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ABSTRACT
Groote and Luttik (1998a) proved that the extension of the theory pCRL with the axioms for branching bisimulation of Van Glabbeek and Weijland (1996) yields a ground complete axiomatisation of branching bisimulation algebras with data, and conditionals and alternative quantification over these, provided that the data part has built-in equality and built-in Skolem functions. In this paper we shall use this result to obtain ground complete axiomatisations of η-bisimulation algebras, delay bisimulation algebras and weak bisimulation algebras with data, conditionals and alternative quantification over data, under the same proviso as before.

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1. Introduction

Groote and Ponse (1994) proposed the specification formalism μCRL to reason formally about the interaction of processes with the data that they communicate. It is an extension of the theory ACP (Bergstra and Klop, 1984) with data-parametrised actions, a conditional construct and alternative quantification over data. The data is specifiable in μCRL as a many-sorted equational specification. In this paper we consider pCRL, which is the extension of the subsystem BPA of ACP with the aforementioned features.

Groote and Luttik (1998b) have shown that pCRL provides a ground complete axiomatisation of strong bisimulation, provided that the data has an equality predicate and Skolem functions built-in. This requirement is known from model theory (cf. Chang and Keisler (1990)) to ensure that a first-order theory is decidable. It was argued by Groote and Luttik (1998b) that it is needed because the operation of alternative quantification may be used to encode any Π^0_4 first-order formula as an identity between process terms in bisimulation semantics: they exhibit for every first-order data formula $\varphi = (\forall x_1)(\exists x_2)(\forall x_3)(\exists x_4)\varphi_0$, where φ_0 is an open formula, pCRL-expressions p and q that are bisimilar if, and only if, φ holds. Hence, any complete axiomatisation of strong bisimilation would involve data of which at least the Π^0_4-fragment is decidable.

The question then arose whether it would suffice to add the laws of Van Glabbeek and Weijland (1996) to pCRL to turn it into a ground complete axiomatisation of branching bisimulation under the same proviso. Groote and Luttik (1998a) showed that this is indeed the case. They used a technique similar to that of Van Glabbeek and Weijland (1996): it is shown that every process term is equal to one without inert silent steps; then the result follows, since on such terms, strong- and branching bisimulation coincide. The difficulty in the presence of an operation for alternative quantification is to determine whether an occurrence of τ is inert or not. Consider the process term $p = \sum_{x:s} \tau p'$; it may have both inert and noninert silent steps. For, there may exist data elements d and d' such that p is branching bisimilar to $p'[x := d]$, while p is not branching bisimilar to $p'[x := d']$. Skolem functions and conditionals are employed to write any such expression as an alternative composition of an inert and a noninert part.

Our goal in the present paper is to find complete axiomatisations of the remaining three congruences in the lattice presented by Van Glabbeek and Weijland (1996): η-bisimulation, delay bisimulation and weak bisimulation. We shall take the same approach as Van Glabbeek and Weijland (1996) and identify subsets of process expressions on which these congruences coincide with branching bisimulation. We shall show that the laws that appear in Van Glabbeek and Weijland (1996) for these congruences (in the case of weak bisimulation, these are Milner's original τ-laws) are sufficient to prove that each process expression is equal to one that is included in such a subset. Consequently, the resulting axiomatisations are complete.

Hennessy and Lin (1996) and Lin (1995) provided complete axiomatisations for extensions of message-passing process algebras and the π-calculus with the silent step in weak bisimulation semantics. In both settings it suffices to add Milner's τ-laws to the axiomatisations of strong bisimulation. These languages include a restricted form of alternative quantification, called *input prefixing*, where quantification and action prefixing is combined into a single construct. Groote and Luttik (1998b) have shown that this form of alternative quantification is less expressive. Parrow and Victor (1998) deal with an extension of their fusion calculus with silent steps in weak bisimulation semantics. Their calculus also contains a single binder instead of input prefixing to express an input mechanism. Surprisingly, however, in their setting it is not possible to just add Milner's τ-laws; they must be replaced by two schemes. Similarly, Fu (1999) shows that the addition of Milner's τ-laws to his χ-calculus is not sufficient to find a complete axiomatisation of weak bisimulation. We conclude that in both cases the extra laws are not due to the separation of quantification and action prefixing, but due to the mobility aspects of the calculi.

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2. Processes and Bisimulations

Let us fix a many-sorted algebra $\mathfrak{D}$ that contains a boolean algebra with precisely two elements $\top$ and $\bot$. This boolean algebra is refered to in the signature Δ of $\mathfrak{D}$ by means of the sort symbol $\mathbb{b}$. The algebra $\mathfrak{D}$ will be the data part of the algebra of processes $\mathfrak{P}$ that we shall now construct.

Let $\mathbb{p}$ be a sort symbol that does not occur in Δ; we shall use it to refer to the process part of $\mathfrak{P}$. We presuppose a set $\mathcal{A}$ of *action declarations* for Δ, i.e., a set of function declarations of the form

$$\mathsf{a}\colon s_1 \cdots s_n \rightarrow \mathbb{p},$$

where $s_1, \ldots, s_n \in \Delta$ are sort symbols. An *atomic action* is an element $a\langle d_1, \ldots, d_n\rangle$, where $a = \mathsf{a}\colon s_1 \cdots s_n \rightarrow \mathbb{p}$ is an action declaration and d_i is an element of $\mathfrak{D}$ of sort s_i, for $1 \leq i \leq n$; let us denote by A the set of all atomic actions that can be constructed in this way from the action declarations in $\mathcal{A}$ and the elements of $\mathfrak{D}$.

