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ABSTRACT

We present a promising technique which is capable of accessing the divergence free component of the partition function for the negative moments of the multi-fractal analysis of data using the wavelet transformation. It is based on implicitly bounding the local logarithmic slope of the wavelet maxima lines between the values of the Hölder exponent of the singularities which are accessible for the wavelet used. The method delivers correct and stable results, illustrated using a test example of the Besicovich measure analysed with the Mexican hat wavelet. The performance of the method is then shown as applied to real-life data.

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1. Introduction

Since its introduction, the multi-fractal approach has quickly gained a wide range of applications in the analysis of natural phenomena. Nevertheless the use of this formalism remained mainly limited to positive measures, due to the fact that the methods of calculating the scaling partition function were not capable of coping with measures locally approaching zero. The problem of divergences is particularly evident in the analysis of fractal functions with 'traditional' approaches, e.g. the structure function approach. As pointed out in [1], attempts to fix this problem by introducing an arbitrary threshold can result in false phase transitions. In addition to this, the range of singularities accessible with traditional methods is very limited [1].

A successful attempt to tackle these problems was developed by Arneodo et al [2, 3] into the wavelet transform based multi-fractal formalism. The range of accessible singularities can be controlled with the type of wavelet used, with its number of vanishing moments. The problem of the partition function locally equal to zero was addressed by a supremum method which would free the wavelet from the working scale, down along the maximum line towards the supremum value of the wavelet transform (not equal to zero) along this maximum line. This approach was reportedly successful and its correctness has been proven for the class of Besicovich measures [2, 3].

Still this approach cannot guarantee that very small, non-zero values contributing to the partition function, resulting e.g. from numerical inaccuracies (noise), will not cause the partition function for the negative moments to diverge. Attempts to solve this problem by introducing arbitrary thresholds are pointless, since they directly affect the partition function for negative moments. Less prominent but still rather limiting are the requirements of very dense sampling of the maxima lines in order to assure connectivity for supremum tracing.

The method we propose in this communication is conceptually and computationally simple, and yet it has proved to provide a means of stable calculation of the partition function in which the offending small values of the WT are 'automatically' removed. It uses the fact that the local logarithmic slope of maxima lines converging to singularities in the analysed function should not exceed certain bounds. The values of these bounds are determined by the range of the Hölder exponents accessible for the wavelet in hand.

2. Analysing Hölder Singularities with the Wavelet Transform

For our purpose, we will use the wavelet transform which has been shown to be a particularly successful tool in assessing the scaling behaviour of functions [4, 2].

Conceptually, the wavelet transform is a convolution product of the signals with the scaled and translated kernel - the wavelet $\psi(x)$. The power given to the normalising factor s, it is often chosen to serve a particular purpose. Here, unless otherwise noted, we will choose the default factor s^{-1}, which conserves the integral $\int dx\,|\psi(x)|$ and thus leaves the L^1 measure invariant.

$$(Wf)(s,b) = \frac{1}{s} \int dx\; f(x)\; U(s,b)\, \psi(x) \; . \tag{2.1}$$

The scaling and translation actions are incorporated as the operator $U(s,b)$; the scale parameter s 'adapts' the width of the wavelet kernel to the *microscopic resolution* required, thus changing its frequency contents, and the location of the analysing wavelet is determined by the parameter b:

$$U(s,b)\psi(x) = \psi(\frac{x-b}{s}) \; , \tag{2.2}$$

where $s, b \in \mathbf{R}$ and $s > 0$ for the continuous version (CWT).

For the purpose of generalisation of the global notion of roughness, like the Hurst exponent, to the multi-fractal formalism extracting the scaling locally, one has to introduce an appropriate local exponent. The relevant concept is known as the Hölder exponent h of the function in x_0. If there exists a polynomial $P_n(x)$ of the degree n such that:

$$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^h \; , \tag{2.3}$$

then h is said to be the local Hölder exponent of the function and it characterises the scaling of the function locally and h lies within the bounds $n < h \leq n+1$. The polynomial P_n corresponds to the Taylor series expansion of f around x_0 up to the order n.

Note: It follows directly that if $h(x_0)$ is equal to a positive integer n, the function f is n times continuously differentiable in x_0. Alternatively, if $n < h(x_0) < n+1$ the function f is continuous and singular in x_0. In this case f is n times differentiable, but its n^{th} derivative is singular in x_0 and the exponent h characterises this singularity. The exponent h, therefore, gives an indication of how regular the function f is in x_0, that is the higher the h, the more regular the local behaviour of the function f.

