Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

On Time-Frequency Analysis and Time-Limitedness

P.J. Oonincx

Probability, Networks and Algorithms (PNA)

PNA-R9720 December 31, 1997

Report PNA-R9720
ISSN 1386-3711

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

On Time-Frequency Analysis and Time-Limitedness

P.J. Oonincx
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT

We study two classical problems, namely the concentration of energy problem and the truncation problem. The first problem deals with time-limited signals that have maximal energy in a certain frequency band. The second problem is about estimating the spectrum of a signal, if this signal is only known at a certain interval. Solutions of the first problem can be used to obtain good solutions for the second one by means of a preprocessing algorithm, called tapering. The truncation problem and the tapering algorithm are also studied for time-scale and time-frequency analysis, using the continuous wavelet transform and the Wigner-Ville representation.

1991 Mathematics Subject Classification: 33C55, 44A15, 45C05, 62M15, 94A12.
Keywords and Phrases: Wavelets, tapers, prolate spheroidal wave functions, Wigner-Ville distribution, spectral analysis, truncation problem.
Note: Work carried out under project PNA 4.2 "Wavelets" and supported financially by the Technology Foundation (STW), project no. CWI44.3403.

1. Introduction

Spectral analysis by means of the Fourier transform, which provides the energy distribution of a signal over all frequencies, offers a valuable tool in signal analysis. However in practice, it is often not sufficient to have only a time or frequency representation, e.g. when we are analysing transient signals. In order to study the behaviour of signals in both time and frequency domain, time-frequency representations can be used. There is not a unique way of representing a signal in both time and frequency, however a general class of representations, satisfying a set of desirable properties has been given by Cohen [2]. Two of the best known time-frequency representations within this Cohen's class are the spectrogram (the squared modulus of the windowed Fourier transform) and the Wigner-Ville representation. Both representations are discussed briefly in this report.

To obtain information about a signal's behaviour both in time and in scale, the wavelet transform can be used. This transform [5, 8] is often used for analysing the time-scale/frequency behaviour of non-stationary signals, like most geophysical signals [6]. In this report we concentrate mostly on this transform to analyse signals in the time-frequency domain.

A central problem in Fourier analysis is the representation of a signal in the frequency domain, if only a segment of this signal is known in the time domain. This truncation can be due to measurement restrictions. However for spectral estimation a noisy signal is sometimes truncated to reduce the effect of the noise. Obviously we cannot represent such a signal in the frequency domain in a proper way. The Fourier transform takes the whole signal into account, while only a short segment is known. This problem is known in literature as the truncation problem [12]. The frequency domain representation must now be seen as an estimation of the signal's behaviour in frequency domain. A method to improve the estimation is to multiply the short segment by some window function. This method is called tapering and is extensively discussed in this report. We also discuss the truncation problem, when using the wavelet transform for a representation of a signal in both time and scale. The method of tapering will be introduced and discussed in combination with the wavelet transform.

We start with some definitions and auxiliary results from Fourier theory.

In dealing with a signal $s \in L^2(I\!R)$, one can consider the spectrum $\hat{s}$ of s given by its Fourier transform

$$\hat{s}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{I\!R} s(t) e^{-i\omega t} dt.$$

The following inversion formula exists

$$s(t) = \frac{1}{\sqrt{2\pi}} \int_{I\!R} \hat{s}(\omega) e^{i\omega t} d\omega.$$

The two integrals converge absolutely, if $s \in \mathcal{S}(I\!R)$, the Schwarz class of rapidly decreasing C^∞-functions on $I\!R$, i.e. for each $k, l \in I\!N$

$$\sup_{\alpha \leq k, \beta \leq l, t \in I\!R} |t^\beta \partial^\alpha s(t)| < \infty.$$

The Fourier transform $s \mapsto \hat{s}$ is a bijection of $\mathcal{S}(I\!R)$ and it can be uniquely extended to a Hilbert space isometry of $L^2(I\!R)$. Preservation of the inner product is expressed by Parseval's formula

$$\int_{I\!R} s_1(t)\overline{s_2(t)} dt = \int_{I\!R} \hat{s}_1(\omega)\overline{\hat{s}_2(\omega)} d\omega, \tag{1.1}$$

for all $s_1,\ s_2 \in L^2(I\!R)$.
As a result we have Plancherel's formula,

$$\int_{I\!R} |s(t)|^2 dt = \int_{I\!R} |\hat{s}(\omega)|^2 d\omega. \tag{1.2}$$

The two equal sides of (1.2) give the energy of a signal $s \in L^2(I\!R)$. For $s \in L^2(I\!R)$, $|\hat{s}(\omega)|^2$ is called the energy spectrum of s.

Definition 1.1
A signal $s \in L^2(I\!R)$ is called time-limited if it is compactly supported, i.e. $s(t) = 0,\ |t| > T$, for a certain T.
A signal $s \in L^2(I\!R)$ is called band-limited if its Fourier transform is compactly supported, i.e. $\hat{s}(\omega) = 0,\ |\omega| > \Omega$, for a certain Ω, which is called the bandwidth.

We define $TL(I\!R)$ and $BL(I\!R)$ to be the spaces of all time-limited and band-limited signals in $L^2(I\!R)$, respectively .

Definition 1.2
A signal s is called of exponential type if it extends to a holomorphic function on $\mathbb{C}$ and if there are two positive constants A and Ω such that

$$|s(z)| < Ae^{\Omega|z|}, \ \ \forall z \in \mathbb{C}.$$

Lemma 1.3
If $s \in BL(\mathbb{R})$, then s is of exponentional type.

Proof
Assume $\hat{s}(\omega) = 0$ for $|\omega| > \Omega$. Then

$$s(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{s}(\omega) e^{i\omega t} d\omega,$$

initially for $t \in \mathbb{R}$, remains well-defined for $t \in \mathbb{C}$, and yields a holomorphic function s on $\mathbb{C}$. Further

$$\begin{aligned} |s(z)| &= \left|\frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{s}(\omega) e^{i\omega z} d\omega\right| \leq \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} |\hat{s}(\omega) e^{i\omega z}| d\omega \leq \frac{e^{\Omega|z|}}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} |\hat{s}(\omega)| d\omega \\ &\leq \frac{e^{\Omega|z|}}{\sqrt{2\pi}} \sqrt{\int_{\mathbb{R}} |\hat{s}(\omega)|^2 d\omega} \cdot \sqrt{\int_{-\Omega}^{\Omega} d\omega} = \sqrt{\frac{\Omega \|s\|_2^2}{\pi}} e^{\Omega|z|}, \ \ \forall z \in \mathbb{C}. \end{aligned}$$

□

Lemma 1.3 can be extended to the Paley-Wiener theorem, a well-known result from Fourier theory; for a complete proof, see [20].

Theorem 1.4 (Paley-Wiener)
If $s \in L^2(\mathbb{R})$ is holomorphic and of exponential type, then $s \in BL(\mathbb{R})$. Conversely, if $s \in BL(\mathbb{R})$, then s is holomorphic and of exponential type.

Since a holomorphic function $s \in L^2(\mathbb{R})$, vanishing at a certain interval, has to be identically zero, the Paley-Wiener theorem immediately yield

Corollary 1.5
$TL(\mathbb{R}) \cap BL(\mathbb{R}) = \{0\}$.

It is clear, that when using the Fourier transform, we hide all information of the signal's behaviour in the time domain. To get information about a signal simultaneously in the frequency domain and the time domain, we may replace the Fourier transform by one of two other integral transformations, namely the windowed Fourier transform (WFT) and the continuous wavelet transform (CWT).

The WFT of $s \in L^2(\mathbb{R})$ is defined by

$$\tilde{s}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} s(u) \overline{g(u-t)} e^{-i\omega u} du, \tag{1.3}$$

for a certain window function $g \in L^2(I\!R)$. Again an inversion formula exists

$$s(t) = \frac{1}{\sqrt{2\pi}\|g\|_2^2} \int\limits_{I\!R^2} \tilde{s}(\omega, u) g(t-u) e^{i\omega t}\, du d\omega.$$

Note that this formula only makes sense for $g \neq 0$ on a set with positive measure. Further, a counterpart of Parseval's relation has been derived

$$(s_1, s_2)_{L^2(I\!R)} = \frac{1}{\|g\|_2^2} (\tilde{s}_1, \tilde{s}_2)_{L^2(I\!R^2)} \quad \forall_{s_1, s_2 \in L^2(I\!R)},$$

which yields analogous to (1.2)

$$\int\limits_{I\!R} |s(t)|^2 dt = \frac{1}{\|g\|_2^2} \int\limits_{I\!R^2} |\tilde{s}(\omega, t)|^2\, dt d\omega \quad \forall_{s \in L^2(I\!R)}.$$

The CWT of $s \in L^2(I\!R)$ is defined by

$$W_\psi[s](a,b) = \frac{1}{\sqrt{|a|}} \int\limits_{I\!R} s(u) \overline{\psi(\frac{u-b}{a})}\, du,$$

for a certain $\psi \in L^2(I\!R)$, which is called the wavelet. Actually the CWT leads to a representation of a signal in the time-scale domain. However, replacing the scale parameter a by the reciprocal frequency $1/\omega$ and the space parameter b by the time parameter t, yields a time-frequency representation similar to the WFT. There exists the inversion formula

$$s(t) = \frac{1}{C_\psi} \int\limits_{I\!R^2} \frac{1}{\sqrt{|a|}} W_\psi[s](a,b) \psi(\frac{t-b}{a}) db \frac{da}{a^2}, \tag{1.4}$$

with $C_\psi = \int\limits_{I\!R} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega$. It is clear, that this formula only holds if $0 < C_\psi < \infty$. Wavelets for which this condition hold are called admissible. In the sequel we only deal with admissible wavelets. Similar to the WFT case, a counterpart of Parseval's relation has been derived

$$(s_1, s_2)_{L^2(I\!R)} = \frac{1}{C_\psi} (W_\psi[s_1], W_\psi[s_2])_{L^2(I\!R^2; a^{-2} da db)} \quad \forall_{s_1, s_2 \in L^2(I\!R)},$$

which yields analogous to (1.2)

$$\int\limits_{I\!R} |s(t)|^2 dt = \frac{1}{C_\psi} \int\limits_{I\!R^2} |W_\psi[s](a,b)|^2\, db \frac{da}{a^2} \quad \forall_{s \in L^2(I\!R)}. \tag{1.5}$$

Amongst others, these results on the WFT and the CWT can be found in e.g. [5, 8].