Next, let $\boldsymbol{\delta}$ and $\boldsymbol{\tau}$ be distinct elements such that $\boldsymbol{\delta}, \boldsymbol{\tau} \notin \mathrm{A}$; we shall abbreviate $\mathrm{A} \cup \{\boldsymbol{\tau}\}$ by $\mathrm{A}_{\boldsymbol{\tau}}$. The set $\mathrm{P} = \bigcup_{n\in\omega} \mathrm{P}^n$ of *processes* is obtained by means of the following recursion

$$\begin{array}{lcl} \mathrm{P}^0 & = & \mathrm{A}_{\boldsymbol{\tau}} \cup \{\boldsymbol{\delta}\} \\ \mathrm{P}^{n+1} & = & \mathrm{P}^n \cup \{\mathbf{p} \cdot \mathbf{q},\ \sum \mathrm{P}' \mid \mathbf{p}, \mathbf{q} \in \mathrm{P}^n,\ \emptyset \neq \mathrm{P}' \subseteq \mathrm{P}^n\}; \end{array}$$

we shall write $\mathbf{p} + \mathbf{q}$ for $\sum\{\mathbf{p}, \mathbf{q}\}$.

$$\mathbf{a} \xrightarrow{\mathbf{a}} \epsilon \qquad \text{for all } \mathbf{a} \in \mathrm{A}_{\boldsymbol{\tau}}$$

$$\frac{\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'}{\mathbf{p}\cdot\mathbf{q} \xrightarrow{\mathbf{a}} \mathbf{p}'\cdot\mathbf{q}} \qquad \frac{\mathbf{p} \xrightarrow{\mathbf{a}} \epsilon}{\mathbf{p}\cdot\mathbf{q} \xrightarrow{\mathbf{a}} \mathbf{q}} \qquad \mathbf{a} \in \mathrm{A}_{\boldsymbol{\tau}} \text{ and } \mathbf{p}, \mathbf{p}', \mathbf{q} \in \mathrm{P}$$

$$\frac{\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{q}}{\sum \mathrm{P}' \xrightarrow{\mathbf{a}} \mathbf{q}} \qquad \mathbf{a} \in \mathrm{A}_{\boldsymbol{\tau}},\ \mathbf{p} \in \mathrm{P}' \subseteq \mathrm{P} \text{ and } \mathbf{q} \in \mathrm{P}^{\epsilon}$$

Table 1: The transition system specification for $\mathfrak{P}$.

We shall now define on our domain of processes the four semantic equivalences that are discussed in Van Glabbeek and Weijland (1996): branching bisimulation, η-bisimulation, delay bisimulation and weak bisimulation. We need to associate with every element of P a transition system. Let ϵ be an element that does not occur in P; we define $\mathrm{P}^{\epsilon} = \mathrm{P} \cup \{\epsilon\}$. The rules in Table 1 define a *transition relation* $\longrightarrow \subseteq \mathrm{P} \times \mathrm{A}_{\boldsymbol{\tau}} \times \mathrm{P}^{\epsilon}$ on P. In the sequel, if we write $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$, then we shall tacitly assume that $\mathbf{p}$ ranges over P, while $\mathbf{p}'$ ranges over P^{ϵ}. Also, if there exists an $\mathbf{a} \in \mathrm{A}_{\boldsymbol{\tau}}$ such that $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$, then we call $\mathbf{p}'$ a *residual* of $\mathbf{p}$. Let us write $\mathbf{p}_0 \Longrightarrow \mathbf{p}_n$ to abbreviate a (possibly empty) sequence of $\boldsymbol{\tau}$-transitions

$$\mathbf{p}_0 \xrightarrow{\boldsymbol{\tau}} \mathbf{p}_1 \xrightarrow{\boldsymbol{\tau}} \dots \xrightarrow{\boldsymbol{\tau}} \mathbf{p}_n \qquad n \geq 0.$$

Definition 2.1 (Weak Bisimulation) A binary relation $\mathcal{R} \subseteq \mathrm{P}^{\epsilon} \times \mathrm{P}^{\epsilon}$ is called a *weak bisimulation relation* (a *w-bisimulation relation*) if it is symmetric and $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ implies

i. if $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$, then either $\mathbf{a} = \boldsymbol{\tau}$ and $\langle \mathbf{p}', \mathbf{q} \rangle \in \mathcal{R}$, or there exist $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}'$ such that $\mathbf{q} \Longrightarrow \mathbf{q}_1 \xrightarrow{\mathbf{a}} \mathbf{q}_2 \Longrightarrow \mathbf{q}'$ and $\langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$; and

ii. $\mathbf{p} \Longrightarrow \epsilon$ implies $\mathbf{q} \Longrightarrow \epsilon$.

We obtain the definitions of the other bisimulation relations as variations on the above definition: if we add to (i) the requirement that $\langle \mathbf{p}, \mathbf{q}_1 \rangle \in \mathcal{R}$, then we obtain the definition of an *η-bisimulation relation*; if we add to (i) the requirement that $\langle \mathbf{p}', \mathbf{q}_2 \rangle \in \mathcal{R}$, then we obtain the definition of a *delay bisimulation relation* (a *d-bisimulation relation*); and if we add to (i) the requirement that both $\langle \mathbf{p}, \mathbf{q}_1 \rangle$ and $\langle \mathbf{p}', \mathbf{q}_2 \rangle$ are members of $\mathcal{R}$, then we obtain the definition of a *branching bisimulation relation* (a *b-bisimulation relation*).

