Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

Heavy-traffic Limit Theorems for the Heavy-tailed GI/G/1 Queue

J.W. Cohen

Probability, Networks and Algorithms (PNA)

PNA-R9719 December 31, 1997

Report PNA-R9719
ISSN 1386-3711

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

Heavy-traffic Limit Theorems for the Heavy-tailed GI/G/1 Queue

J.W. Cohen
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT

The classic $GI/G/1$ queueing model of which the tail of the service time and/or the interarrival time distribution behaves as $t^{-v}S(t)$ for $t \to \infty$, $1 < v < 2$ and $S(t)$ a slowly varying function at infinity, is investigated for the case that the traffic load a approaches one. Heavy-traffic limit theorems are derived for the case that these tails have a similar behaviour at infinity as well as for the case that one of these tails is heavier than the other one. These theorems state that the contracted waiting time $\Delta(a)\mathbf{w}$, with $\mathbf{w}$ the actual waiting time for the stable $GI/G/1$ queue and $\Delta(a)$ the contraction coefficient, converges in distribution for $a \uparrow 1$. Here $\Delta(a)$ is that root of the contraction equation which approaches zero from above for $a \uparrow 1$. The structure of this contraction equation is determined by the character of the two tails. The Laplace-Stieltjes transforms of the limiting distributions are derived. For nonsimilar tails the limiting distributions are explicitly known. For the tails of these distributions asymptotic expressions are derived and compared.

1991 Mathematics Subject Classification: 60K25, 90B22.
Keywords and Phrases: $GI/G/1$ queue, heavy-tailed distributions, Pareto type, second moments infinite, heavy-traffic, limit theorems, actual waiting time, contracted waiting time, limiting distributions, tail asymptotics.
Note: work carried out under project LRD in PNA 2.1.

Introduction

The present study is an extension of a previous study [4] on the classical $GI/G/1$ queueing model with service time distribution $B(t)$ having a heavy-tail of a Pareto-type structure

$$1 - B(t) = \frac{1}{t^{v-1}} S(t) \quad \text{for} \quad t \to \infty,$$

with $1 < v < 2$ and $S(t)$ a slowly varying function at infinity and with the tail of the interarrival time distribution $A(t)$ in some sence less heavy than that of $B(t)$. The case with $A(t)$ heavier than $B(t)$ was also investigated in [4].

In [4] the present-day interest in the performance analysis of queueing models with service time - and/or interarrival time distributions having heavy tails has been exposed, and herefor we refer the interested reader to [4].

In the present study we shall analyse the waiting time distribution for the case that $B(t)$ and $A(t)$ have similar tails, i.e. both $B(t)$ and $A(t)$ have tails which have a 'similar' asymptotic behaviour for $t \to \infty$, for 'similar' see the definition 1.1. of section one. As in [4] it turns out that for similar tails again a contraction coefficient $\Delta(a), 0 < a < 1$, can be defined with $\Delta(a) \downarrow 0$ for $a \uparrow 1$ such that the waiting time $\mathbf{w}$ of the stable GI/G/1 queueing model, i.e. $0 < a < 1$, when multiplied by $\Delta(a)$ converges in distribution for $a \uparrow 1$. The present analysis shows that heavy traffic limit theorems for the contracted waiting time $\Delta(a)\mathbf{w}$ derived in [4] can be formulated also for the classical GI/G/1 queueing model with similar tails. The paper is organized in the following way.

In section 1 we describe the structure of the distributions $A(t)$ and $B(t)$ mainly in terms of their Laplace-Stieltjes (L.S.) transforms $\alpha(\rho)$ and $\beta(\rho)$, see (1.1) , . . . , (1.9). Starting from the description of this structure the concepts: heavier than, similar, identical and nonidentical are defined in definition 1.1. The starting point of the analysis is the relation between the L.S. transform of the stationary distribution $\omega(\rho)$ of the actual waiting time $\mathbf{w}$ and $\chi(\rho)$ which is the L.S. transform of the excess distribution of the idle time $\mathbf{i}$, cf. (1.14), i.e. with $\beta/\alpha = a$,

$$\chi(\bar{\rho}) = \frac{1-\beta(\rho)\alpha(\bar{\rho})}{(\beta-\alpha)\rho}\omega(\rho) \ , \ \mathrm{Re}\ \rho = 0.$$

The solution of this functional equation for the relevant conditions is known and given in (1.15). This functional equation has also been the starting point in [4]. In particular the reader is advised to consult [4] for explicit examples of $A(t)$ and $B(t)$ and the derivation of their L.S. transforms.

The known factor in the functional equation above, i.e. its kernel, is analysed in section 2. Its analysis for the described structure of $\alpha(\rho)$ and $\beta(\rho)$ leads to the definition of the contraction equation, cf. (2.7). This equation is analysed in section 3 and the contraction coefficient $\Delta(a)$ is defined as its root which tends from above to zero for $a \uparrow 1$.

In section 4 the heavy traffic limit theorem is formulated for the contracted waiting time $\Delta(a)\mathbf{w}$ for $a \uparrow 1$ for the case that $1 < \nu < 2$ and $B(t)$ and $A(t)$ have similar tails, see theorem 4.1. This theorem also describes the L.S. transform of the limiting distribution.

In section 5 the analogous theorem 5.1 is derived but now for the case that $\nu = 2$. The limiting distribution has here a very simple form, encountered before in [4].

In sections 6 and 7 the heavy traffic limit theorems are formulated for the case that $B(t)$ has a heavier tail than $A(t)$ and conversely, respectively. These theorems are the same as in [4], however, here the conditions are slightly weaker.

In section 8 an asymptotic expression for the tail of the limiting distribution $W_{\nu-1}(t)$ is derived for the case that $1 < \nu < 2$ and $A(t)$ and $B(t)$ have similar tails. This tail of $W_{\nu-1}(t)$ is compared with that of $R_{\nu-1}(t)$ which occurs whenever the tail of $B(t)$ is heavier than that of $A(t)$. $R_{\nu-1}(t)$ is explicitly known, see herefor [4]. It turns out that the tail of $W_{\nu-1}(t)$ may lie above as well as below that of $R_{\nu-1}(t)$. This is a rather remarkable phenomenon, and corresponds to a similar fact as observed in [9].

In section 8 we compare also in some more detail the behaviour of $\Delta_1(a)\mathbf{w}$, and $\Delta_2(a)\mathbf{w}_2$ with the index 1 refering to the case that $A(t)$ and $B(t)$ have similar tails and the index 2 to that with the tail of $B(t)$ heavier than that of $A(t)$. The reason for this comparison is that the limiting distribution $R_{\nu-1}(t)$ appeared to be a very good approximation for the exact distribution of $\Delta(a)\mathbf{w}$ if a is not close to one and $\nu = 1\frac{1}{2}$, see [8].

The asymptotic expression for the tail of the limiting distribution $I_{\nu-1}(t)$ of the contracted excess time of the idle period $\mathbf{i}$ is also derived in section 8. This limiting distribution is degenerated at zero if $A(t)$ and $B(t)$ have nonsimilar tails where as for similar tails it is a true probability distribution. The asymptotic results described in this section lead to an interesting observation concerning the contracted queueing process.

The three appendices contain same more technical calculations of which the results are needed in the various sections.

1. Definitions

In this study we shall analyse the classical GI/G/1 queueing model with service time and interarrival time distribution $B(t)$ and $A(t)$ both having a heavy tail. The study is an extension of the study of Boxma and Cohen [4].

We first describe the structure of the distributions $A(t)$ and $B(t)$. It is assumed that

$$\alpha := \int_0^\infty t\mathrm{d}A(t) < \infty \ , \ \beta := \int_0^\infty t\mathrm{d}B(t) < \infty, \tag{1.1}$$

$$a := \frac{\beta}{\alpha} < 1,$$

and that they can be represented as follows: a $T > 0$ exists such that,

$$\begin{aligned} 1 - A(t) &= G_{11}(t) + G_{12}(t), \\ 1 - B(t) &= G_{21}(t) + G_{22}(t), \end{aligned} \tag{1.2}$$

with: for a $\delta > 0$,

$$|\int_T^\infty \mathrm{e}^{-\rho t} G_{j1}(t)\mathrm{d}t| < \infty \text{ for } \mathrm{Re}\,\rho > -\delta,\ j = 1, 2\,, \tag{1.3}$$

where the function $G_{j2}(t)$ characterises the dominant term of the righthand sides in (1.2) for $t \to \infty$.

Put

$$c_1 := \alpha/\gamma\ ,\ c_2 := \beta/\gamma\ ,\ \gamma > 0, \tag{1.4}$$

where γ stands for the unit of time. From (1.1) we have

$$c_1 > c_2 > 0,\ ,\ a = \frac{c_2}{c_1} < 1. \tag{1.5}$$

Further, we define: for Re $\rho \geq 0$,

$$\alpha(\rho) = \int_{0-}^\infty \mathrm{e}^{-\rho t}\mathrm{d}A(t)\ ,\ \beta(\rho) := \int_{0-}^\infty \mathrm{e}^{-\rho t}\mathrm{d}B(t). \tag{1.6}$$

The Laplace-Stieltjes (L.S.) transforms are specified as follows.

For Re $\rho \geq 0$,

$$\begin{aligned} 1 - \frac{1 - \alpha(\rho)}{\alpha\rho} &= g_1(\gamma\rho) + C_1(\gamma\rho)^{\nu_1 - 1} L_1(\gamma\rho), \\ 1 - \frac{1 - \beta(\rho)}{\beta\rho} &= g_2(\gamma\rho) + C_2(\gamma\rho)^{\nu_2 - 1} L_2(\gamma\rho), \end{aligned} \tag{1.7}$$

with: for $j = 1, 2$,

i. C_j is a finite positive constant, (1.8)

ii. $1 < \nu_j \leq 2$,

iii. $g_j(\gamma\rho)$ is regular for Re $\rho > -\delta$, $g_j(0) = 0$,

iv. $L_j(\gamma\rho)$ is a regular function of ρ for Re $\rho > 0$, and continuous for Re $\rho \geq 0$, except possibly at $\rho = 0$, with

$L_j(\gamma\rho) \to b_j > 0$ for $|\rho| \to 0$, Re $\rho \geq 0$,

and

$b_j \leq \infty \quad$ for $1 < \nu_j \leq 2$,

$b_j = \infty \quad$ for $\nu_j = 2$,

$$\lim_{x \downarrow 0} \frac{L_j(x\gamma\rho)}{L_j(\gamma\rho)} = 1 \text{ for every } \rho \text{ with Re } \rho \geq 0.$$

It is further assumed that the following limit exists

$$f := \lim_{x \downarrow 0} \frac{L_2(x)}{L_1(x)} \geq 0. \tag{1.9}$$

We make the following remarks concerning the assumptions introduced in (1.7), (1.8) and (1.9).

REMARK 1.1 Many valued functions such as ρ^{ν_j} and e.g. $\log \rho$ are defined by their principal value, so $\rho^{\nu_j} > 0$ for $\rho > 0$, $\log \rho$ is real for $\rho > 0$; also $|\arctan x| < \frac{1}{2}\pi$ for $-\infty < x < \infty$. □

REMARK 1.2 The class of distributions $A(t)$ and $B(t)$, satisfying (1.7) and (1.8) is actually a subclass of those characterised by (1.2). For the former class the tail $1 - A(t)$ behaves as $t^{-\nu_j} S(t)$ for $t \to \infty$, with $S(t)$ a slowly varying function at infinity, $1 < \nu_1 < 2$; see the examples in [4]. □

REMARK: 1.3 Because for $\rho > 0$ the lefthand sides in (1.7) are positive, it follows from (1.8)iii that $L_j(\gamma\rho)$ is real for $\rho > 0$. Note further that the righthand sides of (1.7) are zero for $\rho = 0$, so $x^{\nu_j - 1} L_j(x) \to 0$ for $x \to 0, x > 0$. □

REMARK 1.4 The case for which the limit in (1.9) does not exist will not be considered in the present study. □

REMARK 1.5 We have excluded the case that $C_j = 0$, but note that if $C_1 = 0$ then $A(t)$ has a negative exponential tail since $\delta > 0$. □

We introduce the following nomenclature.

