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ABSTRACT

A theorem on the weak convergence of a properly normalized multivariate continuous local martingale is proved. The time-change theorem used for this purpose allows for short and transparent arguments.

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1. Introduction

In this paper we study the convergence of a d-dimensional continuous local martingale M as 'time' tends to infinity. We suppose that there exist normalizing matrices K_t such that as $t \to \infty$, we have $\|K_t\| \to 0$ and

$$K_t \langle M \rangle_t K_t^T \xrightarrow{\mathrm{P}} \eta\eta^T, \tag{1.1}$$

where η is some random matrix (T denotes transposition and $\|\cdot\|$ is a certain matrix norm, see the beginning of section 2). Our main result, theorem 5.1, states that under this condition, we have weak convergence of the normalized martingale $K_t M_t$ to a mixture of normals.

Recently, a similar result has been reported by Küchler and Sørensen in [8] (see also the book [9]). In their setup, unlike in the present paper, M is a square integrable martingale (not necessarily continuous) with covariance matrices $\Sigma_t = \mathrm{E}\,(M_t M_t^T)$, that determine the normalization in (1.1) via the additional assumption that there exists a positive definite limit of $K_t \Sigma_t K_t^T$ as $t \to \infty$. The latter assumption is typically tedious to verify in practice (see [8], section 4). It seems therefore worthwhile to notice once again that in the special case of our concern, when M is continuous, all we need is assumption (1.1).

The result of Küchler and Sørensen is in fact a Cramér-Wold extension of a one-dimensional result in [3]. We will use the same device in the course of proving theorem 5.1 to reduce the statement of the theorem to a statement about one-dimensional local martingales. However, our basic one-dimensional results rely on totally different arguments than those of [3]. The fact that we focus on continuous martingales allows for using a so-called *time-change device*, by viewing each one-dimensional continuous local martingale as a time-changed Brownian motion. In this way, a statement about continuous local martingales reduces to the corresponding statement about Brownian motions. The time-change device is a quite powerful tool in general and indeed, it leads to short and transparent proofs of the one-dimensional results presented in section 4.

This paper is organized as follows. In section 2 we introduce some notation and recall basic facts about weak convergence in Polish spaces. In section 3 we present some necessary facts concerning continuous local martingales, in particular the time-change theorem mentioned above. A first application of this theorem yields sufficient conditions for a sequence of continuous local martingales to converge to 0, see lemmas 3.3 and 3.4. In section 4 we focus first on the simplest particular case of a one-dimensional martingale whose quadratic variation satisfies condition (1.1) with deterministic η (or rather its one-dimensional analogue (4.1) in section 4). A simple time-change argument yields a limit result in this case, see (4.4) below. In the remainder of section 4 we discuss how to handle the general case of a random limit η in (4.1). This leads us to treating so-called *nested* sequences of local martingales, see corollary 4.2 for a limit result on such sequences that is a consequence of theorem 4.1 on nested Brownian motions. This result provides the main argument in the proof of our theorem 5.1 in section 5.

2. Preliminaries

In the concluding section 5 we need some elementary facts on matrix norms, see e.g. [4]. We denote Euclidean norms by $|\cdot|$. If A is an $n \times m$ matrix, we write $\|A\| = \sup\{|Ax| : x \in \mathbb{R}^m, |x| = 1\}$. We will use the fact that this norm has the following properties (I denotes the identity matrix, B is another matrix such that AB is defined and $x \in \mathbb{R}^m$):

$$\|I\| = 1, \quad \|A^T\| = \|A\|, \quad \|AB\| \leq \|A\|\|B\|, \quad \|Ax\| \leq \|A\||x|.$$