If $\mathbf{p}, \mathbf{q} \in \mathrm{P}^{\epsilon}$, and there is a weak bisimulation that contains the pair $\langle \mathbf{p}, \mathbf{q} \rangle$, then we write $\mathbf{p} \leftrightarroweq_w \mathbf{q}$; similary, we write $\mathbf{p} \leftrightarroweq_b \mathbf{q}$, $\mathbf{p} \leftrightarroweq_d \mathbf{q}$ or $\mathbf{p} \leftrightarroweq_\eta \mathbf{q}$, respectively, to refer to the existence of a branching, a delay or an η-bisimilation that contains the pair $\langle \mathbf{p}, \mathbf{q} \rangle$.

Definition 2.2 An element $\mathbf{p}$ of P we call *η-saturated* if, for all $\mathbf{a} \in \mathrm{A}_{\boldsymbol{\tau}}$, $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}^* \xrightarrow{\boldsymbol{\tau}} \mathbf{p}'$ implies $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$, and all residuals of $\mathbf{p}$ are η-saturated.

Proposition 2.3 If $\mathbf{p}$ and $\mathbf{q}$ are η-saturated, then

i. $\mathbf{p} \leftrightarroweq_\eta \mathbf{q}$ if, and only if, $\mathbf{p} \leftrightarroweq_b \mathbf{q}$; and

ii. $\mathbf{p} \leftrightarroweq_w \mathbf{q}$ if, and only if, $\mathbf{p} \leftrightarroweq_d \mathbf{q}$.

Proof.

i. Since any branching bisimulation relation is an η-bisimulation relation, the implication from right to left is immediate. Let Q be the smallest set that contains $\mathbf{p}$ and $\mathbf{q}$, and that is closed with respect to residuals (i.e., if $\mathbf{p}'$ is a residual of $\mathbf{p} \in Q$, then $\mathbf{p}' \in Q$). Suppose that $\mathbf{p} \leftrightarrows_\eta \mathbf{q}$; we shall prove that the relation

$$\mathcal{R} = \{\langle \mathbf{p}, \mathbf{q} \rangle \mid \mathbf{p}, \mathbf{q} \in Q \text{ and } \mathbf{p} \leftrightarrows_\eta \mathbf{q}\}$$

is a branching bisimulation relation. Suppose that $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ and $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$. The case where $\mathbf{a} = \boldsymbol{\tau}$ and $\mathbf{p}' \leftrightarrows_\eta \mathbf{q}$ is trivial, so suppose that there exist $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}'$ such that $\mathbf{q} \Longrightarrow \mathbf{q}_1 \xrightarrow{\mathbf{a}} \mathbf{q}_2 \Longrightarrow \mathbf{q}'$ and $\langle \mathbf{p}, \mathbf{q}_1 \rangle, \langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$. With induction on the number of $\boldsymbol{\tau}$-transitions in $\mathbf{q}_2 \Longrightarrow \mathbf{q}'$ we find, by η-saturatedness, that $\mathbf{q}_1 \xrightarrow{\mathbf{a}} \mathbf{q}'$. Hence, $\mathcal{R}$ satisfies the first requirement of the definition of branching bisimulation. It is immediate that $\mathcal{R}$ also satisfies the second requirement.

ii. The proof of this item goes in a similar fashion as that of the previous one. □

Definition 2.4 An element $\mathbf{p}$ of P we call *delay saturated* if, for all $\mathbf{a} \in \mathrm{A}_{\boldsymbol{\tau}}$, $\mathbf{p} \xrightarrow{\boldsymbol{\tau}} \mathbf{p}^* \xrightarrow{\mathbf{a}} \mathbf{p}'$ implies $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$, and all residuals of $\mathbf{p}$ are delay saturated.

Proposition 2.5 If $\mathbf{p}$ and $\mathbf{q}$ are delay saturated, then

i. $\mathbf{p} \leftrightarrows_d \mathbf{q}$ if, and only if, $\mathbf{p} \leftrightarrows_b \mathbf{q}$; and

ii. $\mathbf{p} \leftrightarrows_w \mathbf{q}$ if, and only if, $\mathbf{p} \leftrightarrows_\eta \mathbf{q}$.

Proof.

i. Since any branching bisimulation relation is a delay bisimulation relation, the implication from right to left is immediate. Let Q be the smallest set that contains $\mathbf{p}$ and $\mathbf{q}$ and is closed with respect to residuals. Suppose that $\mathbf{p} \leftrightarrows_d \mathbf{q}$; we shall prove that the relation

$$\mathcal{R} = \{\langle \mathbf{p}, \mathbf{q} \rangle \mid \mathbf{p}, \mathbf{q} \in Q \text{ and } \mathbf{p} \leftrightarrows_d \mathbf{q}\}$$

is a branching bisimulation relation. Suppose that $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ and $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$. The case where $\mathbf{a} = \boldsymbol{\tau}$ and $\mathbf{p}' \leftrightarrows_d \mathbf{q}$ is trivial, so suppose that there exist $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}'$ such that $\mathbf{q} \Longrightarrow \mathbf{q}_1 \xrightarrow{\mathbf{a}} \mathbf{q}_2 \Longrightarrow \mathbf{q}'$ and $\langle \mathbf{p}', \mathbf{q}_2 \rangle, \langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$. With induction on the number of $\boldsymbol{\tau}$-transitions in $\mathbf{q} \Longrightarrow \mathbf{q}_1$ we find, by delay saturatedness, that $\mathbf{q} \xrightarrow{\mathbf{a}} \mathbf{q}_2$. Hence, $\mathcal{R}$ satisfies the first requirement of the definition of branching bisimulation. It is immediate that $\mathcal{R}$ also satisfies the second requirement.