With the wavelet of at least m vanishing moments, i.e. orthogonal to polynomials up to degree n (where n is a maximum possible integer, thus $n = m - 1$):

$$\int_{-\infty}^{+\infty} x^n\, \psi(x)\, dx = 0 \qquad \forall n,\; 0 \leq n < m \; ,$$

the wavelet transform of the function f in $x = x_0$ reduces to

$$W^{(m)} f(s, x_0) \sim C \int \psi(x) |s\, x|^{h(x_0)}\, dx \sim C\; |s|^{h(x_0)} \int \psi(x') |x'|^{h(x_0)}\, dx' \; .$$

Therefore, we have the following proportionality of the wavelet transform of the singularity $h \leq n+1$, with the wavelet with m vanishing moments:

$$W^{(m)} f(s, x_0) \sim |s|^{h(x_0)} \; . \tag{2.4}$$

Thus, the continuous wavelet transform can be used for detecting and representing the singularities in the signals even if masked by the polynomial bias.

The continuous wavelet transform described in Eq. 2.1 is an extremely redundant representation. However, it has been shown by Mallat [5] that a representation consisting of only (the modulus of) the maxima lines of the CWT, the wavelet transform modulus maxima representation (WTMM), can be used to characterise the analysed function. In particular, the ability of the CWT to characterise the singularities of the function is fully inherited by WTMM.

It has been shown by Arneodo et al [2] that the WTMM tree is particularly useful for estimating statistical properties of scaling functions through the scaling parameters of the WTMM based multifractal partition function:

$$\mathcal{Z}(q, s) = \sum_{all\ maxima\ at\ scale\ s} \mu_i^q(s) \; \sim s^{\tau(q)} \; , \tag{2.5}$$

where $\mu_i(s)$ is the amplitude of the maximum of the WT at the corresponding scale - the measure contained in the i-th box of the s-coverage, and the partition function is taken over all the maxima at the given scale s. Scaling of this partition function $\mathcal{Z}(q, s) \sim s^{\tau(q)}$ gives the *mass exponents*

$$\tau(q) \sim \frac{\log \mathcal{Z}(q, s)}{\log s} \; , \tag{2.6}$$

from which both the spectrum of generalised dimensions and the spectrum of singularities can be obtained. The latter can be obtained either using the Legendre transformation or the direct method [2]. In both cases, however, estimation of the partition function is necessary and the occurrence of values of WT locally equal to zero will lead to a diverging partition function.

3. Estimating the Hölder Exponent of the Maxima Lines with the Logarithmic Derivative

We will illustrate the ability of the WT maxima method to estimate the singularity exponent on examples. Let us take a single Dirac pulse at $D(1024)$, the saw tooth consisting of an integrated step function at $I(2048)$ and the step function for $S(3072^{-})$ from the right. The Hölder exponent of a Dirac pulse is -1, and each step of integration results in an increase of this exponent by 1. We, therefore, have h=0 for the right sided step function $S(3072^{-})$ and h=1 for the integrated step $I(2048)$. In the maxima of the wavelet transform we obtain the (logarithmic) slopes of the maxima values very closely following the correct values of these exponents, see figure 1.

Indeed, the slope of the maxima lines approaching the singularities reflects precisely the Hölder exponent of these singularities. This, of course, allows for the estimation of the Hölder exponent of these singularities. The integrated step function shown is the highest degree of the singularity that can be accessed with the Mexican hat wavelet that we used. This is due to the fact that it has two vanishing moments, that is it is orthogonal to the polynomials of degree one and two. We would not be able to recover the Hölder exponent of the twice integrated step with the Mexican hat; for accessing singularities in a quadratic polynomial, one has to use the wavelet which is able to filter out the quadratic 'bias', that is the wavelet with three vanishing moments. Such a wavelet can be simply obtained by taking a derivative of the Mexican hat wavelet. We now see that the wavelet used determines the range of the accessible Hölder exponents. The maxima lines of which the slope

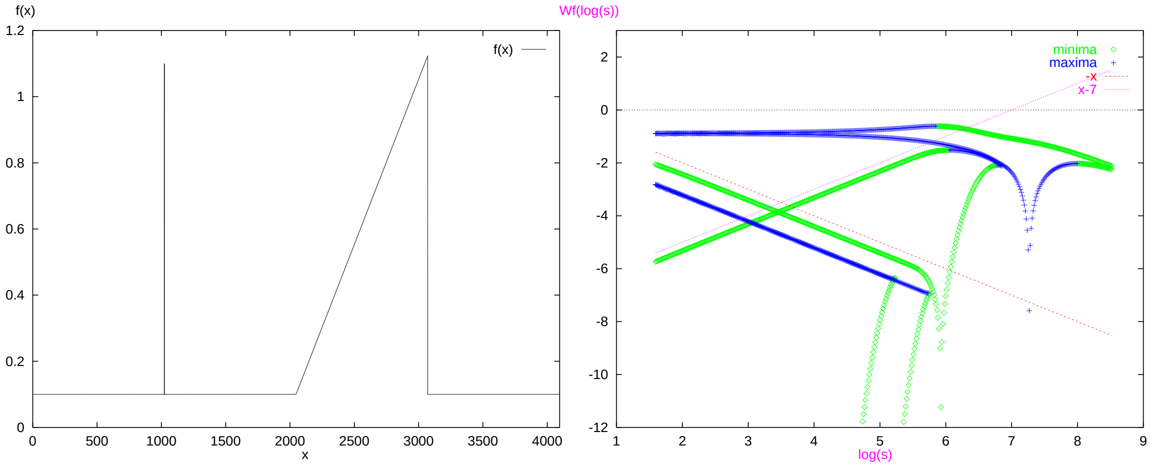

Figure 1: Left: Maxima representation of a single Dirac pulse. Right: The log-log plot of the central maximum, and of its logarithmic derivative. Indicated corresponding theoretical lines are $-x$ and -1. Mexican hat wavelet. Normalisation $1/s$.

exceeds this range can therefore be neglected in the process of calculating the partition function. In the following, we will demonstrate that this requirement leads to the divergence free partition function.

