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A Note on the Superposition of Markov Point Processes

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ABSTRACT
We show that independent superposition of Markov point processes with respect to the same neighbourhood relation preserves the Hammersley–Clifford factorisation up to second order. If the processes are identically distributed, the third order interaction structure is preserved as well. Finally, we prove that the superposition of standardised locally stable Markov point processes converges weakly to a Poisson process.

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1. Introduction

Stoyan, Kendall and Mecke [9] list the following three fundamental operations that can be performed on point patterns:

- thinning;
- clustering;
- superposition.

These operations allow the construction of new, more complex models from simpler ones, and as such are very useful in the modelling of spatial patterns.

Random thinning is the process of deleting each point in a spatial point pattern with a probability that may or may not depend on the other points in the pattern. In the simplest case, each point is deleted independently, with probability $1 - p$ for some $p \in (0, 1)$; more generally, the retention probability is a function of the location to take into account spatial inhomogeneity, or may depend on the pattern to be thinned (for instance, one may remove all points closer than a given distance to another point).

A *cluster process* is a useful model for many natural pheneomena of an aggregated or evolutionary nature. Here, the input pattern is interpreted as a collection of parent points, each giving rise to a cluster of daughter points centred around the parent. The output process is the ensemble of daughters. Note that although the terminology is biological, cluster processes arise in many contexts. For instance the well-known Neyman–Scott process was first proposed to model galaxies in space [6]. More precisely, under this model, the stars are scattered independently according to the same probability distribution around a Poisson 'parent' process.

Finally, the *superposition operator* takes two point processes and forms their union. For more information, see [9] or [4].

The simplest model for point configurations is a stationary Poisson process. If such a process is independently thinned, the result is another Poisson process, possibly inhomogeneous if the retention probability depends on the location. The superposition of two independent Poisson processes is also a Poisson process, and, as we saw above, independent clustering applied to Poisson parents yields a Neyman–Scott process.

In this paper, we shall take the class of Ripley–Kelly Markov point processes [7] as our building blocks. These are generalisations of the Poisson model allowing for local dependence between the points, and are widely used in practice [5]. The effect of independent clustering on the Markov property was investigated by Baddeley, Van Lieshout and Møller [1]. Since an independent thinning can be seen as a cluster process in which each parent has at most a single daughter, their results are valid for the thinning operator as well. It was found, that even a Neyman–Scott process with uniformly bounded clusters is not (in general) a Markov point process [1, Counterexample 1]. However, if the parent process is Markov and the associated clusters are uniformly bounded and almost surely non-empty, then the resulting cluster process satisfies a weaker, connected component Markov property [1, Theorem 2]. As for independent thinning, not even the connected component Markov property is preserved [1, Counterexample 2].

Recently, Chin and Baddeley [3] showed that the class of connected component Markov processes [2] is closed under independent superposition, hence a fortiori superposition of two Ripley–Kelly Markov processes yields a connected component process. Here, we investigate in how far the Ripley–Kelly Markov property is preserved.

The plan of this paper is as follows. In section 2, some key results from the theory of Markov point processes are reviewed. In section 3, the interaction functions of a superposition of independent Markov processes are computed. The results are used to show that the Hammersley–Clifford factorisation is preserved up to second order, and that if the processes are identically distributed, the third order interaction structure is preserved as well. Section 4 considers asymptotical results for the superposition of a large number of independent replicates. Finally, section 5 is devoted to discussion and conclusions.

2. Set-up and notation

Let X be a finite point process on a compact subset A of d-dimensional Euclidean space with non-trivial interior, so that $0 < \mu(A) < \infty$ (writing μ for Lebesgue measure). The realisations of X are finite subsets $\mathbf{x} = \{x_1, \dots, x_n\}$ $(n = 0, 1, \dots)$ of A, also called *configurations*. The class of all configurations will be denoted by $\mathcal{C}$.

In order to define a probability distribution for X, specify its density $p : \mathcal{C} \to [0, \infty)$ with respect to the distribution of a unit rate Poisson process on A.