The report is organised as follows. In Section 2 we deal with two classical problems, namely the problem of maximal energy of time-limited signals within a frequency band, and the truncation problem. Section 3 discusses time-frequency and time-scale representations of signals, focusing on the CWT. For this method we discuss problems when analysing short segments of a non-stationary signal with long duration time. Possible solutions to these problems are considered in Section 4, using known techniques from Section 2.

2. Time Limitedness: Two Classical Problems

The first problem to be considered in this section is the concentration of energy in a certain frequency band of a time-limited signal. So we consider for $s \in TL(\mathbb{R})$ the ratio

$$\beta_s(\Omega) = \frac{\int_{-\Omega}^{\Omega} |\hat{s}(\omega)|^2 \, d\omega}{\int_{\mathbb{R}} |\hat{s}(\omega)|^2 \, d\omega}. \tag{2.1}$$

Here $[-\Omega, \Omega]$ is the frequency band we are looking at. From Corollary 1.5 it is clear that $0 \leq \beta_s(\Omega) < 1$. Therefore it is interesting to study the problem of maximising $\beta_s(\Omega)$ over all $s \in TL(\mathbb{R})$, which we will discuss in this section. In the past this problem has been discussed extensively e.g. by Landau, Pollack and Slepian, see [9, 14, 16, 17].

One may also consider a similar problem, namely how to maximise

$$\alpha_s(T) = \frac{\int_{-T}^{T} |s(t)|^2 \, dt}{\int_{\mathbb{R}} |s(t)|^2 \, dt}$$

over all $s \in BL(\mathbb{R})$, for a certain $T > 0$. Since the Fourier transform is a unitary operator on $L^2(\mathbb{R})$, these two problems are equivalent.

The second problem we consider in this section is the determination of the Fourier transform $\hat{s}(\omega)$ of $s \in L^2(\mathbb{R})$, if s is only known on $[-T, T]$, for a certain fixed $T > 0$. Especially we consider the case in which $s \notin TL(\mathbb{R})$. Then the definition of the Fourier transform implies that $\hat{s}$ cannot be determined exactly; it can only be estimated.

1. The Concentration of Energy Problem

For the first problem we introduce the integral operator $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\mathcal{A}s)(t) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} \frac{\sin(\Omega(t-u))}{(t-u)} s(u) du, \ \forall_{s \in L^2(\mathbb{R})}. \tag{2.2}$$

Observe that $(\mathcal{A}s)\hat{} = \hat{s} \cdot \chi_{[-\Omega,\Omega]}$. Hence $\mathcal{A}$ is a Hermitian projection operator; in fact it is an orthonormal projector. For $T > 0$, let $J : L^2([-T, T]) \to L^2(\mathbb{R})$ be the embedding given by

$$(Js)(t) = \begin{cases} s(t), & \text{if } |t| \leq T, \\ 0, & \text{if } |t| > T. \end{cases}$$

Then the adjoint operator $J^* : L^2(\mathbb{R}) \to L^2([-T, T])$, restricts functions on $\mathbb{R}$ to $[-T, T]$. Now

$$(J^* \mathcal{A} J s)(t) = \sqrt{\frac{2}{\pi}} \int_{-T}^{T} \frac{\sin(\Omega(t-u))}{(t-u)} s(u) du, \quad |t| \leq T,$$

for all $s \in L^2([-T, T])$. Since the integral kernel is in $L^2([-T, T]^2)$, $J^* \mathcal{A} J$ is a Hilbert-Schmidt operator, hence compact.

It is also a positive definite operator. This can be seen as follows. Assume

$$(J^* \mathcal{A} J s, s)_{L^2(\mathbb{R})} = (\mathcal{A} J s, J s)_{L^2(\mathbb{R})} = 0,$$

for some $s \in L^2(\mathbb{R})$. Then $\widehat{Js}(\omega) = 0,\ \omega \in [-\Omega, \Omega]$, by (1.2). Furthermore $\widehat{Js}$ is analytic by Theorem 1.4. Combining these results yields $\widehat{Js} = 0$, and thus also $Js = 0$.

Following Pollack and Slepian [14, 16], we consider possible solutions $s_{\max} \in TL(\mathbb{R})$, with supp $(s_{\max}) = [-T, T]$, that maximise (2.1). Then

$$\beta_{s_{\max}}(\Omega) \cdot (\hat{s}_{\max}, \hat{s}_{\max})_{L^2(\mathbb{R})} = (\chi_{[-\Omega,\Omega]} \cdot \hat{s}_{\max}, \hat{s}_{\max})_{L^2(\mathbb{R})}.$$

Equivalently, using Parseval's theorem,

$$\beta_{s_{\max}}(\Omega) \cdot (s_{\max}, s_{\max})_{L^2(\mathbb{R})} = (\mathcal{A}s_{\max}, s_{\max})_{L^2(\mathbb{R})}.$$

Since $s_{\max}$ is a stationary solution of this equation, it must satisfy

$$(\mathcal{A}s_{\max})(t) = \mu s_{\max}(t), \quad |t| \leq T, \tag{2.3}$$

a homogeneous Fredholm equation of the first kind. Solutions $s \in L^2([-T, T])$ for this equation only exist for a discrete set of real positive values of μ, with the properties that $1 > \mu_1 > \mu_2 > \mu_3 > \ldots$ and $\lim_{n\to\infty} \mu_n = 0$. In general, the eigenvalues of a compact Hermitian operator are not necessary distinct. However, for this particular operator $\mathcal{A}$, Pollack and Slepian have proved [14], that its eigenvalues are distinct. The solutions of (2.3) for $\mu_1, \mu_2, \mu_3, \ldots$ are denoted by $\psi_1, \psi_2, \psi_3, \ldots$. We observe, that we have solved at this moment the problem of maximising (2.1). Namely, $\beta_{s_{\max}} = \mu_1$ and $\beta_{s_{\max}}$ is attained for $s_{\max} = \psi_1$.

It turns out, that the solutions of (2.3), known as *Prolate Spheroidal Wave Functions (PSWF)*, have some nice properties, which we consider in the sequel of this section.

The PSWF $\psi_1, \psi_2, \psi_3, \ldots$ can be chosen to be a real orthogonal complete set in $L^2([-T, T])$. By defining

$$\psi_n(u) = \frac{1}{\mu_n}(\mathcal{A}J\psi_n)(u), \quad |u| > T,$$

for $n \in \mathbb{N}$, we extend the solutions of (2.3) to the whole real axis. We show, that the ψ_n, extended to $\mathbb{R}$, form a real orthogonal set in $L^2(\mathbb{R})$. To do this, we recall, that $\mathcal{A}$ is a projection operator in the way that

$$(\mathcal{A}Js_1, \mathcal{A}Js_2)_{L^2(\mathbb{R})} = (J^*\mathcal{A}Js_1, s_2)_{L^2(\mathbb{R})}.$$

With this property, we derive

$$\begin{aligned}
(\psi_n, \psi_m)_{L^2(\mathbb{R})} &= \frac{1}{\mu_n \mu_m}(\mathcal{A}J\psi_n, \mathcal{A}J\psi_m)_{L^2(\mathbb{R})} = \frac{1}{\mu_n \mu_m}(J^*\mathcal{A}J\psi_n, \psi_m)_{L^2(\mathbb{R})} \\
&= \frac{1}{\mu_n}(\psi_n, \psi_m)_{L^2([-T,T])},
\end{aligned}$$

yielding, that $\{\psi_n \mid n \in \mathbb{N}\}$ is a real orthogonal set in $L^2(\mathbb{R})$. Further it follows immediately from this derivation, that

$$(\psi_n, \psi_m)_{L^2([-T,T])} = \mu_n \delta_{m,n},$$

after orthonormalisation of the PSWF in $L^2(\mathbb{R})$.

To get more insight in the behaviour of the eigenvalues μ_n, we observe that these μ_n are also eigenvalues of the operator $\mathcal{A}(\Omega T)$ given by

$$(\mathcal{A}(\Omega T)w)(t) = \sqrt{\frac{2}{\pi}} \int\limits_{-\Omega T}^{\Omega T} \frac{\sin(t-u)}{(t-u)} w(u)du, \ \forall_{w \in L^2([-\Omega T, \Omega T])}. \tag{2.4}$$

Now we can use the following theorem from [10].

Theorem 2.1
Let $\mathcal{A}(\Omega T)$ be as defined in (2.4) and let $N(\mathcal{A}(\Omega T), \alpha)$, $0 < \alpha < 1$, denote the number of eigenvalues of $\mathcal{A}(\Omega T)$ which are greater than or equal to α. Then

$$N(\mathcal{A}(\Omega T), \alpha) = \frac{2\Omega T}{\pi} + \frac{1}{\pi^2} \log(\frac{1-\alpha}{\alpha}) \log(\Omega T) + o(\log(\Omega T)). \tag{2.5}$$

Theorem 2.1 is useful for considering the distribution of the eigenvalues μ_n for ΩT large. The following theorem was shown by Slepian [18] without rigorous proof. The proof can be established rigorously using Theorem 2.1.

