysis of Theorem 4.2 to the case of $N \geq 2$ on/off sources. Notice that the above Theorem yields the values of $\rho_{i'}$ and $p_{i'}$ (resp. the traffic intensity and the silence probability of source i'), namely:

$$1 - p_{i'} = \rho_{i'} = \sum_{j \neq i} \rho_j.$$

From now on *we assume that* $\rho < 1$. Like in Section 4, we denote

$$\pi_i := \mathbb{E}[P_{i.}] = \frac{\rho_i}{\lambda_i(1-\rho)},\ 1 \leq i \leq N.$$

Theorem 5.3 *Assume that for $t \to \infty$:*

$$\mathbb{P}[A_{i.} > (\alpha_i/\pi_i)t] = (a_i + \mathrm{o}\,(1))l(t)/t^\nu,\ 1 \leq i \leq N, \tag{28}$$

with $a_i \geq 0$ ($1 \leq i \leq N$), $\nu \in (n, n+1)$ (for some $n \geq 1$) and $l(t)$ a slowly varying function. Then:

$$\mathbb{P}[P_{i.} > t] = \left(\frac{(1-\rho)p_i\lambda_i a_i + \rho_i \sum_{j=1}^N p_j \lambda_j a_j}{(1-\rho)\lambda_i} + \mathrm{o}\,(1) \right) \frac{l(t)}{t^\nu},\ 1 \leq i \leq N.$$

Conversely, assume that

$$\mathbb{P}[P_{i.} > t] = (c_i + \mathrm{o}\,(1))l(t)/t^\nu, \quad t \to \infty, \tag{29}$$

with $c_i \geq 0$ ($1 \leq i \leq N$), $\nu \in (n, n+1)$ (for some $n \geq 1$) and $l(t)$ a slowly varying function. Then either all the c_i's are null or all are positive. In the latter case, they necessarily satisfy:

$$\lambda_i c_i - \rho_i \sum_{j=1}^N \lambda_j c_j \geq 0,\ 1 \leq i \leq N, \tag{30}$$

and at least one of the above expressions is non-null. Finally, for $t \to \infty$:

$$\mathbb{P}[A_{i.} > (\alpha_i/\pi_i)t] = \left(\frac{\lambda_i c_i - \rho_i \sum_{j=1}^N \lambda_j c_j}{\lambda_i p_i} + \mathrm{o}\,(1) \right) \frac{l(t)}{t^\nu}.$$

Corollary 5.4 *Under assumption (28) of Theorem 5.3:*

$$\mathbb{P}[P_{\cdot} > t] = \left(\frac{\sum_{i=1}^N p_i \lambda_i a_i}{(1-\rho)\lambda} + \mathrm{o}\,(1) \right) \frac{l(t)}{t^\nu}.$$

Proof:
Immediate in view of (4). ♠

Proof of Theorem 5.3:
We argue by induction on N. By Theorem 4.2, the result is valid for $N = 2$. Assume that it is valid for $N-1$ sources. Then focus on some source i, $1 \le i \le N$, and introduce the fictitious source i'. Easy calculations show that

$$\pi_j^{(i)} = \frac{1-\rho}{1-\rho_{i'}} \pi_j, \; j \neq i, \; \alpha_{i'} = \frac{1-\rho}{1-\rho_{i'}} \pi_{i'} \tag{31}$$

(with the usual notations $\pi_j^{(i)} := \mathbb{E}[P_j^{(i)}]$, $\pi_{i'} := \mathbb{E}[P_{i'}]$). As a first consequence, under Assumption (28), for $j \neq i$:

$$\mathbb{P}[A_{j\cdot} > (\alpha_j/\pi_j^{(i)})t] = \mathbb{P}[A_{j\cdot} > (\alpha_j/\pi_j)\frac{1-\rho_{i'}}{1-\rho} t] = (a_j(\frac{1-\rho}{1-\rho_{i'}})^\nu + \mathrm{o}\,(1)) l(t)/t^\nu.$$

