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Heavy-Traffic Theory for the Heavy-Tailed M/G/1 Queue and ν-stable Lévy Noise Traffic

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ABSTRACT
The workload $\mathbf{v}_t$ of an M/G/1 model with traffic $a < 1$ is analyzed for the case with heavy-tailed message length distributions $B(\tau)$, e.g. $1 - B(\tau) = \mathrm{O}(\tau^{-\nu}), \tau \to \infty, 1 < \nu \leq 2$. It is shown that a factor $\Delta(a)$ exists with $\Delta(a) \downarrow 0$ for $a \uparrow 1$ such that, whenever $\mathbf{v}_t$ is scaled by $\Delta(a)$ and time t by $\Delta_1(a) = \Delta(a)(1-a)$ then $\mathbf{w}_\tau(a) = \Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}$ converges in distribution for $a \uparrow 1$ and every $\tau > 0$. Proper scaling of the traffic load $\mathbf{k}_t$, generated by the arrivals in $[0, t)$, leads to

$$\tilde{\mathbf{w}}_\tau = \max[\mathbf{H}(\tau), \sup_{0<u<\tau} (\mathbf{H}(\tau) - \mathbf{H}(u))], \ \tau > 0,$$

with $\mathbf{H}(\tau) = \mathbf{N}(\tau) - \tau$. Here $\{\mathbf{N}(\tau), \tau \geq 0\}$ with $\nu \neq 2$ is ν-stable Lévy motion, for $\nu = 2$ it is Brownian motions and $\tilde{\mathbf{w}}_\tau$ has the limiting distribution of $\mathbf{w}_\tau(a)$ for $a \uparrow 1$. This relation is analogous to Reich's formula for the M/G/1 model with $a < 1$. The results obtained are generalisations of the diffusion approximation of the M/G/1 model with $B(\tau)$ having a finite second moment.

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1. Introduction

In this study we analyse the workload process $\{\mathbf{v}_t, t \geq 0\}$ of an M/G/1 model with traffic load a and with heavy-tailed message length distribution $B(t)$. For instance

$$1 - B(t) = \mathrm{O}(t^{-\nu}), \ t \to \infty, \ 1 < \nu \leq 2,$$

is an example of the class of message length distributions to be considered, see also [6] for examples. For such an M/G/1 model explicit representations have recently been obtained for the LST of the stationary waiting time distribution; in particular for $\nu = 1\frac{1}{2}$ also for the distribution function, see [1], [7]. In [1] it has been shown that whenever $\mathbf{v}$ is a stochastic variable with distribution the stationary waiting time distribution ($a < 1$) then a unique scaling factor $\Delta(a)$, the contraction factor, exists such that $\Delta(a)\mathbf{v}$ converges in distribution for $a \uparrow 1$ to a true probability distribution. Actually this formulates a heavy traffic theorem for $a \uparrow 1$ for the M/G/1 model with heavy-tailed message length distributions. This result has been extended for the GI/G1 queue, see [8].

In the present study we analyse the workload process $\mathbf{v}_t$ of the heavy-tailed M/G/1 queueing model for $a \uparrow 1$. It will be shown that when the workload $\mathbf{v}_t$ is scaled by $\Delta(a)$ and the time t by $\Delta_1(a) = \Delta(a)(1-a)$ then $\mathbf{w}_\tau(a) := \Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}$ converges in distribution for $a \uparrow 1$ for every $\tau > 0$. It is further shown that the so scaled or contracted workload process $\{\mathbf{w}_\tau(a), \tau \geq 0\}$ converges weakly to a workload process of a "queueing model" of which the input process is described by a stable Lévy motion for $1 < \nu < 2$ and by a Brownian motion for $\nu = 2$.

We continue this introduction with an overview of the various sections. In Section 2 we describe the class of heavy-tailed message length distributions by a characterisation of their LST $\beta(\rho)$. The essential feature is that $\beta(\rho)$ varies regularity at $\rho = 0$ with index $\nu, 1 < \nu \leq 2$. In this section we also consider the busy period $\mathbf{p}(\mathrm{v})$ with initial workload v, in particular when v is a random variable with distribution that of the stationary workload process. It leads to a new functional equation, which is used in Section 3 to study the contracted busy period $\Delta_1(a)\mathbf{p}(\mathrm{v})$ for $a\uparrow 1$. Further Section 2 quotes some known relations concerning the workload $\mathbf{v}_t$.

In Section 3 the contracted variables $\Delta(a)\mathbf{v}, \Delta_1(a)\mathbf{p}_0$ and $\Delta_1(a)\mathbf{p}(\mathrm{v}/\Delta(a))$ are defined, here $\mathbf{v}$ has the stationary waiting time distribution. It is shown that they converge in distribution for $a\uparrow 1$. Their limiting distributions are described.

In Section 4 the variable $\mathbf{w}_\tau(a) = \Delta(a)\mathbf{v}_\tau/\Delta_1(a)$ is analysed by starting from the relations for the $\mathbf{v}_t$-process. It turns out that the scaling of the workload by $\Delta(a)$ and the scaling of time by $\Delta_1(a)$, is the appropriate one. It is shown that $\mathbf{w}_\tau(a)$ converges in distribution for $a\uparrow 1$ for every $\tau > 0$. For $\mathbf{w}_\tau$ a stochastic variable with distribution the limiting distribution of $\mathbf{w}_\tau(a)$ relations are derived which are analogous to those for $\mathbf{v}_t$ mentioned in Section 2. Theorem 4.1 describes these results.

In Section 5 we analyse the contracted version $\Delta(a)\mathbf{k}_{\tau/\Delta_1(a)}$ of the input process $\{\mathbf{k}_t, t > 0\}$, with $\mathbf{k}_t$ the amount of traffic generated by the arrivals in $[0,t)$. We introduce here the virtual backlog $\mathbf{h}_t = \mathbf{k}_t - t$ and the noise-traffic $\mathbf{n}_t = \mathbf{k}_t - at$. The scaled versions of $\mathbf{h}_t$ and $\mathbf{n}_t$ are

$$\mathbf{H}(\tau;a) := \Delta(a)\mathbf{h}_{\tau/\Delta_1(a)}; \mathbf{N}(\tau;a) := \Delta(a)\mathbf{n}_{\tau/\Delta_1(a)},$$

and it shown that

$$\mathbf{H}(\tau;a) = \mathbf{N}(\tau,a) - \tau.$$

$$\mathbf{w}_\tau(a) = \max[\mathbf{H}(\tau;a), \sup_{0<u<\tau} (\mathbf{H}(\tau;a) - \mathbf{H}(u;a))], \ \mathbf{w}_0(a) = 0.$$

It is shown that $\mathbf{H}(\tau;a)$ and $\mathbf{N}(\tau,a)$ converge in distribution for $a\uparrow 1$, their limiting distributions are stable distributions; $\mathbf{N}(\tau)$ is a variable with distribution the limiting one of $\mathbf{N}(\tau;a)$, analogously for $\mathbf{H}(\tau)$. The stochastic process $\{\mathbf{N}(\tau), t \geq 0\}$ appears to be a ν-stable Lévy motion for $1 < \nu < 2$, and a Brownian motion for $\nu = 2$, further,

$$\mathbf{H}(\tau) = \mathbf{N}(\tau) - \tau, \text{ a.s.}$$

With $\tilde{\mathbf{w}}_\tau$ defined by

$$\tilde{\mathbf{w}}_\tau := \max[\mathbf{H}(\tau), \sup_{0<u<\tau} (\mathbf{H}(\tau) - \mathbf{H}(u))],$$

it is shown that the $\{\mathbf{w}_\tau(a), \tau \geq 0\}$ process converges weakly for $a\uparrow 1$ to the $\{\tilde{\mathbf{w}}_t, \tau \geq 0\}$ process.