We continue by recalling some basic facts on weak convergence. For more details, see e.g. [2]. Let $\mathcal{X}$ be a Polish space, i.e. $\mathcal{X}$ is a separable, complete metric space. Let $\mathrm{P}, \mathrm{P}_1, \mathrm{P}_2, \ldots$ be probability measures on the Borel sets of $\mathcal{X}$. We say that P_n *converges weakly* to P if for all bounded continuous functions $f : \mathcal{X} \to \mathbb{R}$, we have $\int f \, \mathrm{dP}_n \to \int f \, \mathrm{dP}$. We denote this by $\mathrm{P}_n \rightsquigarrow \mathrm{P}$. If the probability measures P_n and P are the laws of $\mathcal{X}$-valued random elements X_n and X, we also write $X_n \rightsquigarrow X$ instead of $\mathrm{P}_n \rightsquigarrow \mathrm{P}$. By the *portmanteau theorem* (see [2], theorem 2.1) $X_n \rightsquigarrow X$ is equivalent to $\mathrm{P}(X_n \in B) \to \mathrm{P}(X \in B)$ for all X-continuity sets $B \in \mathcal{B}(\mathcal{X})$. It follows from the definition of weak convergence that if $\mathcal{Y}$ is another Polish space and $\phi : \mathcal{X} \to \mathcal{Y}$ is continuous, then $X_n \rightsquigarrow X$ implies $\phi(X_n) \rightsquigarrow \phi(X)$. This result is known as the *continuous mapping theorem*. We will also use some results about weak convergence in product spaces. Let $\mathcal{X} \times \mathcal{Y}$ be the product of two Polish spaces. In order to prove that

$$(X_n, Y_n) \rightsquigarrow (X, Y) \tag{2.1}$$

in $\mathcal{X} \times \mathcal{Y}$, it suffices to verify that for all X-continuity sets $A \in \mathcal{B}(\mathcal{X})$ and Y-continuity sets $B \in \mathcal{B}(\mathcal{Y})$ we have

$$\mathrm{P}(X_n \in A, Y_n \in B) \to \mathrm{P}(X \in A, Y \in B)$$

(see [2], theorem 3.1). Another useful result is *Slutsky's lemma* (theorem 4.4 of [2]). It states that if Y is deterministic, then (2.1) is equivalent to $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$.

A second mode of convergence used below is convergence in probability. If $X, X_1, X_2, \ldots$ are random elements with values in a metric space $(\mathcal{X}, d)$, then we say that X_n *converges in probability* to X if

$$\forall \varepsilon > 0 : \quad \mathrm{P}(d(X_n, X) > \varepsilon) \to 0.$$

We denote this mode of convergence by the symbol $\xrightarrow{\mathrm{P}}$. Of course, this only makes sense either if all the random elements are defined on the same probability space, or if X is deterministic. This convergence depends on the metric d only through the topology it induces on $\mathcal{X}$. So we may replace d by any other equivalent metric. The two modes of convergence coincide if the limit is deterministic, i.e. if X is deterministic, we have

$$X_n \rightsquigarrow X \iff X_n \xrightarrow{\mathrm{P}} X.$$

We will consider continuous random processes as random elements of the space $C[0,\infty)$. This is the space of all continuous functions $f : [0,\infty) \to \mathbb{R}$, endowed with the metric

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \left(\max_{t \le n} |f(t) - g(t)| \wedge 1 \right).$$

Under this metric, $C[0,\infty)$ is a Polish space. By $C_0^+[0,\infty)$ we denote the subspace of $C[0,\infty)$ that consists of all functions f that are non-decreasing and start in 0, i.e. $f(0) = 0$. Since $C_0^+[0,\infty)$ is closed in $C[0,\infty)$, it is a Polish space. We will use the fact that the following maps are continuous ($\mathcal{X}$ is an arbitrary Polish space):

$$\phi : C[0,\infty) \times [0,\infty) \to \mathbb{R}, \quad \phi(f,t) = f(t), \tag{2.2}$$

$$\psi : C[0,\infty) \times C_0^+[0,\infty) \to C[0,\infty), \quad \psi(f,g) = f \circ g. \tag{2.3}$$

$$\xi : C[0,\infty) \times C_0^+[0,\infty) \times \mathcal{X} \to C[0,\infty) \times \mathcal{X}, \quad \xi(f,t,x) = (f(t),x). \tag{2.4}$$

We will also need the following characterization of convergence in $C_0^+[0,\infty)$. For a proof, see e.g. [6], theorem VI.2.15.