Theorem 2.2
Let μ_n, $n \in \mathbb{N}$, be the eigenvalues of $\mathcal{A}(\Omega T)$. Then for all $\epsilon > 0$, there exists an $M \in \mathbb{N}$, such that

1. $\mu_n < \epsilon$, if $n \geq (1+\delta)\frac{2\Omega T}{\pi}$,
2. $1 - \mu_n < \epsilon$, if $1 \leq n \leq (1-\delta)\frac{2\Omega T}{\pi}$,
3. $|\mu_n - (1+e^{\pi\theta})^{-1}| < \epsilon$, if $n \sim \frac{2\Omega T}{\pi} + \frac{\theta}{\pi}\log(\Omega T)$,

for $\Omega T > M$, and $1 \geq \delta > 0$ arbitrary small and fixed. In the third statement, $\theta \in \mathbb{R}$ is an arbitrary parameter.

Proof
From Theorem 2.1 we get

$$N(\mathcal{A}(\Omega T), \mu_n) = \frac{2\Omega T}{\pi} + \frac{1}{\pi^2} \log(\frac{1-\mu_n}{\mu_n}) \log(\Omega T) + g_n(\Omega T),$$

with $\lim_{\Omega T \to \infty} \frac{g_n(\Omega T)}{\log(\Omega T)} = 0$. So

$$\log(\frac{1-\mu_n}{\mu_n}) = \frac{\pi^2 N(\mathcal{A}(\Omega T), \mu_n) - 2\Omega T\pi - \pi^2 g_n(\Omega T)}{\log(\Omega T)},$$

yielding

$$\mu_n = (1 + e^{\frac{\pi^2 N(\mathcal{A}(\Omega T), \mu_n) - 2\Omega T \pi}{\log(\Omega T)}} \cdot e^{-\frac{\pi^2 g_n(\Omega T)}{\log(\Omega T)}})^{-1}. \tag{2.6}$$

Substituting $N(\mathcal{A}(\Omega T), \mu_n) = (1+\delta)\frac{2\Omega T}{\pi}$ into (2.6) yields

$$\lim_{\Omega T \to \infty} \mu_n = \lim_{\Omega T \to \infty} (1 + e^{\frac{2\delta\pi\Omega T}{\log(\Omega T)}} \cdot e^{-\frac{g_n(\Omega T)}{\log(\Omega T)}})^{-1} = 0.$$

Since the eigenvalues are ordered in descending order, statement 1 follows immediately from this result. Equivalently we derive statement 2 by substituting $N(\mathcal{A}(\Omega T), \mu_n) = (1-\delta)\frac{2\Omega T}{\pi}$ into (2.6). Finally, by taking $N(\mathcal{A}(\Omega T), \mu_n) = \frac{2\Omega T+\theta}{\pi}$, for some $\theta \in \mathbb{R}$, in Eq. (2.6), statement 3 is achieved. □

Obviously, nearly the first $\frac{2\Omega T}{\pi}$ eigenvalues are close to unity, and nearly all others are close to 0, for ΩT large. Furthermore the number of eigenvalues not close to 0 or 1 grows like $\log(\Omega T)$, for ΩT large.

Of course we could derive much more properties of the PSWF and the eigenvalues of $\mathcal{A}(\Omega T)$. However for the relation with the other problems, discussed in this report, we only need the mutual orthogonality of the PSWF and the fact that approximately the first $\frac{2\Omega T}{\pi}$ eigenvalues are close to unity, for ΩT large. This means, that almost all energy of the corresponding PSWF $\psi_1, \dots \psi_{2\Omega T/\pi}$ is contained in the frequency band $[-\Omega, \Omega]$. For more properties of the PSWF, one may consult e.g. [9], [14] and [16].

At the end of this discussion, we briefly consider the case, when we are dealing with a discrete-time signal. For this discussion, we follow Slepian [17]. Without loss of generality we consider a band-limited signal, with bandwidth Ω_0, sampled with time intervals $\Delta t \leq 1/2\Omega_0$. Then we can consider also the problem of maximising the ratio $\beta_s(\Omega)$ as defined in (2.1), with $0 < \Omega < \Omega_0$. Solutions to this problem provide the discrete prolate spheroidal sequences (DPSS) $\nu_k \in l^2(\{-N, \dots, N\}),\ k = 0, \dots, 2N$, with $N = T/\Delta t$. These sequences ν_k are the eigenvectors of the matrix eigenvalue problem given by

$$\sqrt{\frac{2}{\pi}}\Delta t \sum_{l=-N}^{N} A_{n,l} = \mu_k \nu_k(n), n = -N, \dots, N, \tag{2.7}$$

with A the $(N \times N)$ Toeplitz matrix given by

$$A_{n,l} = \begin{cases} \Omega, & \text{if } n = l \\ \frac{\sin(\Omega(n-l))}{(n-l)}, & \text{if } n \neq l \end{cases} \tag{2.8}$$

Comparing (2.7) and (2.8) with (2.2) and (2.3), we see that this eigenvalue problem follows from the eigenvalue problem in the continuous case by approximating the integral operator $\mathcal{A}$ by means of Riemann sums. Slepian proved, that both the eigenvectors ν_k and the eigenvalues μ_k satisfy similar conditions as the PSWF ψ_k and their corresponding eigenvalues λ_k. Here we only mention the existence of this discrete-time problem. A detailed discussion of this problem can be found in literature [13, 17, 19].

2. The Truncation Problem

For the second problem we assume $s \in L^2(\mathbb{R})$. However, s is only known at a certain interval $[-T, T]$. The problem, we are now dealing with, is to determine the Fourier transform $\hat{s}$ in terms of the segment $s \cdot \chi_{[-T,T]}$. Since s is not necessarily zero outside this interval, the Fourier transform can only be estimated. Several methods can be used to estimate $\hat{s}$, see e.g. [12]. Here we discuss a method, called tapering, which is based on using window functions.

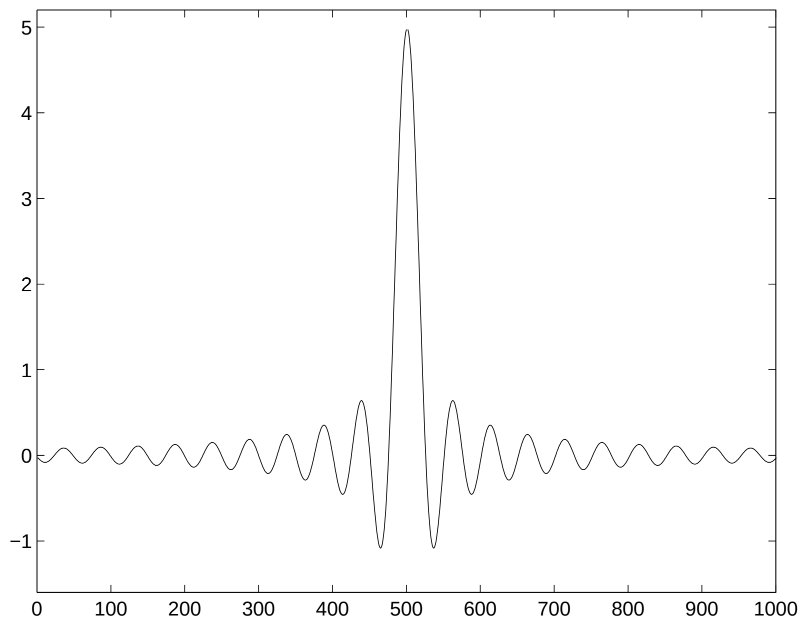

Figure 1: The kernel D_T

If we compute the Fourier transform of $s_1 = s \cdot \chi_{[-T,T]}$, we get

$$\hat{s}_1(\omega) = \frac{1}{\sqrt{2\pi}}(\hat{s} * \hat{\chi}_{[-T,T]})(\omega) = \frac{1}{\sqrt{2\pi}}(\hat{s} * D_T)(\omega), \tag{2.9}$$

where $D_T(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega T)}{\omega}$.
Having a look at the kernel D_T in Figure 1, we see that it consists of a broad main lobe around its center and some smaller side lobes. This means, that due to the convolution product with D_T, contents of one frequency band can be transported into another frequency band via the side lobes of D_T. This phenomena is called spectral leakage, [13]. Observe, that this phenomena does not automatically lead to bad results concerning the estimation of the spectrum. Also the behaviour of the signal should be taken into account. Following [13], an indication whether the estimation $\hat{s}_1$ is biased by spectral leakage is given by the dynamic range of s, given by the ratio

$$\mathcal{R}(s) = 10 \cdot \log(\frac{\sup_{\omega \in I\!R} |\hat{s}(\omega)^2}{|} \inf_{\omega \in I\!R} |\hat{s}(\omega)|^2). \tag{2.10}$$

The bias in $\hat{s}_1$ can be attributed to spectral leakage if s is a signal with high dynamic range. Especially $\hat{s}_1$ is badly biased at those frequencies ω_0, for which $|\hat{s}(\omega_0)|^2 / \sup_{\omega \in I\!R} |\hat{s}(\omega)|^2$ is small. Note, that to compute $\mathcal{R}(s)$, the unknown spectrum $\hat{s}$ of s is needed. In practice, if no knowledge of s is available outside the interval $[-T, T]$, it is hard to say whether the bias in $\hat{s}_1$ is due to spectral leakage. However for the spectra of geophysical data, high dynamical ranges often appear.

A technique to reduce spectral leakage is to replace D_T in (2.9) by some appropriate kernel D_w. In the time-domain this means that we have to multiply s by some appropriate window function $w \in L^2([-T,T])$, called a taper. An estimate of $\hat{s}$ is then given by the Fourier transform of $s_w = s \cdot w$, namely

$$\hat{s}_w(\omega) = \frac{1}{\sqrt{2\pi}}(\hat{s} * \hat{w})(\omega) = \frac{1}{\sqrt{2\pi}}(\hat{s} * D_w)(\omega). \tag{2.11}$$

Besides the fact, that an appropriate taper w should minimise the bias in $\hat{s}_w$ due to spectral leakage, we also require that $\hat{s}_w$ is asymptotically unbiased, i.e.

$$\lim_{T\to\infty} \hat{s}_w(\omega) = \hat{s}(\omega), \quad \forall\, \omega \in \mathbb{R}. \tag{2.12}$$

It is obvious that $w = \chi_{[-T,T]}$ satisfies (2.12). For other tapers a sufficient condition, such that it satisfies (2.12), is given in the following theorem.