ii. The proof of this item goes in a similar fashion as that of the previous one. □

The four relations on P that we have just defined are equivalence relations (cf. Basten (1996) and Van Glabbeek and Weijland (1996)), but they are not congruences with respect to the operations $\cdot$ and $\sum$. The standard counterexample runs as follows: if $\mathbf{a}$ and $\mathbf{b}$ are distinct atomic actions, then $\boldsymbol{\tau} \cdot \mathbf{a} \leftrightarrows_b \mathbf{a}$, but $\boldsymbol{\tau} \cdot \mathbf{a} + \mathbf{b} \not\leftrightarrows_w \mathbf{a} + \mathbf{b}$. The following definitions enable us to define the four largest congruences that are, respectively, contained in $\leftrightarrows_b$, $\leftrightarrows_\eta$, $\leftrightarrows_d$ and $\leftrightarrows_w$.

Definition 2.6 (Rootedness) A relation $\mathcal{R}$ is

1. *w-rooted with respect to* $\mathbf{p}$ if $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ and $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$ implies that there exist $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}'$ such that $\mathbf{q} \Longrightarrow \mathbf{q}_1 \xrightarrow{\mathbf{a}} \mathbf{q}_2 \Longrightarrow \mathbf{q}'$ and $\langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$;

2. *η-rooted with respect to* $\mathbf{p}$ if $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ and $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$ implies that there exist $\mathbf{q}_1$ and $\mathbf{q}'$ such that $\mathbf{q} \Longrightarrow \mathbf{q}_1 \xrightarrow{\mathbf{a}} \mathbf{q}'$ and $\langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$; and

3. *d-rooted with respect to* $\mathbf{p}$ if $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ and $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$ implies that there exist $\mathbf{q}_1$ and $\mathbf{q}'$ such that $\mathbf{q} \xrightarrow{\mathbf{a}} \mathbf{q}_1 \Longrightarrow \mathbf{q}'$ and $\langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$;

4. *b-rooted with respect to* $\mathbf{p}$ if $\langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}$ and $\mathbf{p} \xrightarrow{\mathbf{a}} \mathbf{p}'$ implies that there exists $\mathbf{q}'$ such that $\mathbf{q} \xrightarrow{\mathbf{a}} \mathbf{q}'$ and $\langle \mathbf{p}', \mathbf{q}' \rangle \in \mathcal{R}$.

We shall write $\mathbf{p} \leftrightarroweq_{rw} \mathbf{q}$ if there exists a weak bisimulation relation that contains the pair $\langle \mathbf{p}, \mathbf{q} \rangle$ and is w-rooted with respect to both $\mathbf{p}$ and $\mathbf{q}$. Similarly, we define the rooted versions $\leftrightarroweq_{r\eta}$, $\leftrightarroweq_{rd}$ and $\leftrightarroweq_{rb}$ of $\leftrightarroweq_{\eta}$, $\leftrightarroweq_{d}$ and $\leftrightarroweq_{b}$, respectively.

Theorem 2.7 The relations $\leftrightarroweq_{r*}$, for $* \in \{w, \eta, d, b\}$, are congruences on P with respect to the operations $\cdot$ and $\sum$.

Proof. Using that $\leftrightarroweq_{*}$ is an equivalence relation, it is straightforward to prove that $\leftrightarroweq_{r*}$ is an equivalence relation. It remains to verify that $\leftrightarroweq_{r*}$ has the substitution property for $\cdot$ and $\sum$.
If $\mathbf{p}_1 \leftrightarroweq_{*} \mathbf{q}_1$ and $\mathbf{p}_2 \leftrightarroweq_{*} \mathbf{q}_2$, and $\mathcal{R}_1$ and $\mathcal{R}_2$ are the witnessing $*$-bisimulation relations, then

$$\{\langle \mathbf{p} \cdot \mathbf{p}_2, \mathbf{q} \cdot \mathbf{q}_2 \rangle, \langle \mathbf{p} \cdot \mathbf{q}_2, \mathbf{q} \cdot \mathbf{p}_2 \rangle \mid \langle \mathbf{p}, \mathbf{q} \rangle \in \mathcal{R}_1\} \cup \mathcal{R}_2$$

is a $*$-bisimulation relation that is $*$-rooted with respect to $\mathbf{p}_1 \cdot \mathbf{p}_2$ and $\mathbf{q}_1 \cdot \mathbf{q}_2$. Hence, $\mathbf{p}_1 \cdot \mathbf{p}_2 \leftrightarroweq_{r*} \mathbf{q}_1 \cdot \mathbf{q}_2$.
If $\emptyset \neq \mathrm{P}', \mathrm{P}'' \subseteq \mathrm{P}$ and $\mathrm{P}'/{\leftrightarroweq_{r*}} = \mathrm{P}''/{\leftrightarroweq_{r*}}$, then for all $\mathbf{p}' \in \mathrm{P}'$ there exists $\mathbf{p}'' \in \mathrm{P}''$ and a $*$-bisimulation relation $\mathcal{R}_{\mathbf{p}'}$ that is $*$-rooted with respect to $\mathbf{p}'$ and $\mathbf{p}''$ and contains the pair $\langle \mathbf{p}', \mathbf{p}'' \rangle$, and for all $\mathbf{p}'' \in \mathrm{P}''$ there exists $\mathbf{p}' \in \mathrm{P}'$ and a $*$-bisimulation relation $\mathcal{R}_{\mathbf{p}''}$ that is $*$-rooted with respect to $\mathbf{p}'$ and $\mathbf{p}''$ and contains the pair $\langle \mathbf{p}'', \mathbf{p}' \rangle$. Arbitrary unions of $*$-bisimulation relations are $*$-bisimulation relations. Hence, the relation