Lemma 2.1. *Let $f, f_1, f_2, \ldots \in C_0^+[0,\infty)$ and suppose that D is a dense subset of $[0,\infty)$. If for all $t \in D$ we have $f_n(t) \to f(t)$, then $f_n \to f$ in $C_0^+[0,\infty)$.*

3. The time-change device

In this paper we consider local martingales $M = \{M_t\}_{t\ge 0}$ with continuous sample paths $t \mapsto M_t$. All local martingales M are assumed to start in 0, i.e. $M_0 = 0$. Throughout this section and the next, local martingales are one-dimensional. We assume that all filtrations satisfy the so-called *usual conditions*. So if $\{\mathcal{F}_t\}_{t\ge 0}$ is a filtration on $(\Omega, \mathcal{F}, \mathrm{P})$, we assume that $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$, for all $t \ge 0$ and that $\mathcal{F}_0$ contains all P-null events in $\mathcal{F}$. This technical assumption assures the existence of the quadratic variation process $\langle M \rangle$ of a continuous local martingale M. Recall that this is the unique (up to indistinguishabilty) continuous, non-decreasing process starting in 0 such that $M^2 - \langle M \rangle$ is again a local martingale. So we can consider a one-dimensional continuous local martingale M as a random element of $C[0,\infty)$ and its quadratic variation $\langle M \rangle$ as a random element of $C_0^+[0,\infty)$. For these and other basic facts about continuous local martingales, see e.g. [7], chapter 1. The following well-known theorem will play a central role in the sequel. It states that each continuous local martingale can be embedded in a Brownian motion. A proof of this theorem can be found in [7], theorem 3.4.6 and problem 3.4.7.

Theorem 3.1 (Time-Change Theorem). *Let $M = (M_t, \mathcal{F}_t : t \geq 0)$ be a continuous local martingale and for $s \geq 0$, define*

$$\tau_s = \inf\{t \geq 0 : \langle M \rangle_t > s\}, \quad \mathcal{G}_s = \mathcal{F}_{\tau_s}.$$

The underlying probability space can be suitably extended in order to support a Brownian motion W with respect to the filtration $\{\mathcal{G}_t\}$, such that a.s.

$$M_t = W_{\langle M \rangle_t}, \quad \forall\, t \geq 0.$$

Remark 3.2. See [7], remark 4.1 on p. 169 for the exact construction of the extended probability space. It is important to note that the extension does not change the law of the local martingale. In this paper we study properties of sequences of such laws and therefore we may assume that each continuous local martingale M in question is embedded in a Brownian motion W in the sense of the above theorem. We will call W *the Brownian motion corresponding to* M.

We now apply the time-change device to prove the following simple lemma. It gives a sufficient condition for a sequence of continuous local martingales to converge to 0.

Lemma 3.3. *For each $n \in \mathbb{N}$, let M^n be a continuous local martingale. If $\langle M^n \rangle \rightsquigarrow 0$, then $M^n \rightsquigarrow 0$.*

Proof. Let W^n be the Brownian motion corresponding to M^n. Of course, the weak limit of the sequence W^n is again a Brownian motion W. By Slutsky's lemma, it then follows from the assumption that $(W^n, \langle M^n \rangle) \rightsquigarrow (W, 0)$. Since $M^n = \psi(W^n, \langle M^n \rangle)$, where ψ is the continuous map defined by (2.3), the proof is complete by the continuous mapping theorem. □

Next we present a lemma that is usually proved by using the Lenglart inequality (see [7], problem 1.5.25). The alternative proof provided below is based on a time-change argument.