Theorem 2.3
Let $s \in L^1(\mathbb{R})$ and $w_T \in L^\infty(\mathbb{R})$, continuous at zero, with the properties, that $w_T(0) = 1$ and $w(t) = w_T(t \cdot T)$. Then

$$\lim_{T\to\infty} \|\hat{s}_{w_T} - \hat{s}\|_\infty = 0.$$

Proof
We derive

$$\begin{aligned}
|\hat{s}_{w_T}(\omega) - \hat{s}(\omega)| &= \Big|\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (w_T(t) - 1)\cdot s(t) e^{-i\omega t} dt\Big| \\
&\le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |w_T(t) - 1| \cdot |s(t)| dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |w(t/T) - 1| \cdot |s(t)| dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{|t|\le M} |w(t/T) - 1| \cdot |s(t)| dt + \\
&\quad \frac{1}{\sqrt{2\pi}} \int_{|t|> M} |w(t/T) - 1| \cdot |s(t)| dt \\
&\le \frac{1}{\sqrt{2\pi}} \sup_{|t|\le M} |w(t/T) - 1| \cdot \|s\|_1 + \frac{1}{\sqrt{2\pi}}(\|w_T\|_\infty + 1) \cdot \int_{|t|>M} |s(t)| dt.
\end{aligned}$$

Let $\varepsilon > 0$. Now choose $M > 0$, such that $\int_{|t|>M} |s(t)| dt \le \varepsilon\sqrt{\pi}/\sqrt{2}(\|w_T\|_\infty + 1)$. Further take T large, so that $\sup_{|t|\le M} |w(t/T) - 1| \le \varepsilon\sqrt{2\pi}/2\|s\|_1$. Then we have

$$\sup_{\omega\in\mathbb{R}} |\hat{s}_{w_T}(\omega) - \hat{s}(\omega)| \le \varepsilon.$$

□

We observe, that [4] also mentions properties of good data tapers, as described in Theorem 2.3. However there, s belongs to a certain class of stationary processes.

In choosing an optimal taper to reduce spectral leakage, we need a measure for the bias in the estimate. In the literature [1, 4, 13] most descriptions of the leakage phenomena deal with stationary stochastic processes. A theoretical description of leakage in this manner can be found in [4]. Here we advance, that tapers w_m that minimise spectral leakage should satisfy

$$\|\hat{s}_{w_m} - \hat{s}\|_\infty = \inf_{w \in L^2([-T,T])} \|\hat{s}_w - \hat{s}\|_\infty. \tag{2.13}$$

Since spectral leakage is a local phenomena, we have chosen a minimisation of the bias in the L^∞-norm. By looking at other norms, we would sum the biases at different frequencies. If we are only interested in a certain frequency band $Y \in I\!R$, optimal tapers $w \in L^2([-T,T])$ should minimise

$$\sup_{\omega \in Y} |\hat{s}_w(\omega) - \hat{s}(\omega)|.$$

It is hard to find a taper that satisfies (2.13), however we can give an upper bound for $\|\hat{s}_w - \hat{s}\|_\infty$, which controls the bias. In the following theorem we derive such an upper bound.

Theorem 2.4
Let $s \in L^1(I\!R)$ and $w, \hat{w} \in L^1(I\!R)$, with $w(0) = 1$. Then

$$\forall_{\varepsilon>0}\, \exists_{\Omega>0} \;:\; \|\hat{s}_w - \hat{s}\|_\infty \le \varepsilon + \sqrt{\frac{2}{\pi}} \|\hat{s}\|_\infty \int\limits_{|u|>\Omega} |\hat{w}(u)| du. \tag{2.14}$$

Proof
We derive

$$\begin{aligned}
|\hat{s}_w(\omega) - \hat{s}(\omega)| &= |\hat{s}_w(\omega) - w(0)\hat{s}(\omega)| \\
&= \frac{1}{\sqrt{2\pi}} \Big| \int\limits_{I\!R} \hat{s}(\omega - u)\hat{w}(u) - \hat{s}(\omega)\hat{w}(u) du \Big| \\
&\le \frac{1}{\sqrt{2\pi}} \int\limits_{I\!R} |\hat{s}(\omega - u) - \hat{s}(\omega)| \cdot |\hat{w}(u)| du \\
&\le \frac{1}{\sqrt{2\pi}} \int\limits_{-\Omega}^{\Omega} |\hat{s}(\omega - u) - \hat{s}(\omega)| \cdot |\hat{w}(u)| du \\
&\quad + \frac{1}{\sqrt{2\pi}} \int\limits_{|u|>\Omega} |\hat{s}(\omega - u) - \hat{s}(\omega)| \cdot |\hat{w}(u)| du.
\end{aligned}$$

Since we assumed $s \in L^1(I\!R)$, it can be proved that $\hat{s} \in C_0(I\!R)$, the supremum-normed Banach space of all continuous functions on $I\!R$, that vanish at infinity, see [15]. So let $\varepsilon > 0$, then there exists an $\Omega > 0$, such that $|\hat{s}(\omega - u) - \hat{s}(\omega)| \le \varepsilon\sqrt{2\pi}/\|\hat{w}\|_1$, for $|u| < \Omega$. By choosing such an Ω, we get

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\Omega}^{\Omega} |\hat{s}(\omega - u) - \hat{s}(\omega)| \cdot |\hat{w}(u)| du \le \frac{\varepsilon}{\|\hat{w}\|_1} \int\limits_{-\Omega}^{\Omega} |\hat{w}(u)| du \le \epsilon.$$

Further

$$\int_{|u|>\Omega} |\hat{s}(\omega-u)-\hat{s}(\omega)|\cdot|\hat{w}(u)|du \leq \int_{|u|>\Omega} (|\hat{s}(\omega-u)|+|\hat{s}(\omega)|)\cdot|\hat{w}(u)|du$$
$$\leq 2\|\hat{s}\|_\infty \int_{|u|>\Omega} |\hat{w}(u)|du.$$

Taking the supremum over all frequencies ω completes the proof.
□

We observe, that a taper w that minimises the upper bound, given by (2.14), and that satisfies the assumptions of Theorem 2.4, should have a spectral amplitude $|\hat{w}(u)|$, which is well localised in a small frequency band $|u| < \Omega$. Note, that in the maximal energy problem, we searched for signals, with a well localised energy spectrum in some frequency band $[-\Omega, \Omega]$. It is easy to verify [21], that when considering a sampled band-limited signal, we can also derive

$$\forall_{\varepsilon>0}\,\exists_{\Omega>0}\;:\;\|\hat{s}_w-\hat{s}\|_\infty \leq \varepsilon + 2\sqrt{\frac{\pi-\Omega}{\pi}}\|\hat{s}\|_\infty \left(\int_{\Omega<|u|\leq\pi} |\hat{w}(u)|^2 du\right)^{1/2}, \tag{2.15}$$

for $s, w \in L^1(I\!\!R)\cap L^2(I\!\!R)$, and $w(0) = 1$. Tapers that minimise the upper bound (2.15) are the DPSS, the solutions of (2.7), following from (2.1) in the discrete case.

One can also be interested in the bias $|\hat{s}_w(\omega)|^2 - |\hat{s}(\omega)|^2$. An upper bound for this bias is given in a corollary of Theorem 2.4.

Corollary 2.5
Let $s \in L^1(I\!\!R)$ and $w, \hat{w} \in L^1(I\!\!R)$, with $w(0) = 1$. Then

$$\forall_{\varepsilon>0}\,\exists_{\Omega>0}\;:\;|\hat{s}_w(\omega)|^2-|\hat{s}(\omega)|^2 \leq (1+\frac{1}{\sqrt{2\pi}}\|\hat{w}\|_1)\cdot\|\hat{s}\|_\infty\cdot$$
$$\left(\varepsilon+\sqrt{\frac{2}{\pi}}\|\hat{s}\|_\infty\int_{|u|>\Omega}|\hat{w}(u)|du\right). \tag{2.16}$$

Proof
If $f \in L^\infty(I\!\!R)$ and $g \in L^1(I\!\!R)$, then

$$\|f*g\|_\infty \leq \|f\|_\infty\cdot\|g\|_1,$$

see e.g. [21]. Therefore

$$\|\hat{s}_w\|_\infty = \frac{1}{\sqrt{2\pi}}\|\hat{s}*\hat{w}\|_\infty \leq \frac{1}{\sqrt{2\pi}}\|\hat{s}\|_\infty\cdot\|\hat{w}\|_1,$$

using that $\hat{s} \in C_0(I\!\!R)$. With this result we derive

$$|\hat{s}_w(\omega)|^2-|\hat{s}(\omega)|^2 \leq (|\hat{s}_w(\omega)|+|\hat{s}(\omega)|)\cdot(|\hat{s}_w(\omega)|-|\hat{s}(\omega)|)$$
$$\leq (\|\hat{s}_w\|_\infty+\|\hat{s}\|_\infty)\cdot\|\hat{s}_w-\hat{s}\|_\infty$$
$$\leq \|\hat{s}\|_\infty(1+\frac{1}{\sqrt{2\pi}}\|\hat{w}\|_1)\cdot\|\hat{s}_w-\hat{s}\|_\infty.$$

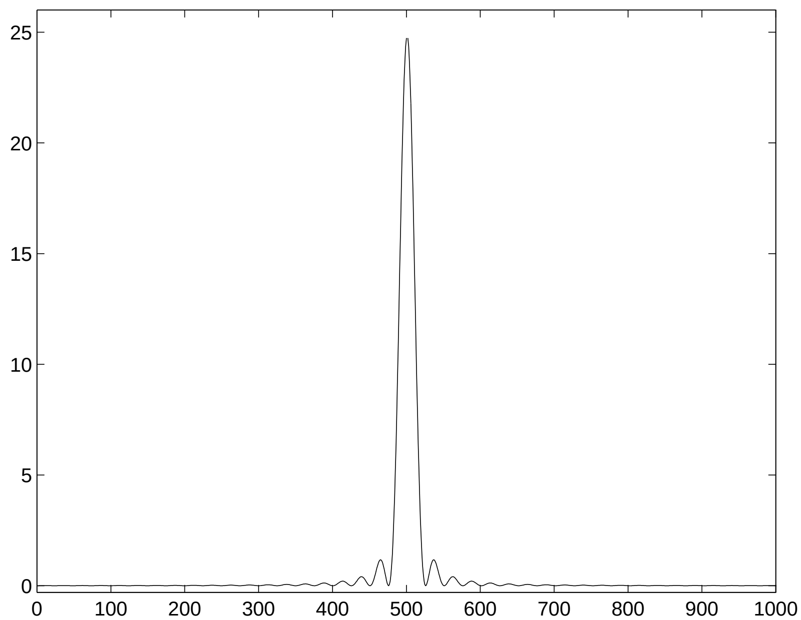

Figure 2: The Spectrum of the Bartlett taper.