Since a typical activity period $A_{i'\cdot}$ of source i' is a typical busy period of the system without source i, the induction hypothesis (in the form of Corollary 5.4) then shows that:

$$\mathbb{P}[A_{i'\cdot} > t] = \left(\frac{\sum_{j\neq i} p_j \lambda_j a_j^{(i)}}{(1-\rho_{i'})\lambda_{i'}} + \mathrm{o}\,(1) \right) \frac{l(t)}{t^\nu}, \text{ with } a_j^{(i)} = a_j(\frac{1-\rho}{1-\rho_{i'}})^\nu, \; j \neq i.$$

Using (31) again, we finally obtain:

$$\mathbb{P}[A_{i'\cdot} > (\alpha_{i'}/\pi_{i'})t] = \mathbb{P}[A_{i'\cdot} > \frac{1-\rho}{1-\rho_{i'}} t] = (a_{i'} + \mathrm{o}\,(1)) l(t)/t^\nu,$$

with:

$$a_{i'} = \left(\frac{1-\rho}{1-\rho_{i'}} \right)^{-\nu} \frac{\sum_{j\neq i} p_j \lambda_j a_j^{(i)}}{(1-\rho_{i'})\lambda_{i'}} = \frac{\sum_{j\neq i} p_j \lambda_j a_j}{(1-\rho_{i'})\lambda_{i'}}.$$

Now we have two sources i and i' which satisfy the conditions of Theorem 4.2, so we can directly conclude that:

$$\begin{aligned}\mathbb{P}[P_{i.} > t] &= \left(\frac{(1-\rho_{i'})p_i\lambda_i a_i + \rho_i p_{i'}\lambda_{i'} a_{i'}}{(1-\rho)\lambda_i} + \mathrm{o}\,(1)\right)\frac{l(t)}{t^\nu} \\ &= \left(\frac{(1-\rho)p_i\lambda_i a_i + \rho_i \sum_{j=1}^N p_j\lambda_j a_j}{(1-\rho)\lambda_i} + \mathrm{o}\,(1)\right)\frac{l(t)}{t^\nu}.\end{aligned}$$

Since the argument is valid for all i, this completes the induction for the direct part of the theorem.

Conversely, under assumption (29) a typical busy period $P_{i'.}$ starting with an activity period of source i' satisfies:

$$\mathbb{P}[P_{i'.} > t] = (c_{i'} + \mathrm{o}\,(1))l(t)/t^\nu,$$

with

$$c_{i'} = \sum_{j\neq i}\frac{\lambda_j}{\lambda_{i'}}c_j$$

by Formula (25). From Theorem 4.2 we obtain:

$$\begin{aligned}\mathbb{P}[A_{i.} > (\alpha_i/\pi_i)t] &= \left(\frac{(1-\rho_i)(\lambda_i c_i) - \rho_i(\lambda_{i'}c_{i'})}{\lambda_i p_i} + \mathrm{o}\,(1)\right)\frac{l(t)}{t^\nu} \\ &= \left(\frac{\lambda_i c_i - \rho_i \sum_{j=1}^N \lambda_j c_j}{\lambda_i p_i} + \mathrm{o}\,(1)\right)\frac{l(t)}{t^\nu}.\end{aligned}$$

Moreover, constants c_i and $c_{i'}$ must satisfy:

$$(1-\rho_i)(\lambda_i c_i) - \rho_i(\lambda_{i'}c_{i'}) \geq 0.$$

By applying this argument to all i, we complete the induction for the converse part. In view of conditions (30), if $c_j > 0$ for some j, then $c_j > 0$ for all j. In this case, by summing all these inequalities and using $\rho < 1$, we see that at least one of them must be strict. ♠

5.3 Special cases

(i) $\mathbf{r_i = 1,\ i = 1, \ldots, N}$

Formula (27) reduces to results of [20] for the case of N identical sources, of [15] for the case of N non-identical sources, and of [9] for the case of $N = \infty$ identical sources (Cohen obtained his result via a renewal-theoretic approach). In the latter case, with $N \to \infty$, $\lambda_i \equiv \lambda \to 0$ such that $N\lambda = \Lambda$, Formula (27) reduces to

$$\pi_1[\omega] \;=\; \alpha_1[\omega + \Lambda(1 - \pi_1[\omega])],$$

the equation for the busy period LST in an $M/G/1$ queue with arrival rate Λ and service time LST $\alpha_1[\cdot]$. Note that the result of De Meyer and Teugels [17] for the tail behaviour of the busy period in an $M/G/1$ queue now immediately yields the result of Theorem 5.3.