Let us denote the maxima scale and position coordinates with $(s_m(t), b_m(t))$, where t is the parametrisation variable. If we use the scale s for the parametrization of the maximum line, $(s_m(t) \equiv s$, in this case, the maximum position coordinate $b_m(s))$ is an unknown function of scale. Let us, however, neglect this dependence in the following, assuming a small deviation of the location of the maximum from the location of the singularity from which it originates.

We will use the logarithmic derivative along the maxima lines as follows,

$$\begin{aligned} \frac{\delta ln(WT(s_m, b_m))}{\delta ln(s)} &= \\ &= \frac{\delta}{\delta S} ln(WT(exp(S_m), b_m)) = \\ &= \frac{1}{(WT(exp(S_m), b_m))} \frac{\delta}{\delta S}(WT(exp(S_m), b_m)) \ , \end{aligned} \tag{3.1}$$

which for the wavelet transform of the Hölder singularity, Eq 2.4 directly reduces to

$$\frac{1}{exp(S^{h(x_0)})} \frac{\delta \ exp(S^{h(x_0)})}{\delta S} = \frac{1}{exp(S^{h(x_0)})} \ h(x_0) \ exp(S^{h(x_0)}) = h(x_0) \ .$$

The logarithmic derivative can be evaluated for the entire $WT(s, b)$, per point of the scale axis and position, using a 'slope' wavelet, Eq: 3.2. Such a wavelet allows evaluating the scale derivative $\frac{\delta}{\delta S} \ WT(exp(S), b)$ at arbitrary locations, in particular allowing sparse sampling of the maxima lines along scale. Thus, no costly sampling of the scale axis required for ensuring maxima continuity for slope determination is necessary.

$$
\begin{aligned}
\frac{\delta}{\delta S}\,(WT(exp(S),b))\;dx &= \\
&= \frac{\delta}{\delta S}\int f(x)\;(\frac{1}{exp(S)}\psi(\frac{x-b}{exp(S)}))\;dx = \\
&= \int f(x)\;\frac{\delta}{\delta S}\;(\frac{1}{exp(S)}\psi(\frac{x-b}{exp(S)}))\;dx = \\
&= \int f(x)\;\psi_{DS}(\frac{x-b}{s})\;dx\;. \qquad (3.2)
\end{aligned}
$$

The slope wavelet for the Mexican hat wavelet (where we denote $\tilde{x} = x - b$) is therefore,

$$
\begin{aligned}
\frac{\delta}{\delta S}\,(\frac{1}{exp(S)}\;\psi(\frac{\tilde{x}}{exp(S)})) &= \\
&= \frac{\delta}{\delta S}\;(\frac{1}{exp(S)}\;e^{\,(-\frac{1}{2}\;(\tilde{x}/exp(S))^2)}((\tilde{x}/exp(S))^2-1)) = \\
&= \frac{1}{exp(S)}\;e^{\,(-\frac{1}{2}\;(\tilde{x}/exp(S))^2)}((\tilde{x}/exp(S))^4-4\;(\tilde{x}/exp(S))^2+1) = \\
&= \frac{1}{s}\;e^{\,(-\frac{1}{2}\;(\tilde{x}/s)^2)}((\tilde{x}/s)^4-4\;(\tilde{x}/s)^2+1)\;. \qquad (3.3)
\end{aligned}
$$