Substituting (2.14) into this last result completes the proof.
□

Also other desirable properties of tapers can be taken into account, see [1]. However, here we are only interested in tapers, that satisfy (2.12) and (2.13).
Some tapers, which are often used for spectrum estimation, are

1. The Bartlett taper:
$$w(t) = (1 - |t|/T) \cdot \chi_{[-T,T]}(t),$$

2. The Tukey taper:
$$w(t) = (1/2 + 1/2\cos(\pi t/T)) \cdot \chi_{[-T,T]}(t),$$

3. The Hamming taper:
$$w(t) = (0.54 + 0.46\cos(\pi t/T)) \cdot \chi_{[-T,T]}(t),$$

4. The p%-cosine taper:
$$w(t) = \chi_{[-\alpha T,\alpha T]}(t) + (1/2 + 1/2\cos(\pi\frac{t-\alpha T}{(1-\alpha)T})) \cdot (\chi_{[-T,-\alpha T]}(t) + \chi_{[\alpha T,T]}(t)),$$

   with $\alpha = 1 - p/100$,

5. The Blackman-Harris taper:

$$w(t) = (0.42 + 0.5\cos(\pi t/T) + 0.08\cos(2\pi t/T)) \cdot \chi_{[-T,T]}(t).$$

In Figure 2, the spectrum of the Bartlett taper has been depicted. Comparing this figure with Figure 1, we see that in the case of a Bartlett taper less energy is contained in the side lobes, compared to D_T. However, although these tapers have good overall properties, none of the tapers mentioned above are optimal in a certain sense, like the minimisation of (2.13). In [7], an overview is given of all kinds of discrete tapers with their properties. These discrete tapers can be obtained by sampling analog tapers [12], the tapers we consider in this report.

3. Time-Frequency and Time-Scale Analysis

To investigate the behaviour of a non-stationary signal s, we would like to get information about s both in the frequency and in the time domain. To achieve this, we might use a Cohen class time-frequency representation [3]. Here we mention two well-known members of this class, namely

- The Spectrogram:

$$P_g[s](\omega, t) = |\tilde{s}(\omega, t)|^2, \tag{3.1}$$

  with $\tilde{s}$ the WFT of s, as defined in (1.3),

- The Wigner-Ville distribution:

$$WV[s](\omega, t) = \frac{1}{2\pi} \int_{\mathbb{R}} s(t + p/2)\overline{s(t - p/2)} e^{-i\omega p} dp. \tag{3.2}$$

The following relation [3] exists between these two representations

$$P_g[s](\omega, t) = \int_{\mathbb{R}^2} WV[s](f, u) WV[g](f - \omega, u - t)\, df du. \tag{3.3}$$

We observe, that such a convolution type relation exists for all time-frequency representations of the Cohen's class.

Another approach is to investigate the behaviour of s in the time-scale plane. For this purpose we use a scalogram T_ψ, depending on a wavelet ψ, defined by

$$T_\psi[s](a, b) = |W_\psi[s](a, b)|^2. \tag{3.4}$$

To derive a relation between the scalogram and the Wigner-Ville distribution, we use Moyal's formula [11]:

$$|(s_1, s_2)_{L^2(\mathbb{R})}|^2 = 2\pi \int_{\mathbb{R}^2} WV[s_1](\omega, t) WV[s_2](\omega, t)\, d\omega dt.$$

Further we can write

$$T_\psi[s](a, b) = |W_\psi[s](a, b)|^2 = |(s, \psi_{a,b})_{L^2(\mathbb{R})}|^2,$$

with $\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi(\frac{t-b}{a})$. Applying Moyal's formula on the previous result and using scaling and translation properties of the Wigner-Ville distribution [3], we arrive at the following relation

$$\begin{aligned} T_\psi[s](a,b) &= 2\pi \int_{\mathbb{R}^2} WV[s](\omega,t)WV[\psi_{a,b}](\omega,t)\, d\omega dt \\ &= 2\pi \int_{\mathbb{R}^2} WV[s](\omega,t)WV[\psi](a\omega, \frac{t-b}{a})\, d\omega dt. \end{aligned} \tag{3.5}$$

When considering the behaviour of segments of a non-stationary signal, the Wigner-Ville representation might not be an appropriate tool, since it weights all parts of the signal equally and is therefore highly nonlocal. Furthermore, the spectrogram has a uniform resolution in frequency space, which is cumbersome when analysing multi-component signals, consisting of components with varying durations and frequency contents. In order to determine the behaviour of segments of a multi-component signal, as described before, we shall concentrate in this report mainly on the wavelet transform.

The main problem we are dealing with, is to determine $W_\psi[s](a,b)$ and $T_\psi[s](a,b)$, for $-T \leq b \leq T$, if $s \in L^2(\mathbb{R})$ is only known within $[-T,T]$. Although the wavelet transform is acting locally on s, the following lemma shows that $W_\psi[s](a,b)$ can only be estimated in this case.

Lemma 3.1
Let ψ have support $[t_1,t_2]$ and $s_1 = s\cdot\chi_{[-T,T]}$, with $s \in L_2(\mathbb{R})$. Then

$$W_\psi[s_1](a,b) = W_\psi[s](a,b), \quad -T \leq b \leq T,$$

if one of the following conditions on a holds

1. $a \in [\frac{-b-T}{t_2}, \frac{T-b}{t_2}]\backslash\{0\}$, if $t_1 \geq 0$,
2. $a \in [\frac{b-T}{|t_1|}, \frac{b+T}{|t_1|}]\backslash\{0\}$, if $t_2 \leq 0$,
3. $a \in [\max(\frac{b-T}{|t_1|}, \frac{-b-T}{t_2}), \min(\frac{b+T}{|t_1|}, \frac{T-b}{t_2})]\backslash\{0\}$, if $t_1 < 0 < t_2$.

Proof
We write

$$W_\psi[s](a,b) - W_\psi[s_1](a,b) = \frac{1}{\sqrt{|a|}}\left(\int_{-\infty}^{-T} + \int_{T}^{\infty}\right)s(t)\overline{\psi(\frac{t-b}{a})}\, dt.$$

If ψ has support $[t_1,t_2]$, then $W_\psi[s](a,b) - W_\psi[s_1](a,b) = 0$, if $\frac{t-b}{a} \notin [t_1,t_2]$, $\forall |t| \geq T$. This is equivalent with

$$[at_1+b, at_2+b] \cap \mathbb{R}\backslash[-T,T] = \emptyset \qquad \text{for } a > 0, \tag{3.6}$$

$$[at_2+b, at_1+b] \cap \mathbb{R}\backslash[-T,T] = \emptyset \qquad \text{for } a < 0. \tag{3.7}$$

Assuming $-T \leq b \leq T$, we can distinguish three cases, both for (3.6) and for (3.7).
For $t_1 \geq 0$, (3.6) changes into $at_2 + b \leq T$ and (3.7) into $at_2 + b \geq -T$. Taking these results together yields $(-b-T)/t_2 \leq a < 0$ and $0 < a \leq (T-b)/t_2$.
For $t_2 \leq 0$, (3.6) becomes $at_1 + b \geq -T$ and (3.7) becomes $at_1 + b \leq T$. Together these results yield $(T-b)/t_1 \leq a < 0$ and $0 < a \leq (-T-b)/t_1$.

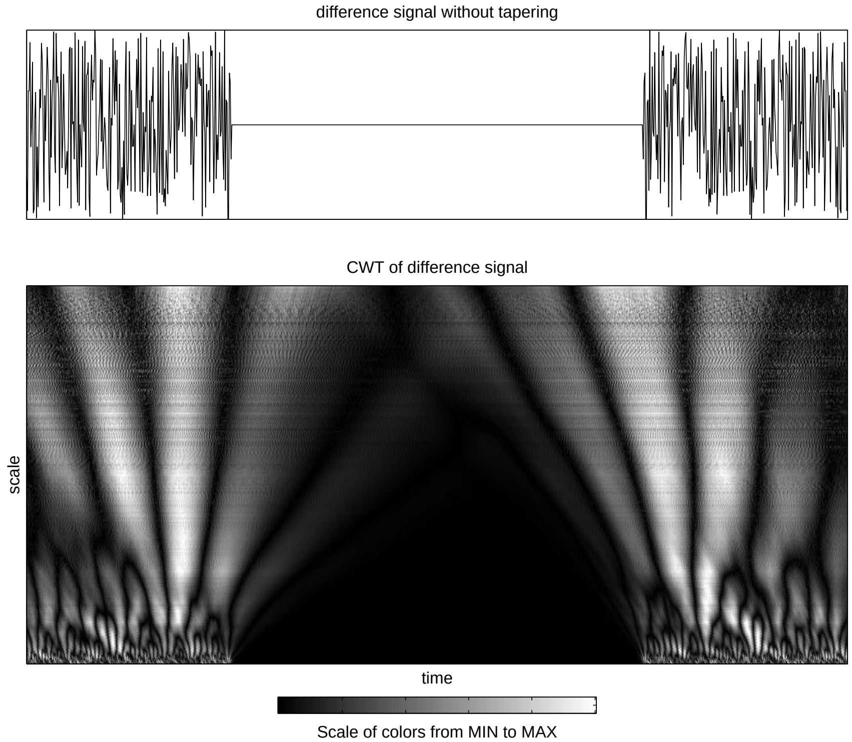

Figure 3: Area without bias

Now take $t_1 < 0 < t_2$. Then (3.6) is equivalent with $at_1 + b \geq -T \wedge at_2 + b \leq T$, which can be written as $a \leq \min((-b-T)/t_1, (T-b)/t_2)$. Further (3.7) is equivalent with $at_1 + b \leq T \wedge at_2 + b \geq -T$, which can be written as $a \geq \max((T-b)/t_1, (-b-T)/t_2)$
□