(ii) N identical sources

Again Formula (27) reduces to one equation:

$$\begin{aligned}\pi_1[\omega] &= \alpha_1[r_1\omega + (N\lambda_1 r_1 - \lambda_1)(1 - \pi_1[\omega])] \\ &= \beta_1[\omega + \frac{N\lambda_1 r_1 - \lambda_1}{r_1}(1 - \pi_1[\omega])]. \end{aligned} \tag{32}$$

Here $\beta_1[\omega] := \alpha_1[r_1\omega]$ is the LST of $B_{1.} := r_1 A_{1.}$. Equation (32) immediately implies that the busy period in this special case is distributed like the busy period in an $M/G/1$ queue with arrival rate $(N\lambda_1 r_1 - \lambda_1)/r_1$ and service times B_{1n}; and again the tail behaviour of the busy period follows immediately from the $M/G/1$ result of De Meyer and Teugels [17].

As above, one can let $N \to \infty$ such that $N\lambda = \Lambda$.

(iii) $\mathbf{N = 1}$

Formula (27), or alternately Formula (32), now reduces to:

$$\pi_1[\omega] = \alpha_1[r_1\omega + \lambda_1(r_1 - 1)(1 - \pi_1[\omega])]. \tag{33}$$

Consider an ordinary $M/G/1$ queue Q^* with arrival rate λ_1 and service times $(r_1 - 1)A_{1n}$. The LST $\pi^*[\omega]$ of its busy period distribution satisfies:

$$\pi^*[\omega] = \alpha_1[(r_1 - 1)\omega + \lambda_1(r_1 - 1)(1 - \pi^*[\omega])]. \tag{34}$$

The number of customers served in one busy period of Q^* equals in distribution the number of active periods in one busy period of the fluid queue with one source. To see this, observe that after each active period A_{1n} the buffer content has increased with $(r_1 - 1)A_{1n}$. But the *length* of the busy period of the fluid queue equals in distribution $r_1/(r_1 - 1)$ times the length of the busy period of Q^*, since it equals r_1 times the sum of the lengths of the active periods. Hence $\pi_1[\omega] = \pi^*[\frac{r_1}{r_1-1}\omega]$, which is confirmed by comparing (33) and (34).

5.4 Dominating sources

The reasoning in Subsection 5.1 reveals that, as far as busy period analysis is concerned, it is possible to aggregate sources and thus to concentrate on just two sources. In this subsection our aim is even more drastic: To completely remove all sources except those with the heaviest-tailed activity period distributions.

By a time-scaling argument, all the previous results can be easily extended to the situation in which the outflow rate is c instead of 1. The stability condition is then $\rho < c$, and the formulas for the mean busy periods become:

$$\pi_i = \frac{\rho_i}{\lambda_i(c-\rho)}, \quad 1 \le i \le N.$$

Under condition (28) of Theorem 5.3, we finally obtain:

$$\begin{cases} \mathbb{P}[P_{i.} > t] = \left(\dfrac{(c-\rho)p_i\lambda_i a_i + \rho_i \sum_{j=1}^N p_j \lambda_j a_j}{(c-\rho)\lambda_i} + \mathrm{o}\,(1) \right) \dfrac{l(t)}{t^\nu}, \ 1 \le i \le N, \\ \mathbb{P}[P_{.} > t] = \left(\dfrac{c\sum_{i=1}^N p_i \lambda_i a_i}{(c-\rho)\lambda} + \mathrm{o}\,(1) \right) \dfrac{l(t)}{t^\nu}. \end{cases}$$

Now call source i a *dominating* source if $a_i > 0$, resp. a *dominated* source if $a_i = 0$, assuming that the set N^* of dominating sources is not empty. The question arises of evaluating the contribution of the dominated sources to the tail behaviour of the busy periods, since all the associated terms a_i cancel in the above formulas. Notice that these sources contribute to the total traffic intensity ρ. So a natural idea consists in comparing our model of N sources with the "model N^*" where all the dominated sources are deleted (the total traffic intensity is thus $\rho^* := \sum_{i\in N^*} \rho_i$) *and* the outflow rate is reduced from c to $c* := c - \sum_{j\notin N^*} \rho_j$.