For the M/G/1-model the workload process $\mathbf{v}_t$ is described by Reich's formula

$$\mathbf{v}_t = \max[\mathbf{h}_t, \sup_{0<u<t} (\mathbf{h}_t - \mathbf{h}_u)], \ \mathbf{v}_0 = 0.$$

For the contracted M/G/1 model, i.e. work scaled by $\Delta(a)$ and time by $\Delta_1(a)$, the relation for $\mathbf{v}_t$ transforms into that of $\mathbf{w}_\tau(a)$ and here the noise traffic $\mathbf{N}_(\tau;a)$ is seen to be the equivalent to the noise traffic $\mathbf{n}_t$ for the unscaled M/G/1 model. This holds also in the limit for $a\uparrow 1$, and for this limiting case the ν-stable Lévy motion $\{\mathbf{N}(\tau), \tau \geq \nu\}, 1 < \nu \leq 2$, represents the noise traffic of the limiting M/G/1 model.

In appendix A an asymptotic expression for the tail of the limiting distribtuion of $\Delta_1(a)\mathbf{p}_0$ for $a\uparrow 1$ is derived.

The results obtained in the present study are extensions of the heavy traffic theory for the M/G/1 model with $B(t)$ having a finite second moment. For such an M/G/1 model it is known, that the diffusion approximation provides good results from the numerical viewpoint, even for traffic a not so close to one. For the heavy-tailed M/G/1 model the equivalent approximation is here the process with traffic input the noise-traffic $\mathbf{N}(\tau)$, the ν-stable Lévy motion. It is conjectured that such an approach will yield a good approximation. This is motivated by the numerical results in [9] for the waiting time distribution when it is approximated by the distribution of the contracted waiting time.

2. The model

We consider the M/G/1-queue with arrival rate λ and message length distribution $B(t)$ with

$$\beta := \int_0^\infty \tau \mathrm{d}B(\tau) < \infty.$$

and

$$a := \lambda\beta < 1. \tag{2.1}$$

In our analysis β will be taken as the unit of time. We consider message length distributions of the following type. For a finite $T > 0$ it is assumed that: for $t > T$,

$$1 - B(t) = \mathcal{G}_1(t) + \mathcal{G}_2(t). \tag{2.2}$$

Here $\mathcal{G}_2(t)$ is a function with the property

$$|\int_T^\infty \mathrm{e}^{-\rho t}\mathrm{d}\mathcal{G}_2(t)| < \infty \text{ for Re } \rho > -\delta, \text{ for a } \delta > 0, \tag{2.3}$$

and the function $\mathcal{G}_1(t)$ is the dominant part of the tail of $1 - B(t)$ for $t \to \infty$ with, (2.2);

$$\mathcal{G}_1(t) = \mathrm{O}(t^{-\nu}),\ t \to \infty,\ 1 < \nu \leq 2, \tag{2.4}$$

is an example of a heavy-tailed message length distribution.

Denote by $\beta(\rho)$ the Laplace-Stieltjes transform (LST) of $B(t)$,

$$\beta(\rho) := \int_{0-}^\infty \mathrm{e}^{-\rho t}\mathrm{d}B(t),\ \text{Re } \rho \geq 0. \tag{2.5}$$

For distributions of the type (2.2) the LST $\beta(\rho)$ can be described by

$$1 - \frac{1-\beta(\rho)}{\rho} = g(\rho) + c\rho^{\nu-1}L(\rho) \text{ for Re } \rho \geq 0, \tag{2.6}$$

with

i $c > 0$ a constant
ii $1 < \nu \leq 2$,
iii $g(\rho)$ is a regular function of ρ for Re $\rho > -\delta$, (2.7)
iv $L(\rho)$ is regular for Re $\rho > 0$, and continuous for Re $\rho \geq 0$, except possibly at $\rho = 0$, $L(\rho) \to b > 0$ for $|\rho| \to 0$, Re $\rho \geq 0$ with $b = \infty$ if $\nu = 2$,

$$\lim_{x \downarrow 0} \frac{L(\rho x)}{L(x)} = 1 \text{ for Re } \rho \geq 0,\ \rho \neq 0,$$

Remark 2.1. The principal value of $\rho^{\nu-1}$ is defined so that $\rho^{\nu-1} > 0$ for $\rho > 0$. □

For examples of distributions $B(t)$ of the type (2.3) and with LST as given by (2.6), see [1]; cf. further [2], vol. I, p. 467, 501.

Note, that (2.6) and (2.7) imply, that: for Re $\rho \geq 0$,

$$g(\rho) \to 0 \text{ and } \rho^{\nu-1}L(\rho) \to 0 \text{ for } |\rho| \to 0. \tag{2.8}$$

The contraction coefficient $\Delta(a)$ is defined as that zero of

$$x^{\nu-1}L(x) = \frac{1-a}{ac},\ x > 0, \tag{2.9}$$

which tends to zero for $a \uparrow 1$. From (2.6),...,(2.9), it is not difficult to see that $\Delta(a)$ is well defined and that it is a simple zero. Note, herefore, that the lefthand side of (2.6) is an increasing function of $\rho \in [0,\infty)$, and that $\Delta(a)$ is real for $(1-a)/ac < 1$ or for $0 < 1-a << 1$.

Our present analysis mainly concerns the workload $\mathbf{v}_t$ at time $t > 0$. It is well known that for $a < 1$ this stochastic variable $\mathbf{v}_t$ converges in distribution for $t \to \infty$. The limiting distribution is the stationary waiting time distribution. Let $\mathbf{v}$ be a stochastic variable with distribution this stationary distribution, then, as is well known, we have for Re $\rho \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\} = \frac{1-a}{1-a\frac{1-\beta(\rho)}{\rho}} = \frac{1}{1+\frac{a}{1-a}\{1-\frac{1-\beta(\rho)}{\rho}\}}. \tag{2.10}$$

Denote by $\mathbf{p}$ the duration of a busy period and by $\mathbf{p}(\mathrm{v})$ that of an initial busy period, i.e.

$$\mathbf{p}(\mathrm{v}) := \inf_{t\geq 0}\{t : \mathbf{v}_t = 0|\mathbf{v}_0 = \mathrm{v}\},\ \mathrm{v} \geq 0. \tag{2.11}$$

It is known, cf. [3], p. 261, formula (4.94), that: for Re $\rho \geq 0$, $\mathrm{v} \geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}(\mathrm{v})}\} := \mathrm{e}^{-\frac{\rho\mathrm{v}}{1-a}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}_0}\}}, \tag{2.12}$$

with $\mathbf{p}_0$ a stochastic variable for which holds,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}_0}\} = 1-a+a\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}}\}}{\rho\mathrm{E}\{\mathbf{p}\}},\ \mathrm{Re}\ \rho \geq 0, \tag{2.13}$$

note that

$$\mathrm{E}\{\mathbf{p}\} = \frac{1}{1-a}.$$

Since $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}}\}$, Re $\rho \geq 0$, is for $a < 1$ the unique zero of

$$\mu = \beta(\rho + a(1-\mu)),\ |\mu| < 1, \tag{2.14}$$

a simple calculation shows that

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}_0}\} = \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}(\mathbf{v})}\} = \mathrm{E}\{\mathrm{e}^{-\frac{\rho}{1-a}\mathbf{v}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}_0}\}}\},\ \mathrm{Re}\ \rho \geq 0. \tag{2.15}$$

The probabilistic interpretation of the relation (2.15) is the following. For the stationary $\mathbf{v}_t$-process $\mathbf{v}_t$ and $\mathbf{v}$ have the same distribution. Hence $\mathbf{p}_0 = 0$ if $\mathbf{v}_t = 0$ and if $\mathbf{v}_t > 0$ then $\mathbf{p}_0$ has the distribution of the overshoot distribution of $\mathbf{p}$; and consequently $\mathbf{p}_0$ and $\mathbf{p}(\mathbf{v})$ should have the same distribution.