DEFINITION 1.1 The tail of $A(t)$ is said to be *heavier* than that of $B(t)$ whenever one of the following cases occurs.

$$\begin{aligned} &\text{i.} \quad \nu_1 < \nu_2, \\ &\text{ii.} \quad \nu_1 = \nu_2\ ,\ b_1 = \infty,\ b_2 < \infty, \\ &\text{iii.} \quad \nu_1 = \nu_2,\ b_1 = \infty,\ b_2 = \infty \text{ and } f = 0; \end{aligned} \tag{1.10}$$

analogously, the tail of $B(t)$ is called heavier than that of $A(t)$ if in (1.10) ν_1 and ν_2, b_1 and b_2 are interchanged and $f = 0$ is replaced by $f = \infty$.

Whenever, $A(t)$ is not heavier than $B(t)$, and $B(t)$ not heavier than $A(t)$ it is said that $A(t)$ and $B(t)$ have *similar* tails, and they have *identical* tails if $C_1 = C_2$ and $L_1(\gamma\rho) \equiv L_2(\gamma\rho)$ for Re $\rho \geq 0$, whereas for $C_1 L_1(\gamma\rho) \not\equiv C_2 L_2(\gamma\rho)$ for a ρ with Re $\rho > 0$ and $f = 1$ the tails are said to be *pseudo-identical*. Similar tails are said to be nonidentical if they are not idential or pseudo-identical.

So far for the characterisation of the L.S. transforms $\alpha(\rho)$ and $\beta(\rho)$.

Because $a < 1$, cf. (1.1), the GI/G/1 queueing model possesses a unique stationary waiting distribution $W(t)$, say. By $\mathbf{i}$ we shall denote the idle time of a busy cycle of the waiting time process and by $\mathbf{w}$ a stochastic variable with distribution $W(t)$.

Put: for Re $\rho \geq 0$,

$$\omega(\rho) := \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}}\} \equiv \int_{0-}^{\infty} \mathrm{e}^{-\rho t} \mathrm{d}W(t), \tag{1.11}$$

$$\chi(\rho) := \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{i}}\}}{\rho \mathrm{E}\{\mathbf{i}\}}, \tag{1.12}$$

so that

$$\begin{aligned} \chi(-\rho) &= \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho \mathbf{i}}\}}{-\rho \mathrm{E}\{\mathbf{i}\}} && \text{for Re } \rho \leq 0, \\ &= \chi(\bar{\rho}) && \text{for Re } \rho = 0. \end{aligned} \tag{1.13}$$

It is well known, cf. [1], that:

i. $\chi(\bar{\rho}) = \dfrac{1-\beta(\rho))\alpha(\bar{\rho})}{(\beta-\alpha)\rho}\omega(\rho) \quad \text{for Re}\ \rho = 0;$ (1.14)

ii. $\omega(\rho)$ is regular for Re $\rho > 0,$ continuous and uniformly bounded by one for Re $\rho \geq 0;$

iii. $\chi(-\rho)$ is regular for Re $\rho < 0,$ continuous and uniformly bounded by one for Re $\rho \leq 0$ with $\chi(\bar{\rho})$ its boundary value at Re $\rho = 0;$

iv. $\chi(0) = \omega(0) = 1.$

The conditions (1.14) formulate the Riemann Boundary Value Problem for the functions $\omega(\rho)$ and $\chi(\rho))$, cf. [1]. For literature concerning this boundary value problem cf. [1], [2], [3] and [4].

In [1] the solution of the boundary value problem (1.14) has been given, whenever $\alpha(\rho)$ and $\beta(\rho)$ satisfy the conditions (26) of [1]. It is not difficult to show that $\alpha(\rho)$ and $\beta(\rho)$ as given by (1.1), (1.7) and (1.8) indeed satisfy those conditions. Consequently, it follows from theorem 4 of [1] that

$$\begin{aligned} &\omega(\rho) = \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\} = \mathrm{e}^{H(\rho)}, &&\text{Re}\ \rho > 0,\\ &\chi(-\rho) = \frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{i}}\}}{-\rho\mathrm{E}\{\mathbf{i}\}} = \mathrm{e}^{H(\rho)}, &&\text{Re}\ \rho < 0, \end{aligned} \tag{1.15}$$

where

$$H(\rho) := \frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \frac{1-\beta(\xi)\alpha(\bar{\xi})}{(\beta-\alpha)\xi}\}\frac{\rho}{\xi-\rho}\frac{\mathrm{d}\xi}{\xi},$$

with the integral defined as a principal value integral at infinity and as a principal value singular Cauchy integral at ρ if Re $\rho = 0$, cf. [1].

2. On the kernel $k(\rho; c_1, c_2)$

In this section we analyse the kernel

$$k(\rho; c_1, c_2) := \frac{1-\beta(\rho)\alpha(\bar{\rho})}{(\beta-\alpha)\rho}\ ,\ \text{Re}\ \rho = 0, \tag{2.1}$$

of the boundary value problem (1.4) with $\alpha(\rho)$ and $\beta(\rho)$ given by (1.1), (1.7), (1.8) and (1.9) and with, cf. (1.8)ii,

$$\nu_1 = \nu_2 = \nu\ ,\ 1 < \nu \leq 2. \tag{2.2}$$

A lengthly algebraic computation shows that (1.7), (1.8) and (2.2) lead to: for Re $\rho = 0$,

$$\begin{aligned} k(\rho; c_1, c_2) = \quad & \frac{1}{c_2-c_1}[c_2 - c_1 + c_1 g_1(\gamma\bar{\rho}) - c_2 g_2(\gamma\rho) - \{1-g_1(\gamma\bar{\rho})\}\{1-g_2(\gamma\rho)\}c_1c_2\gamma\bar{\rho}\\ & +\{1-g_2(\gamma\rho)\}c_2c_1C_1L_1(\gamma\bar{\rho})(\gamma\bar{\rho})^\nu - \{1-g_1(\gamma\bar{\rho})\}c_1c_2C_2L_2(\gamma\rho)(\gamma\rho)^\nu\\ & -c_1c_2C_1C_2L_1(\gamma\bar{\rho})L_2(\gamma\rho)\frac{|\gamma\rho|^{2\nu}}{\gamma\rho}\\ & -\{c_2C_2L_2(\gamma\rho) + (-1)^\nu c_1C_1L_1(\gamma\bar{\rho})\}(\gamma\rho)^{\nu-1}]. \end{aligned} \tag{2.3}$$

Put: for Re $\rho = 0$,

$$h(\rho; c_1, c_2) \ := \frac{1}{c_2-c_1}[c_2 - c_1 - (\gamma\rho)^{\nu-1}\{c_2C_2L_2(\gamma\rho)\ + (-1)^\nu c_1C_1L_1(\gamma\bar{\rho})\}]. \tag{2.4}$$

With

$$\rho = xr \ , \ x > 0, \tag{2.5}$$

we obtain from (2.4) and (2.5) for $x > 0$, Re $r = 0$,

$$\begin{aligned} h(xr; c_1, c_2) \ &= \frac{1}{c_2 - c_1}[c_2 - c_1 - x^{\nu-1}(\gamma r)^{\nu-1} c_2 C_2 \frac{L_2(x\gamma r)}{L_2(x)} L_2(x) \\ &+ x^{\nu-1}(\gamma\bar{\rho})^{\nu-1} c_1 C_1 \frac{L_1(x\gamma\bar{r})}{L_1(x)} L_1(x)]. \end{aligned} \tag{2.6}$$

The equation

$$c_1 - c_2 = x^{\nu-1}|c_2 C_2 L_2(x) + (-1)^\nu c_1 C_1 L_1(x)| \ , x > 0, \tag{2.7}$$

will be called the *contraction equation*, it will be studied in the next section.

The function $h(\rho; c_1, c_2)$ introduced in (2.4) may be considered as the principal term for $\rho \to 0$, Re $\rho = 0$ of the function $k(\gamma\rho; c_1, c_2)$, cf. (2.3), note that $1 < \nu \leq 2$.

It is noted that (2.7) may be rewritten as:

$$c_1 - c_2 = x^{\nu-1}\{c_2^2 C_2^2 L_2^2(x) + c_1^2 C_1^2 L_1^2(x) + 2(\cos\nu\pi) c_1 c_2 L_1(x) L_2(x)\}^{\frac{1}{2}}, \tag{2.8}$$

for $x > 0$.

3. The contraction equation

Put

$$L(x) := c_2 C_2 L_2(x) + (-1)^\nu c_1 C_1 L_1(x), \ x > 0. \tag{3.1}$$

then the contraction equation, defined in (2.7) becomes for $1 < \nu \leq 2$,

$$c_1 - c_2 = x^{\nu-1}|L(x)|, \ x > 0. \tag{3.2}$$

The righthand side in (3.2) is positive for $x > 0$, and tends to zero for $x \downarrow 0$, cf. remarks 1.1 and 1.3. Because $c_1 > c_2$, cf. (1.1) and (4.4), it follows for

$$1 - \frac{c_2}{c_1} << 1 \textit{ or equivalently } 1 - a << 1,$$

that the contraction equation (3.2) has a unique positive root with the property that it tends to zero for $a \uparrow 1$. Henceforth this root will be denoted by $\Delta(a)$, so

$$\begin{aligned} &c_1 - c_2 = \Delta^{\nu-1}(a)|L(\Delta(a))| && \text{for } 1 - a << 1, \\ &\Delta(a) \downarrow 0 && \text{for } a \uparrow 1, \ i.e. \, c_1 = 1, c_2 \to a. \end{aligned} \tag{3.3}$$

With

$$\rho = r\Delta(a), \tag{3.4}$$

we consider the function $k(\gamma\rho; c_1, c_2)$, Re $\rho = 0$ for $a \uparrow 1$. From (2.3) we have: for Re $r = 0$,

$$\begin{aligned}
&k(r\Delta(a);c_1,c_2) = [1+\frac{1}{\Delta}\{c_1g_1(\gamma\bar{r}\Delta)-c_2g_2(\gamma r\Delta)\}\frac{\Delta}{c_2-c_1}\\
&+\{1-g_1(\gamma\bar{r}\Delta)\}\{1-g_2(\gamma r\Delta)\}c_1c_2\gamma\bar{r}\frac{\Delta}{c_2-c_1}\\
&+\{1-g_2(\gamma r\Delta)\}c_1c_2C_1\frac{L_1(\gamma\bar{r}\Delta)}{L_1(\Delta)}(\gamma\bar{r})^{\nu}\frac{L_1(\Delta)}{L(\Delta)}\frac{\Delta^{\nu}L(\Delta)}{c_2-c_1}\\
&-\{1-g_1(\gamma\bar{r}\Delta)\}c_1c_2C_2\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}(\gamma r)^{\nu}\frac{L_2(\Delta)}{L(\Delta)}\frac{\Delta^{\nu}L(\Delta)}{c_2-c_1}\\
&-c_1c_2C_1C_2\frac{L_1(\gamma\bar{r}\Delta)}{L_1(\Delta)}\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}\frac{|\gamma r|^{2\nu}}{\gamma r}\frac{L_1(\Delta)L_2(\Delta)}{L^2(\Delta)}\frac{\Delta^{2\nu-1}L^2(\Delta)}{c_2-c_1}\\
&-\{c_2C_2\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}\frac{L_2(\Delta)}{|L(\Delta)|}+(-1)^{\nu}c_1C_1\frac{L_1(\gamma\bar{r}\Delta)}{L_1(\Delta)}\frac{L_1(\Delta)}{|L(\Delta)|}\}\frac{\Delta^{\nu-1}|L(\Delta)|}{c_2-c_1}(\gamma r)^{\nu-1}]_{\Delta=\Delta(a)}.
\end{aligned} \tag{3.5}$$