Lemma 3.4. *For each $n \in \mathbb{N}$, let M^n be a continuous local martingale. Suppose that all the M^n are defined on the same probability space. If for some dense $D \subseteq [0, \infty)$, we have*

$$\langle M^n \rangle_t \xrightarrow{\text{P}} 0, \quad \forall\, t \in D, \tag{3.1}$$

then $M^n \rightsquigarrow 0$.

Proof. From assumption (3.1) it follows by lemma 2.1 that we have $\langle M^n \rangle \xrightarrow{\text{P}} 0$. This is equivalent to $\langle M^n \rangle \rightsquigarrow 0$, so the statement follows from the preceding lemma. □

4. Nested local martingales

In section 5 our main result concerning the limiting behavior of a continuous local martingale will be proved. The main argument used in the course of this proof will be presented at the end of this section, see corollary 4.2 concerning nested sequences of continuous local martingales. In order to explain why it is necessary to treat such nested sequences, we first consider the following special case.

Let M be a one-dimensional continuous local martingale. Suppose that for a certain non-negative number η and positive numbers k_t

$$\frac{\langle M\rangle_t}{k_t} \xrightarrow{\mathrm{P}} \eta \quad \text{as } t \to \infty. \tag{4.1}$$

Let W be the Brownian motion corresponding to M. For each $t \geq 0$ define the process W^t by putting $W^t_s = W_{k_t s}/\sqrt{k_t}$, for all $s \geq 0$. Then the scaling property of Brownian motion implies that each process W^t is again a Brownian motion. For all $t \geq 0$ we have

$$\frac{M_t}{\sqrt{k_t}} = W^t_{\frac{\langle M\rangle_t}{k_t}}.$$

Each W^t is a Brownian motion, so we have $W^t \rightsquigarrow B$, where B is a Brownian motion. Since η is deterministic, we have by Slutsky's lemma the implication

$$\left[W^t \rightsquigarrow B, \quad \frac{\langle M\rangle_t}{k_t} \xrightarrow{\mathrm{P}} \eta\right] \Rightarrow \left[\left(W^t, \frac{\langle M\rangle_t}{k_t}\right) \rightsquigarrow (B, \eta)\right]. \tag{4.2}$$

By the continuous mapping theorem it thus follows that for any continuous map ϕ

$$\phi\left(W^t, \frac{\langle M\rangle_t}{k_t}\right) \rightsquigarrow \phi(B, \eta). \tag{4.3}$$

In the special case of the map ϕ defined by (2.2) the left hand side is equal to $M_t/\sqrt{k_t}$ and the right hand side equals B_η, so (4.3) yields

$$\frac{M_t}{\sqrt{k_t}} \rightsquigarrow N(0, \eta). \tag{4.4}$$

Hence in this simple case the time-change device already gives us a desired result, a central limit theorem for the normalized martingale $M_t/\sqrt{k_t}$. But when η is random, the matter is more complicated. We can no longer use Slutsky's lemma to justify the implication (4.2). For this purpose we will prove theorem 4.1 which tells us that thanks to the special *nesting relation* between the Brownian motions W^t, they are asymptotically independent of $\langle M\rangle_t / k_t$. This means that in the case of a random η the implication (4.2) also holds, with B a Brownian motion that is independent of η.

It will be convenient to formulate the nesting condition in terms of filtrations. For all $n \in \mathbb{N}$, let $\{\mathcal{F}^n_t\}_{t\geq 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Following [3], we call the sequence $\{\mathcal{F}^n_t\}_{t\geq 0}$ *nested* if there exists a sequence $t_n \downarrow 0$ such that

$$\mathcal{F}^n_{t_n} \subseteq \mathcal{F}^{n+1}_{t_{n+1}}$$

for all $n \in \mathbb{N}$, and

$$\bigvee_{n=1}^{\infty} \mathcal{F}^n_{t_n} = \bigvee_{n=1}^{\infty} \mathcal{F}^n_{\infty}.$$