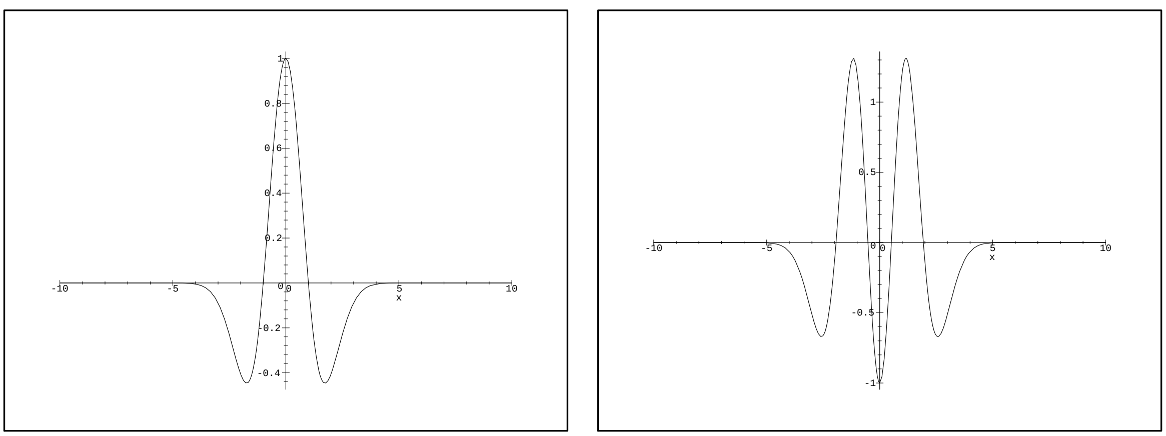

Figure 2: Mexican hat wavelet (left) and the corresponding slope wavelet (right). Note the change of sign between the plot and the formula.

In figure 3, we illustrate the logarithmic derivative calculated from 3.1 for the isolated Dirac pulse. The slope of -1 which would be expected is, however, distorted at extremally small and large scales - the finite scale effects (affecting the slope of the maximum line) are reproduced by the logarithmic derivative. These are caused by the under-sampling of the wavelet for the small scale and the cutting-off of the tails of the infinitely supported Mexican hat wavelet for the large scale. Ideally, the WT lines should follow the two theoretical lines shown.

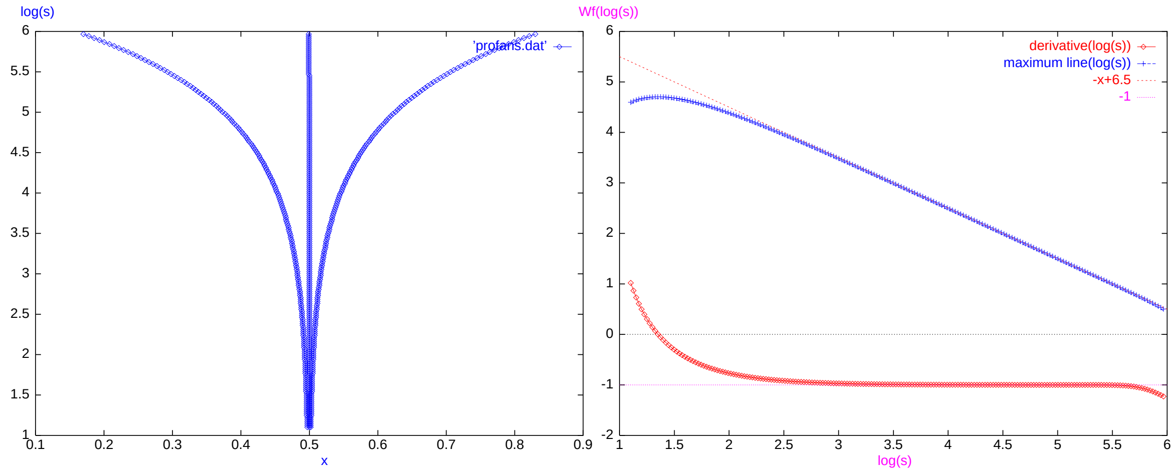

Figure 3: Left: Maxima representation of a single Dirac pulse. Right: The log-log plot of the central maximum, and of its logarithmic derivative. Indicated corresponding theoretical lines are $-x$ and -1. Mexican hat wavelet. Normalisation $1/s$.

4. Bounding the Hölder Exponent of the Maxima Line

We can, therefore, take for a fact that the logarithmic derivative is capable of locally estimating the Hölder exponent of the maxima lines originating in the singularities [4]. But the range of the Hölder exponents accessible for a given wavelet is not infinite. It begins with minimal negative value for the strongest degree of the tempered distribution accesible (-1 for the Dirac pulse), and ends with m - the number of vanishing moments of the wavelet. It is thus possible to select the parts of the maxima lines which correspond with the accessible range of Hölder exponents for a particular wavelet used.

These values can be used as bounds for the relevant slopes of the wavelet transform maxima corresponding with the accessible Hölder exponents. This 'automatically' restricts the values of the maxima lines to non-zero ones - the logarithmic derivative, Eq 3.1 would reach $-\infty$ for values of maximum of the wavelet transform approaching 0. Additionally, these bounds also put thresholds on the effect of finite size distortions of the maxima lines.

Some degree of tolerance may actually be necessary for accommodating local fluctuations of the actual slope (due to the finite size effects.) In addition, it may prove necessary to lower the upper bound below m in order to filter out regular C^{∞} bias [3]. The minimum value of the accesible Hölder exponent can be determined by the strongest singularity of the (tempered) distribution to be accessed. In most practical cases this will not be lower than -1, but it can also be exactly determined from the length of the data sample.

It is important to note that bounding the logarithmic derivative does not impose a threshold on the (small) values of the WT - arbitrary small values are still possible as long as the logarithmic slope remains within the imposed bounds.