From (1.8)iii we obtain: for $\Delta(a)\downarrow 0$,

$$|g_j(\gamma r\Delta(a))| = \mathrm{O}(\Delta(a)),\ j=1,2; \tag{3.6}$$

and from (1.8),

$$\frac{L_j(\gamma r\Delta(a))}{L_j(\Delta(a))}\to 1 \text{ for } j=1,2. \tag{3.7}$$

To investigate the behaviour of $k(\gamma r\Delta a;c_1,c_2)$ for $\Delta(a)\downarrow 0$ we first consider the case that, see definition 1.1,

$$A(t)\ and\, B(t)\ have\ similar\ tails. \tag{3.8}$$

It follows from (1.8)iv, (1.9), (3.1) and (3.8) that

$$\frac{L_j(\Delta(a))}{L(\Delta(a))}, j=1,2\ has\ a\ finite\ limit\ for\ \Delta(a)\downarrow 0. \tag{3.9}$$

From the definition of $\Delta(a)$, cf. (3.3) and (1.8), it follows that: for $1<\nu\leq 2$ and $a\uparrow 1, j=1,2,$

$$\begin{aligned}
&\text{i.}\quad \frac{\Delta(a)}{c_2-c_1} = \frac{\Delta^{\nu-1}(a)L(\Delta(a))}{c_2-c_1}\frac{\Delta^{2-\nu}(a)}{L(\Delta(a))}\to 0,\\
&\text{ii.}\quad \frac{\Delta^{\nu}(a)}{c_2-c_1} = \frac{\Delta(a)^{\nu-1}L(\Delta(a))}{c_2-c_1}\frac{\Delta(a)}{L(\Delta(a))}\to 0,\\
&\text{iii.}\quad \frac{\Delta^{2\nu-1}(a)}{c_2-c_1}L^2(\Delta(a)) = \frac{\Delta(a)^{\nu-1}L(\Delta(a))}{c_2-c_1}\Delta^{\nu}(a)L(\Delta(a))\to 0.
\end{aligned} \tag{3.10}$$

Hence from (1.9) (2.6), (2.7), (3.1), (3.2), (3.6), ..., (3.10) it is seen that the folllowing limit exists and that: for Re $r=0$,

$$\begin{aligned}
&\lim_{a\uparrow 1} k(r\Delta(a);c_1,c_2) =\\
&1+(\gamma r)^{\nu-1}\lim_{a\uparrow 1}[c_2C_2\frac{L_2(\Delta(a))}{|L(\Delta(a))|}+(-1)^{\nu}c_1C_1\frac{L_1(\Delta(a))}{|L(\Delta(a))|}] =\\
&1+\lim_{a\uparrow 1}[c_2C_2\frac{L_2(\Delta(a))}{|L(\Delta(a))|}(\gamma r)^{\nu-1}-c_1C_1\frac{L_1(\Delta(a))}{|L(\Delta(a))|}(\gamma\bar{r})^{\nu-1}].
\end{aligned} \tag{3.11}$$

Put, cf. (1.9); for $j=1,2$,

$$d_j := C_j\lim_{a\uparrow 1}\frac{L_j(\Delta(a))}{|L(\Delta(a))|},\ c_1=1, c_2=a, \tag{3.12}$$

so that for $0 < f < \infty$,

$$\begin{aligned} &0 < d_1 = \frac{C_1}{D} \quad \text{and } 0 < d_2 = \frac{C_2 f}{D}, \\ &D := [C_1^2 + f^2 C_2^2 + 2(\cos \nu\pi) f C_1 C_2]^{\frac{1}{2}}, \\ &d_1 + d_2 > 1. \end{aligned} \tag{3.13}$$

Consequently, from (3.11): for $\operatorname{Re} r = 0$,

$$\lim_{a \uparrow 1} k(r\Delta(a); c_1, c_2) = 1 + d_2(\gamma r)^{\nu-1} - d_1(\gamma \bar{r})^{\nu-1}. \tag{3.14}$$

The relation (3.14) holds for the case that $A(t)$ and $B(t)$ have similar tails, for the case of nonsimilar tails see sections 6 and 7.

4. The heavy-traffic theorem for similar tails and $1 < \nu < 2$.

In this section we shall investigate the solution (1.15) of the boundary value problem (1.14) for the case $a \uparrow 1$ with $A(t)$ and $B(t)$ having similar tails with $1 < \nu < 2$. Put: for $\operatorname{Re} r > 0$,

$$\rho = r\Delta(a), \tag{4.1}$$

with $\Delta(a)$ that zero of the contraction equation (3.2) which tends to zero for $a \uparrow 1$. Note that, cf. (1.11),

$$\omega(r\Delta(a)) = \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}}\} \text{ for } \operatorname{Re} r \geq 0. \tag{4.2}$$

It then follows from (1.15) and (4.2) that: for $\operatorname{Re} r > 0$,

$$\omega(r\Delta(a)) = \mathrm{e}^{H(r\Delta(a))}, \tag{4.3}$$

with

$$H(r\Delta(a)) = \frac{1}{2\pi \mathrm{i}} \int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \frac{1 - \beta(\xi)\alpha(\bar{\xi})}{(\beta - \alpha)\xi}\} \frac{r\Delta(a)}{\xi - r\Delta(a)} \frac{\mathrm{d}\xi}{\xi}. \tag{4.4}$$

It is seen by using (1.4) and (2.1) that: for $\operatorname{Re} r > 0$,

$$H\{\frac{r}{\gamma}\Delta(a)\} = \frac{1}{2\pi \mathrm{i}} \int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log k(\xi\Delta(a)/\gamma; c_1, c_2)\} \frac{r}{\xi - r} \frac{\mathrm{d}\xi}{\xi}, \tag{4.5}$$

with, cf. (3.11); for $\operatorname{Re} \xi = 0$,

$$\lim_{a \uparrow 1} k(\xi\Delta(a)/\gamma; c_1, c_2) = 1 + d_2 \xi^{\nu-1} - d_1 \bar{\xi}^{\nu-1}. \tag{4.6}$$

Define: for $\operatorname{Re} r > 0$,

$$\Phi(\frac{r}{\gamma}) := \frac{1}{2\pi \mathrm{i}} \int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\{1 + d_2\xi^{\nu-1} - d_1\bar{\xi}^{\nu-1}\} \frac{r}{\xi - r} \frac{\mathrm{d}\xi}{\xi}. \tag{4.7}$$

Obviously: for $r > 0$,

$$\Phi(\frac{r}{\gamma}) = \lim_{a \uparrow 1} H(\frac{r}{\gamma}\Delta(a)), \tag{4.8}$$

if the limit and the integral in (4.5) can be interchanged.

We first consider $\Phi(\frac{r}{\gamma})$. We have: for $r>0$,

$$
\begin{aligned}
\Phi(\frac{r}{\gamma}) \quad &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \log\{1+[d_2\mathrm{e}^{\frac{\nu-1}{2}\pi\mathrm{i}} - d_1\mathrm{e}^{-\frac{\nu-1}{2}\pi\mathrm{i}}]s^{\nu-1}\}\frac{r}{\mathrm{i}s-r}\frac{\mathrm{d}s}{s} \\
&+\frac{1}{2\pi\mathrm{i}}\int_\infty^0 \log\{1+[d_2\mathrm{e}^{-\frac{\nu-1}{2}\pi\mathrm{i}} - d_1\mathrm{e}^{\frac{\nu-1}{2}\pi\mathrm{i}}]s^{\nu-1}\}\frac{r}{-\mathrm{i}s-r}\frac{\mathrm{d}s}{s} = \\
&-\frac{1}{2\pi\mathrm{i}}\int_0^\infty \log\{1+s^{\nu-1}[(d_2-d_1)\cos\frac{\nu-1}{2}\pi \\
&-\mathrm{i}(d_2+d_1)\sin\frac{\nu-1}{2}\pi]\}(-r)\frac{r-\mathrm{i}s}{r^2+s^2}\frac{\mathrm{d}s}{s} \\
&+\frac{1}{2\pi\mathrm{i}}\int_0^\infty \log\{1+s^{\nu-1}[(d_2-d_1)\cos\frac{\nu-1}{2}\pi \\
&+\mathrm{i}(d_2+d_1)\sin\frac{\nu-1}{2}]\}(-r)\frac{r+\mathrm{i}s}{r^2+s^2}\frac{\mathrm{d}s}{s} = \\
&\frac{-1}{2\pi\mathrm{i}}\int_0^\infty \log\{\frac{1+[(d_2-d_1)\cos\frac{\nu-1}{2}\pi+\mathrm{i}(d_2+d_1)\sin\frac{\nu-1}{2}\pi]s^{\nu-1}}{1+[(d_2-d_1)\cos\frac{\nu-1}{2}\pi-\mathrm{i}(d_2+d_1)\sin\frac{\nu-1}{2}\pi]s^{\nu-1}}\}\frac{r^2}{r^2+s^2}\frac{\mathrm{d}s}{s} \\
&-\frac{1}{2\pi\mathrm{i}}\int_0^\infty \log[\{1+[(d_2-d_1)s^{\nu-1}\cos\frac{\nu-1}{2}\pi\}^2 \\
&+\{(d_2+d_1)s^{\nu-1}\sin\frac{\nu-1}{2}\pi\}^2]\frac{\mathrm{i}rs}{r^2+s^2}\frac{\mathrm{d}s}{s}.
\end{aligned}
\tag{4.9}
$$

Put

$$
\begin{aligned}
&A := (d_2+d_1)\sin\frac{\nu-1}{2}\pi > 0, \\
&B := (d_2-d_1)\cos\tfrac{\nu-1}{2}\pi, \\
&C := d_1^2+d_2^2+2(d_2-d_1)\cos(\nu-1)\pi = (d_1-d_2)^2+4d_1d_2\sin^2\tfrac{\nu-1}{2}\pi > 0.
\end{aligned}
\tag{4.10}
$$

It follows from (4.9) and (4.10) that: for $r>0$,

$$
\begin{aligned}
\Phi(\frac{r}{\gamma}) \quad &= -\frac{1}{\pi}\int_0^\infty \{\arctan\frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}}\}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s} \\
&-\frac{1}{2\pi}\int_0^\infty \log\{1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}\}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s}.
\end{aligned}
\tag{4.11}
$$

Because of $1<\nu<2$ it is seen that: for $r>0, s>0$,

$$
1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)} > 1 \,\text{for}\; s>0, \tag{4.12}
$$

and

$$
-\frac{\pi}{2} \le \arctan\frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}} \le \frac{\pi}{2}, \tag{4.13}
$$

so that both integrals in (4.11) exist and are finite.