$$\{\langle \textstyle\sum \mathrm{P}', \sum \mathrm{P}'' \rangle, \langle \sum \mathrm{P}'', \sum \mathrm{P}' \rangle\} \cup \bigcup \{\mathcal{R}_{\mathbf{p}'} \mid \mathbf{p}' \in \mathrm{P}'\} \cup \bigcup \{\mathcal{R}_{\mathbf{p}''} \mid \mathbf{p}'' \in \mathrm{P}''\}$$

is a $*$-bisimulation relation that is $*$-rooted with respect to $\sum \mathrm{P}'$ and $\sum \mathrm{P}''$. We conclude that $\leftrightarroweq_{r*}$ is a congruence on P with respect to the operations $\cdot$ and $\sum$. □

With respect to η-saturated processes, η-rootedness coincides with b-rootedness and w-rootedness coincides with d-rootedness; and with respect to d-saturated processes, d-rootedness coincides with b-rootedness and w-rootedness coincides with η-rootedness. Hence, we easily find the following corollaries to Propositions 2.3 and 2.5.

Corollary 2.8 If $\mathbf{p}$ and $\mathbf{q}$ are η-saturated, then $\mathbf{p} \leftrightarroweq_{r\eta} \mathbf{q}$ iff $\mathbf{p} \leftrightarroweq_{rb} \mathbf{q}$.

Corollary 2.9 If $\mathbf{p}$ and $\mathbf{q}$ are delay saturated, then $\mathbf{p} \leftrightarroweq_{rd} \mathbf{q}$ iff $\mathbf{p} \leftrightarroweq_{rb} \mathbf{q}$.

Corollary 2.10 If $\mathbf{p}$ and $\mathbf{q}$ are both η- and delay saturated, then $\mathbf{p} \leftrightarroweq_{rw} \mathbf{q}$ iff $\mathbf{p} \leftrightarroweq_{rb} \mathbf{q}$.

3. Ground Complete Axiomatisations

Groote and Luttik (1998b) presented the equational theory pCRL to reason formally about processes and data. They showed that, provided that the data has built-in equality and built-in Skolem functions, their proof system is complete for strong bisimulation. They also showed in a later work (Groote and Luttik (1998a)) that the addition of the laws of Van Glabbeek and Weijland (1996) suffices to find a complete proof system for branching bisimulation. In this section we shall employ this latter result to show that it suffices to add Milner's τ-laws to the proof system for pCRL to find a system that is complete for weak bisimulation. We shall obtain complete proof systems for η-bisimulation and delay bisimulation as intermediate results.

We assume given an ω-complete many-sorted equational specification $\mathsf{D} = \langle \Delta, \mathrm{E} \rangle$ (where E is a set of Δ-equations) of the data part $\mathfrak{D}$ of $\mathfrak{P}$. That is, $\mathsf{D} \vdash s \approx t$ if, and only if, $\mathfrak{D} \models s \approx t$, for any Δ-terms s and t that may involve elements from some presupposed infinite set $\mathcal{V}$ of data variables (formally, this would be family of infinite sets of variables, indexed by the sorts in Δ). Let us denote by Π the signature that is the extension of Δ with the sort $\mathbb{p}$, the action declarations in $\mathcal{A}$, and

1. constants δ and τ of sort $\mathbb{p}$;

(A1)	$x + y$	$\approx y + x$	(A6)	$x + \delta$	$\approx x$
(A2)	$x + (y + z)$	$\approx (x + y) + z$	(A7)	$\delta \cdot x$	$\approx \delta$
(A3)	$x + x$	$\approx x$			
(A4)	$(x + y) \cdot z$	$\approx x \cdot z + y \cdot z$			
(A5)	$x \cdot (y \cdot z)$	$\approx (x \cdot y) \cdot z$			

Table 2: The axioms for process algebras with deadlock.

2. binary function declarations $(\_ + \_)\colon \mathbb{p}\mathbb{p} \to \mathbb{p}$ and $(\_ \cdot \_)\colon \mathbb{p}\mathbb{p} \to \mathbb{p}$;
3. a ternary function declaration $(\_ \lhd \_ \rhd \_)\colon \mathbb{p}\mathbb{b}\mathbb{b} \to \mathbb{p}$; and
4. a binder declaration $\sum : \mathbb{p}$.

We call Π a pCRL-signature. The constant δ and the binary operations $+$ and $\cdot$ are from the theory ACP (Bergstra and Klop (1984)); $+$ stands for alternative composition, and $\cdot$ stands for sequential composition. In Table 2 we have listed the axioms of ACP that concern these operations. The conditional $(\_ \lhd \_ \rhd \_)$ (read $p \lhd c \rhd q$ as "if c then p else q") and the binder $\sum$ to quantify over data stem from μCRL (Groote and Ponse (1994)), an extension of ACP that allows reasoning about the combination of processes and data. Terms over Π will be considered modulo α-conversion. Π-terms that do not contain variables of sort $\mathbb{p}$, we shall call $\mathbb{p}$-*ground*.