As an example test case, let us take the well defined multi-fractal Besicovich measure with $p = 0.3, q = 0.7$. Calculating the partition function from the maxima of the standard wavelet transform with the Mexican hat leads to the locally diverging scaling function, see figure 4. These divergences in the negative moments occur where the maxima lines locally approach zero due to the local second derivative of the measure reaching zero. This effect is repeated with consistent regularity, corresponding with the renormalisation of the support of the measure - in our case factor $1/3$ for the Cantor set support.

Since the Mexican hat wavelet has two vanishing moments; $m = 2$ we should be able to approach singularities within the range $-1 < h < 1$ of the Hölder exponent. Let us, however, first put some

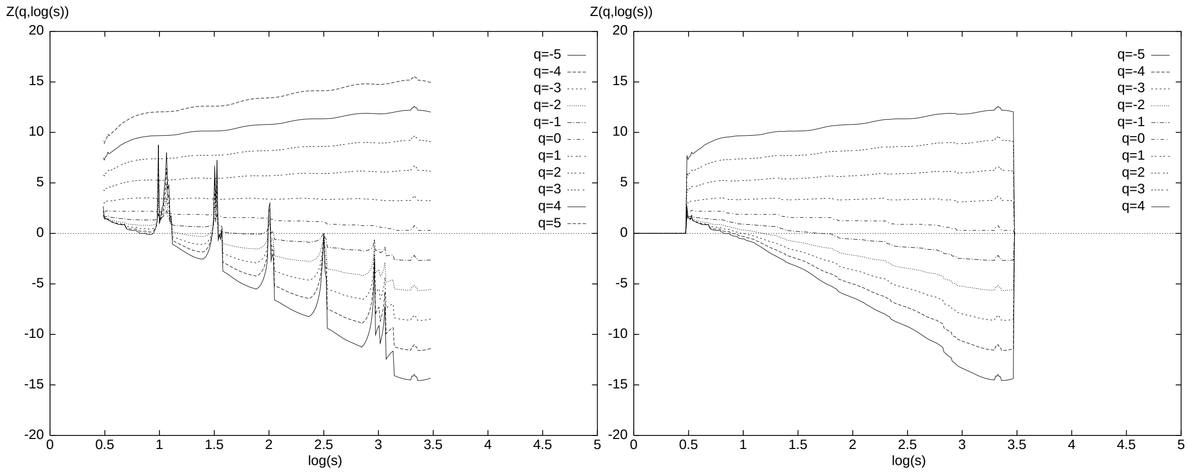

Figure 4: Left: Moments of the partition function without bounding. Right: Moments with bounding $2 < h < 2$. Mexican hat wavelet. Normalisation 1.

rather liberal bounds, from the range $-2 < h < 2$, on the local logarithmic derivative of the maxima lines. Calculating the partition function with these maxima results in an excellent continuity of the scaling function for the negative moments. This continuity gives a visual verification of the correctness of the selection between the relevant and irrelevant maxima - due to the discrete character of the renormalisation of the Cantor set, the scaling function for negative moments without the bounds imposed consists of both diverging and correct parts in a repetitive fashion.

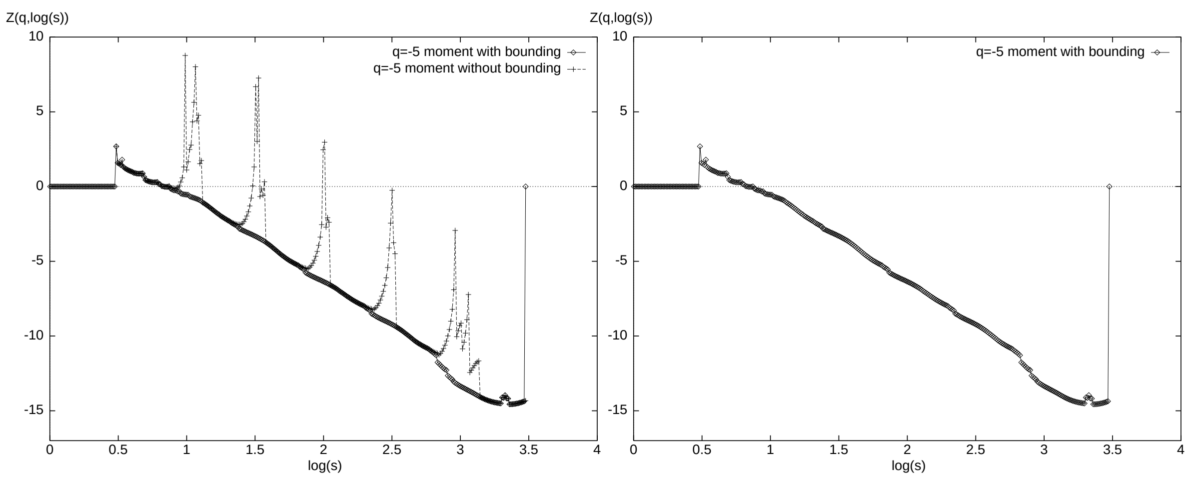

Figure 5: Left: Overlayed results of the moment -5 with and without bounding. Right: Moment -5 as it should be.