We next show that the relation (4.8) holds. By inserting in (4.5) the expression (3.5) for $k(r\Delta(a);c_1,c_2)$ it is readily seen by transforming the integral in (4.5) to an integral over $[0,\infty)$ (cf. the transformation of (4.7) into (4.9)) that the integral in (4.5) converges uniformly for $0<\Delta(a)<<1$; Hence from (4.12) and (4.13) it is seen that (4.8) holds and it follows that the following limit exists, and cf. (4.2), (4.3), (4.7) and (4.8),

$$\begin{aligned}\lim_{a\uparrow 1}\omega(\frac{r}{\gamma}\Delta(a)) &\equiv \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma}\}\\ &=\mathrm{e}^{\Phi(\frac{r}{\gamma})},\quad \mathrm{Re}\ r>0,\end{aligned}\tag{4.14}$$

with $\Phi(\frac{r}{\gamma})$ given by (4.7) for Re $r>0$ and by (4.11) for $r>0$. Because $\omega(r\Delta(a))=1$ for $r=0$, cf. (1.14)iii and (1.15) and $\Phi(r/\gamma)\to 0$ for $r\to 0$, it follows from Feller's continuity theorem for the L.S. transforms of probability distributions, cf. [7], p. 431, that the *contracted* waiting $\Delta(a)\mathbf{w}/\gamma$ converges in distribution for $a\uparrow 1$. Hence we have the following heavy traffic limit theorem.

THEOREM 4.1. *Whenever the interarrival and service time distribution $A(t)$ and $B(t)$ have similar tails with $1<\nu<2$ then the contracted waiting time $\Delta(a)\mathbf{w}$ converges in distribution for $a\uparrow 1$ and*

$$\text{i.}\quad \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma}\}=\mathrm{e}^{\Phi(r/\gamma)},\ \mathrm{Re}\ r\geq 0,\tag{4.15}$$

where

$$\begin{aligned}
\text{i.}\quad \Phi(\frac{r}{\gamma}) &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\{1+d_2\xi^{\nu-1}-d_1\bar{\xi}^{\nu-1}\}]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi},\ \mathrm{Re}\ r>0,\\
\text{ii.}\quad \Phi(\frac{r}{\gamma}) &= -\frac{1}{2}\log\{1+d_2r^{\nu-1}-d_1\bar{r}^{\nu-1}\}\\
&\quad+\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\{1+d_2\xi^{\nu-1}-d_1\bar{\xi}^{\nu-1}\}]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi},\mathrm{Re}\ r=0,\\
\text{iii.}\quad \Phi(\frac{r}{\gamma}) &= -\frac{1}{\pi}\int_0^\infty\{\arctan\frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}}\}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s}\\
&\quad-\frac{1}{2\pi}\int_0^\infty\log\{1+B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}\}\frac{\mathrm{d}s}{1+s^2},\ r\geq 0,
\end{aligned}\tag{4.16}$$

here d_1 and d_2 are given by (3.13), for A, B and C see (4.10), for the contraction coefficient $\Delta(a)$, see section 3.

PROOF. The statement concerning the convergence has been proved above, see below (4.14). For (4.16)i, see (4.7) and (4.14); (4.16)ii follows from (4.16)i by using the Plemelj-Sokhotski formula, cf. [1], for (4.16)iii, cf. (4.11). □

REMARK 4.1. It should be noted that (4.16)i and ii is the analytic continuation of (4.16)iii into Re $r>0$. □

We conclude this section with the analysis of the excess variable $\tilde{\mathbf{i}}$ of the idle time $\mathbf{i}$, i.e. $\tilde{\mathbf{i}}$ is the nonnegative stochastic variable with distribution function given by

$$\Pr\{\tilde{\mathbf{i}}<t\}=\frac{1}{\mathrm{E}\{\mathbf{i}\}}\int_0^t[1-\Pr\{\mathbf{i}\leq t\}]\mathrm{d}t,\ t\geq 0.\tag{4.17}$$

It is wellknown that: for Re $\rho \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\tilde{\mathbf{i}}}\} = \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{i}}\}}{\rho\mathrm{E}\{\mathbf{i}\}} = \chi(\rho), \tag{4.18}$$

for the last equality sign in (4.18) see (1.12). From (1.15) we have

$$\chi(\rho) = \mathrm{e}^{H(-\rho)}, \text{ Re } \rho > 0, \tag{4.19}$$

and, as before, cf. (4.7), it is shown that: for $1 < \nu < 2$, Re $r \geq 0$,

$$\lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\tilde{\mathbf{i}}/\gamma}\} = \lim_{a\uparrow 1} \chi(r\Delta(a)/\gamma) = \mathrm{e}^{\Phi(-r/\gamma)}, \tag{4.20}$$

with

$$\Phi(-r/\gamma) = \frac{-1}{2\pi\mathbf{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\{1 + d_2\xi^{\nu-1} - d_1\bar{\xi}^{\nu-1}\} \frac{r}{\xi+r}\frac{\mathrm{d}\xi}{\xi}. \tag{4.21}$$

Analogous to the derivation of (4.11) we obtain for: $r > 0$,

$$\begin{aligned}\Phi(-\frac{r}{\gamma}) \quad &= -\frac{1}{\pi}\int_0^\infty \arctan \frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s} \\ &+\frac{1}{2\pi}\int_0^\infty \log\{1+2B(rs)^{\nu-1} + C(rs)^{2(\nu-1)}\}\frac{\mathrm{d}s}{1+s^2}.\end{aligned} \tag{4.22}$$

Note that $\Phi(-r/\gamma) \to 0$ for $r \to 0$. It follows readily that we have proved the following

Corollary 4.1. *For the conditions of the theorem 4.1 the stochastic variable $\Delta(a)\tilde{\mathbf{i}}$ converges in distribution for $a \uparrow 1$ and with $\Phi(-r/\gamma)$ given by (4.22):*

$$\lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r(a)\tilde{\mathbf{i}}/\gamma)} = \mathrm{e}^{\Phi(-r/\gamma)}, \; r > 0. \tag{4.23}$$

5. The heavy-traffic theorem for similar tails with $\nu = 2$.

In this section we investigate the case with $A(t)$ and $B(t)$ have similar tails and $\nu = 2$.

With $\Delta(a)$ the zero of the contraction equation (3.2) with $\nu = 2$, i.e. of

$$c_1 - c_2 = xL(x) \; , \; x > 0,$$

and $L(x) \to \infty$ for $x \downarrow 0$, cf. (1.8)iv, (1.9) and (3.1) for $\nu = 2$, we have as in section 4: for Re $r > 0$,

$$\lim_{a\uparrow 1} H(\frac{r}{\gamma}\Delta(a)) \; = \; \frac{1}{2\pi\mathrm{i}} \int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1+d_2\xi - d_1\bar{\xi}\}]\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}. \tag{5.1}$$

Because we consider the case $\nu = 2$, the similar analysis which has led to (4.11) shows that: for $r > 0$,

$$\begin{aligned}\lim_{a\uparrow 1} H(r\Delta(a)/\gamma) \quad &= -\frac{1}{\pi}\int_0^\infty \{\arctan Ars\}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s} \\ &+\frac{1}{2\pi}\int_0^\infty \log\{1+A^2(rs)^2\}\frac{\mathrm{d}s}{1+s^2},\end{aligned} \tag{5.2}$$

note that: for $\nu = 2$, cf. (4.10),

$$A = d_1 + d_2 = 1,\ B = 0,\ C = (d_1 + d_2)^2 = A^2 = 1, \tag{5.3}$$

with d_1 and d_2 given by (3.13), since for $\nu = 2$ we have $0 < f < \infty$, cf. definition 1.1.

The integrals occurring in (5.2) have been calculated in appendix B. From (5.2), and (b.1), (b2), (b.18) of appendix B it is seen that

$$\lim_{a\uparrow 1} H(\frac{r}{\gamma}\Delta(a)\} = -\log\{1+r\},\ r > 0. \tag{5.4}$$

Consequently, it follows from (1.15) that

$$\begin{aligned}\hat{\omega}(r) \quad &:= \lim_{a\uparrow 1} \omega(r\Delta(a)/\gamma) = \lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma)}\} \\ &= \frac{1}{1+r} \text{ for } r > 0,\end{aligned} \tag{5.5}$$

Because the righthand side of (5.5) is continuous in $r = 0$, (5.5) also holds for $r \geq 0$. Moreover both sides of (5.5) are regular for Re $r > 0$, continuous for Re $r \geq 0$ and so by analytic continuation:

$$\hat{\omega}(r) = \frac{1}{1+r} \text{ for Re } r \geq 0. \tag{5.6}$$

By applying again Feller's continuity theorem we obtain as in the preceeding section the following heavy traffic limit theorem.

THEOREM 5.1. *Whenever $A(t)$ and $B(t)$ have similar tails with $\nu = 2$ then $\Delta(a)\mathbf{w}$ converges in distribution for $a \uparrow 1$ and*

$$\begin{aligned}&\text{i.}\quad \lim_{a\uparrow 1} \mathrm{E}\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma}\} = \frac{1}{1+r},\ \text{Re } r \geq 0, \\ &\text{ii.}\quad \lim_{a\uparrow 1} \Pr\{\Delta(a)\mathbf{w} < \gamma t\} = 1 - \mathrm{e}^{-t},\ t \geq 0.\end{aligned} \tag{5.7}$$

REMARK 5.1. It is noted that the result for $\nu = 2$ is identical to that in [4], cf. section 5 of [4], although the conditions for its validity in [4] differ slightly from those required here, which are some what weaker. □

REMARK 5.2. To calculate $\chi(\Delta(a)r)$ for $a \uparrow 1$, replace in (5.2) r by $-r$ and use the results in appendix B; it is then seen that $H(-r\Delta(a)/\gamma) \to 0$ for $a \uparrow 1$ and so from (1.15) it is seen that

$$\lim_{a\uparrow 1} \chi(r\Delta(a)) = 1 \text{ for } r \geq 0. \tag{5.8}$$

□

6. THE TAIL OF $B(t)$ IS HEAVIER THAN THAT OF $A(t)$

In this section we consider the case that the tail of $B(t)$ is heavier than the one of $A(t)$; so that definition 1.1 shows that one of the following cases occur:

$$\begin{aligned}&\text{i.}\quad \nu_1 > v_2, \\ &\text{ii.}\quad \nu_1 = \nu_2, \quad b_1 < \infty, b_2 = \infty, \\ &\text{iii.}\quad \nu_1 = \nu_2, \quad b_1 = \infty, b_2 = \infty,\ f = \infty.\end{aligned} \tag{6.1}$$

With $k(\gamma\rho; c_1, c_2)$ as defined in (2.1) we have, cf. (2.3), for Re $\rho = 0$,

$$
\begin{aligned}
&k(\rho;c_1,c_2)= \\
&\frac{1}{c_2-c_1}[c_2-c_1+c_1g_1(\gamma\bar\rho)-c_2g_2(\gamma\rho)-\{1-g_1(\gamma\bar\rho)\}\{1-g_2(\gamma\rho)\}c_1c_2\gamma\bar\rho \\
&+\{1-g_2(\gamma\rho)\}c_2c_1C_1L_1(\gamma\bar\rho)(\gamma\bar\rho)^{\nu_1}-\{1-g_1(\gamma\bar\rho)\}c_1c_2C_2L_2(\gamma\rho)(\gamma\rho)^{\nu_2} \\
&-c_1c_2C_1C_2L_1(\gamma\bar\rho)L_2(\gamma\rho)\frac{(\gamma\bar\rho)^{\nu_1}(\gamma\rho)^{\nu_2}}{\gamma\rho} \\
&-\{c_2C_2L_2(\gamma\rho)(\gamma\rho)^{\nu_2-1}-c_1C_1L_1(\gamma\bar\rho)(\gamma\bar\rho)^{\nu_1-1}\}].
\end{aligned}
\tag{6.2}
$$

For the present case the equation

$$
c_1-c_2=x^{\nu_2-1}c_2C_2L_2(x),\ x>0,\ 1<\nu_2\le 2, \tag{6.3}
$$

will be called the *contraction equation*, and by $\Delta(a)$ we shall again denote that zero of (6.3) which goes to zero for $\frac{c_2}{c_1}=a\uparrow 1$; this zero of (6.3) is unique.