From (2.15) it is seen that

$$z(\rho) := \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{p}_0}\},\ \mathrm{Re}\ \rho \geq 0, \tag{2.16}$$

satisfies for $a < 1$,

$$\begin{aligned} &i. \quad z = \mathrm{E}\{\mathrm{e}^{-\frac{\rho}{1-a}\mathbf{v}z}\}, |z| \leq 1 \text{ for Re } \rho \geq 0,\\ &ii \quad z = 1 \text{ for } \rho = 0. \end{aligned} \tag{2.17}$$

It is simple to see that (2.17) has for $\rho \geq 0$ a unique solution, and that it is positive. From

$$\frac{\mathrm{d}z}{\mathrm{d}\rho}[1-a+\mathrm{E}\{\mathbf{v}\mathrm{e}^{-\frac{\rho}{1-a}\mathbf{v}z}\}] = -z\mathrm{E}\{\mathrm{e}^{-\frac{\rho\mathbf{v}}{1-a}z}\}, \rho \geq 0,$$

it is seen that

$$\frac{\mathrm{d}z}{\mathrm{d}\rho} \neq 0 \text{ for } \rho \geq 0.$$

This relation implies that this solution is regular for $\rho > 0$, continuous for $\rho \geq 0$. Consequently $z(\rho), \rho \geq 0$, cf. (2.16), is the unique solution of (2.17). Since the righthand side of (2.17) is regular for $\mathrm{Re}\,\rho > 0$, continuous for $\mathrm{Re}\,\rho \geq 0$ whenever $\mathrm{Re}\,z > 0$, it follows by analytic continuation that $z(\rho), \mathrm{Re}\,\rho \geq 0$, is the unique solution of (2.17).

We quote some further relations from [3]. From [3], p. 260, formula (4.92), it is easily derived that: for $\mathrm{Re}\,\rho \geq 0, \mathrm{v}_0 \geq 0$,

$$\int_0^\infty \mathrm{e}^{-\rho t} \Pr\{\mathbf{v}_t = 0 | \mathbf{v}_0 = \mathrm{v}_0\} \mathrm{d}t = \frac{1-a}{\rho} \frac{\mathrm{e}^{-\frac{\rho \mathrm{v}_0}{1-a}} \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{p}_0}\}}{\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{p}_0}\}}. \tag{2.18}$$

Further, from [3], p. 262, formula (4.99): for $\mathrm{Re}\,\rho \geq 0, t \geq 0, \mathrm{v}_0 \geq 0$,

$$\begin{aligned} \int_{0-}^\infty \mathrm{e}^{-\rho\sigma} \mathrm{d} \Pr\{\mathbf{v}_t < \sigma | \mathbf{v}_0 = \mathrm{v}_0\} &= \mathrm{e}^{-\rho \mathrm{v}_0 + \rho t\{1 - a\frac{1-\beta(\rho)}{\rho}\}} \\ -\rho U_1(t - \mathrm{v}_0) \int_0^{t-\mathrm{v}_0} &\mathrm{e}^{\rho\{1-a\frac{1-\beta(\rho)}{\rho}\}(t-\mathrm{v}_0-u)} \Pr\{\mathbf{v}_{u+\mathrm{v}_0} = 0 | \mathbf{v}_0 = \mathrm{v}_0\} \mathrm{d}u, \end{aligned} \tag{2.19}$$

with

$$U_1(x) = 0 \text{ for } x < 0,\ U_1(x) = 1 \text{ for } x \geq 0. \tag{2.20}$$

3. On the contracted variables

In this section we consider the contracted variables $\Delta(a)\mathbf{v}, \Delta_1(a)\mathbf{p}(\mathrm{v})$ and $\Delta_1(a)\mathbf{p}(\mathbf{v})$ for $a \uparrow 1$, here

$$\Delta_1(a) := \Delta(a)(1-a). \tag{3.1}$$

From theorem 5.1 of [1] we have with $\beta(\rho)$ as given in (2.6) for the M/G/1 model the following statement.

Theorem 3.1 *For the M/G/1 queue $\Delta(a)\mathbf{v}$ converges in distribution for $a \uparrow 1$. The limiting distribution $R_{\nu-1}(t)$ is given by*

$$\begin{aligned} R_{\nu-1}(t) &= \sum_{n=0}^\infty (-1)^n \frac{t^{n(\nu-1)}}{\Gamma(n(\nu-1)+1)},\ t \geq 0, \\ &= \frac{1}{\pi} \sum_{n=1}^H (-1)^{n-1} \frac{\Gamma(n(\nu-1)) \sin n(\nu-1)\pi}{t^{n(\nu-1)}} + \mathrm{O}(t^{-(H+1)(\nu-1)}) \text{ for } t \to \infty, \end{aligned} \tag{3.2}$$

for every finite $H \in \{1, 2, \dots\}$, and for $\mathrm{Re}\,r \geq 0$,

$$\int_0^\infty \mathrm{e}^{-rt} \mathrm{d}R_{\nu-1}(t) = \lim_{a \uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{v}}\} = \frac{1}{1 + r^{\nu-1}}. \tag{3.3}$$

Proof. The theorem is a special case of that in [1], and so the proof here is slightly simpler. From (2.7)iii and (2.8) it follows that

$$g(\rho) = \mathrm{O}(|\rho|) \text{ for } \mathrm{Re}\,\rho \geq 0, |\rho| \to 0. \tag{3.4}$$

From the definition of $\Delta(a)$ it is readily seen that

$$\frac{\Delta(a)}{1-a} \to 0 \text{ for } a \uparrow 1. \tag{3.5}$$

Put in (2.10),

$$\rho = r\Delta(a), \ \text{Re}\, r \geq 0,$$

so that we have from (2.6), (2.7), (2.10) and the definition of $\Delta(a)$ that: for Re $r \geq 0, 0 < 1-a << 1$,

$$\begin{aligned} \text{E}\{\text{e}^{-r\Delta(a)\mathbf{v}}\} &= [1 + \frac{a}{1-a}\{1 - \frac{1-\beta(r\Delta(a))}{r\Delta(a)}\}]^{-1} \\ &= [1 + \frac{a}{1-a} g(r\Delta(a)) + \frac{ac}{1-a}\Delta(a)^{\nu-1} L(\Delta(a)) r^{\nu-1} \frac{L(r\Delta(a))}{L(\Delta(a))}]^{-1} \\ &= [1 + \frac{a}{1-a} g(r\Delta(a)) + r^{\nu-1}\frac{L(r(\Delta(a))}{L(\Delta(a))}]^{-1}. \end{aligned} \tag{3.6}$$

From (2.7)iv, (3.4) and (3.5) it is seen that: for $a \uparrow 1$,

$$\frac{a}{1-a} g(r\Delta(a)) \to 0 \text{ and } \frac{L(r\Delta(a))}{L(\Delta(a))} \to 1, \ \text{Re}\, r \geq 0, r \neq 0.$$

So the statement (3.3) follows. From this, from Feller's continuity theorem for LST's of probability distributions with support $[0, \infty)$, and from the continuity in $r = 0$ of the last member of (3.3), it follows that $\Delta(a)\mathbf{v}$ converges in distribution for $a \uparrow 1$. For the proof of the statements (3.2) concerning $R_{\nu-1}(t)$ the reader is referred to [1]. □

Next we consider the contracted variable

$$\Delta_1(a)\mathbf{p}_0,$$

with $\Delta_1(a) = \Delta(a)(1-a)$. Put

$$\rho = r\Delta_1(a), \ \text{Re}\, r \geq 0, \ 0 < 1-a << 1,$$

then it follows from (2.15) with

$$x(r; a) := \text{E}\{\text{e}^{-r\Delta_1(a)\mathbf{p}_0}\}, \ \text{Re}\, r \geq 0, \ 0 < 1-a << 1, \tag{3.7}$$

that $x(r, a)$ satisfies for $0 < 1-a << 1$,

$$\begin{aligned} &i. \quad x = \text{E}\{\text{e}^{-r\Delta(a)\mathbf{v}x}\}, \text{Re}\, r \geq 0, \\ &ii. \quad x = 1 \text{ for } r = 0. \end{aligned} \tag{3.8}$$