Any sequence $t_n \downarrow 0$ for which these conditions are satisfied is called an N-sequence. If for example $\mathcal{F}^n_t = \mathcal{F}_{a_n t}$, where a_n is some sequence converging to ∞, then the filtrations are nested and $t_n = 1/\sqrt{a_n}$ is an N-sequence. A sequence of adapted processes $X^n = (X^n_t, \mathcal{F}^n_t : t \geq 0)$ on $(\Omega, \mathcal{F}, \mathrm{P})$ is called nested if the corresponding filtrations $\{\mathcal{F}^n_t\}_{t\geq 0}$ are nested. As in [3] and [5], this nesting condition will lead to so-called *stable* limit results. Suppose the r.v.'s X_n and X are all defined on the same probability space. If $X_n \rightsquigarrow X$, then the convergence is called *stable* if for each r.v. Y on the same probability space we also have joint weak convergence of the pair (X_n, Y). See e.g. [1] for more details on stable convergence.

We will obtain a limit result for nested continuous local martingales as a corollary of the following theorem concerning nested Brownian motions, that turn out to be asymptotically independent of any other random element.

Theorem 4.1. *Let $W^n = (W^n_t, \mathcal{F}^n_t : t \geq 0)$ be a sequence of Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. If for all $n \in \mathbb{N}$ there exists an $\{\mathcal{F}^n_t\}$-stopping time τ_n such that*

(i) $\tau_n \xrightarrow{\mathrm{P}} 0$,

(ii) $\mathcal{F}^n_{\tau_n} \subseteq \mathcal{F}^{n+1}_{\tau_{n+1}}\ \forall n \in \mathbb{N}$,

(iii) $\bigvee_{n=1}^{\infty} \mathcal{F}^n_{\tau_n} = \bigvee_{n=1}^{\infty} \mathcal{F}^n_{\infty}$,

then, for all random elements X on $(\Omega, \mathcal{F}, \mathrm{P})$ with values in a Polish space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, we have $(W^n, X) \rightsquigarrow (W, X)$, where W is a Brownian motion that is independent of X.

Proof. For $n \in \mathbb{N}$ define the process V^n by

$$V^n_t = \begin{cases} W^n_t - W^n_{\tau_n}, & t > \tau_n \\ 0, & t \leq \tau_n. \end{cases}$$

Observe that $W^n - V^n = \{W^n_{\tau_n \wedge t}\}_{t\geq 0}$ is a continuous local martingale, with quadratic variation process $\{\tau_n \wedge t\}_{t\geq 0}$. It then follows from (i) and lemma 3.4 that $W^n - V^n \rightsquigarrow 0$. Hence, by Slutsky's lemma, it suffices to show that $(V^n, X) \rightsquigarrow (W, X)$, where W is a Brownian motion, independent of X. We will show that for all W-continuity sets $A \in \mathcal{B}(C[0,\infty))$ and X-continuity sets $B \in \mathcal{B}(\mathcal{X})$, we have

$$\mathrm{P}(V^n \in A, X \in B) \to \mathrm{P}(W \in A)\mathrm{P}(X \in B)$$

(recall that this is sufficient, see section 2). The fact that $W^n - V^n \rightsquigarrow 0$ implies in particular that V^n converges weakly to a Brownian motion. Hence, by the portmanteau theorem, we have

$$\mathrm{P}(V^n \in A) \to \mathrm{P}(W \in A),$$

for all W-continuity sets $A \in \mathcal{B}(C[0,\infty))$. In view of the inequality

$$\begin{aligned} |\mathrm{P}(V^n \in A,& X \in B) - \mathrm{P}(W \in A)\mathrm{P}(X \in B)| \leq \\ &|\mathrm{P}(V^n \in A, X \in B) - \mathrm{P}(V^n \in A)\mathrm{P}(X \in B)| + \\ &|\mathrm{P}(V^n \in A)\mathrm{P}(X \in B) - \mathrm{P}(W \in A)\mathrm{P}(X \in B)| \end{aligned}$$

it remains to show that $|\mathrm{P}(V^n \in A, X \in B) - \mathrm{P}(V^n \in A)\mathrm{P}(X \in B)| \to 0$.