To illustrate Lemma 3.1, we see in Figure 3 the CWT of some difference signal $s - s_1$ using the Daubechies wavelet D_4, see [5]. It is depicted, that $W_\psi[s](a,b) - W_\psi[s_1](a,b) = 0$, for (a,b) within the triangle, defined by $0 < a \leq \min(\frac{b+T}{|t_1|}, \frac{T-b}{t_2})$. Also we see that, outside this triangle, some bias exists in $\{(a,b) \mid 0 < a, \, -T \leq b \leq T\}$, due to the fact that s_1 is unknown outside $[-T,T]$. Further, we observe that the larger the support of the analysing wavelet is, the smaller the area without bias becomes. Actually, for not compactly supported wavelets this bias can be noticed everywhere in $I\!R$.

In the next theorems, we derive relations both between the CWT of s_1 and the CWT of s, and between the scalogram of s_1 and the scalogram of s.

Theorem 3.2
Let $s \in L^2(I\!R)$. Then

$$W_\psi[s \cdot \chi_{[-T,T]}](a,b) = \int\limits_{I\!R^2} K_T(u,v;a,b) W_\psi[s](u,v) \, dv \frac{du}{u^2}, \tag{3.8}$$

with $K_T(u,v;a,b) = \overline{W_\psi[\psi_{a,b} \cdot \chi_{[-T,T]}](u,v)}/C_\psi$.

Proof
Using the definition of the CWT we can write

$$W_\psi[s \cdot \chi_{[-T,T]}](a,b) = \int_{I\!R} s(t)\chi_{[-T,T]}(t)\overline{\psi_{a,b}(t)}\; dt.$$

This can be rewritten, with inversion formula (1.4), as

$$\begin{aligned}
W_\psi[s \cdot \chi_{[-T,T]}](a,b) &= \int_{I\!R} \chi_{[-T,T]}(t)\overline{\psi_{a,b}(t)}\;(\frac{1}{C_\psi}\int_{I\!R^2} \frac{1}{\sqrt{|u|}} W_\psi[s](u,v)\psi(\frac{t-v}{u})\; dv\frac{du}{u^2})\; dt \\
&= \frac{1}{C_\psi}\int_{I\!R^2} W_\psi[s](u,v)\int_{I\!R}\frac{1}{\sqrt{|u|}}\chi_{[-T,T]}(t)\;\overline{\psi_{a,b}(t)}\;\psi(\frac{t-v}{u})\; dt\; dv\frac{du}{u^2} \\
&= \int_{I\!R^2} W_\psi[s](u,v)\overline{\frac{1}{\sqrt{|u|}}\frac{1}{C_\psi}\int_{I\!R}\chi_{[-T,T]}(t)\;\psi_{a,b}(t)\;\overline{\psi(\frac{t-v}{u})}\; dt}\; dv\frac{du}{u^2} \\
&= \int_{I\!R^2}\frac{1}{C_\psi}\overline{W_\psi[\psi_{a,b}\cdot\chi_{[-T,T]}](u,v)}\; W_\psi[s](u,v)\; dv\frac{du}{u^2} \\
&= \int_{I\!R^2} K_T(u,v;a,b)\; W_\psi[s](u,v)\; dv\frac{du}{u^2},
\end{aligned}$$

with $K_T(u,v;a,b) = \overline{W_\psi[\psi_{a,b} \cdot \chi_{[-T,T]}](u,v)}/C_\psi$.
□

To derive a relation for the difference of $W_\psi[s](a,b)$ and $W_\psi[s\cdot\chi_{[-T,T]}](a,b)$, we prove the following theorem.

Theorem 3.3: *Reproducing kernel property*
Let $s \in L^2(I\!R)$. Then

$$W_\psi[s](a,b) = \int_{I\!R^2} K(u,v;a,b)W_\psi[s](u,v)\; dv\frac{du}{u^2}, \tag{3.9}$$

with $K(u,v;a,b) = \overline{W_\psi[\psi_{a,b}](u,v)}/C_\psi$.

Proof
The proof follows the proof of Theorem 3.2, with $\chi_{[-T,T]}$ replaced by 1.
□

By taking the difference of (3.8) and (3.9), we arrive with some straightforward computations at an expression for the difference between the CWT of s and the CWT of s_1.

Corollary 3.4
Let $s \in L^2(I\!R)$. Then

$$W_\psi[s](a,b) - W_\psi[s \cdot \chi_{[-T,T]}](a,b) = \int\limits_{I\!R^2} G_T(u,v;a,b) W_\psi[s](u,v)\ dv \frac{du}{u^2}, \tag{3.10}$$

with $G_T(u,v;a,b) = \overline{W_\psi[\psi_{a,b} - \psi_{a,b} \cdot \chi_{[-T,T]}](u,v)}/C_\psi$.

An expression for the difference of the scalogram of s and the scalogram of s_1 can be derived easily from (3.5), namely

$$\begin{aligned} T_\psi[s](a,b) - T_\psi[s \cdot \chi_{[-T,T]}](a,b) &= 2\pi \int\limits_{I\!R^2} WV[s](\omega,t) WV[\psi](a\omega, \frac{t-b}{a})\ d\omega dt - \\ & \quad 2\pi \int\limits_{I\!R^2} WV[s \cdot \chi_{[-T,T]}](\omega,t) WV[\psi](a\omega, \frac{t-b}{a})\ d\omega dt \\ &= 2\pi \int\limits_{I\!R^2} \delta_T[s](\omega,t) WV[\psi](a\omega, \frac{t-b}{a})\ d\omega dt, \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} \delta_T[s](\omega,t) &= WV[s](\omega,t) - WV[s \cdot \chi_{[-T,T]}](\omega,t) \\ &= WV[s](\omega,t) - (WV[s] *_\omega WV[\chi_{[-T,T]}])(\omega,t), \end{aligned} \tag{3.12}$$

with $*_\omega$ denoting the convolution product in the frequency domain. For the last result we used some elementary properties of the Wigner-Ville representation [3].

The time-frequency representation $\delta_T[s](\omega,t)$ does not only play a role when comparing the scalograms or the Wigner-Ville representations of s and s_1 with each other. Also $\delta_T[s](\omega,t)$ appears when considering the difference of the spectrograms of s and s_1. From (3.3) we can derive in a straightforward way

$$P_g[s](\omega,t) - P_g[s \cdot \chi_{[-T,T]}](\omega,t) = \int\limits_{I\!R^2} \delta_T[s](f,u) WV[g](f-\omega, u-t)\ df du. \tag{3.13}$$

At the end of this section we derive an expression for $\delta_T[s](\omega,t)$, in order to compute the differences of the discussed time-frequency/scale representations of s and s_1. For this, we need the following property of Wigner-Ville representations [3]

$$\begin{aligned} WV[s_1+s_2](\omega,t) &= WV[s_1](\omega,t) + WV[s_2](\omega,t) + \\ & \quad \frac{1}{\pi}\mathrm{Re}\{\int\limits_{I\!R} s_2(t+p/2)\overline{s_1(t-p/2)} e^{-i\omega p} dp\}. \end{aligned} \tag{3.14}$$

Theorem 3.5
Let $s \in L^2(I\!R)$ and let $\delta_T[s](\omega,t) = WV[s](\omega,t) - WV[s\cdot\chi_{[-T,T]}](\omega,t)$. Then

$$\delta_T[s](\omega,t) = \frac{1}{2\pi}\Big(\int_{-\infty}^{-2(T+|t|)} + \int_{2(T+|t|)}^{\infty}\Big)s(t+p/2)\overline{s(t-p/2)}e^{-i\omega p}dp + \begin{cases} \frac{1}{\pi}\mathrm{Re}\{\int_{2(T-t)}^{2(T+t)} s(t+p/2)\overline{s(t-p/2)}e^{-i\omega p}dp\}, & \text{if } 0 \le t \le T, \\ \frac{1}{\pi}\mathrm{Re}\{\int_{-2(T-t)}^{-2(T+t)} s(t+p/2)\overline{s(t-p/2)}e^{-i\omega p}dp\}, & \text{if } -T \le t < 0, \end{cases} \tag{3.15}$$

with $s_1 = s\cdot\chi_{[-T,T]}$ and $s_2 = s - s_1$.