First notice that the mean busy periods in model N^* are given by:

$$\pi_i^* = \frac{\rho_i}{\lambda_i(c^* - \rho^*)} = \frac{\rho_i}{\lambda_i(c-\rho)} = \pi_i, \quad i \in N^*,$$

so they are unchanged. Under condition (28) of Theorem 5.3, we thus have:

$$\begin{aligned}
\mathbb{P}[P_{i\cdot}^* > t] &\overset{t\to\infty}{\sim} \left(\frac{(c^*-\rho^*)p_i\lambda_i a_i + \rho_i \sum_{j\in N^*} p_j\lambda_j a_j}{(c^*-\rho^*)\lambda_i}\right)\frac{l(t)}{t^\nu} \\
&= \left(\frac{(c-\rho)p_i\lambda_i a_i + \rho_i \sum_{j=1}^{N} p_j\lambda_j a_j}{(c-\rho)\lambda_i}\right)\frac{l(t)}{t^\nu} \\
&\overset{t\to\infty}{\sim} \mathbb{P}[P_{i\cdot} > t], \quad i \in N^*.
\end{aligned}$$

Hence the busy periods $P_{i\cdot}$ and $P_{i\cdot}^*$ have just the same tail behaviour: as far as the tail behaviour is concerned, all the dominated sources are replaced by a constant stream of rate $\sum_{j\notin N^*}\rho_j$. The outcome is not as nice if we compare $P_\cdot$ and $P_\cdot^*$; setting $\lambda^* = \sum_{i\in N^*}\lambda_i$, we obtain:

$$\mathbb{P}[P_\cdot^* > t] \overset{t\to\infty}{\sim} \left(\frac{c^*\sum_{i\in N^*} p_i\lambda_i a_i}{(c^*-\rho^*)\lambda^*}\right)\frac{l(t)}{t^\nu} = \frac{c^*\lambda}{c\lambda^*}\left(\frac{c\sum_{i=1}^{N} p_i\lambda_i a_i}{(c-\rho)\lambda}\right)\frac{l(t)}{t^\nu},$$

hence $(\lambda^*/c^*)\mathbb{P}[P_\cdot^* > t] \sim (\lambda/c)\mathbb{P}[P_\cdot > t]$.

Remark 5.5 *The result about the dominated sources is reminiscent of a result of Jelenkovic and Lazar [14]. They study the tail behaviour of the buffer content distribution in a fluid queue with one source with heavy-tailed activity periods and several sources with exponential activity periods. They observe that this tail of the buffer content distribution is equivalent with that of a fluid queue with* only *the one source with the heavy-tailed activity period distribution, and in which the exponential sources are taken into account by reducing the outflow rate.*

6 Conclusion

In this paper we have considered a fluid queue fed by N independent on/off sources with exponentially distributed silence periods and with the inflow rate of each source being at least as large as the outflow rate of the buffer.

We have determined the LST's of the distributions of the busy periods that start with an activity period of source i, $i = 1, \ldots, N$. We have also related the tail behaviour of the busy period distributions to the tail behaviour of the activity period distributions: The tails of all busy period distributions are regularly varying of index $-\nu$ iff the heaviest of the tails of the activity period distributions is regularly varying of index $-\nu$.

One implication of this result is the following. Suppose that the output of the fluid queue Q_1 feeds into another infinite-capacity fluid queue Q_2. That output process constitutes an on/off process with $\exp(\lambda_1 + \ldots + \lambda_N)$ distributed silence periods and with activity periods that are busy periods of Q_1. If the tails of the distributions of these busy periods are regularly varying of index $-\nu$ with $\nu \in (1, 2)$, then the output process of Q_1 is LRD – just like the input process of Q_1. This should be compared with the result about propagation of LRD in networks of quasi-reversible queues mentioned in [1].

A next step is to study Q_2 in isolation. If the outflow rate of Q_2 is larger than one, then its buffer will never fill. If the outflow rate of Q_2 is at most equal to one, then Sections 4 and 5 of the present paper yield its busy period behaviour (and the buffer content behaviour follows from, e.g., [4]). To close the circle, we end by observing that, in the present study, the traffic from the N on/off sources may be viewed as outputs from N independent fluid queues with outflow rates $r_1, \ldots, r_N$.