This conjecture can be verified in figure 5 left, showing in the same plot the scaling function for the moment $q = -5$, with and without the $2 < h < 2$ exponent bound. Indeed, the divergences disappear exactly where they should, while the parts of the line where there are no divergences remain unaffected. The resulting scaling function for the moment $q = -5$ alone is shown in figure 5 right.

We can compare other individual moments of the partition function in order to verify how they are affected by the bounding.

In figure 6, we show the moments 0 and 5 overlayed, from both a non-bounded and a bounded calculation. The scaling function for the moment 5 remains practically unaffected, which shows that large contributions to the analysed measure remain unaffected. For the moment 0, the plot of the partition function is, however, shifted, but retains the slope. This is an obvious consequence of removing parts of the maxima lines - there is less measure in the support. Still it has the same scaling exponent - the result gives the same correct fractal dimension of the support of the measure.

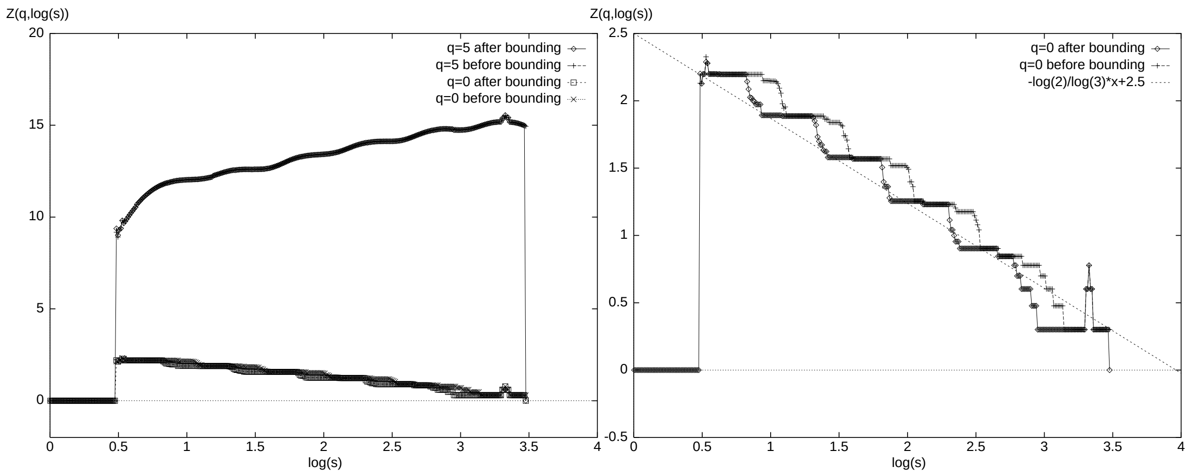

Figure 6: Left: Partition function moments 0 and 5 remain virtually unaffected (overlayed results). Right: Close-up showing moment 0 - the slope is retained.

In order to visualise the effect of slope bounding on the individual maxima lines, we plotted in figure 7 the maxima lines before and after bounding. One can check that for the relatively liberal bounds, the effect of the bounding procedure used results in removing the tips of the lines, to some depth. Of course, determining this depth is crucial and cannot be a priori decided.

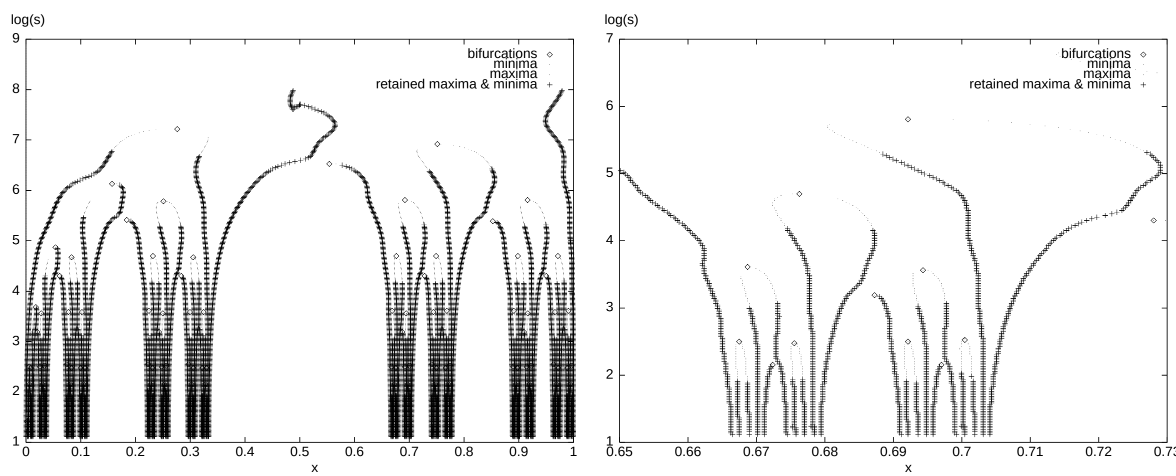

Figure 7: Left: Maxima lines showing tips of the lines (thin dotted lines) automatically removed. Right: Close-up. Mexican hat wavelet. $-2 < h < 2$. The bifurcation points marking the beginnings of the original maxima lines are visible.