Proof
Substituting $s_1 = s\cdot\chi_{[-T,T]}$ and $s_2 = s - s_1$ into (3.14) yields

$$WV[s](\omega,t) - WV[s\cdot\chi_{[-T,T]}](\omega,t) = WV[s_2](\omega,t) + \frac{1}{\pi}\mathrm{Re}\{\int_{I\!R} s_2(t+p/2)\overline{s_1(t-p/2)}e^{-i\omega p}dp\}.$$

Writing out $WV[s_2](\omega,t)$ gives

$$\begin{aligned} WV[s_2](\omega,t) &= \frac{1}{2\pi}\int_{I\!R} s_2(t+p/2)\overline{s_2(t-p/2)}e^{-i\omega p}dp \\ &= \frac{1}{2\pi}\int_{I_1(t)\cup I_2(t)} s(t+p/2)\overline{s(t-p/2)}e^{-i\omega p}dp, \end{aligned}$$

with $I_1(t) = \{p \in I\!R \mid t+p/2 \le -T \wedge t-p/2 \ge T\}$ and $I_2(t) = \{\, p \in I\!R \mid t+p/2 \ge T \wedge t-p/2 \le -T\}$. These integration domains can be rewritten as $I_1(t) = (-\infty, \min(-2T-2t, -2T+2t)] = (-\infty, -2T-2|t|]$ and $I_2(t) = [\max(2T-2t, 2T+2t), \infty) = [2T+2|t|, \infty)$, yielding the first term of the right hand side of (3.15).
Further writing out $\int_{I\!R} s_2(t+p/2)\overline{s_1(t-p/2)}e^{-i\omega p}dp$ gives

$$\int_{I\!R} s_2(t+p/2)\overline{s_1(t-p/2)}e^{-i\omega p}dp = \int_{I_3(t)\cap I_4(t)} s(t+p/2)\overline{s(t-p/2)}e^{-i\omega p}dp,$$

with $I_3(t) = \{p \in I\!R \mid t+p/2 \le -T \vee t+p/2 \ge T\}$ and $I_4(t) = \{\, p \in I\!R \mid -T \le t-p/2 \le T\}$. These sets can be rewritten as $I_3(t) = (-\infty, -2T-2t] \cup [2T-2t, \infty)$ and $I_4(t) = [-2T+2t, 2T+2t]$. Taking the intersection of $I_3(t)$ and $I_4(t)$ yields

$$I_3(t)\cap I_4(t) = \begin{cases} [2T-2t, 2T+2t], & \text{if } 0 \le t \le T, \\ [-2T+2t, -2T-2t], & \text{if } -T \le t < 0. \end{cases}$$

Substituting this result into the domain of the preceding integral completes the proof.
□

We observe, that the integrals appearing in (3.15) are so-called pseudo Wigner-Ville representations of s, [3].

4. Tapered Wavelet Analysis

Assume the signal $s \in L^2(I\!R)$ is only known within $[-T, T]$. Then from Lemma 3.1 it is clear, that we have to estimate $W_\psi[s](a, b)$ for $(a, b) \in I\!R^2$, outside one of the regions, defined in Lemma 3.1. Now the idea is to treat this truncation problem in the same way as in Section 2, where we considered the truncation problem when using the Fourier transform. Therefore, in order to reduce the bias in the estimate, we multiply s by a taper $w \in L^2([-T, T])$, before taking the CWT of the signal. However, observe that the estimate is always unbiased in a region, as defined in Lemma 3.1, if the wavelet is compactly supported. This observation yields the method of Tapered Wavelet Analysis (TWA):

1. If ψ has support $[t_1, t_2]$. Then define V by the subset of $\{(a, b) \;|a \neq 0, -T \leq b \leq T\}$ for which
$$W_\psi[s \cdot \chi_{[-T,T]}](a, b) = W_\psi[s](a, b),$$
cf. Lemma 3.1. Further, let $V^* = (I\!R^+ \times I\!R)\backslash V$. Then an estimate for $W_\psi[s](a, b)$ is given by
$$\widetilde{W}_\psi[s](a, b) = W_\psi[s](a, b) \cdot \chi_V + W_\psi[s \cdot w](a, b) \cdot \chi_{V^*}, \tag{4.1}$$

2. If ψ is not compactly supported. Then an estimate for $W_\psi[s](a, b)$ is given by
$$\widetilde{W}_\psi[s](a, b) = W_\psi[s \cdot w](a, b), \tag{4.2}$$

with $w \in L^2([-T, T])$ a taper, appropriate for the CWT. If we define $V = \emptyset$, if ψ is not compactly supported, then obviously the estimate of $W_\psi[s]$ is only biased outside V, for all ψ. So in all cases the taper only affects the biased values of the CWT.

Again the question arises which conditions w has to satisfy.

As in Section 2, the first condition on the taper is that the TWA has to be asymptotically unbiased. This can be written as

$$\lim_{T\to\infty} \widetilde{W}_\psi[s](a, b) = W_\psi[s](a, b), \;\; \forall\, (a, b) \in V^*. \tag{4.3}$$

In the same fashion, we derived sufficient conditions on a taper in Theorem 2.3, we come to a sufficient conditions on a taper, such that it satisfies (4.3).

Theorem 4.1
Let $s \in L^2(I\!R)$ and $w_T \in L^\infty(I\!R)$, continuous at zero, with the properties, that $w_T(0) = 1$ and $w(t) = w_T(t \cdot T)$. Then
$$\lim_{T\to\infty} \|W_\psi[s \cdot w_T] - W_\psi[s]\|_\infty = 0.$$

Proof
We derive

$$\begin{aligned} |W_\psi[s \cdot w_T](a, b) - W_\psi[s](a, b)| &= |(s \cdot w_T - s, \psi_{a,b})| \leq \|s \cdot w_T - s\|_2 \cdot \|\psi_{a,b}\|_2 \\ &= \|s \cdot w_T - s\|_2 \cdot \|\psi\|_2. \end{aligned}$$

So
$$\|W_\psi[s \cdot w_T] - W_\psi[s]\|_\infty \leq \|s \cdot w_T - s\|_2 \cdot \|\psi\|_2.$$

To complete the proof, we will show

$$\lim_{T\to\infty} \|s \cdot w_T - s\|_2 = 0,$$

following the proof of Theorem 2.3.
We compute

$$\begin{aligned}
\int_{I\!R} |w_T(t) - 1|^2 \cdot |s(t)|^2 dt &= \int_{I\!R} |w(t/T) - 1|^2 \cdot |s(t)|^2 dt \\
&= \int_{|t|\le M} |w(t/T) - 1|^2 \cdot |s(t)|^2 dt + \\
&\quad \int_{|t|>M} |w(t/T) - 1|^2 \cdot |s(t)|^2 dt \\
&\le \sup_{|t|\le M} |w(t/T) - 1|^2 \cdot \|s\|_2^2 + (\|w_T\|_\infty + 1)^2 \cdot \int_{|t|>M} |s(t)|^2 dt.
\end{aligned}$$

Let $\varepsilon > 0$. Now choose $M > 0$, such that $\int_{|t|>M} |s(t)|^2 dt \le \varepsilon^2/2(\|w_T\|_\infty + 1)^2$. Further take T large, so that $\sup_{|t|\le M} |w(t/T) - 1|^2 \le \varepsilon^2/2\|s\|_2^2$. Then we have

$$\|s \cdot w_T - s\|_2^2 \le \varepsilon^2.$$

□

Observe, that a taper satisfying (4.3), by definition, satisfies

$$\lim_{T\to\infty} T_\psi[s \cdot w](a,b) = T_\psi[s](a,b), \quad \forall\, (a,b) \in V^*. \tag{4.4}$$

A measure for the bias in the TWA can be given in a similar way as in (2.13). In this report we search for optimal tapers w_m for the TWA, in the sense that

$$\|W_\psi[s \cdot w_m] - W_\psi[s]\|_\infty = \inf_{w\in L^2([-T,T])} \|W_\psi[s \cdot w] - W_\psi[s]\|_\infty. \tag{4.5}$$

In the case, we are only interested in a certain region $Y \subset V^*$, optimal tapers $w \in L^2([-T,T])$ should minimise

$$\sup_{(a,b)\in Y} |W_\psi[s \cdot w](a,b) - W_\psi[s](a,b)|. \tag{4.6}$$

Also now it is hard to find a taper that satisfies (4.5) or (4.6). In the following theorem, we derive an upper bound for (4.6), which controls the bias.

Theorem 4.2
Let $s \in L^1(I\!R), \psi \in L^\infty(I\!R)$ and $w, \hat{w} \in L^1(I\!R)$, with $w(0) = 1$. Further let $Y \subset V^*$ be compact.

Then

$$\forall_{\varepsilon>0}\,\exists_{\Omega>0}\ :\quad \sup_{(a,b)\in Y} |W_\psi[s\cdot w](a,b) - W_\psi[s](a,b)| \leq \varepsilon + \sqrt{\frac{2}{a_m \pi}} \|\hat{s}\|_\infty \|\psi\|_\infty \int\limits_{|u|>\Omega} |\hat{w}(u)|\ du, \tag{4.7}$$

with $a_m = \min\{|a| \mid \exists_{b\in I\!R}\ :\ (a,b)\in Y\}$.

Proof
We observe that we can write $W_\psi[s](a,b)$ also as a convolution product, namely

$$W_\psi[s](a,b) = (s * \check{\psi}_a)(b),$$

where $\check{\psi}_a(t) = \overline{\psi(-t/a)}/\sqrt{|a|}$. With the convolution product notation, we derive

$$\begin{aligned} |W_\psi[s\cdot w](a,b) - W_\psi[s](a,b)| &= |(s_w * \check{\psi}_a)(b) - (s * \check{\psi}_a)(b)| \\ &\leq \|(s_w - s) * \check{\psi}_a\|_\infty \\ &\leq \|s_w - s\|_1 \cdot \|\check{\psi}_a\|_\infty \\ &\leq \|\hat{s}_w - \hat{s}\|_\infty \cdot \|\check{\psi}_a\|_\infty, \end{aligned}$$

using Young's inequality, see e.g. [21].
By definition $\|\check{\psi}_a\|_\infty = \|\psi\|/\sqrt{a_m}$, with $a_m = \min\{|a| \mid \exists_{b\in I\!R}\ :\ (a,b)\in Y\}$. Now the proof is established by substituting (2.14) into this result.
□

We see that tapers that minimise the upper bound in (2.14) also minimise the upper bound in (4.7). However in the proof of Theorem 4.2 we neglected the possible scaling behaviour of the signal and the taper. Therefore one might expect better estimations of the bias if the scale is taken into account more precisely. Further research on this topic has to be done. A possible starting point for estimations that depend more on scaling behaviour is given in the following theorem.