Let $\sim$ be a symmetric relation on A. A point process X is said to be *Markov with respect to* $\sim$ if its density $p(\cdot)$ is *hereditary*, that is $p(\mathbf{x}) > 0$ implies $p(\mathbf{y}) > 0$ for all $\mathbf{y} \subseteq \mathbf{x}$, and satisfies the following Markov property. Let $\mathbf{x}$ be a configuration such that $p(\mathbf{x}) > 0$. Then for any $a \in A$, the likelihood ratio

$$\lambda(a \mid \mathbf{x}) = \frac{p(\mathbf{x} \cup \{a\})}{p(\mathbf{x})} \tag{2.1}$$

depends only on a and on $\{x_i \in \mathbf{x} : a \sim x_i\}$, the set of *neighbours* of a. The function $\lambda(\cdot \mid \cdot)$ is called the *Papangelou conditional intensity*. If $\lambda(\cdot \mid \cdot)$ is uniformly bounded, X is said to be *locally stable*.

The Hammersley–Clifford theorem [7] provides a factorisation of $p(\cdot)$ into local interaction functions. Recall that a clique is a configuration C for which all its members are neighbours, $c_i \sim c_j$ for all $c_i, c_j \in C$. By convention, the empty set and singletons are cliques. Then $p(\cdot)$ defines a Markov point process if and only if it can be written as

$$p(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{y}) \tag{2.2}$$

where $\phi(\mathbf{y}) = 1$ unless $\mathbf{y}$ is a clique.

The interested reader is referred to Ripley and Kelly [7] or Van Lieshout [5] for more details on Markov point processes.

3. Superposition

Let X_1 and X_2 be independent Markov point process with respect to some neighbourhood relation $\sim$ on A, and write $p_i(\cdot)$ for the density of X_i with respect to the distribution of a unit rate Poisson process on A (as defined in Section 2). Then the superposition $X_s = X_1 \cup X_2$ is absolutely continuous with respect to the reference Poisson process as well, with density

$$\begin{aligned} p_s(\mathbf{x}) &= e^{-\mu(A)} \sum_{\mathbf{x}_1, \mathbf{x}_2} p_1(\mathbf{x}_1)\, p_2(\mathbf{x}_2) \\ &= e^{-\mu(A)} \sum_{\mathbf{x}_1, \mathbf{x}_2} \left[\prod_{\mathbf{u} \subseteq \mathbf{x}_1} \phi_1(\mathbf{u}) \prod_{\mathbf{v} \subseteq \mathbf{x}_2} \phi_2(\mathbf{v}) \right] \end{aligned} \tag{3.1}$$

for $\mathbf{x} \in \mathcal{C}$. Here $\phi_i(\cdot)$ denote the interaction functions of X_i, $i \in \{1, 2\}$, and the sum ranges over all ordered partitions of $\mathbf{x}$ in two components $\mathbf{x}_1$ and $\mathbf{x}_2$. By Theorem 3 in Chin and Baddeley [3], the superposition density $p_s(\cdot)$ factorises into a product over terms associated with each $\sim$-connected component (cf. Baddeley and Møller [2]). Here we will show that in general $p_s(\cdot)$ fails to satisfy the Hammersley–Clifford factorisation (2.2), but that the pair-interactions vanish (as do the third order interactions if X_1 and X_2 are identically distributed).

Counterexample 1 Let $\sim$ be a symmetric neighbourhood relation on A. Let X_1 and X_2 be independent identically distributed Strauss processes [8] with density

$$p(\mathbf{x}) = \alpha \gamma^{s(\mathbf{x})},$$

where $\gamma \in (0, 1)$, and $s(\mathbf{x})$ denotes the number of neighbour pairs in $\mathbf{x}$. Suppose A is sufficiently large to allow for a configuration $\mathbf{x} = \{a, b, c, d\}$ such that $a \sim b \sim c \sim d$ are the only related points (see Figure 1). Then the Papangelou conditional intensity $\lambda_s(\cdot \mid \cdot)$ of the superposition $X_s = X_1 \cup X_2$ satisfies

$$\lambda_s(d \mid \{a,b,c\}) = \frac{2\gamma^4 + 12\gamma^2 + 2}{2\gamma^2 + 4\gamma + 2} \neq \frac{2\gamma^2 + 4\gamma + 2}{4} = \lambda_s(d; \{a,c\}),$$

hence X_s is not Markovian with respect to the given relation.