As in Section 2 it is shown that $x(r; a)$, Re $r \geq 0$, is the unique solution of (3.8). From Theorem 3.1 we know that $\Delta(a)\mathbf{v}$ converges in distribution for $a \uparrow 1$. Let $\mathbf{w}$ be a stochastic variable with distribution the limiting distribution $R_{\nu-1}(t)$ of $\Delta(a)\mathbf{v}$ for $a \uparrow 1$. As in Section 2 it is readily seen that the equation

$$\begin{aligned} &i. \quad y = \text{E}\{\text{e}^{-r\mathbf{w}y}\}, \quad \text{Re}\, r \geq 0, \\ &ii. \quad y = 1 \qquad\qquad \text{for } r = 0. \end{aligned} \tag{3.9}$$

has for $r \geq 0$ a unique solution $y(r)$. This solution is real and regular for $r > 0$, continuous for $r \geq 0$; note that (2.9)i implies that for $r \geq 0$,

$$\frac{\text{d}y}{\text{d}r} \neq 0. \tag{3.10}$$

From (3.7) it is seen that the functions $x(r;a), \mathrm{Re}\, r \geq 0$ form a class of regular functions, which are uniformly bounded by one. Let $a_n, n = 1,2,\dots,$ with $0 < 1 - a_n << 1$ be a sequence with $a_n \uparrow 1$ for $n \to \infty$. From the regularity property just mentioned it follows that the sequence $a_n, n = 1,2,\dots,$ contains a subsequence $a_{n_j} j = 1,2,\dots,$ such that the sequence $x(r, a_{n_j}), \mathrm{Re}\, r \geq 0$, converges uniformly to a limit $\tilde{y}(r)$ which is also a uniformly bounded function and which is regular for $\mathrm{Re}\, r > 0$. cf. [4], p. 153. By dominated convergence it follows from (3.8) and (3.9) that $\tilde{y}(r)$ satisfies (3.9). Because this conclusion holds for any sequence $a_n, n = 1,2,\dots,$ as defined above, it follows that the following limit

$$y(r) := \lim_{a \uparrow 1} x(r;a), \mathrm{Re}\, r \geq 0, \tag{3.11}$$

exists, that it satisfies (3.9) and is regular for $\mathrm{Re}\, r > 0$. Above it has been shown that the equation (3.9) has a unique solution for $r \geq 0$, and that it is regular for $r > 0$, and so by analytic continuation it is seen that $y(r)$, as defined in (3.11) is the unique solution of (3.9), and $y(r)$ is regular for $\mathrm{Re}\, r > 0$, and continuous for $\mathrm{Re}\, r \geq 0$.

THEOREM 3.2 *The stochastic variabel $\Delta_1(a)\mathbf{p}_0$, converges in distribution for $a \uparrow 1$, and its LST $y(r), \mathrm{Re}\, r \geq 0$, is the unique solution of*

$$\begin{aligned} y = \frac{1}{1 + (ry)^{\nu - 1}}, &\quad \mathrm{Re}\, r \geq 0, \\ y = 1 &\quad \text{for } r = 0. \end{aligned} \tag{3.12}$$

PROOF. Above it has been shown that $y(r), \mathrm{Re}\, r \geq 0$, is the unique solution of (3.9), so by using Theorem 3.1 and by noticing that $R_{\nu-1}(t)$ is the distribution of $\mathbf{w}$ the statement (3.12) follows from (3.3) and (3.9). From (3.7) and (3.11) it follows that $x(r,a)$, the LST of $\Delta_1(a)\mathbf{p}_0$, converges for $a \uparrow 1$ to $y(r), \mathrm{Re}\, r \geq 0$. Because $y(r)$ is continuous for $r \geq 0$, application of Feller's continuity theorem shows that $\Delta_1(a)\mathbf{p}_0$ converges in distribution for $a \uparrow 1$. □

COROLLARY 3.1 *The stochastic variable $\Delta_1(a)(\mathbf{p}(\mathrm{v}/\Delta(a)), \mathrm{v} > 0$, converges in distribution for $a \uparrow 1$ and*

$$\lim_{a \uparrow 1} \mathrm{E}\{\mathrm{e}^{-r\Delta_1(a)\mathbf{p}(\mathrm{v}/\Delta(a))} = \mathrm{e}^{-r\mathrm{v}y(r)}, \ \mathrm{Re}\, r \geq 0, \ \mathrm{v} > 0. \tag{3.13}$$

PROOF. From (2.12) we have with $\rho = \Delta_1(a)$ for $\mathrm{Re}\, r \geq 0, 0 < 1 - a << 1$, that

$$\mathrm{E}\{\mathrm{e}^{-r\Delta_1(a)\mathbf{p}(\mathrm{v}/\Delta(a))}\} = \mathrm{e}^{-r\mathrm{v}\mathrm{E}\{\mathrm{e}^{-r\Delta_1(a)\mathbf{p}_0}\}}.$$

From this relation the proof follows similarly to that of the proof of theorem 3.2. □

REMARK 3.1. The solution $y(r)$ of (3.12) is obviously a function of $r^{\nu-1}$. Put: for $\mathrm{Re}\, r \geq 0$,

$$z(r^{\nu-1}) := y(r), \tag{3.14}$$

so that $z(s)$ is the unique solution of

$$\begin{aligned} z = &\quad \frac{1}{1 + sz^{\nu-1}}, \ \mathrm{Re}\, s \geq 0, \\ z = 1 &\quad \text{for } s = 0. \end{aligned} \tag{3.15}$$

Hence it follows for $z(s)$ a solution of

$$z = 1 - sz^{\nu}, \tag{3.16}$$

that

$$\frac{\mathrm{d}z}{\mathrm{d}s}[1-\nu s z^{\nu-1}] = -z^{\nu},$$

So it is readily seen from

$$1+\nu s_{\mathrm{v}} z^{\nu-1}(s_{\mathrm{v}}) = 0 \;\rightarrow\; z(s_{\mathrm{v}}) = \frac{\nu}{\nu-1} \text{ and } s_{\mathrm{v}} = -\frac{1}{\nu}(\frac{\nu-1}{\nu})^{\nu-1}, \tag{3.17}$$

that $s = s_{\mathrm{v}}$ is the only branch point of the solution $z(s)$ of (3.15). Since $z(s)$ is regular for Re $s > 0$, continuous for Re $s \geq 0$, it follows that $z(s)$ can be continued analytically from out Re $s \geq 0$ into the complex s-plane slitted along the halfline $(-\infty, s_{\mathrm{v}})$.

Denote by $P_{\nu-1}(t)$ the limiting distribution of the distribution of $\Delta_1(a)\mathbf{p}_0$ for $a \uparrow 1$. In appendix A the following asymptotic expression for $P_{\nu-1}(t), t \to \infty$, is derived.

For every finite $H \in \{1, 2, \ldots, \}$ and $t \to \infty$,

$$1 - P_{\nu-1}(t) = \sum_{n=1}^{H} (-1)^{n-1} t^{-n(\nu-1)} \frac{\Gamma(n\nu+1)\sin\{n(\nu-1)\pi\}}{n!n(\nu-1)(n(\nu-1)+1)} + \mathrm{O}(t^{-(H+1)(\nu-1)}). \tag{3.18}$$

4. On the contracted $\mathbf{v}_t$-process

In this section we analyze the contracted $\mathbf{v}_t$-process. Before defining it we first consider the expression for

$$\omega(\rho,\theta) := \int_0^\infty \mathrm{e}^{-\theta t}\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{v}_t} | \mathbf{v}_0 = 0\}\mathrm{d}t, \tag{4.1}$$

with Re $\rho \geq 0$, Re $\theta > 0$.