Let us repeat the procedure for somewhat stronger bounds, for example $-1.2 < h < 1.2$, see figure 8.

The results will not be all that different. For the minimal allowable range $-1 \leq h \leq 1$ we will see some distortions (grossly due to the finite size effects, which are partly due to numerical inaccuracy of the estimation of the derivative).

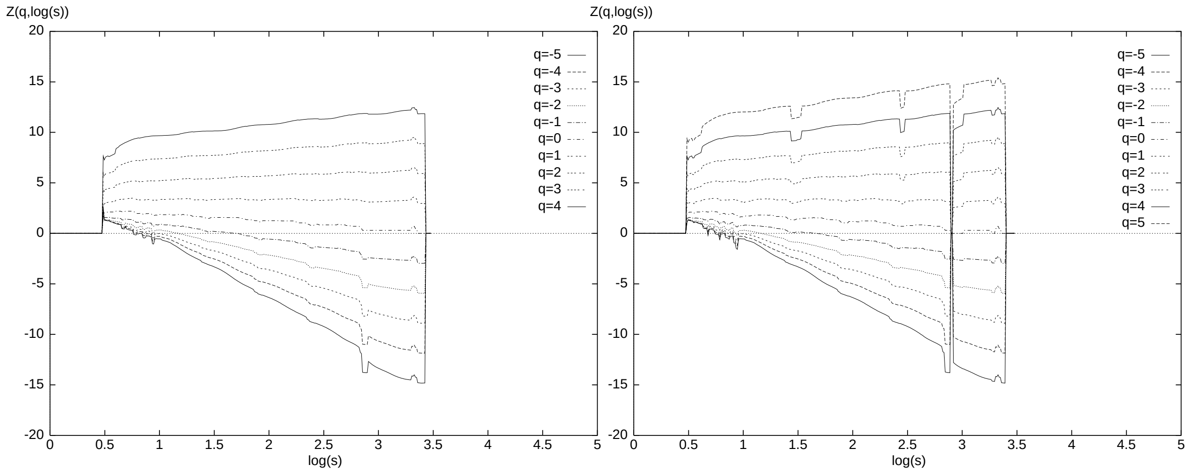

Figure 8: Left: Moments for $-1.2 < h < 1.2$ Right: Moments for $-1 \leq h \leq 1$. Mexican hat wavelet. Normalisation 1.

Interestingly, for the range $-1.2 < h < 1.2$ (and tighter), one can see that not only the tips of the maxima lines are affected, see figure 9.

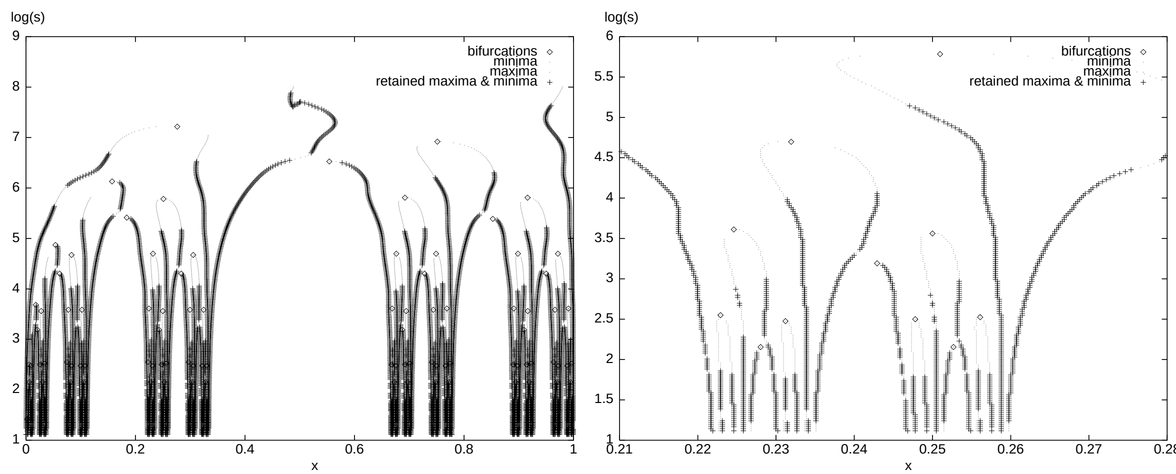

Figure 9: Left: Maxima lines showing parts of the lines (thin dotted lines) automatically removed. Right: Close-up. Mexican hat wavelet. $-1.2 < h < 1.2$.