For notational convenience, put $\mathcal{G} = \bigvee_{n=1}^{\infty} \mathcal{F}_{\infty}^{n}$. From assumptions (ii) and (iii) it follows, by the Martingale Convergence Theorem, that for all $B \in \mathcal{B}(\mathcal{X})$

$$\mathrm{P}(X \in B \,|\, \mathcal{F}_{\tau_n}^{n}) \xrightarrow{L^1} \mathrm{P}(X \in B \,|\, \mathcal{G}).$$

Consequently, we have for all $A \in \mathcal{B}(C[0,\infty))$ and $B \in \mathcal{B}(\mathcal{X})$

$$\left| \mathrm{E}\left[1_{\{V^n \in A\}} \mathrm{P}(X \in B \,|\, \mathcal{F}_{\tau_n}^{n})\right] - \mathrm{E}\left[1_{\{V^n \in A\}} \mathrm{P}(X \in B \,|\, \mathcal{G})\right] \right| \to 0.$$

By the strong Markov property, V^n is independent of $\mathcal{F}_{\tau_n}^{n}$. This implies that the first expectation in the preceding display is equal to $\mathrm{P}(V^n \in A)\mathrm{P}(X \in B)$. The $\mathcal{G}$-measurability of V^n implies that the second expectation is equal to $\mathrm{P}(V^n \in A, X \in B)$. □

Corollary 4.2. *Let $(M_t^n, \mathcal{F}_t^n : t \geq 0)$ be a nested sequence of continuous local martingales and suppose that there exists an N-sequence t_n such that*

$$\langle M^n \rangle_{t_n} \xrightarrow{\mathrm{P}} 0.$$

Let $t \geq 0$ be fixed. If there exists a non-negative random variable C, such that

$$\langle M^n \rangle_{t} \xrightarrow{\mathrm{P}} C,$$

then, for each random element X defined on $(\Omega, \mathcal{F}, \mathrm{P})$ with values in some Polish vector space $\mathcal{X}$, we have

$$(M_t^n, X) \rightsquigarrow (W_C, X),$$

where W is a Brownian motion that is independent of (C, X).

Proof. Let $(W_t^n, \mathcal{G}_t^n : t \geq 0)$ be the Brownian motion corresponding to $(M_t^n, \mathcal{F}_t^n : t \geq 0)$ and define $\tau_n = \langle M^n \rangle_{t_n}$. Then τ_n is a $\{\mathcal{G}_t^n\}$-stopping time (see the time-change theorem). By construction, all conditions of the preceding theorem are satisfied. It then follows from this theorem and Slutsky's lemma that

$$(W^n, \langle M_t^n \rangle, X) = (W^n, C, X) + (0, \langle M_t^n \rangle - C, 0) \rightsquigarrow (W, C, X),$$

where W is a Brownian motion that is independent of the pair (C, X). Now write $(M_t^n, X) = \xi(W^n, \langle M_t^n \rangle, X)$, with ξ the continuous map defined in (2.4) and apply the continuous mapping theorem. We get $(M_t^n, X) = \xi(W^n, \langle M_t^n \rangle, X) \rightsquigarrow \xi(W, C, X) = (W_C, X)$. □

5. The main theorem

The following theorem is the main result of the paper. The conditions of the theorem involve matrices K_t of which we require that $\|K_t\| \to 0$. Since all norms on a Euclidean space generate the same topology, this is equivalent to the condition that each entry of K_t converges to 0. As usual, $N_d(0, \Sigma)$ denotes a d-dimensional normal distribution with mean vector 0 and covariance matrix Σ.