Theorem 4.3
Let $s \in L^2(I\!R)$ and $w \in L^2([-T,T])$. Then

$$W_\psi[s](a,b) - W_\psi[s\cdot w](a,b) = \int\limits_{I\!R^2} G_w(u,v;a,b) W_\psi[s](u,v)\ dv \frac{du}{u^2}, \tag{4.8}$$

with $G_w(u,v;a,b) = \overline{W_\psi[\psi_{a,b} - \psi_{a,b}\cdot w](u,v)}/C_\psi$.

Proof
Following the proof of Theorem 3.2, with $\chi_{[-T,T]}$ replaced by w, we get

$$W_\psi[s\cdot w](a,b) = \int\limits_{I\!R^2} K_w(u,v;a,b) W_\psi[s](u,v)\ dv \frac{du}{u^2}, \tag{4.9}$$

with $K_w(u,v;a,b) = \overline{W_\psi[\psi_{a,b}\cdot w](u,v)}/C_\psi$. Taking the difference of $W_\psi[s]$ and $W_\psi[s\cdot w]$ by using (3.9) and (4.9) completes the proof.

□

One may also be interested in the bias appearing in the scalogram, due to tapering after truncation, namely $T_\psi[s\cdot w](a,b) - T_\psi[s](a,b)$. An upper bound for this bias is given in a corollary of Theorem 4.2.

Corollary 4.4
Let $s \in L^1(I\!R), \psi \in L^\infty(I\!R)$ and $w, \hat{w} \in L^1(I\!R)$, with $w(0) = 1$. Further let $Y \subset V^*$ be compact. Then

$$\begin{aligned}\forall_{\varepsilon>0}\,\exists_{\Omega>0}\ :\qquad & \sup_{(a,b)\in Y} |T_\psi[s\cdot w](a,b) - T_\psi[s](a,b)| \\ & \leq (1+\|w\|_\infty)\cdot\|s\|_1\cdot\|\psi\|_\infty/\sqrt{a_m})\,\cdot \\ & (\varepsilon + \sqrt{\frac{2}{a_m\pi}}\|\hat{s}\|_\infty\|\psi\|_\infty \int\limits_{|u|>\Omega} |\hat{w}(u)|\,du), \qquad (4.10)\end{aligned}$$

with $a_m = \min\{|a| \mid \exists_{b\in I\!R}\ :\ (a,b)\in Y\}$.

Proof
We derive

$$\begin{aligned}|T_\psi[s_w](a,b) - T_\psi[s](a,b)| &= (|W_\psi[s_w](a,b)| + |W_\psi[s](a,b)|)\,\cdot \\ & \quad (|W_\psi[s_w](a,b)| - |W_\psi[s](a,b)|) \\ &\leq (|W_\psi[s_w](a,b)| + |W_\psi[s](a,b)|)\,\cdot \\ & \quad |W_\psi[s_w](a,b) - W_\psi[s](a,b)|.\end{aligned}$$

Using the convolution product notation , we get

$$\begin{aligned}|W_\psi[s_w](a,b)| &= |(s_w * \check{\psi}_a)(b)| \leq \|s_w * \check{\psi}_a\|_\infty \\ &\leq \|s_w\|_1\cdot\|\check{\psi}_a\|_\infty \leq \|s\|_1\cdot\|w\|_\infty\cdot\|\psi\|_\infty/\sqrt{a_m},\end{aligned}$$

and in the same fashion

$$|W_\psi[s](a,b)| \leq \|s\|_1\cdot\|\psi\|_\infty/\sqrt{a_m}.$$

Substituting (4.7) into the previous result completes the proof.
□

For arbitrary tapers $w \in L^2([-T,T])$, we can also derive another formula for the difference of the scalogram of a signal s and the tapered signal $s\cdot w$, similar to (3.11) and (3.12), namely

$$\begin{aligned}T_\psi[s](a,b) - T_\psi[s\cdot w](a,b) &= 2\pi\int\limits_{I\!R^2} WV[s](\omega,t)WV[\psi](a\omega,\frac{t-b}{a})\,d\omega dt\, - \\ & \quad 2\pi\int\limits_{I\!R^2} WV[s\cdot w](\omega,t)WV[\psi](a\omega,\frac{t-b}{a})\,d\omega dt \\ &= 2\pi\int\limits_{I\!R^2} \delta_w[s](\omega,t)WV[\psi](a\omega,\frac{t-b}{a})\,d\omega dt, \qquad (4.11)\end{aligned}$$

with

$$\delta_w[s](\omega,t) = WV[s](\omega,t) - WV[s\cdot w](\omega,t)$$
$$= WV[s](\omega,t) - (WV[s] *_\omega WV[w])(\omega,t). \quad (4.12)$$

Finally we come to a result, which relates the differences in the energy spectrum, the Wigner-Ville representation and the scalogram of a signal s and the tapered signal $s \cdot w$.

Theorem 4.5
Let $s \in L^2(I\!R)$ and $w \in L^2([-T,T])$. Let further

$$\delta_w[s](\omega,t) = WV[s](\omega,t) - WV[s_w](\omega,t).$$

Then

1. $|\hat{s}(\omega)|^2 - |\hat{s}_w(\omega)|^2 = \int_{I\!R} \delta_w[s](\omega,t)dt,$
2. $T_\psi[s](a,b) - T_\psi[s_w](a,b) = 2\pi \int_{I\!R^2} \delta_w[s](\omega,t) WV[\psi_{a,b}](\omega,t)\ d\omega dt,$

with $s_w = s \cdot w$.

Proof
First we observe, that the Wigner-Ville representation satisfies the marginal

$$\int_{I\!R} WV[s](\omega,t)dt = |\hat{s}(\omega)|^2,$$

see [3]. Therefore

$$|\hat{s}(\omega)|^2 - |\hat{s}_w(\omega)|^2 = \int_{I\!R} WV[s](\omega,t)dt - \int_{I\!R} WV[s_w](\omega,t)dt.$$

The second statement has been derived already in (4.11)
□

From this theorem it follows, that the bias appearing in the Wigner-Ville representation, is a measure for the bias both in the energy spectrum and in the scalogram. Therefore further research on truncation problems and tapering, when using Wigner-Ville representations can be very useful to study the described problems in this report.

5. Concluding remarks

In this report we discussed the problem, that shows up when analysing segments of a signal with a Fourier or wavelet transform. Then the analysis can only be an estimation of the frequency or scaling behaviour of the signal. To improve this estimate, preprocessing the segments with a taper before taking the Fourier transform can be useful. In this report we introduced the tapering algorithm also in combination with the wavelet transform. Upper bounds for the errors in the estimates and sufficient conditions on appropriate tapers have been derived both for the Fourier analysis and the wavelet analysis. To analyse the energy spectra and the scalograms of segments, we derived relations between the truncation problem for the Wigner-Ville representation of a segment and the Fourier and wavelet analysis of such a segment.

Also the maximal energy problem has been revisited. We have studied properties of the solutions to this problem and we have shown how this problem is related to the truncation problem.

Results on tapers for wavelet analysis in this report neglect the possible scaling behaviour of an analysed segment and the taper. Therefore it is an aim of further research to find optimal (signal dependent) tapers for wavelet analysis of segments of a signals. Another aim of research is to study the truncation problem for the Cohen's class time-frequency representations, since a direct link between the energy spectrum, the scalogram and the Wigner-Ville representation exists.

ACKNOWLEDGEMENTS
The author wishes to thank T.H. Koornwinder (University of Amsterdam), N.M. Temme and R.A. Zuidwijk (both CWI) for the helpful discussions and remarks.

REFERENCES

1. P. Bloomfield, *Fourier Analysis of Time Series: An Introduction*, J. Wiley & Sons, New York, 1976
2. L. Cohen, "Generalized phase-space distribution functions", *J. Math. Phys.*, 7, 781-786, 1966
3. L. Cohen, *Time-Frequency Analysis*, Prentice Hall, New Jersey, 1995
4. R. Dahlhaus, "Nonparametric High Resolution Spectral Estimation", *Probab. Th. Rel. Fields*, 85, 147-180, 1990
5. I. Daubechies, *Ten lectures on Wavelets*, SIAM, Philadelphia, 1992
6. E. Foufoula-Georgiou & P. Kumar, *Wavelets in Geophysics*, Academic Press, London, 1994
7. F. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform", *Proc. IEEE*, 66, 51-83, 1978
8. G. Kaiser, *A friendly Guide to Wavelets*, Birkhäuser, Boston, 1994
9. H.J. Landau & H.O. Pollack, "Prolate spheroidal wave functions, Fourier analysis and uncertainty II", *Bell System Tech. J.*, 40, 65-84, 1961
10. H.J. Landau & H. Widom, "Eigenvalue distribution of time and frequency limiting", *J. Math. Anal. and Appl.*, 77, 469-481, 1980
11. J.E. Moyal, "Quantum Mechanics as a statistical theory", *Proc. Camb. Phil. Soc.*, 45, 99-124, 1949
12. A. Papoulis, *Signal Analysis*, McGraw-Hill, New York, 1977
13. D. Percival & A. Walden, *Spectral Analysis for Physical Applications*, Cambridge University Press, Cambridge, 1993

14. H.O. Pollack & D. Slepian, “Prolate spheroidal wave functions, Fourier analysis and uncertainty I”, *Bell System Tech. J.*, 40, 43-64, 1961
15. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973
16. D. Slepian, “Some comments on Fourier analysis, uncertainty and modelling”, *SIAM Review*, 25 (3), 379-393, 1983
17. D. Slepian, “Prolate spheroidal wave functions, Fourier analysis, and uncertainty-V: The discrete case”, *Bell Syst. Techn. J.*, 57, 1371-1429, 1978
18. D. Slepian, “Some asymptotic expansions for prolate spheroidal wave functions”, *J. Math. and Physics*, 44, 99-140, 1965
19. D.J. Thomson, “Spectrum estimation and harmonic analysis”, *Proc. IEEE, 70*, 1055-1096, 1982
20. R. Young, *An introduction to nonharmonic Fourier series*, Academic Press, New York, 1980
21. A.C. Zaanen, *Continuity, Integration and Fourier Theory*, Springer, Berlin, 1989