With

$$
\rho=r\Delta(a)
$$

we have from (6.2): for Re $r=0$,

$$
\begin{aligned}
&k(r\Delta(a);c_1,c_2)=[1+\frac{1}{\Delta}\{c_1g_1(\gamma\bar r\Delta)-c_2g_2(\gamma r\Delta)\}\frac{\Delta}{c_2-c_1} \\
&+\{1-g_1(\gamma\bar r\Delta)\}\{1-g_2(\gamma r\Delta)\}c_1c_2\gamma\bar r\frac{\Delta}{c_2-c_1} \\
&+\{1-g_2(\gamma r\Delta)\}c_2c_1C_1\frac{L_1(\gamma\bar r\Delta)}{L_1(\Delta)}(\gamma\bar r)^{\nu_1}\frac{L_1(\Delta)}{L_2(\Delta)}\frac{\Delta^{\nu_1}L_2(\Delta)}{c_2-c_1} \\
&-\{1-g_1(\gamma\bar r\Delta)\}c_1c_2\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}(\gamma r)^{\nu_2}\frac{\Delta^{\nu_2}L_2(\Delta)}{c_2-c_1} \\
&-c_1c_2C_1C_2\frac{L_1(\gamma\bar r\Delta)}{L_1(\Delta)}\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}\frac{L_1(\Delta)}{L_2(\Delta)}\frac{(\gamma\bar r)^{\nu_1}(\gamma r)^{\nu_2}}{\gamma r}\frac{\Delta^{\nu_1+\nu_2-1}L_2^2(\Delta)}{c_2-c_1} \\
&-c_2C_2\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}(\gamma r)^{\nu_2-1}\frac{\Delta^{\nu_2-1}L_2(\Delta)}{c_2-c_1} \\
&+c_1C_1\frac{L_1(\gamma\bar r\Delta)}{L_1(\Delta)}\frac{L_1(\Delta)}{L_2(\Delta)}(\gamma\bar r)^{\nu_1-1}\frac{\Delta^{\nu_1-1}L_2(\Delta)}{c_2-c_1}]_{\Delta=\Delta(a)}.
\end{aligned}
\tag{6.4}
$$

From (6.3) and the definition of $\Delta(a)$, we have

$$
\frac{\Delta(a)^{\nu_2-1}c_2C_2L_2(\Delta(a))}{c_1-c_2}\to 1 \text{ for } \Delta(a)\to 0. \tag{6.5}
$$

We consider the relation (6.4) for $\Delta(a)\downarrow 0$. As in section 3 it is shown for all three cases of (6.1) that (1.8), (1.9) and (6.5) imply that the following limit exists and that: for $1<\nu_2\le 2$ and Re $r=0$,

$$
\lim_{a\uparrow 1}k(r\Delta(a);c_1;c_2)=1+(\gamma r)^{\nu_2-1}. \tag{6.6}
$$

As in section 4 we obtain from (6.6): for Re $r>0$,

$$
\begin{aligned}
\Phi(\frac{r}{\gamma}) \quad &:= \lim_{a\uparrow 1}\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty}\{\log k(\xi\Delta(a)/\gamma;c_1c_2)\}\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi} \\
&=\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}\log\{1+\xi^{\nu_2-1}\}\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi} \\
&=-\log\{1+r^{\nu_2-1}\}.
\end{aligned}
\tag{6.7}
$$

The last equality in (6.7) is easily derived by contour integration in the righthalf ξ-plane by noting that the logarithm in the integrand is regular for Re $\xi > 0$ and continuous for Re $\xi \geq 0$, since $\nu_2 > 1$, see also [4].

REMARK 6.1. A simple calculation shows that $-\Phi(r/\gamma)$ in (6.7) can be also written as (4.4) with $d_1 = 1, d_2 = 0$, i.e. $A = \sin \frac{\nu-1}{2}\pi, B = \cos \frac{\nu-1}{2}\pi, C = A^2$. □

As in section 4 and 5 the following theorem is proved.

THEOREM 6.1. *Whenever the tail of $B(t)$ is heavier than that of $A(t)$ then $\Delta(a)\mathbf{w}$ converges in distribution for $a \uparrow 1$ and*

$$\text{i.} \quad \lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma}\} = \frac{1}{1+r^{\nu_2-1}}, \mathrm{Re}\ r \geq 0, \tag{6.8}$$

$$\text{ii.} \quad \lim_{a\uparrow 1} \Pr\{\Delta(a)\mathbf{w} < \gamma t\} = R_{\nu_2-1}(t)\ , t \geq 0\, , 1 < \nu_2 \leq 2,$$

with the contraction coefficient $\Delta(a)$ that root of the contraction equation (6.3) as defined below (6.3).

REMARK 6.2. The distribution $R_{\nu-1}(t)$ and its asymptotic series for $t \to \infty$ are explicitly known, see herefor the study [4]. Theorem 6.1 remains true for all $\nu_1 > \nu_2$, as it is readily seen from the analysis given above. Theorem 6.1 has been obtained for the first time in [4], however, the conditions for its validity are here somewhat weaker than in [4]. □

REMARK 6.3. From (5.7) it is seen that $\Phi(r/\gamma) = 0$ for $r < 0$, note that the integral in (6.7) has no pole in Re $\xi > 0$ if $r < 0$. So from (1.15) we obtain

$$\lim_{a\uparrow 1} \chi(r\Delta(a)) = 1 \text{ for } r \geq 0. \tag{6.9}$$

□

7. THE TAIL OF $A(t)$ IS HEAVIER THAN THAT OF $B(t)$

In this section we consider the case that the tail of $A(t)$ is heavier than that of $B(t)$. From definition 1.1 it is seen that one of the following cases occur:

$$\text{i.} \quad \nu_2 > \nu_1, \tag{7.1}$$

$$\text{ii.} \quad \nu_2 = \nu_1, b_1 < \infty, b_2 = \infty,$$

$$\text{ii.} \quad \nu_2 = \nu_1, b_1 = \infty, b_2 = \infty\ ,\ f = 0.$$

For the present case the equation

$$c_1 - c_2 = x^{\nu_1-1}c_1C_1L_1(x), x > 0, \tag{7.2}$$

will be called the *contraction equation* and by $\Delta(a)$ we shall again denote that zero of (7.2) which tends to zero for $c_2 \uparrow c_1$ or equivalently $a \uparrow 1$.

With

$$\rho = r\Delta(a),$$

we have from (7.2),

$$\frac{\Delta^{\nu_1-1}(a)c_1C_1L_1(\Delta(a))}{c_1 - c_2} \to 1 \ \text{ for } \Delta(a) \downarrow 0. \tag{7.3}$$

As in (6.2) we now have: for Re $r=0$,

$$
\begin{aligned}
&k(r\Delta(a);c_2,c_2)=[1+\frac{1}{\Delta}\{c_1g_1(\gamma\bar{r}\Delta)-c_2g_2(\gamma r\Delta)\}\frac{\Delta}{c_2-c_1}\\
&+\{1-g_1(\gamma\bar{r}\Delta)\}\{1-g_2(\gamma r\Delta)\}c_1c_2\gamma\bar{r}\frac{\Delta}{c_2-c_1}\\
&+\{1-g_1(\gamma\bar{r}\Delta)\}\{1-g_2(\gamma r\Delta)\}c_1c_2\gamma\bar{r}\frac{\Delta}{c_2-c_1}\\
&-\{1-g_1(\gamma\bar{r}\Delta)\}c_1c_2C_2\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}(\gamma r)^{\nu_2}\frac{L_2(\Delta)}{L_1(\Delta)}\frac{\Delta^{\nu_2}L_1(\Delta)}{c_2-c_1}\\
&+\{1-g_2(\gamma r\Delta)\}c_1c_2C_1\frac{L_1(\gamma\bar{r}\Delta)}{L_1(\Delta)}(\gamma\bar{r})^{\nu_1}\frac{\Delta^{\nu_1}L_1(\Delta)}{c_2-c_1}\\
&-c_1c_2C_1C_2\frac{L_1(\gamma\bar{r}\Delta)}{L_2(\Delta)}\frac{L_2(\gamma r\Delta)}{L_1(\Delta)}\frac{L_2(\Delta)}{L_1(\Delta)}\frac{(\gamma\bar{r})^{\nu_1}(\gamma r)^{\nu_2}}{\gamma r}\frac{\Delta^{\nu_1+\nu_2-1}L_1^2(\Delta)}{c_2-c_1}\\
&-\{c_2C_2\frac{L_2(\gamma r\Delta)}{L_2(\Delta)}\frac{L_2(\Delta)}{L_1(\Delta)}(\gamma r)^{\nu_2-1}\frac{\Delta^{\nu_2-1}L_1(\Delta)}{c_2-c_1}\\
&-c_1C_1\frac{L_1(\gamma\bar{r}\Delta)}{L_1(\Delta)}(\gamma\bar{r})^{\nu_1-1}\frac{\Delta^{\nu_1-1}L_1(\Delta)}{c_2-c_1}\}]_{\Delta=\Delta(a)}.
\end{aligned}
\tag{7.4}
$$

We consider the relation (7.4) for $a\uparrow 1$. Analogously to the derivation of (6.6) it is seen that: for Re $r=0$,

$$
\lim_{a\uparrow 1}k(r\Delta(a);c_1,c_2)=1+(\gamma\bar{r})^{\nu_1-1}. \tag{7.5}
$$

It follows as before that: for Re $r>0$,

$$
\begin{aligned}
\Phi(\frac{r}{\gamma})\quad &:=\lim_{a\uparrow 1}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\log\{k(\chi\Delta(a)/\gamma;c_1,c_2)\}\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}\\
&=\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty}\log\{1-\bar{\xi}^{\nu_1-1}\}\frac{r}{\xi-r}\frac{\mathrm{d}\xi}{\xi}\\
&=\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty}\log\{1-\xi^{\nu_1-1}\}\frac{r}{\xi+r}\frac{\mathrm{d}\xi}{\xi}.
\end{aligned}
\tag{7.6}
$$

To calculate the integral in (7.6) write: for Re $r>0$,

$$
\begin{aligned}
\Phi(\frac{r}{\gamma})\quad &=\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}^{+\mathrm{i}\infty}\log\{\frac{1-\xi^{\nu_1-1}}{1-\xi}\}\frac{r}{\xi+r}\frac{\mathrm{d}\xi}{\xi}+\\
&\frac{1}{2\pi\mathrm{i}}\int_{\xi=-\mathrm{i}\infty}^{\mathrm{i}\infty}\log\{1-\xi\}\frac{r}{\xi+r}\frac{\mathrm{d}\xi}{\xi}.
\end{aligned}
\tag{7.7}
$$

The logarithm in the first integral in (7.7) is regular for Re $\xi>0$, and continuous for Re $\xi\geq 0$, since $\nu_1>1$. Because Re $\xi>0$ it follows readily by contour integration in the right half ξ-plane that this integral is zero. The logarithm in the second integral of (7.7) is regular for Re $\xi<0$, continuous for Re $\xi\leq 0$. Because $\xi=-r$ is a simple pole of this second integral which is also a principal value singular Cauchy integral, it follows readily by contour integration in the left half ξ-plane that this second integral is equal to $-\log\{1+r\}$. Hence: for Re $r>0$,

$$\Phi(\frac{r}{\gamma}) = \log \frac{1}{1+r}. \tag{7.8}$$

As in the preceding sections 4, 5 and 6 we obtain the following

THEOREM 7.1. *Whenever the tail of $A(t)$ is heavier than that of $B(t)$ then $\Delta(a)\mathbf{w}$ converges in distribution for $a \uparrow 1$ and*

$$\begin{aligned} &\lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma}\} = \frac{1}{1+r} \text{ for Re } r \geq 0, \\ &\lim_{a\uparrow 1} \Pr\{\Delta(a)\mathbf{w} < \gamma t\} = 1 - \mathrm{e}^{-t} \text{ for } t \geq 0, \end{aligned} \tag{7.9}$$

with $\Delta(a)$ that root of the contraction equation (7.2) which tends to zero for $a \uparrow 1$.