From (2.19) with $\mathbf{v}_0 = 0$ we obtain: for Re $\rho = 0$, Re $\theta > 0$,

$$\omega(\rho,\theta) = \frac{1}{\theta - \rho\{1 - a\frac{1-\beta(\rho)}{\rho}\}}\{1 - \rho\int_0^\infty \mathrm{e}^{-\theta t}\,\mathrm{Pr}\,\{\mathbf{v}_t = 0|v_0 = 0\}\mathrm{d}t\}, \tag{4.2}$$

with cf. (2.18): for Re $\theta \geq 0$,

$$\int_0^\infty \mathrm{e}^{-\theta t}\,\mathrm{Pr}\,\{\mathbf{v}_t = 0|\mathbf{v}_0 = 0\}\mathrm{d}t = \frac{1-a}{\theta}\mathrm{E}^{-1}\{\mathrm{e}^{-\theta \mathbf{p}_0}\}. \tag{4.3}$$

Put for $0 < 1 - a << 1$,

$$\begin{aligned} &\rho = r\Delta(a), \mathrm{Re}\, r \geq 0\,,\, t = \frac{\tau}{\Delta_1(a)}, \\ &\theta = s\Delta_1(a) = s\Delta(a)(1-a), \mathrm{Re}\, s \geq 0, \end{aligned} \tag{4.4}$$

and

$$\mathbf{w}_\tau(a) := \Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}, \tau \geq 0. \tag{4.5}$$

The $\{\mathbf{w}_\tau(a), \tau \geq 0\}$ process will be called the *contracted* $\mathbf{v}_t$-process. It follows that: for Re $r \geq 0$, Re $s > 0$,

$$\omega(r\Delta(a), s\Delta_1(a)) = \frac{1}{\Delta_1(a)}\int_0^\infty \mathrm{e}^{-s\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau(a)}\}\mathrm{d}\tau, \tag{4.6}$$

where we have deleted the conditioning event $\mathbf{v}_0 = 0$ in the righthand side of (4.6). Put for Re $r \geq 0$, Re $s > 0, 0 < 1 - a << 1$,

$$\Omega(r,s;a) := \int_0^\infty \mathrm{e}^{-s\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{W}_\tau(a)}\}\mathrm{d}\tau. \tag{4.7}$$

It follows from (4.2), (4.3) and (4.6) that: for $\mathrm{Re}\, r \geq 0, \mathrm{Re}\, s > 0, 0 < 1-a << 1$,

$$\Omega(r,s;a) = [s - r\{1 + \frac{a}{1-a}[1 - \frac{1-\beta(r\Delta(a))}{r\Delta(a)}]\}]^{-1}[1 - \frac{r}{s}\mathrm{E}^{-1}\{\mathrm{e}^{-s\Delta_1(a)\mathbf{p}_0}\}]. \tag{4.8}$$

In the proof of Theorem 2.1 it has been shown that

$$\frac{a}{1-a}\{1 - \frac{1-\beta(r\Delta(a))}{r\Delta(a)}\} \to r^{\nu-1} \text{ for } a\uparrow 1 \text{ and } \mathrm{Re}\, r \geq 0. \tag{4.9}$$

From Theorem 3.1 we have: for $\mathrm{Re}\, s \geq 0$,

$$\lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-s\Delta_1(a)\mathbf{p}_0}\} = y(s), \tag{4.10}$$

with $y(r), \mathrm{Re}\, r \geq 0$, the unique zero of (3.12).

From (4.8), (4.9), (4.10) and Theorem 3.2 we obtain: for $\mathrm{Re}\, r \geq 0, \mathrm{Re}\, s > 0$,

$$\begin{aligned}\Omega(r,s) := \lim_{a\uparrow 1}\Omega(r,s;a) &= \frac{1}{s - r(1+r^{\nu-1})}\{1 - \frac{r}{s}y^{-1}(s)\} \\ &= \frac{1}{s}\frac{sy(s)-r}{s-r(1+r^\nu - 1)}[1 + (sy(s))^{\nu-1}].\end{aligned} \tag{4.11}$$

Note that the definition (4.7) of $\Omega(r,s;a)$ implies that $\Omega(r,s;a)$ is regular for $\mathrm{Re}\, s > 0$, since $\mathrm{E}\{\mathrm{e}^{-r\mathbf{W}_\tau(a)}\}$ is bounded for $\mathrm{Re}\, r \geq 0$.

Note also that by using (3.12)

$$s_0 = r_0(1 + r_0^{\nu-1}) \Leftrightarrow \frac{r_0}{s_0} = 1 + s_0^{\nu-1}(\frac{r_0}{s_0})^{\nu-1} \Leftrightarrow y(s_0) = \frac{r_0}{s_0}. \tag{4.12}$$

By using the inversion formula for the Laplace transformation we have from (4.7): for $\mathrm{Re}\, r \geq 0, \varepsilon_1 > 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{W}_\tau(a)}\} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\varepsilon_1}^{\mathrm{i}\infty+\varepsilon_1} \mathrm{e}^{s\tau}\Omega(r,s;a)\mathrm{d}s, \tag{4.13}$$

from which it follows by using (4.11) and dominated convergence that : for $\mathrm{Re}\, r \geq 0, \varepsilon_1 > 0$,

$$\begin{aligned}&\lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\mathbf{W}_\tau(a)}\} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\varepsilon_1}^{\mathrm{i}\infty+\varepsilon_1} \mathrm{e}^{s\tau}\Omega(r,s)\mathrm{d}s \\ &= \tfrac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\varepsilon_1}^{\mathrm{i}\infty+\varepsilon_1} \mathrm{e}^{st}\tfrac{1}{s-r(1+r^{\nu-1})}\{1 - \tfrac{r}{s}y^{-1}(s)\}\mathrm{d}s.\end{aligned} \tag{4.14}$$

The righthand side of (4.14) is obviously a continuous function of r for $r \geq 0$, and it is seen that it converges for $r \downarrow 0$ to

$$\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\varepsilon_1}^{\mathrm{i}\infty+\varepsilon_1} \mathrm{e}^{s\tau}\frac{\mathrm{d}s}{s} = 1 \text{ for } \tau > 0.$$

Hence it follows from Feller's continuity theorem for the LST of a distribution with support contained in $[0,\infty)$ that

$\mathbf{w}_\tau(a)$ converges for $a \uparrow 1$ in distribution for every $\tau > 0$. (4.15)

Denote by $\mathbf{w}_\tau$ a stochastic variable with distribution the limiting distribution of $\mathbf{w}_\tau(a)$ for $a \uparrow 1$.

From (4.7), (4.11) and (4.15) we then obtain by using dominated convergence that: for $\mathrm{Re}\, r \geq 0, \mathrm{Re}\, s > 0$,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-s\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\}\mathrm{d}\tau &= \Omega(r,s) = \\ &= \frac{1}{s}\frac{sy(s)-r}{s-r(1+r^{\nu-1})}[1+(sy(s))^{\nu-1}].\end{aligned} \tag{4.16}$$

Because $y(0)=1$, see Theorem 3.2, we obtain

$$\lim_{s\downarrow 0}\int_0^\infty s\mathrm{e}^{-s\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\}\mathrm{d}\tau = \frac{1}{1+r^{\nu-1}} \text{ for } \mathrm{Re}\, r \geq 0. \tag{4.17}$$

From (4.5) and (4.15) we have: for $\mathrm{Re}\, r \geq 0, \tau > 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\} = \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau(a)}\} = \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}}\},$$

from which it follows, since uniform convergence and Theorem 3.1 imply, that for $\mathrm{Re}\, r \geq 0$,

$$\lim_{\tau\uparrow\infty}\ \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{v}_{\tau/\Delta_1(a)}}\} = \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\Delta(a)\mathbf{v}}\} = \frac{1}{1+r^{\nu-1}},$$

$$\lim_{\tau\to\infty}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\} = \frac{1}{1+r^{\nu-1}}. \tag{4.18}$$