7 Appendix

Proof of Lemma 2.2:

Here we treat only the case $C = 0$. Since $l(t)/t^\nu$ is equivalent (when $t \to \infty$) to a non-increasing function (see Theorem 1.5.4 of [3]), we may assume a random variable Y such that $\mathbb{P}[Y > t] \sim l(t)/t^\nu$, and we denote by $\gamma[\omega]$ its LST. Then $\mathbb{E}[Y^n] < \infty$, and *a fortiori* $\mathbb{E}[X^n] < \infty$ if $\mathbb{P}[X > t] = \mathrm{o}\,(\mathbb{P}[Y > t])$. Like in [17], let us now set:

$$f_0(t) = \mathbb{P}[X > t], \quad f_{k+1}(t) = \int_t^\infty f_k(u)du,$$

and similarly define $g_k(t)$ from $g_0(t) = \mathbb{P}[Y > t]$. It is easily checked that

$f_k(0) = \mathbb{E}[X^k]/k!$; hence under the conditions of the Lemma, $f_k(\cdot)$ and $g_k(\cdot)$ are well defined for $k \leq n$. Moreover, a straightforward induction yields:

$$\phi_n[\omega] = \omega^{n+1} \int_0^\infty e^{-\omega t} f_n(t) dt,$$

with a similar formula for $\gamma_n[\omega]$.

(i)⇒(ii) If $f_0(t) = \mathrm{o}\,(g_0(t))$ $(t \to \infty)$, then $\int_t^\infty f_0(u)du = \mathrm{o}\,(\int_t^\infty g_0(u)du)$. An immediate induction then shows that $f_n(t) = \mathrm{o}\,(g_n(t))$. For any $\epsilon > 0$, there exists a $T > 0$ such that $f_n(t) \leq \epsilon g_n(t)$ for $t \geq T$, hence:

$$\begin{aligned} \phi_n[\omega] &\leq \omega^{n+1} \left(\int_0^T e^{-\omega t} f_n(t) dt + \epsilon \int_T^\infty e^{-\omega t} g_n(t) dt \right) \\ &\leq \omega^{n+1} \int_0^T f_n(t) dt + \epsilon \gamma_n[\omega]. \end{aligned}$$

By Lemma 2.2 in case $C > 0$: $\omega^{n+1} = \mathrm{o}\,(\gamma_n[\omega])$ $(\omega \to 0)$, so we obtain:

$$\limsup_{\omega \to 0} \phi_n[\omega]/\gamma_n[\omega] \leq \epsilon.$$

Since this result is valid for any $\epsilon > 0$, the proof is complete.

(ii)⇒(i) By Lemma 2.2 in case $C > 0$:

$$\gamma_n[\omega] \sim (-1)^n \Gamma(1-\nu) l(1/\omega) \omega^\nu \sim (-1)^n \Gamma(1-\nu) g_0(1/\omega), \quad \omega \to 0. \quad (35)$$

Since f_n is decreasing:

$$\phi_n[\omega] \geq \omega^{n+1} \int_0^{1/\omega} e^{-\omega t} f_n(t) dt \geq \omega^n e^{-1} f_n(1/\omega).$$

So **(ii)** and (35) imply that $\omega^n f_n(1/\omega) = \mathrm{o}\,(g_0(1/\omega))$, or equivalently:

$$f_n(t) = \mathrm{o}\,(t^n g_0(t)), \quad t \to \infty.$$

Now write:

$$f_n(t) \geq \int_t^{2t} f_{n-1}(u) du \geq t f_{n-1}(2t),$$

which in view of the previous formula yields: $f_{n-1}(2t) = \mathrm{o}\,(t^{n-1} g_0(t))$, or equivalently:

$$f_{n-1}(t) = \mathrm{o}\,(t^{n-1} g_0(t)), \quad t \to \infty,$$

since g_0 is regularly varying. By repeating this argument we finally obtain: $f_0(t) = \mathrm{o}\,(g_0(t))$ $(t \to \infty)$, which completes the proof. ♠

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