REMARK 7.1. Theorem 7.1 also holds for all $\nu_2 > \nu_1$, as it may be seen from the analysis above. This theorem has also been obtained in [4], however, the present theorem holds for somewhat weaker conditions than that in [4]. □

REMARK 7.2. It is simply seen that the integrals in (7.7) are both zero for $r < 0$. Hence from (1.15) it is shown that

$$\lim_{a\uparrow 1} \chi(r\Delta(a)) = 1 \text{ for } r > 0. \tag{7.10}$$

□

8. THE ASYMPTOTIC EXPRESSION FOR THE TAIL OF THE LIMITING DISTRIBUTION, $1 < \nu < 2$

In this section we shall derive an asymptotic expression for $t \to \infty$ for the limiting distribution $W_{\nu-1}(t)$ with $1 < \nu < 2$, where

$$1 - W_{\nu-1}(t) := \lim_{a\uparrow 1} \Pr\{\Delta(a)\mathbf{w} \geq t\}. \tag{8.1}$$

From theorem 4.1 we have: for $r > 0$,

$$\hat{\omega}(r) := \lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{w}/\gamma}\} = \mathrm{e}^{\Phi(r/\gamma)}. \tag{8.2}$$

Hence: for Re $r > 0$,

$$\int_{0-}^{\infty} \mathrm{e}^{-rt} \mathrm{d}W_{\nu-1}(t) = \mathrm{e}^{\Phi(r)},$$

so that

$$\int_{0}^{\infty} \mathrm{e}^{-rt}\{1 - W_{\nu-1}(\gamma t)\}\mathrm{d}t = \frac{1 - \mathrm{e}^{\Phi(r/\gamma)}}{r}, r \geq 0. \tag{8.3}$$

From (4.16) and (a.20) of appendix A we have: for $1 < \nu < 2, r \downarrow 0$,

$$\Phi(r/\gamma) = -\max(d_1, d_2) r^{\nu-1}\{1 + \mathrm{O}(r^{(\nu-1)})\}. \tag{8.4}$$

Consequently, from (8.3) and (8.4) we obtain for $1 < \nu < 2, r \downarrow 0$,

$$\int_{0}^{\infty} \mathrm{e}^{-rt}\{1 - W_{\nu-1}(\gamma t)\}\mathrm{d}t = \max(d_1, d_2) r^{\nu-2}\{1 + \mathrm{O}(r^{(\nu-1)})\}. \tag{8.5}$$

By applying theorem 2 of [5], vol. II. p. 159, we obtain from (8.5): for $1 < \nu < 2, t \to \infty$,

$$1 - W_{\nu-1}(\gamma t) = \frac{\max(d_1, d_2)}{\Gamma(2-\nu)} \frac{1}{t^{\nu-1}} \{1 + \mathrm{O}(\frac{1}{t^{\nu-1}})\}, \tag{8.6}$$

with d_1 and d_2 given by (3.13).

For the case that the tail of $B(t)$ is heavier than that of $A(t)$, it is seen from (6.8) that: for $t \to \infty$,

$$1 - R_{\nu_2-1}(\gamma t) = \frac{1}{\Gamma(2-\nu_2)} \frac{1}{t^{\nu_2-1}} \{1 + \mathrm{O}(\frac{1}{t^{\nu_2-1}})\}. \tag{8.7}$$

Hence with $\nu_2 = \nu$ it follows that $W_{\nu-1}(t)$ and $R_{\nu-1}(t)$ have 'similar' tails, and for $t \to \infty$.

$$1 - W_{\nu-1}(\gamma t) = \max(d_1, d_2)\{1 - R_{\nu-1}(\gamma t)\}\{1 + \mathrm{O}(\frac{1}{t^{\nu-1}})\}. \tag{8.8}$$

We have, cf. (3.13),

$$\begin{aligned} d_1 + d_2 &= \frac{C_1 + fC_2}{D} > 1, \\ |C_1 - fC_2| < D &< C_1 + fC_2 \quad \text{for } 1 < \nu < 2, \\ D &= \{C_1^2 + f^2 C_2^2\}^{1/2} \quad \text{for } \nu = 1\frac{1}{2}. \end{aligned} \tag{8.9}$$

Because of (8.8) it is of interest to consider $\max(d_1, d_2)$ as a function of $\nu \in (1,2)$; note that (8.9) implies that $d_j, j = 1, 2,$ can be very large. In appendix C it is shown with

$$\delta_{1,2} := \frac{1}{3}[1 - 4\cos\nu\pi \pm 4\sqrt{(\cos\nu\pi - 1)(\cos\nu\pi + \frac{1}{2})}], \tag{8.10}$$

that

$$\begin{aligned} &\text{i.} \quad 1\frac{1}{3} < \nu < 2 \Rightarrow d_1 + d_2 < 2, \\ &\text{ii.} \quad 1 < \nu < 1\frac{1}{3}, \delta_1 < \frac{fC_2}{C_1} < \delta_2 \Rightarrow d_1 + d_2 < 2, \\ &\text{iii.} \quad 1 < \nu < 1\frac{1}{3}, 0 < \frac{fC_2}{C_1} < \delta_1 \text{ or } \frac{fC_2}{C_1} > \delta_2 \Rightarrow d_1 + d_2 > 2, \\ &\text{iv.} \quad \nu = 1\frac{1}{3}, \frac{fC_2}{C_1} = 1 \Rightarrow d_1 = d_2 = 1,\ d_1 + d_2 = 2. \end{aligned} \tag{8.11}$$

Note that

$$\frac{d_2}{d_1} = \frac{fC_2}{C_1}, \tag{8.12}$$

and

$$0 < \delta_1 < 1 < \delta_2. \tag{8.13}$$

Hence it is seen that for all $1 < \nu < 2, \nu \neq 1\frac{1}{3}$, the tail of the limiting distribution $W_{\nu-1}(\gamma t)$ may lie above as well as below that of the limiting distribution $R_{\nu-1}(\gamma t)$, note that the latter arises whenever the tail of $B(t)$ is heavier than that of $A(t)$. This leads to a remarkable conclusion, when comparing the case that $B(t)$ has a heavier tail than $A(t)$ to the case that they have similar tails. In the latter case $W_{\nu-1}(\gamma t)$ and $R_{\nu-1}(\gamma t)$ have similar tails and that of $W_{\nu-1}(\gamma t)$ may be lighter as well as heavier than that of $R_{\nu-1}(\gamma t)$. A similar phenomenon has been observed in [9]. So far for the comparison of the tails of the limiting distributions $W_{\nu-1}(t)$ and $R_{\nu-1}(t)$ for the contracted waiting times $\Delta_1(a)\mathbf{w}(a)$

and $\Delta_2(a)\mathbf{w}_2$, of which the first one refers to the case that $B(t)$ and $A(t)$ have similar tails and the latter one to that with the tail of $B(t)$ heavier than that of $A(t)$.
Numerical results for the case $\nu = 1\frac{1}{2}$, cf. [4], [8], have indicated that for $0 < 1-a << 1$, the limiting distribution for the contracted waiting time $\Delta(a)\mathbf{w}$ leads to a very good approximation for the distribution of $\mathbf{w}$ even for moderate values of a, i.e. a not so close to one. Therefore let us compare the tails of the distributions

$$W_{\nu-1}(\frac{t}{\Delta_1(a)}) \text{ and } R_{\nu-1}(\frac{t}{\Delta_2(a)}),$$

with $\Delta_1(a)$ and $\Delta_2(a)$ defined by the contraction equations, cf. (3.2) and (6.3) with $c_1 = 1, c_2 = a$, i.e. for $\Delta_j(a) > 0, j = 1,2$,

$$\begin{aligned} 1-a \quad &= \Delta_1^{\nu-1}(a)|L(\Delta_1(a))| = \Delta_1^{\nu-1}(a)|aC_2L_2(\Delta_1(a)) + (-1)^\nu C_1L_1(\Delta_1(a))|, \\ &= \Delta_1^{\nu-1}(a)aC_2L_2(\Delta_1(a))|1 + (-1)^\nu \frac{C_1L_1(\Delta_1(a))}{aC_2L_2(\Delta_1(a))}. \end{aligned} \tag{8.14}$$

$$1-a \;=\; \Delta_2^{\nu-1}(a)aC_2L_2(\Delta_2(a)). \tag{8.15}$$

Note cf. (1.8)iv. and (1.9), that for $\varepsilon > 0, 0 < x << 1$,

$$\frac{L_2(\varepsilon x)}{L_1(\varepsilon x)} = \frac{L_2(\varepsilon x)}{L_2(x)}\frac{L_1(x)}{L_1(\varepsilon x)}\frac{L_2(x)}{L_1(x)} \sim \frac{L_2(x)}{L_1(x)} \sim f. \tag{8.16}$$

Let us approximate (8.14), cf. (8.16), and with $1-a << 1$, by:

$$1-a \sim \Delta_1^{\nu-1}aC_2L_2(\Delta_1(a))|1 + (-1)^\nu \frac{C_1}{fC_2}|. \tag{8.17}$$

Hence from (8.16) and (8.17) and (3.13),

$$\frac{\Delta_1(a)}{\Delta_2(a)} \sim |1 + (-1)^\nu \frac{C_1}{fC_2}|^{\frac{-1}{\nu-1}} = (\frac{1}{d_2})^{\frac{1}{\nu-1}}. \tag{8.18}$$

From (8.8), (8.9) and (8.18) we have: for $t \to \infty$,

$$\Pr\{\Delta_1(a)\mathbf{w}_1/\gamma \geq t\} = \max(d_1, d_2)\Pr\{\Delta_2(a)\mathbf{w}/\gamma \geq t\} =$$

$$\Pr\{\Delta_2(a)\mathbf{w}_1/\gamma \geq \frac{\Delta_2(a)}{\Delta_1(a)}t\} = \frac{\max(d_1,d_2)}{\Gamma(2-\nu)}\frac{1}{t^{\nu-1}}\{1 + \mathrm{O}(\frac{1}{t^{\nu-1}})\},$$

so

$$\begin{aligned} &\Pr\{\Delta_2(a)\mathbf{w}_1/\gamma \geq \tau\} = \frac{\max(d_1,d_2)}{\Gamma(2-\nu)}\{\frac{\Delta_2(a)}{\Delta_1(a)}\}^{\nu-1}\frac{1}{\tau^{\nu-1}}\{1 + \mathrm{O}((\frac{\Delta_2(a)}{\Delta_1(a)}\frac{1}{\tau})^{\nu-1})\} \\ &= \frac{\max(d_1,d_2)/d_2}{\Gamma(2-\nu)}\frac{1}{\tau^{\nu-1}}\{1 + \mathrm{O}(\frac{d_2^{-1}}{\tau^{\nu-1}})\}. \end{aligned}$$

Hence we obtain the interesting result:

$$\lim_{t\to\infty}\lim_{a\uparrow 1}\frac{\Pr\{\Delta_1(a)\mathbf{w}_1 \geq \gamma t\}}{\Pr\{\Delta_2(a)\mathbf{w}_2 \geq \gamma t\}} = \max(1, \frac{d_2}{d_1}) = \max(1, f\frac{C_2}{C_1}), \tag{8.19}$$

which compares the case of similar tails ($\mathbf{w}_1$) to that that with $B(\cdot)$ having a heavier tail than $A(\cdot)(\mathbf{w}_2)$.