This result also follows from (4.17) by using a wellknown Abel theorem for the Laplace transform when it is known that $\mathbf{w}_\tau$ converges in distribution for $\tau \to \infty$. From (4.3), (4.4) and (4.5) we obtain: for $\mathrm{Re}\, s > 0, 0 < 1-a << 1$,

$$\int_0^\infty \mathrm{e}^{-st}\,\Pr\{\mathbf{w}_t(a)=0|\mathbf{v}_0=0\}\mathrm{d}t = \frac{1-a}{s}\mathrm{E}^{-1}\{\mathrm{e}^{-s\Delta_1(a)\mathbf{p}_0}\}. \tag{4.19}$$

Hence, by using the inversion formula for the Laplace Transform, we have: for $t > 0, 0 < 1-a << 1$, and $\varepsilon > 0$,

$$\Pr\{\mathbf{w}_\tau(a)=0|\mathbf{v}_0=0\} = \frac{1-a}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon}\frac{\mathrm{e}^{s\tau}}{s}\mathrm{E}^{-1}\{\mathrm{e}^{-s\Delta_1(a)\mathbf{p}_0}\}\mathrm{d}s, \tag{4.20}$$

From Theorem (3.2) we have: for $\mathrm{Re}\, s \geq 0$,

$$\lim_{a\uparrow 1}\mathrm{E}^{-1}\{\mathrm{e}^{-s\Delta_1(a)\mathbf{p}_0}\} = y^{-1}(s),$$

note that (3.12) implies that $y(s) \neq 0$ for $\mathrm{Re}\, s \geq 0$. Hence from (4.13) by using dominated convergence it is seen that for $\tau \geq 0$, the following limit exists and

$$\lim_{a\uparrow 1}\frac{1}{a-1}\Pr\{\mathbf{w}_\tau(a)=0|\mathbf{v}_0=0\} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon}\frac{\mathrm{e}^{s\tau}}{s}\frac{\mathrm{d}s}{y(s)}. \tag{4.21}$$

From the results derived above it is seen that the following theorem results.

THEOREM 4.1. *For the M/G/1 queueing model with traffic load $a<1$, with message length distribution characterized by (3.6) and $\Delta(a)$ that zero of $acx^{\nu-1}L(x)/(1-a)=1, x>0$, which tends to zero for $a\uparrow 1$, holds for the workload $\mathbf{v}_t, t\geq 0$ that:*

i. *the contracted workload $\mathbf{w}_\tau(a):=\Delta_1(a)\mathbf{v}_{\tau/\Delta_1(a)}$ with $\Delta_1(a)=\Delta(a)(1-a)$ for $0<(1-a)/ac$ <1 converges in distribution for $a\uparrow 1$ and every $\tau>0$;*

ii. *with $\mathbf{w}_\tau$ a stochastic variable with distribution the limiting distribution of $\mathbf{w}_\tau(a)$* for $a\uparrow 1$, *holds:*

for $\operatorname{Re} r\geq 0, \tau\geq 0, \varepsilon>0$,

$$\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\}=\frac{1}{2\pi\mathrm{i}}\int\limits_{-\mathrm{i}\infty+\varepsilon}^{\mathrm{i}\infty+\varepsilon}\frac{\mathrm{e}^{s\tau}}{s-r(1+r^{\nu-1})}\{1-\frac{r}{s}y^{-1}(s)\}\mathrm{d}s, \tag{4.22}$$

here $y(s)$ is the LST of $\lim_{a\uparrow 1}\Delta_1(a)\mathbf{p}_0$, cf. Theorem 3.2;

iii. *for* $\operatorname{Re} r\geq 0$,

$$\lim_{\tau\to\infty}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\}=\frac{1}{1+r^{\nu-1}},$$

and $\mathbf{w}_\tau$ for $\tau\to\infty$ and $\Delta(a)\mathbf{v}$ for $a\uparrow 1$ have the same limiting distribution $R_{\nu-1}(t)$, see Theorem 2.1,

iv. *for* $\operatorname{Re} r\geq 0,\ \operatorname{Re} s>0$,

$$\int\limits_0^\infty \mathrm{e}^{-s\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{w}_\tau}\}\mathrm{d}\tau=\frac{1}{s-r(1+r^{\nu-1})}\{1-\frac{r}{s}y^{-1}(s)\}.$$

REMARK 4.1. The statements have actually been proved for the case that $\mathbf{v}_0=0$, i.e. $\mathbf{w}_0=0$, so that the expectations in (3.24) should be conditional expectations, the conditional event being $\mathbf{w}_0=0$. However, the analysis above needs hardly any change whenever $\mathbf{v}_0$ or $\mathbf{w}_0$ are positive, and it is then seen that the resulting limiting distributions are independent of $\mathbf{w}_0$. □

REMARK 4.2. In the relations (4.24) the average message length distribution β has been taken as the time unit, see below (2.1). When we do not make this convention then τ should be replaced by τ/β and $\mathbf{w}_\tau$ by $\mathbf{w}_{\tau/\beta}/\beta$. □

5. THE CONTRACTED INPUT PROCESS

In this section we consider the input process $\{\mathbf{k}_t, t\geq 0\}$ of the M/G/1-queue. Here $\mathbf{k}_t$ is the total amount of work generated by the arrivals in $[0,t), \mathbf{k}_0=0$. The virtual backlog $\mathbf{h}_t$ and the noise-traffic $\mathbf{n}_t$ at time t are defined by

$$\begin{aligned}\mathbf{h}_t&=\mathbf{k}_t-t,\\ \mathbf{n}_t&=\mathbf{k}_t-a\,t.\end{aligned} \tag{5.1}$$

The virtual waiting time or workload $\mathbf{v}_t$ at time t is given by Reich's formula, see [3], p. 170: with $\mathbf{v}_0=\mathrm{v}_0\geq 0$,

$$\mathbf{v}_t=\max[\mathbf{v}_0+\mathbf{h}_t,\ \sup_{0<u<t}(\mathbf{h}_t-\mathbf{h}_u)]. \tag{5.2}$$

It is wellknown that: for $\operatorname{Re}\rho\geq 0,\ t\geq 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{k}_t}\} = \sum_{n=0}^{\infty}\frac{(at)^n}{n!}\mathrm{e}^{-at}\beta^n(\rho) = \mathrm{e}^{-a\rho t\frac{1-\beta(\rho)}{\rho}}. \tag{5.3}$$

From the last three relations it follows that: for $\mathrm{Re}\,\rho = 0, t \geq 0$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{n}_t}\} &= \mathrm{e}^{a\rho t\{1-\frac{1-\beta(\rho)}{\rho}\}},\\ \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{h}_t}\} &= \mathrm{e}^{\rho t\{1-a\frac{1-\beta(\rho)}{\rho}\}}.\end{aligned} \tag{5.4}$$

Note that

$$\mathrm{E}\{\mathbf{n}_t\} = \mathrm{E}\{\mathbf{k}_t - at\} = 0,\ t \geq 0. \tag{5.5}$$

We introduce the following scaling, cf. (4.4).

$$\begin{aligned}&\rho = r\Delta(a),\ \mathrm{Re}\,r \geq 0,\\ &t = \tau/\Delta_1(a).\end{aligned} \tag{5.6}$$

With this scaling we define the contracted versions of $\mathbf{n}_t, \mathbf{h}_t$ and $\mathbf{v}_t$,

$$\begin{aligned}\mathbf{N}(\tau;a) &:= \Delta(a)\mathbf{n}_{\tau/\Delta_1(a)},\\ \mathbf{H}(\tau;a) &:= \Delta(a)\mathbf{h}_{\tau/\Delta_1(a)},\\ \mathbf{w}_\tau(a) &:= \Delta(a)\mathbf{v}_{\tau\Delta_1(a)}.\end{aligned} \tag{5.7}$$