Theorem 5.1. *Let $(M_t, \mathcal{F}_t : t \geq 0)$ be a d-dimensional continuous local martingale. If there exist invertible, non-random $d \times d$-matrices $(K_t : t \geq 0)$ such that as $t \to \infty$*

(i) $K_t \langle M \rangle_t K_t^T \xrightarrow{\mathrm{P}} \eta\eta^T$
where η is a random $d \times d$-matrix,

(ii) $\|K_t\| \to 0$,

then, for each $\mathbb{R}^k$-valued random vector X defined on the same probability space as M, we have

$$(K_t M_t, X) \rightsquigarrow (\eta Z, X), \quad \text{as } t \to \infty,$$

where $Z \sim N_d(0, I)$ and Z is independent of (η, X).

Remark 5.2. In terms of stable convergence (see section 4), we may reformulate the statement of the theorem as follows:

$$K_t M_t \rightsquigarrow V \quad \text{(stably)},$$

where V has characteristic function $u \mapsto \mathrm{E}\, \exp[-\frac{1}{2} u^T \eta\eta^T u]$.

Proof of theorem 5.1. First observe that for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$, we have

$$\mathrm{E}\, e^{ix^T \eta Z + iy^T X} = \mathrm{E}\, e^{-\frac{1}{2} x^T \eta\eta^T x + iy^T X}.$$

So in terms of characteristic functions we have to prove that for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$

$$\mathrm{E}\, e^{ix^T K_t M_t + iy^T X} \to \mathrm{E}\, e^{-\frac{1}{2} x^T \eta\eta^T x + iy^T X} \quad \text{as } t \to \infty.$$

That is, we need to prove

$$(x^T K_t M_t, Y) \rightsquigarrow (x^T \eta Z, Y) \quad \text{as } t \to \infty, \tag{5.1}$$

for all $x \in \mathbb{R}^d$ and all real-valued random variables Y, where $Z \sim N_d(0, I)$ and Z is independent of (η, Y).

Let a_n be an arbitrary sequence so that $a_n \to \infty$. We introduce the one-dimensional continuous processes M^n as follows:

$$M_t^n = x^T K_{a_n} M_{a_n t}, \quad t \geq 0.$$

Observe that for all $n \in \mathbb{N}$, M^n is a continuous local martingale with respect to the filtration $\{\mathcal{F}_{a_n t}\}$ and that

$$\langle M^n \rangle_t = x^T K_{a_n} \langle M \rangle_{a_n t} K_{a_n}^T x, \quad t \geq 0. \tag{5.2}$$

In this notation (5.1) reduces to

$$(M_1^n, Y) \rightsquigarrow (x^T \eta Z, Y). \tag{5.3}$$

In order to prove (5.3) we will show that every subsequence a_{l_n} of a_n has a further subsequence a_{k_n}, such that

$$\left(M_1^{k_n}, Y\right) \rightsquigarrow (x^T \eta Z, Y).$$

We can choose a subsequence a_{k_n} of a_{l_n} and numbers $0 < t_n \downarrow 0$, so that

$$a_{k_n} t_n \uparrow \infty \quad \text{and} \quad \|K_{a_{k_n}} K_{a_{k_n} t_n}^{-1}\| \to 0. \tag{5.4}$$

Indeed, since $\|K_{a_{l_n}}\| \to 0$ and $1 = \|I\| \leq \|K_{a_{l_n}}\| \|K_{a_{l_n}}^{-1}\|$, we have $\|K_{a_{l_n}}^{-1}\| \to \infty$. So we can choose the subsequence a_{k_n} in such a way that the following inequalities are satisfied:

$$\|K_{a_{k_n}}\| \leq \frac{1}{n\|K_{a_{l_n}}^{-1}\|} \quad \text{and} \quad a_{k_n} \geq n a_{l_n}. \tag{5.5}$$

Now put $t_n = a_{l_n}/a_{k_n}$. By the second of the inequalities we have $t_n \leq 1/n$, so $t_n \downarrow 0$. Moreover, $a_{k_n} t_n = a_{l_n}$, which tends to ∞ cf. the first condition in (5.4). As for the second condition in (5.4), it is satisfied as well since by the inequality in (5.5)

$$\|K_{a_{k_n}} K_{a_{k_n} t_n}^{-1}\| \leq \|K_{a_{k_n}}\| \|K_{a_{k_n} t_n}^{-1}\| \leq \frac{1}{n},$$

which means that the sequences a_{k_n} and t_n possess the desired properties.