In Table 3 we have listed the axioms for the conditional and alternative quantification over data. The schemes Sum3 and Sum4 define an axiom for every instantiation of p and q with Π-terms of sort $\mathbb{p}$ in variables from $\mathcal{V}$. The scheme Sum1 defines an axiom for every instantiation of p with a pCRL-term in which the variable v does not occur freely; similar remarks can be made about the schemes Sum5 and Sum12. We assume that the data algebra has a built-in equality predicate for every sort $s \in \Delta$, i.e., a binary operation $[\_ =_s \_]$ such that, for all elements d and d' of sort s of $\mathfrak{D}$, $[d =_s d] = \mathsf{T}$, and $[d =_s d'] = \bot$ if $d \neq d'$. The scheme Ae defines an axiom for every action declaration $\mathsf{a}\colon s_1 \cdots s_n \to \mathbb{p}$ in $\mathcal{A}$; $\bar{u}$ and $\bar{w}$ refer to lists of n variables $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$, such that u_i and w_i are of sort s_i.

The constants δ and τ and the function declarations $+$ and $\cdot$ are interpreted as their boldface variants, defined in the previous section, in the algebra $\mathfrak{P}$. Suppose that α is a *valuation*, a total function from $\mathcal{V}$ to the elements of $\mathfrak{D}$. We extend α to a homomorphism $\bar{\alpha}$ of the algebra of Π-terms to elements of $\mathfrak{P}$, with the following interpretation for the conditional construct and the binder $\sum$:

$$\begin{aligned} \bar{\alpha}(p \lhd b \rhd q) &= \begin{cases} \bar{\alpha}(p) & \text{if } \bar{\alpha}(b) = \mathsf{T}; \\ \bar{\alpha}(q) & \text{if } \bar{\alpha}(b) = \bot; \text{ and} \end{cases} \\ \bar{\alpha}(\textstyle\sum_{x:s} p) &= \textstyle\sum \{\bar{\alpha}_{[x:=d]}(p) \mid d \text{ an element of } \mathfrak{D} \text{ of sort } s\}, \end{aligned}$$

where $\bar{\alpha}_{[x:=d]}$ refers to the homomorphic extension of the valuation $\alpha_{[x:=d]}$ (which is obtained from α by sending x to d).

In view of Theorem 2.7 of the previous section, the relations $\leftrightarrows_{r*}$, for $* \in \{w, \eta, d, b\}$ are congruences with respect to all the operations induced on $\mathfrak{P}$ by the function declarations in Π; we denote the quotients of $\mathfrak{P}$ over these congruences by $\mathfrak{P}/{\leftrightarrows_{r*}}$.

To reason formally with the axioms of Tables 2 and 3, we use a system of equational logic, extended with a congruence rule for binders (see Groote and Luttik (1998b)). In the setting of pCRL, this rules takes the form

$$\frac{p \approx q}{\sum_{x:s} p \approx \sum_{x:s} q};$$

(Cond1)	$x \lhd \top \rhd y$	$\approx x$
(Cond2)	$x \lhd \bot \rhd y$	$\approx y$
(Cond3)	$x \lhd b \rhd y$	$\approx x \lhd b \rhd \delta + y \lhd \neg b \rhd \delta$
(Cond4)	$(x \lhd b_1 \rhd \delta) \lhd b_2 \rhd \delta$	$\approx x \lhd b_1 \wedge b_2 \rhd \delta$
(Cond5)	$(x \lhd b_1 \rhd \delta) + (x \lhd b_2 \rhd \delta)$	$\approx x \lhd b_1 \vee b_2 \rhd \delta$
(Cond6)	$(x \lhd b \rhd \delta)y$	$\approx xy \lhd b \rhd \delta$
(Cond7)	$(x + y) \lhd b \rhd \delta$	$\approx x \lhd b \rhd \delta + y \lhd b \rhd \delta$
(Sca)	$(x \lhd b \rhd \delta)(y \lhd b \rhd \delta)$	$\approx (xy \lhd b \rhd \delta)$

(Ae) $\mathsf{a}(\bar{u}) \lhd [\bar{u} = \bar{w}] \rhd \delta \approx \mathsf{a}(\bar{w}) \lhd [\bar{u} = \bar{w}] \rhd \delta$

(Sum1)	$\sum_v p$	$\approx p$	if $v \notin FV(p)$
(Sum3)	$\sum_v p$	$\approx \sum_v p + p$	
(Sum4)	$\sum_v (p + q)$	$\approx \sum_v p + \sum_v q$	
(Sum5)	$(\sum_v p)q$	$\approx \sum_v pq$	if $v \notin FV(q)$
(Sum12)	$(\sum_v p) \lhd c \rhd \delta$	$\approx \sum_v p \lhd c \rhd \delta$	if $v \notin FV(c)$

Table 3: The axioms of pCRL for conditionals and alternative quantification over data.

we shall write $p\mathsf{CRL}(\mathsf{D}, \mathcal{A}) \vdash t_1 \approx t_2$ if the equation $t_1 \approx t_2$ is derivable from the axioms listed in Tables 2 and 3, and the axioms of D by means of equational logic thus extended.

For notational convenience, we shall make use of a unit ϵ for $\cdot$. We stress that it is only used to make our presentation here shorter, and that it is not an element of the signature.

Process terms may be thought of as having the form defined below.

Definition 3.1 Let A be the set of action terms, and let BT be the set of boolean terms.

We inductively define the set of *basic terms* as follows:

1. δ is a basic term;
2. if p is a basic term or $p = \epsilon$, then $\sum_{\bar{x}:\bar{s}} a \cdot p \lhd b \rhd \delta$ (with $a \in A \cup \{\tau\}$ and $b \in BT$) is a basic term; and
3. if p and q are basic terms, then $p + q$ is a basic term.