We continue this section with the derivation of an asymptotic result for the limiting distribution $\tilde{I}_{\nu-1}(t)$ of $\Delta(a)\tilde{\mathbf{i}}$ for $a \uparrow 1$, cf. corollary 4.1. From this corollary we have: for $r > 0, 1 < \nu < 2$,

$$\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}\tilde{I}_{\nu-1}(t) = \mathrm{e}^{\Phi(-r/\gamma)}. \tag{8.20}$$

From appendix A, cf. (a.6), (a.13), (a.17) and (a.18) it is seen that: for $r \downarrow 0, r > 0$,

$$\Phi(-r/\gamma) = -\min(d_1, d_2) r^{\nu-1}\{1 + \mathrm{O}(r^{\nu-1})\}. \tag{8.21}$$

Hence

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-rt}\{1 - \tilde{I}_{\nu-1}(t)\}\mathrm{d}t &= \frac{1}{r}\{1 - \mathrm{e}^{\Phi(-r/\gamma)}\} \\ &= \min(d_1, d_2) r^{\nu-2}\{1 + \mathrm{O}(r^{\nu-1})\} \text{ for } r \downarrow 0.\end{aligned} \tag{8.22}$$

From which it follows by using theorem 2 of [5], vol. II, p. 159 that: for $t \to \infty, 1 < \nu < 2$,

$$1 - \tilde{I}_{\nu-1}(\gamma t) = \frac{\min(d_1, d_2)}{\Gamma(2-\nu)} \frac{1}{t^{\nu-1}}\{1 + \mathrm{O}(\frac{1}{t^{\nu-1}})\}. \tag{8.23}$$

The relations (8.20) and (8.23) lead to an interesting conclusion. Note that they apply for the case that $A(t)$ and $B(t)$ have similar heavy tails and $1 < \nu < 2$. It is seen that the limiting distribution $\tilde{I}_{\nu-1}(t)$ is not a degenerated distribution. However, if $B(t)$ has a heavier tail than $A(t)$ or conversely then it is seen from sections 6 and 7, cf. (6.9) and (7.10), that the limiting distribution of $\Delta(a)\tilde{\mathbf{i}}$ is a degenerated one at zero. The same holds for the case $\nu = 2$ discussed in section 5.

We conclude the present section with a more detailed consideration of the case of similar tails in terms of the ratio d_2/d_1. For this case we have, cf. (3.13),

$$0 < \frac{d_2}{d_1} < \infty.$$

From definiton (1.1) and (3.12) it is seen that the tails of $A(t)$ and $B(t)$ are nonidentical if $d_2 \neq d_1$, and identical or pseudo-identical if $d_2 = d_1 = 1$.

Whenever

$$1 < \frac{d_2}{d_1} < \infty, \tag{8.24}$$

so that

$$\max(d_1, d_2) = d_2 \ , \ \min(d_1, d_2) = d_1,$$

then for $t \to \infty$, (8.6) shows that $1 - W_{\nu-1}(t)$ is mainly influenced by by the tail of $B(t)$, whereas $1 - \tilde{I}_{\nu-1}(t)$ is mainly dominated by that of $A(t)$, cf. (8.23).

Whenever

$$0 < \frac{d_2}{d_1} < 1,$$

then $1 - W_{\nu-1}(t), t \to \infty$, is mainly dominated by the tail of $A(t)$, whereas $1 - \tilde{I}_{\nu-1}(t)$ by that of $B(t)$. But note that in (8.19) the tail of $B(t)$ as well as that of $A(t)$ is present with that of $A(t)$ dominating because $d_1 > d_2$. Note that these conclusions do not depend on ν because d_2/d_1 is independend of ν.

The asymptotic results discussed above indicate that the contracted queueing process is strongly dominated by the heavier tail of $A(t)$ and $B(t)$ if these distributions have nonsimilar tails. Whenever they have similar tails it appears that the contracted queuing process preserves more queuing aspects of the stable model because for similar tails the contracted excess time $\Delta(a)\tilde{\mathbf{i}}$ has a true limiting distribution. In this limiting distribution and in that of the contracted waiting time the characteristics d_1 and d_2 of both tails are present. For $1 < \nu < 1\frac{1}{3}$ the coefficients in the asymptotic expressions for the tails of these distributions may strongly vary.

Acknowledgement. With much pleasure the author remembers Prof. Boxma's extensive comments and views on the heavy tailed queueing model.

Appendix A

In this appendix we first calculate the integral

$$\int_0^\infty \frac{s^\mu}{1+s^2}\mathrm{d}s \ \text{ with } \ -1<\mu<1. \tag{a.1}$$

Note that

$$\int_{-\infty}^0 \frac{s^\mu}{1+s^2}\mathrm{d}s = \int_0^\infty \frac{\mathrm{e}^{\mu\pi\mathrm{i}}\sigma^\mu}{1+\sigma^2}\mathrm{d}\sigma = \mathrm{e}^{\mu\pi\mathrm{i}}\int_0^\infty \frac{\sigma^\mu}{1+\sigma^2}\mathrm{d}\sigma. \tag{a.2}$$

Hence

$$\int_{-\infty}^\infty \frac{s^\mu}{1+s^2}\mathrm{d}s = \{1+\mathrm{e}^{\mu\pi\mathrm{i}}\}\int_0^\infty \frac{s^\mu}{1+s^2}\mathrm{d}s. \tag{a.3}$$

The integral in the lefthand side is well defined, its integrand is regular for Im $s>0$, continuous for Im $s\geq 0$, except for a single pole at $s=\mathrm{i}$. By contour integration in the upper half s-plane, it is seen that: for $|\mu|<1$,

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{s^\mu}{1+s^2}\mathrm{d}s = \mathrm{Re}\ \frac{s^\mu}{2^s}|_{s=\mathrm{i}} = \mathrm{Re}\ \frac{\mathrm{i}^\mu}{2\mathrm{i}}. \tag{a.4}$$

Hence from (a.3) and (a.4) we have: for $|\mu|<1$,

$$\int_{-\infty}^\infty \frac{s^\mu}{1+s^2}\mathrm{d}s = \frac{\pi}{2}\sec\frac{1}{2}\mu\pi. \tag{a.5}$$

To investigate $\Phi(r/\gamma)$ for $r\downarrow 0$, cf. (4.16)iii, put for $r\geq 0, 1<\nu<2$,

$$F(r,s) := \frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}},$$

we have: for $r\geq 0$,

$$\begin{aligned} I_1(r) &:= \frac{1}{\pi}\int_0^\infty \{\arctan\frac{A(rs)^{\nu-1}}{1+B(rs)^{\nu-1}}\}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s} \\ &= \frac{Ar^{\nu-1}}{\pi}\int_0^\infty\{\frac{\arctan F(r,s)}{F(r,s)}\frac{1}{1+B(rs)^{\nu-1}}\}\frac{s^{\nu-2}}{1+s^2}\mathrm{d}s. \end{aligned} \tag{a.6}$$

Because $rs\geq 0$ we have

$$0\leq \frac{\arctan F(r,s)}{F(r,s)}\leq 1. \tag{a.7}$$

Hence if $B\geq 0$ it follows because of dominated convergence and by using (a.5) that

$$\begin{aligned}&\lim_{r\downarrow 0}\int\limits_0^\infty \frac{\arctan F(r,s)}{F(r,s)}\frac{1}{1+B(rs)^{\nu-2}}\frac{s^{\nu-2}}{1+s^2}\mathrm{d}s\\ &=\int\limits_0^\infty \frac{s^{\nu-2}}{1+s^2}\mathrm{d}s=\frac{1}{2}\pi\sec\frac{2-\nu}{2}\pi=\frac{1}{2}\pi\{\sin\frac{\nu-1}{2}\pi\}^{-1},\end{aligned}\tag{a.8}$$

since $1<\nu<2$.

Hence from (4.10), (a.6) and (a.8): for $B\geq 0, 1<\nu<2$,

$$I_1(r)\sim\frac{1}{2}(d_1+d_2)r^{\nu-1}\{1+\mathrm{O}(r^{\nu-1})\}\text{ for } r\downarrow 0.\tag{a.9}$$

Next consider the case $B<0$. Denote by

$$\sigma_0:=\ \frac{1}{r}(\frac{-1}{B})^{\nu-1},\tag{a.10}$$

the zero of $1+B(rs)^{\nu-1}$ in $rs>0$.
Write with $0<\varepsilon<<1$,

$$I_1(r,\varepsilon):=\ \frac{Ar^{\nu-1}}{\pi}[\int\limits_0^{\sigma_0-\varepsilon}+\int\limits_{\sigma_0-\varepsilon}^{\sigma_0+\varepsilon}+\int\limits_{\sigma_0+\varepsilon}^{\infty}]\{\frac{\arctan F(r,s)}{F(r,s)}\frac{1}{1+B(rs)^{\nu-1}}\frac{s^{\nu-2}}{1+s^2}\mathrm{d}s\}.\tag{a.11}$$

It is readily verified that the second integral in (a.11) tends to zero for $\varepsilon\downarrow 0$, uniformly in $r>0$. Hence by letting $r\downarrow 0$ we obtain from (a.7) by using dominated convergence,

$$\int\limits_0^{\sigma_0-\varepsilon}\cdots+\int\limits_{\sigma_0+\varepsilon}^{\infty}\cdots\Rightarrow\{\int\limits_0^{\sigma_0-\varepsilon}+\int\limits_{\sigma_0+\varepsilon}^{\infty}\}\frac{s^{\nu-2}}{1+s^2}\mathrm{d}s,$$

uniformly in $\varepsilon>0$. By letting $\varepsilon\downarrow 0$ we obtain again (a.9) but now for $B<0$. Hence we have: for $1<\nu<2$ and $r\downarrow 0$,

$$I_1(r)=\frac{1}{2}(d_1+d_2)r^{\nu-1}\{1+\mathrm{O}(r^{2(\nu-1)})\},\tag{a.12}$$

see also remark A.1 below.
Put: for $r>0, 1<\nu<2$,

$$I_2(r):=\frac{1}{2\pi}\int\limits_0^\infty\log\{1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}\}\frac{\mathrm{d}s}{1+s^2}.\tag{a.13}$$

From the last line in (4.9) and from (4.10) it is seen that: for $s>0, r>0$,

$$1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}>0.\tag{a.14}$$

Write: for $rs>0, B\neq 0$,

$$I_2(r)=\frac{2}{2\pi}Br^{\nu-1}\int\limits_0^\infty\frac{\log\{1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}\}}{2B(rs)^{\nu-1}}\frac{s^{\nu-1}}{1+s^2}\mathrm{d}s.\tag{a.15}$$

The first factor of the integral is in absolute value uniformly bounded in $r\geq 0$.