We are going to apply corollary 4.2 to the local martingales M^{k_n}. We saw already that M^{k_n} is a continuous local martingale w.r.t. the filtration $\{\mathcal{F}_{a_{k_n} t}\}$, so it is clear that the M^{k_n} are nested. By the first relation in (5.4), t_n is an N-sequence. Moreover, by (5.2) we have

$$\begin{aligned} \| \left\langle M^{k_n} \right\rangle_{t_n} \| &= \|x^T K_{a_{k_n}} \langle M \rangle_{a_{k_n} t_n} K_{a_{k_n}}^T x\| \\ &= \|x^T (K_{a_{k_n}} K_{a_{k_n} t_n}^{-1}) K_{a_{k_n} t_n} \langle M \rangle_{a_{k_n} t_n} K_{a_{k_n} t_n}^T (K_{a_{k_n}} K_{a_{k_n} t_n}^{-1})^T x\| \\ &\leq \|K_{a_{k_n}} K_{a_{k_n} t_n}^{-1}\|^2 \|K_{a_{k_n} t_n} \langle M \rangle_{a_{k_n} t_n} K_{a_{k_n} t_n}^T\| |x|^2. \end{aligned}$$

So it follows by the second relation in (5.4) and by assumption (i) that $\left\langle M^{k_n} \right\rangle_{t_n} \xrightarrow{\mathrm{P}} 0$.

The preceding paragraph shows that the assertion of corollary 4.2 can be applied to the local martingales M^{k_n}. To this end, observe that by assumption (i)

$$\left\langle M^{k_n} \right\rangle_1 = x^T K_{a_{k_n}} \langle M \rangle_{a_{k_n}} K_{a_{k_n}}^T x \xrightarrow{\mathrm{P}} x^T \eta \eta^T x.$$

It then follows from the corollary that

$$\left(M_1^{k_n}, Y\right) \rightsquigarrow (W_{x^T \eta \eta^T x}, Y),$$

where W is a Brownian motion, independent of $(x^T \eta \eta^T x, Y)$. Finally, use the independence of W and $(x^T \eta \eta^T x, Y)$ to see that $(W_{x^T \eta \eta^T x}, Y)$ has the same distribution as $(x^T \eta Z, Y)$, where $Z \sim N_d(0, I)$ and Z is independent of (η, Y). □

References

1. D.J. Aldous and G.K. Eagleson (1978), On Mixing and Stability of Limit Theorems, *Ann. Prob.* **6**, 325–331.
2. P. Billingsley (1968), *Convergence of Probability Measures*, Wiley.
3. D. Feigin (1985), Stable Convergence of Semimartingales, *Stoch. Proc. Appl.* **19**, 125–134.
4. I. Gohberg and S. Goldberg (1981), *Basic Operator Theory*, Birkhäuser.
5. P.G. Hall and C.C. Heyde (1980), *Martingale Limit Theory and its Application*, Academic Press.
6. J. Jacod and A.N. Shiryaev (1987), *Limit Theorems for Stochastic Processes*, Springer.
7. I. Karatzas and S.E. Shreve (1991), *Brownian Motion and Stochastic Calculus*, 2nd Edition, Springer.
8. U. Küchler and M. Sørensen (1996), *A Note on Limit Theorems for Multivariate Martingales*, Research Report no. 350, dept. theor. statist., University of Aarhus.
9. U. Küchler and M. Sørensen (1997), *Exponential Families of Stochastic Processes*, Springer.