It follows from (5.4) and (5.7) that: for $0 < 1-a << 1, \mathrm{Re}\,r = 0, \tau \geq 0$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;a)}\} &= \mathrm{e}^{\frac{ar\tau}{1-a}\{1-\frac{1-\beta(r\Delta(a))}{r\Delta(a)}\}},\\ \mathrm{E}\{\mathrm{e}^{-r\mathbf{H}(\tau;a)}\} &= \mathrm{e}^{\frac{r\tau}{1-a}\{1-a\frac{1-\beta(r\Delta(a))}{r\Delta(a)}\}},\\ \mathbf{H}(\tau;a) &= \mathbf{N}(\tau;a) - \tau;\end{aligned} \tag{5.8}$$

and from (5.3),

$$\mathbf{w}_\tau(a) = \max[\Delta(a)\mathrm{v}_0 + \mathbf{H}(\tau;a), \sup_{0<u<\tau}(\mathbf{H}(\tau;a) - \mathbf{H}(u;a))]. \tag{5.9}$$

Theorem 5.1. *$\mathbf{N}(\tau;a)$ and $\mathbf{H}(\tau;a)$ both converge in distribution for $a\uparrow 1$ for every $\tau > 0$ and: for $\mathrm{Re}\,r = 0$,*

$$\begin{aligned}&i. \quad \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau;a)}\} = \mathrm{e}^{\tau r^\nu},\\ &ii. \quad \lim_{a\uparrow 1}\mathrm{E}\{\mathrm{e}^{-r\mathbf{H}(\tau;a)}\} = \mathrm{e}^{r\tau(1+r^{\nu-1})}.\end{aligned} \tag{5.10}$$

Proof. The relations (5.10) follow from (5.8) by using the same arguments as in the proof of Theorem 3.1. Because the righthand sides in (5.10) are both continuous at $r = 0$, it follows from the continuity theorem for characteristic functions that $\mathbf{N}(\tau;a)$, and similarly $\mathbf{H}(\tau;a)$, converges in distribution for $a\uparrow 1$. □

Let $\mathbf{N}(\tau)$ and $\mathbf{H}(\tau) = \mathbf{N}(\tau) - \tau$, be stochastic variables for which holds: for $\mathrm{Re}\,r = 0, \tau \geq 0, 1 < \nu \leq 2$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}_{tau}}\} &= \mathrm{e}^{\tau r^\nu},\\ \mathrm{E}\{\mathrm{e}^{-r\mathbf{H}(\tau)}\} &= \mathrm{e}^{r\tau(1+r^{\nu-1})} = \mathrm{e}^{r\tau}\mathrm{E}\{\mathrm{e}^{-r\mathbf{N}(\tau)}\}.\end{aligned} \tag{5.11}$$

The distributions of $\mathbf{N}(\tau)$ and $\mathbf{H}(\tau)$ may be characterized by using the notation in [5]. p. 11. Here $\mathcal{S}_\nu(\sigma,1,\mu)$ stands for the distribution of a stochastic variable $\mathbf{x}$ with characteristic function:

$$\mathrm{E}\{\mathrm{e}^{\mathrm{i}\theta\mathbf{X}}\} = \mathrm{e}^{-\sigma^\nu|\theta|^\nu\{1-\mathrm{i}\frac{\theta}{|\theta|}\tan\frac{1}{2}\nu\pi\}+\mathrm{i}\mu\theta}, \tag{5.12}$$

with

$$\sigma \geq 0, 1 < \nu \leq 2, \mu \text{ and } \theta \text{ both real.} \tag{5.13}$$

The distribution $\mathcal{S}_\nu(\sigma,1,\mu)$ belongs to the class of stable distributions. From (5.4) and (5.12) it follows, cf. [5], p. 15 and 51, that: for $\tau \geq 0, 1 < \nu \leq 2$,

$$\begin{aligned} &\mathcal{S}_\nu([-\tau\cos\frac{1}{2}\nu\pi]^{1/\nu},1,0) \quad \text{is the distribution of } \mathbf{N}(\tau),\\ &\mathcal{S}_\nu([-\tau\cos\frac{1}{2}\nu\pi]^{1/\nu},1,-\tau) \quad \text{is the distribution of } \mathbf{H}(\tau). \end{aligned} \tag{5.14}$$

Note that (5.11) implies that $\mathbf{N}(\tau)$ and $\sigma^{1/\nu}\mathbf{N}(\tau/\sigma), \sigma > 0$, have the same distribution, i.e. $\mathbf{N}(\tau)$ is self-similar with index $1/\nu$.

From the definition of $\mathbf{N}(\tau)$ it is readily seen that the process $\{\mathbf{N}(\tau), \tau \geq 0\}$ has stationary independent increments because

$$\mathbf{N}(\tau_2) - \mathbf{N}(\tau_1) \text{ and } \mathbf{N}(\tau_4) - \mathbf{N}(\tau_3)$$

are independent for $0 \leq \tau_1 < \tau_2 \leq \tau_3 < \tau_4$, and

$$\mathbf{N}(\tau_2) - \mathbf{N}(\tau_1) \stackrel{\mathrm{d}}{=} \mathbf{N}(\tau_2 - \tau_1), \tau_2 > \tau_1 \geq 0, \tag{5.15}$$

here $\stackrel{\mathrm{d}}{=}$ stands for "have the same distribution".

It is readily seen, cf. [5], p.113 and 349, that the $\mathbf{N}(\tau)$ process is a ν-stable Lévy motion for $1 < \nu < 2$, for $\nu = 2$ it is Brownian motion.

The same arguments show that the process $\{\mathbf{H}(\tau), \tau \geq 0\}$ has stationary, independent increments.

Next, we introduce the contracted M/G/1 model. Its workload process is the process $\{\mathbf{w}_\tau(a), \tau \geq 0\}$ with $\mathbf{w}_\tau(a)$ as given by (5.9). Its virtual backlog is given by $\mathbf{H}(\tau;a)$ and its noise traffic by $\mathbf{N}(\tau;a)$. From (5.1) and (5.7) we have for $0 < a < 1, \tau \geq 0$,

$$\begin{aligned} &\mathrm{E}\{\mathbf{N}(\tau;a)\} = 0\ ,\ \mathrm{E}\{\mathbf{H}(\tau;a)\} = -\tau,\\ &\mathbf{H}(\tau;a) = \mathbf{N}(\tau;a) - \tau. \end{aligned} \tag{5.16}$$

We next consider the $\tilde{M}/\tilde{G}/1$ model with noise traffic $\mathbf{N}(\tau)$, virtual backlog $\mathbf{H}(\tau) = \mathbf{N}(\tau) - \tau$ and workload $\tilde{\mathbf{w}}_\tau$ at time τ defined by

$$\tilde{\mathbf{w}}_\tau = \max[\mathbf{H}(\tau), \sup_{0<u<\tau}(\mathbf{H}(\tau) - \mathbf{H}(u))]. \tag{5.17}$$

The question arises whether this $\tilde{M}/\tilde{G}/1$ model can be considered as the "limit" for $a \uparrow 1$ of the above described contracted M/G/1 model, i.e. whether the $\{\mathbf{w}_\tau(a), \tau \geq 0\}$ process converges weakly for $a \uparrow 1$ to the $\{\tilde{\mathbf{w}}_\tau, \tau \geq 0\}$ processes. This is indeed the case and we now sketch its proof. It has been shown above that the $\{\mathbf{N}(\tau;a), \tau > 0\}$ process has homogeneous independent increments, i.e. the distribution of $\mathbf{N}(\tau_1;a) - \mathbf{N}(\tau_2 : a)$ is a function of $\tau_2 - \tau_1$. From the results above it is readily seen that the finite dimensional distributions of the increments of the $\{\mathbf{N}(\tau;a), \tau > 0\}$ process converge weakly to those of the $\mathbf{N}_{(\tau)}, \tau > 0\}$ process, which is also a process with homogeneous independent increments. The analogous statements hold for the $\mathbf{H}(\tau;a)$ and the $\mathbf{H}(\tau)$-process. The functional $\sup_{0<u<\tau}\{\mathbf{H}(\tau;a) - \mathbf{H}(u;a)\}$ is a continuous functional of the $\mathbf{H}(\tau;a)$ process and it satisfies the conditions of Corollary 3.2 of [11]. It is readily verified that this corollary, when applied to the $\mathbf{H}(\tau;a)$

and $\mathbf{H}(\tau)$ process, shows that the just mentioned functional converges for $a\uparrow 1$ in the Skorokhod topology to the analogous functional of the $\mathbf{H}(\tau)$-process. It then follows from (5.9) and (5.17) that the $\{\mathbf{w}(\tau;a),\tau>0\}$ process converges in the Skorokhod topology to the $\{\tilde{\mathbf{w}}(\tau),\tau>0\}$ process.