Lemma 3.2 (Basic Term Lemma) For every $\mathbb{p}$-ground process term p there exists a basic term q such that $p\mathsf{CRL}(\mathsf{D}, \mathcal{A}) \vdash p \approx q$.

Proof. Straightforward by induction on the number of symbols in p. □

(T1)	$x\tau$	$\approx x$
(T2)	τx	$\approx \tau x + x$
(T3)	$x(\tau y + z)$	$\approx x(\tau y + z) + xy$
(B)	$x(\tau(y + z) + y)$	$\approx x(y + z)$

Table 4: Milner's three τ-laws and the branching bisimulation law of Van Glabbeek and Weijland (1996).

In Table 4 we have listed Milner's τ-laws and the law for branching bisimulation of Van Glabbeek and Weijland (1996). Groote and Luttik (1998b) have proved the following completeness theorem:

Theorem 3.3 If $\mathfrak{D}$ has built-in equality and built-in Skolem functions, then for all $\mathbb{p}$-ground process terms p and q, $p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T1}, \mathrm{B} \vdash p \approx q$ iff $\mathfrak{P}/\underline{\leftrightarrow}_{rb} \models p \approx q$.

The branching bisimulation law is derivable in the system $p\mathsf{CRL}+\mathrm{T1}, \mathrm{T2}$.

Proposition 3.4 $p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T1}, \mathrm{T2} \vdash \mathrm{B}$.

Proof. We have the following derivation

$$\begin{aligned} x(\tau(y+z)+y) &\approx x(\tau(y+z)+y+z+y) && \text{(by T2)} \\ &\approx x(\tau(y+z)+y+z) && \text{(by A3)} \\ &\approx x(\tau(y+z)) && \text{(by T2)} \\ &\approx x(y+z) && \text{(by T1)} \end{aligned}$$

□

We call a $\mathbb{p}$-ground process term p *delay saturated* if $\bar{\alpha}(p)$ is delay saturated, for all valuations α of $\mathcal{V}$ in $\mathfrak{D}$.

Theorem 3.5 For every $\mathbb{p}$-ground process term p there exists a delay saturated basic term q such that $p\mathsf{CRL}+\mathrm{T2} \vdash p \approx q$.

Proof. We may assume by Lemma 3.2 that p is a basic term; we prove the theorem by induction on the structure of p.
If $p = \delta$, then p is delay saturated.
Suppose that $p = \sum_{\bar{x}:\bar{s}} ap' \lhd b \rhd \delta$. If $p' = \epsilon$, then p is delay saturated, so suppose that p' is a delay saturated basic term. If $a \neq \tau$, then p is delay saturated. If $a = \tau$, then we derive

$$\begin{aligned} p = \textstyle\sum_{\bar{x}:\bar{s}} \tau p' \lhd b \rhd \delta &\approx \textstyle\sum_{\bar{x}:\bar{s}} (\tau p' + p') \lhd b \rhd \delta && \text{(by T2)} \\ &\approx \textstyle\sum_{\bar{x}:\bar{s}} \tau p' \lhd b \rhd \delta + \sum_{\bar{x}:\bar{s}} p' \lhd b \rhd \delta && \text{(by Cond7 and Sum4),} \end{aligned}$$

and this latter term is delay saturated. Since the alternative composition of delay saturated basic terms is delay saturated, the proof is complete. □

As an immediate consequence of this theorem we find that $p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T1}, \mathrm{T2}$ is a ground complete axiomatisation of $\mathfrak{P}/\underline{\leftrightarrow}_{rd}$.

Corollary 3.6 If $\mathfrak{D}$ has built-in equality and built-in Skolem functions, then

$$p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T1}, \mathrm{T2} \vdash p \approx q \text{ if, and only if, } \mathfrak{P}/\underline{\leftrightarrow}_{rd} \models p \approx q,$$

for all $\mathbb{p}$-ground process terms p and q.

Proof. It is straightforward to verify that the axiom T2 is valid in the algebra $\mathfrak{P}/\underline{\leftrightarrow}_{rd}$, hence the implication from left to right follows from Theorem 3.3. The proof of the converse is as follows. By the previous theorem we may assume that p and q are delay saturated, so, by Corollary 2.9, if $\mathfrak{P}/\underline{\leftrightarrow}_{rd} \models p \approx q$, then $\mathfrak{P}/\underline{\leftrightarrow}_{rb} \models p \approx q$, whence, by Theorem 3.3, $p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T1}, \mathrm{B} \vdash p \approx q$, and, using Proposition 3.4, $p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T1}, \mathrm{T2} \vdash p \approx q$. □

We call a $\mathbb{p}$-ground process term p *η-saturated* if $\bar{\alpha}(p)$ is η-saturated, for all valuations α of $\mathcal{V}$ in $\mathfrak{D}$. We shall prove that any process term is provably equal to an η-saturated basic term, using axioms T3 and B. We need the following generalised form of T3.