The question is naturally whether this also works for real life signals. In our tests, we obtained very good results with a high level of stability. Let us illustrate this point with two plots of the partition function moments for $-1 < h < 1$ and $-2 < h < 2$, see figure 10:

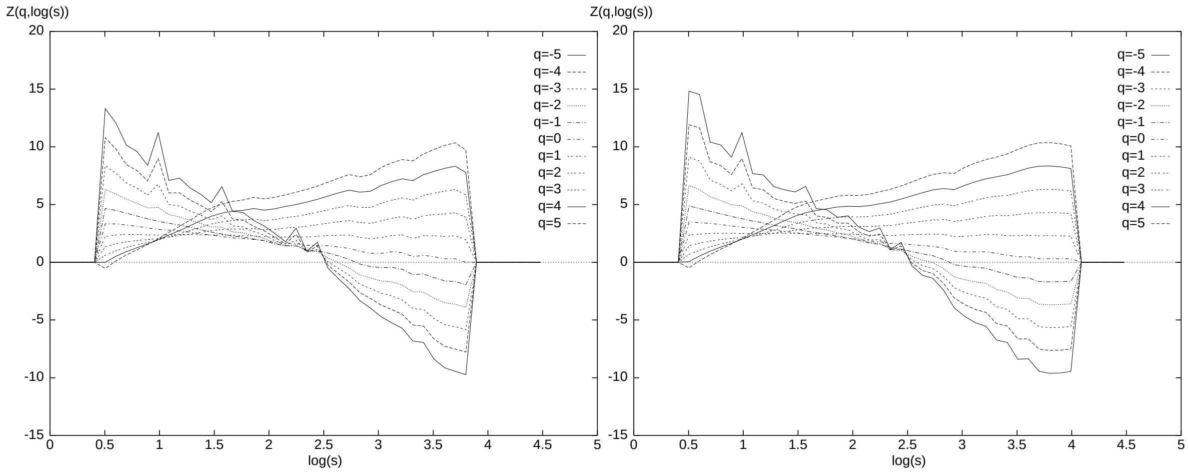

Figure 10: Left: Moments for $-1 < h < 1$ Right: Moments for $-2 \leq h \leq 2$. Mexican hat wavelet. Normalisation 1.

The stability of the moments can be verified in figure 11, showing overlayed moments for $q = -5$ with three thresholds $-1 < h < 1$, $-2 < h < 2$, and $-5 < h < 5$. 'Last but not least', the figure shows the plot of the $\tau(q)$ for $-2 < h < 2$.

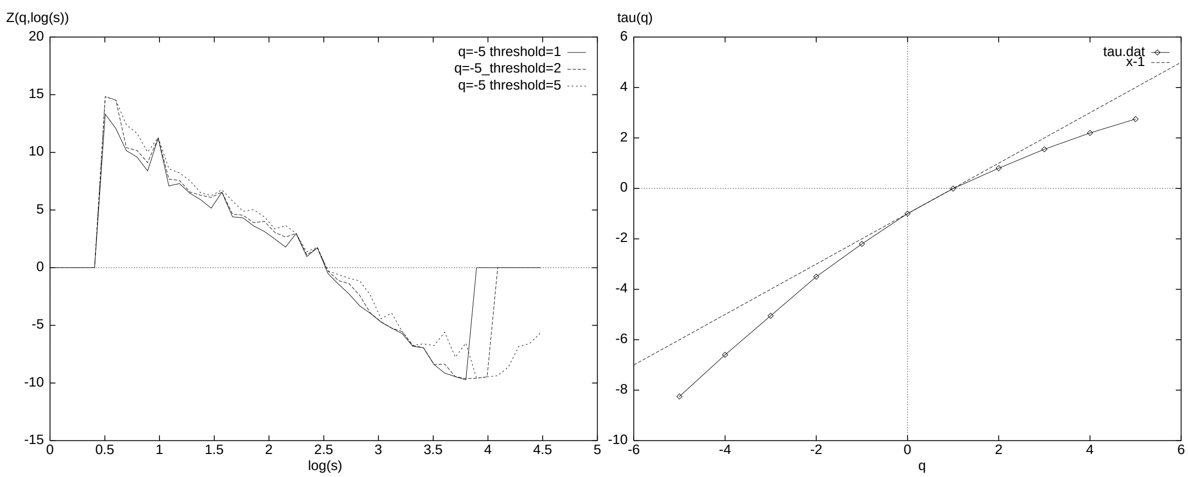

Figure 11: Left: Overlayed $q = -5$ moment for three thresholds $-1 < h < 1$, $-2 < h < 2$ and $-5 < h < 5$. Right: $\tau(q)$ for the moments from figure 10. Scale range $1.7 - 3.7$. Mexican hat wavelet. $-1.2 < h < 1.2$.

5. Closing

We presented a novel technique which is capable of filtering out the diverging part of the partition function for the negative moments of the wavelet based multi-fractal analysis. It uses straightforward bounding of the local logarithmic slope of the wavelet maxima lines with the appropriate ranges of the Hölder exponent (accessible for the wavelet in hand). The estimation of the logarithmic slope is performed using a slope wavelet which is both inexpensive and robust, thus allowing sparse sampling of the scale axis. The entire procedure has been shown to be stable and deliver correct results.

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