Hence by using (a.5), we have:

$$\lim_{r\downarrow 0}\int_0^\infty \frac{\log\{1+2B(rs)^{\nu-1}+C(rs)^{2(\nu-1)}\}}{2B(rs)^{\nu-1}}\frac{s^{\nu-1}}{1+s^2}\mathrm{d}s$$
$$= \pm\frac{1}{2}\pi\sec\frac{1}{2}(\nu-1)\pi \text{ for } \pm B>0,\ 1<\nu<2. \tag{a.16}$$

So from (4.10), (a.15) and (a.16): for $r\downarrow 0, B\neq 0$,

$$\begin{aligned} I_2(r) &= \frac{1}{2}|B|r^{\nu-1}\sec\frac{1}{2}(\nu-1)\pi\{(1+\mathrm{O}(r^{\nu-1})\} \\ &= \frac{1}{2}|d_2-d_1|r^{\nu-1}\{1+\mathrm{O}(r^{\nu-1})\}. \end{aligned} \tag{a.17}$$

Next we consider $I_2(r)$ for $r\downarrow 0$ with $B=0$, so that: for $r>0$,

$$\begin{aligned} I_2(r) &= \frac{1}{2\pi}\int_0^\infty \log\{1+C(rs)^{2(\nu-1)}\}\frac{\mathrm{d}s}{1+s^2} \\ &= \frac{Cr^{\nu-1}}{2\pi}\int_0^\infty \frac{\log\{1+C(rs)^{2(\nu-1)}\}}{C(rs)^{\nu-1}}\frac{s^{\nu-1}}{1+s^2}\mathrm{d}s. \end{aligned} \tag{a.18}$$

Again the first factor of the integrand is uniformly bounded in $r\geq 0$ and tends to zero for $r\downarrow 0$. Hence, for $B=0$ and $r\downarrow 0$:

$$I_2(r)=o(r^{\nu-1}). \tag{a.19}$$

Hence from (a.6) and (a.19) we obtain: for $r\downarrow 0$,

$$\begin{aligned} I_1(r)+I_2(r) &= \frac{1}{2}(d_1+d_2+|d_2-d_1|)r^{\nu-1}\{1+\mathrm{O}(r^{\nu-1})\} \\ &= \max(d_1,d_2)r^{\nu-1}\{1+\mathrm{O}(r^{\nu-1})\}. \end{aligned} \tag{a.20}$$

Remark A.1. Actually, we have not proved the character of the order terms in (a.12) and (a.17). A finer asymptotic analysis proves the character of these order terms. The proof is fairly standard, but rather laboriously and lengthly and has therefore been omitted. Note: to derive the asymptotic series for

$$\int_0^\infty \{\arctan s^{\nu-1}\}\frac{r^2}{r^2+s^2}\frac{\mathrm{d}s}{s},$$

write for $0<r<1$,

$$\int_0^\infty \{\arctan s^{\nu-1}\}\frac{r^2}{r^2+s^2}\frac{\mathrm{d}s}{s} = \{\int_0^r+\int_r^1+\int_1^\infty\}\{\arctan s^{\nu-1}\}\{\frac{r^2}{r^2+s^2}\}\frac{\mathrm{d}s}{s},$$

and use in the various integrals the series expansion for $\arctan x$ with $|x|<1$, and $|x|>1$ and for $r^2/(r^2+s^2)$ for $|r/s|<1$ and for $|s/r|<1$. □

Appendix B

In this appendix we shall calculate, the integrals, cf. (5.2); for $r>0$,

$$J_1(r) := \frac{1}{\pi}\int_0^\infty [\arctan(Ars)]\frac{1}{1+s^2}\frac{\mathrm{d}s}{s}, \tag{b.1}$$

$$J_2(r) := \frac{1}{2\pi} \int_0^\infty \log\{1 + A^2(rs)^2\} \frac{\mathrm{d}s}{1+s^2}. \tag{b.2}$$

Obviously, we have

$$J_1(r) = \frac{1}{2\pi} \int_{-\infty}^\infty [\arctan(Ars)] \frac{1}{1+s^2} \frac{\mathrm{d}s}{s}, \tag{b.3}$$

$$J_2(r) = \frac{1}{4\pi} \int_{-\infty}^\infty [\log\{1 + A^2(rs)^2\}] \frac{\mathrm{d}s}{1+s^2}. \tag{b.4}$$

Remark 4.1. We define the principal values of the logarithms, note that $A > 0$,

$$\log\{1 - \mathrm{i}Ars\} \text{ and } \log\{1 + \mathrm{i}A(rs)\},\ r \geq 0,$$

as follows; note that the value of $\lim \mathrm{e}^{H(\frac{r}{\gamma})}, a \uparrow 1$ is not influenced by this definition.
For $s > 0$,

$$\log\{1 \mp \mathrm{i}Ars\} = \log\{1 + A^2(rs)^2\}\mathrm{e}^{\mp \arctan Ars}. \tag{b.5}$$

With this definition of their principal values we have: for $r > 0$,

$$\begin{aligned} &\log\{1 - \mathrm{i}Ars\} \quad \text{is uniquely defined on the slitted } s\text{-plane with slit} \\ &\qquad \{s : s = x + \mathrm{i}(Ar)^{-1}, -\infty < x \leq 0\}, \\ &\log\{1 + \mathrm{i}Ars\} \quad \text{is uniquely defined on the slitted } s\text{-plane with slit} \\ &\qquad \{s : s = x - \mathrm{i}(Ar)^{-1}, -\infty < x \leq 0\}. \end{aligned} \tag{b.6}$$

□

We have, cf. [6], p.114, for Im $s = 0$,

$$\arctan Ars = \frac{\mathrm{i}}{2} \log \frac{1 - \mathrm{i}Ars}{1 + \mathrm{i}Ars}. \tag{b.7}$$

Hence with: for $r > 0$,

$$J_{11}(r) := \frac{\mathrm{i}}{4\pi} \int_{-\infty}^{+\infty} \log\{1 - \mathrm{i}Ars\} \frac{1}{1+s^2} \frac{\mathrm{d}s}{s}, \tag{b.8}$$

$$J_{21}(r) := \frac{\mathrm{i}}{4\pi} \int_{-\infty}^{+\infty} \log\{1 + \mathrm{i}Ars\} \frac{1}{1+s^2} \frac{\mathrm{d}s}{s}, \tag{b.9}$$

we have

$$J_1(r) = J_{11}(r) - J_{12}(r). \tag{b.10}$$

Obviously $\log\{1 - \mathrm{i}Ars\}$ is regular for Im $s > 0$, continuous for Im $s \geq 0$. Further the integrand in the expression for $J_{11}(r)$ has only a single pole in Im $s \geq 0$, viz. $s = \mathrm{i}$. Hence it is readily seen by contour integration in the upper half s-plane that

$$J_{11}(r) = 2\pi\mathrm{i} \frac{\mathrm{i}}{4\pi} \frac{\log\{1 - \mathrm{i}Ars\}}{2s^2}\Big|_{s=\mathrm{i}} = \frac{1}{4} \log\{1 + Ar\}. \tag{b.11}$$

Analogously, via integration in the lower s-plane we obtain

$$J_{12}(r) = -2\pi\mathrm{i}\frac{\mathrm{i}}{4\pi}\frac{\log\{1+\mathrm{i}Ars\}}{2s^2}|_{s=-\mathrm{i}} = -\frac{1}{4}\log\{1+Ar\}. \tag{b.12}$$

Hence from (b.10): for $r \geq 0$,

$$J_1(r) = \frac{1}{2}\log\{1+Ar\}. \tag{b.13}$$

Put: for $r \geq 0$,

$$\begin{aligned} J_{21}(r) &:= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\log\{1-\mathrm{i}Ars\}\frac{\mathrm{d}s}{1+s^2}, \\ J_{22}(r) &:= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\log\{1+\mathrm{i}Ars\}\frac{\mathrm{d}s}{1+s^2}, \end{aligned} \tag{b.14}$$

so that for $r > 0$,

$$J_2(r) = J_{21}(r) + J_{22}(r). \tag{b.15}$$

As above we have

$$\begin{aligned} J_{21}(r) &= 2\pi\mathrm{i}\frac{1}{4\pi}\frac{\log\{1-\mathrm{i}Ars\}}{2s}|_{s=\mathrm{i}} = \frac{1}{4}\log\{1+Ar\}, \\ J_{22}(r) &= -2\pi\mathrm{i}\frac{1}{4\pi}\frac{\log 1+Ars\}}{2s}|_{s=-\mathrm{i}} = \frac{1}{4}\log\{1+Ar\}, \end{aligned} \tag{b.16}$$

so from (b.15),

$$J_2(r) = \frac{1}{2}\log\{1+Ar\}. \tag{b.17}$$

Hence from (b.13) and (b.17): for $r > 0$,

$$J_1(r) + J_2(r) = \log\{1+Ar\}. \tag{b.18}$$

Appendix C

In this appendix we consider the conditions for the validity of the inequalities, cf. (8.11),

$$d_1 + d_2 \gtreqless 2 \text{ for } 1 < \nu < 2. \tag{c.1}$$

From (3.13) we have

$$\frac{1}{2}(d_1+d_2) > 1 \iff 3x^2 + 2(1-4\cos\nu\pi)x + 3 < 0, \tag{c.2}$$

with

$$x = \frac{fC_2}{C_1} = \frac{d_2}{d_1}. \tag{c.3}$$

Write

$$3x^2 - 2(1-4\cos\nu\pi)x + 3 = 3(x-\delta_1)(x-\delta_2), \tag{c.4}$$

$$\delta_{1,2} := \frac{1}{3}(1-4\cos\nu\pi) \mp \frac{4}{3}\sqrt{(\cos\nu\pi - 1)(\cos\nu\pi + \frac{1}{2})}.$$

Because: for $1 < \nu < 2$,

$$\cos \nu\pi - 1 < 0,$$

$$\cos \nu\pi + \tfrac{1}{2} > 0 \quad \text{for } \nu > 1\tfrac{1}{3};$$

it follows that

$$1\frac{1}{3} < \nu < 2 \;\Rightarrow\; \frac{1}{2}(d_1 + d_2) < 1. \tag{c.5}$$

Further since $\delta_1 \delta_2 = 1$,

$$1 < \nu < 1\frac{1}{3} \Rightarrow 0 < \delta_1 < 1 < \delta_2, \tag{c.6}$$

and so

$$\begin{aligned} &1 < \nu < 1\frac{1}{3}, \delta < \frac{fC_2}{C_1} < \delta_2 &&\Rightarrow \frac{1}{2}(d_1 + d_2) < 1, \\ &1 < \nu < 1\frac{1}{3} \text{ and } 0 < \frac{fC_2}{C_1} < \delta_1 \quad \text{or } \frac{fC_2}{C_1} > \delta_2 &&\Rightarrow \frac{1}{2}(d_1 + d_2) > 1. \end{aligned} \tag{c.7}$$

References

1. Cohen, J.W., Complex functions in queueing theory, *Archiv Elektr. Uebertragung*, **47**, (1993), 300-331 (Pollaczek Mémorial Volume).
2. Cohen, J.W., On the M/G/1 queue with heavy-tailed service time distributions, Technical Report, PNA-R9702, CWI, Amsterdam, 1997.
3. Cohen, J.W., A heavy-traffic theorem for the GI/G/1 queue with a Pareto service time distribution, Technical Report, PNA-R9709, CWI, Amsterdam (1997), to appear in J. Appl. Math. and Stoch. Analysis, special issue in honour of R. Syski.
4. Boxma, O.J., Cohen, J.W., Heavy-traffic analysis for the GI/G/1 queue with heavy tailed distributions, Technical Report, PNA-R9710, CWI, Amsterdam, 1997.
5. Doetsch, G., Handbuch der Laplace Transformation, Vol. I, II, III, Birkhäuser Verlag, Basel, 1950-56.
6. Dwight, H.B., Tables of Integrals and other Mathematical Data. The Macmillian Company, New York, 1947.
7. Feller, W., An Introduction to Probability Theory and its Applications, Vol. II, Wiley, New York, 1971.
8. Boxma, O.J., Cohen, J.W., The M/G/1 queue with heavy-tailed service time distributions, to appear.
9. Krishnan, K.R., Neidhardt, A.L., Erramilli, A., Scaling analysis in traffic management of self-similar processes, Teletraffic Contributions for the Information Age, eds. V. Ramaswami, P.E. Wirth, Vol. 26, p. 1087–1097, Elsevier, New York, 1997.