Lemma 3.7 If $q = \sum_{\bar{x}:\bar{s}} \tau q' \lhd b \rhd \delta$ is a basic term, then

$$p\mathsf{CRL}(\mathsf{D}, \mathcal{A})+\mathrm{T3} \vdash p(q+r) \approx p(q+r) + \textstyle\sum_{\bar{x}:\bar{s}} pq' \lhd b \rhd \delta.$$

Proof. Suppose that

$$p(q+r) \lhd b \rhd \delta \approx p(q+r) \lhd b \rhd \delta + pq' \lhd b \rhd \delta; \qquad (*)$$

then the lemma follows:

$$\begin{aligned} p(q+r) &\approx \textstyle\sum_{\bar{x}:\bar{s}} p(q+r) && \text{(by SUM1)}\\ &\approx \textstyle\sum_{\bar{x}:\bar{s}} (p(q+r) \lhd b \rhd \delta + p(q+r) \lhd \neg b \rhd \delta) && \text{(by COND1 and COND5)}\\ &\approx \textstyle\sum_{\bar{x}:\bar{s}} (p(q+r) + pq' \lhd b \rhd \delta) && \text{(by *, COND1 and COND5)}\\ &\approx p(q+r) + \textstyle\sum_{\bar{x}:\bar{s}} (pq' \lhd b \rhd \delta) && \text{(by SUM4 and SUM1)} \end{aligned}$$

(note that the application of SUM1 is allowed, since we may assume, by α-conversion, that $\{\bar{x}\} \cap (FV(p) \cup FV(r)) = \emptyset$). We prove (*) by means of the following derivation:

$$\begin{aligned} p(q+r) \lhd b \rhd \delta &\approx p(q + \tau q' \lhd b \rhd \delta + r) \lhd b \rhd \delta && \text{(by SUM3)}\\ &\approx p(\tau q' + q + r) \lhd b \rhd \delta && \text{(by SCA, COND5 and COND4)}\\ &\approx (p(\tau q' + q + r) + pq') \lhd b \rhd \delta && \text{(by T3)}\\ &\approx p(q+r) \lhd b \rhd \delta + pq' \lhd b \rhd \delta && \text{(by COND7).} \end{aligned}$$

□

THEOREM 3.8 For every $\mathbb{p}$-ground process term p there exists an η-saturated basic term q such that $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T3 $\vdash p \approx q$.

PROOF. By Lemma 3.2 it suffices to prove the theorem for basic terms; we proceed by induction on their structure.

If $p = \delta$, then p is η-saturated.

Suppose that $p = \sum_{\bar{x}:\bar{s}} ap' \lhd b \rhd \delta$. If $p' = \epsilon$, then p is η-saturated. If p' is an η-saturated basic term, then there exist disjoint finite sets I and J such that

1. $p' = \sum_{i \in I} p_i + \sum_{j \in J} p_j$;
2. $p_i = \sum_{\bar{x}_i:\bar{s}_i} \tau p_i' \lhd b_i \rhd \delta$, for all $i \in I$; and
3. $p_j = \sum_{\bar{x}_j:\bar{s}_j} a_j p_j' \lhd b_j \rhd \delta$, with $a_j \neq \tau$, for all $j \in J$.

We may assume by α-conversion that the variables in $\bar{x}$ are all different from the variables in the $\bar{x}_i$, for all $i \in I$, so with Lemma 3.7, COND7, SUM12, COND4 and SUM4, we derive that

$$p \approx p + \sum_{i \in I} \textstyle\sum_{\bar{x}:\bar{s}} \sum_{\bar{x}_i:\bar{s}_i} ap_i' \lhd b \wedge b_i \rhd \delta.$$

The righthand side of this equation is an η-saturated basic term.

Since the alternative composition of η-saturated basic terms is η-saturated, this completes the proof.

□

Consequently, $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T1, T3, B is a ground complete axiomatisation of $\mathfrak{P}/\underline{\leftrightarrow}_{r\eta}$.

COROLLARY 3.9 If $\mathfrak{D}$ has built-in equality and built-in Skolem functions, then

$$p\mathsf{CRL}(\mathsf{D},\mathcal{A})\text{+T1, T3, B} \vdash p \approx q \text{ if, and only if, } \mathfrak{P}/\underline{\leftrightarrow}_{r\eta} \models p \approx q,$$

for all $\mathbb{p}$-ground process terms p and q.

PROOF. It is straightforward to verify that the axiom T3 is valid in the algebra $\mathfrak{P}/\underline{\leftrightarrow}_{rd}$, hence the implication from left to right follows from Theorem 3.3. The proof of the converse is as follows. By the previous theorem we may assume that p and q are η-saturated, so, by Corollary 2.8, if $\mathfrak{P}/\underline{\leftrightarrow}_{r\eta} \models p \approx q$, then $\mathfrak{P}/\underline{\leftrightarrow}_{rb} \models p \approx q$, whence, by Theorem 3.3, $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T1, B $\vdash p \approx q$; and $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T1, T2 $\vdash p \approx q$. □

We shall now prove our main result: $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T1–T3 is a ground complete axiomatisation of $\mathfrak{P}/\leftrightarrows_{rw}$.

THEOREM 3.10 If $\mathfrak{D}$ has built-in equality and built-in Skolem functions, then

$$p\mathsf{CRL}(\mathsf{D},\mathcal{A})\text{+T1–T3} \vdash p \approx q, \text{ if, and only if, } \mathfrak{P}/\leftrightarrows_{rw} \models p \approx q,$$

for all $\mathbb{p}$-ground process terms p and q.

PROOF. The implication from left to right is immediate by Theorems 3.3, 3.6 and 3.9; we prove the other implication. Since applications of T2 preserve η-saturatedness, we may, by Theorems 3.5 and 3.8, assume that p and q are both η- and delay saturated basic terms. Hence, by Corollary 2.10, if $\mathfrak{P}/\leftrightarrows_{rw} \models p \approx q$, then $\mathfrak{P}/\leftrightarrows_{rb} \models p \approx q$, whence, by Theorem 3.3, $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T1, B $\vdash p \approx q$; and, using Proposition 3.4, $p\mathsf{CRL}(\mathsf{D},\mathcal{A})$+T1–T3 $\vdash p \approx q$. □

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