In Section 4 it has been shown that $\mathbf{w}_\tau(a)$ converges in distribution for $a\uparrow 1$, so that from the definition of $\mathbf{w}_\tau, \tau\geq 0$, cf. (4.22)i, it is seen that

$$\tilde{\mathbf{w}}_\tau \overset{\mathrm{d}}{=} \mathbf{w}_\tau. \tag{5.18}$$

Consequently, the statements (4.22)ii, iii and iv also hold for $\tilde{\mathbf{w}}_\tau$.

By using the notation

$$\mathbf{y}_u \underset{u\to u_0}{\overset{\mathrm{d}}{\longrightarrow}} \mathbf{y}$$

to express that $\mathbf{y}_u$ converges in distribution for $u\to u_0$ with limiting distribution that of $\mathbf{y}$, the limit results obtained above may be written as follows.

From Theorem 3.1, from (4.5) and (4.24)iii and $1<\nu\leq 2$,

$$\begin{aligned} &\mathbf{w}_\tau(a) = \Delta(a)\mathbf{v}_{\tau/\Delta_1(a)} \underset{a\uparrow 1}{\overset{\mathrm{d}}{\longrightarrow}} \mathbf{w}_\tau \underset{\tau\to\infty}{\overset{\mathrm{d}}{\longrightarrow}} \mathbf{w}_\infty, \\ &\mathbf{w}_\tau(a) \underset{\tau\to\infty}{\overset{\mathrm{d}}{\longrightarrow}} \Delta(a)\mathbf{v} \underset{a\uparrow 1}{\overset{\mathrm{d}}{\longrightarrow}} \mathbf{w}_\infty, \end{aligned} \tag{5.19}$$

and for $1<\nu<2$,

$$\mathbf{w}_\tau(a) \underset{a\uparrow 1}{\overset{\mathrm{d}}{\longrightarrow}} \tilde{\mathbf{w}}_\tau \underset{\tau\to\infty}{\overset{\mathrm{d}}{\longrightarrow}} \mathbf{w}_\infty, \tag{5.20}$$

here $\mathbf{w}_\infty$ has the distribution $R_{\nu-1}(t)$, cf. (3.2).

Appendix A

In this Appendix we derive an asymptotic expression for the limiting distribution $P_{\nu-1}(t)$ of the distribution of $\Delta_1(a)\mathbf{p}_0$ for $a\uparrow 1$, cf. Theorem 3.2.

With $z(s)$ as defined in Remark 3.1 put

$$x(s) := z(s) - 1, \tag{a.1}$$

so that

$$\begin{aligned} &x(s) = (-s)(1+x(s))^\nu, \\ &x(0) = 0. \end{aligned} \tag{a.2}$$

Let C be the circle with center at $x=0$ and radius x_{v}, with

$$x_{\mathrm{v}} := x(s_{\mathrm{v}}) = z(s_{\mathrm{v}}) - 1 = \frac{1}{\nu-1}, \tag{a.3}$$

and s_{v} as defined in (3.17). It is readily seen that: for $|s|<|s_{\mathrm{v}}|$,

$$|s||1+x|^{\nu} < |x| \text{ for } |x| < |x_v|. \tag{a.4}$$

Because x and $1+x$ are both regular functions of x for $|x| < |s_{\mathrm{v}}|$, it follows from Rouché's theorem that the equation $x = (-s)(1+x)^{\nu}$ has a unique root inside the circle with center at $x = 0$ and radius $|s_{\mathrm{v}}| - \varepsilon$ for $0 < \varepsilon << 1$. Hence, we can apply Langrange's theorem [10], p.133, for the derivation of a power series for $x(s)$. It results that: for $|s| < |s_{\mathrm{v}}|$,

$$x(s) = \sum_{n=0}^{\infty} (-1)^n \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} (1+t)^{n\nu}|_{t=0} = \sum_{n=1}^{\infty} (-1)^n s^n \frac{\Gamma(n\nu+1)}{n!\Gamma(n(\nu-1)+2)}.$$

Hence from (3.14), (a.1) and the definition of $y(r)$, cf. Theorem 3.2, we have

$$y(r) = 1 + \sum_{n=1}^{\infty} (-1)^n r^{n(\nu-1)} \frac{\Gamma(n\nu+1)}{n!\Gamma(n(\nu-1)+2)}, \tag{a.5}$$

for

$$0 \le \mathrm{Re}\, r^{\nu-1} < -s_{\mathrm{v}},$$

or

$$\frac{1-y(r)}{r} = \sum_{n=1}^{\infty} (-1)^{n-1} r^{n(\nu-1)-1} \frac{\Gamma(n\nu+1)}{n!\Gamma(n(\nu-1)+2)}. \tag{a.6}$$

Since $y(r)$ is the LST of $P_{\nu-1}(t)$, we have: for $\mathrm{Re}\, r \ge 0$,

$$\frac{1-y(r)}{r} = \int_0^{\infty} \mathrm{e}^{-rt} \{1 - P_{\nu-1}(t)\} \mathrm{d}t. \tag{a.7}$$

From (a.6) and (a.7) we derive the asymptotic relation for $1 - P_{\nu-1}(t) \to \infty$, by using Theorem 2 of [2], vol.II. p.159. The function $y(r)$ is regular for $\mathrm{Re}\, r > 0$, continuous for $\mathrm{Re}\, r \ge 0$ and $r = 0$ is a branch point of $y(r)$. From (3.12) it is seen that $y(r)$ can be continued analytically into $\{r : |r| > 0, -\frac{1}{2}\pi - \omega < \arg r < \frac{1}{2}\pi + \omega\}$ for a $\omega \in (0, \frac{1}{2}\pi)$ and that $y(r)$ is absolutely integrable on every finite interval with $|\arg r| < u$, $|u| < \frac{1}{2}\pi + \omega$. From the just mentioned Theorem of [2] and from (a.6) and (a.7) it follows that: for $t \to \infty$ and every finite $H \in \{1, 2, \dots\}$,

$$\begin{aligned} 1 - P_{\nu-1}(t) \quad &= \sum_{n=1}^{H} (-1)^{n-1} t^{-n(\nu-1)} \frac{\Gamma(n\nu+1)}{n!\Gamma(n(\nu-1)+2)} \quad \frac{1}{\Gamma(1-n(\nu-1))} \\ &+ \mathrm{O}(t^{-(H+1)(\nu-1)}). \end{aligned} \tag{a.8}$$

Using the relation

$$\Gamma^{-1}(1-z) = \Gamma(z) \frac{\sin \pi z}{z}$$

it is seen that (a.8) may be rewritten as:

$$\begin{aligned} 1 - P_{\nu-1}(t) \quad &= \frac{1}{\pi} \sum_{n=1}^{H} (-1)^{n-1} t^{-n(\nu-1)} \frac{\Gamma(n\nu+1) \sin\{n(\nu-1)\pi\}}{n!n(\nu-1)(n(\nu-1)+1)} \\ &+ \mathrm{O}(t^{-(H+1)(\nu-1)}) \text{ for } t \to \infty, \end{aligned} \tag{a.9}$$

and so (a.9) is the asymptotic expression for the tail of the distribution $P_{\nu-1}(t)$.

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