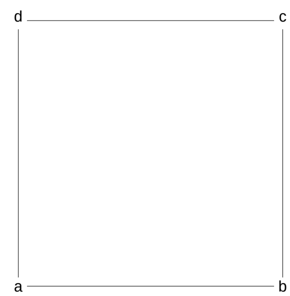

Figure 1: Neighbourhood graph on $\mathbf{x} = \{a,b,c,d\}$.

Surprisingly, one has to consider sets of four points in the above counterexample; the pair and triple 'interactions' do respect the neighbourhood relation even for a pairwise interaction model such as the Strauss process. To make this statement more precise, define 'interaction functions' recursively as follows.

Definition 1 *Let X_1 and X_2 be independent Markov point process on A with respect to the relation $\sim$, and let X_s be the superposition of X_1 and X_2. Recursively define*

$$\phi_s(\emptyset) = p_s(\emptyset),$$

and

$$\phi_s(\mathbf{x}) = \frac{p_s(\mathbf{x})}{\prod_{\mathbf{x} \neq \mathbf{y} \subset \mathbf{x}} \phi_s(\mathbf{y})} \tag{3.2}$$

for configurations $\mathbf{x} \in \mathcal{C}$ of cardinality $n(\mathbf{x}) > 1$ (setting $0/0 = 0$ if $\mathbf{x}$ is a $\sim$-clique and 1 otherwise).

Lemma 1 *Let X_1 and X_2 be independent Markov point process with respect to $\sim$ on A. Then the superposition process $X_s = X_1 \cup X_2$ is hereditary and $\phi_s(\cdot)$ is well-defined.*

Proof: Suppose $p_s(\mathbf{x})$ is strictly positive for some configuration $\mathbf{x}$. By (3.1), a partition $\mathbf{x}_1 \cup \mathbf{x}_2 = \mathbf{x}$ exists for which $p_1(\mathbf{x}_1) > 0$ and $p_2(\mathbf{x}_2) > 0$. Since X_1 and X_2 are Markov point processes, $p_i(\mathbf{x}_i \cap \mathbf{y})$, $i \in \{1, 2\}$, is strictly positive for every configuration $\mathbf{y} \subseteq \mathbf{x}$. Therefore

$$p_s(\mathbf{y}) \geq e^{-\mu(A)} p_1(\mathbf{x}_1 \cap \mathbf{y})\, p_2(\mathbf{x}_2 \cap \mathbf{y}) > 0$$

for all $\mathbf{y} \subseteq \mathbf{x}$, hence p_s is hereditary.

To show that $\phi_s(\cdot)$ is well-defined, suppose $\prod_{\mathbf{x} \neq \mathbf{y} \subset \mathbf{x}} \phi_s(\mathbf{y}) = 0$. Let $\mathbf{y}$ be a proper subset of $\mathbf{x}$ such that $\phi_s(\mathbf{y}) = 0$ and that is 'smallest' in the sense that no subconfiguration $\mathbf{y} \neq \mathbf{z} \subset \mathbf{y}$ has a vanishing interaction function $\phi(\mathbf{z}) = 0$. By (3.2), $p_s(\mathbf{y}) = 0$. Finally, as X_s is hereditary, $p_s(\mathbf{x}) = 0$ and $\phi_s(\mathbf{x})$ is well-defined. □

Theorem 1 *Let X_1 and X_2 be independent Markov point processes with respect to the same symmetric relation $\sim$ on A. Then, for configurations $\mathbf{x} \in \mathcal{C}$ containing at most two points, the Hammersley–Clifford factorisation (2.2) holds, that is*

$$p_s(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi_s(\mathbf{y})$$

where $\phi_s(\mathbf{x}) = 1$ whenever $\mathbf{x}$ is not a $\sim$-clique.

In words, Theorem 1 claims that independent superposition of Markov point processes results in a process having a similar local dependence structure up to second order.

Proof: Since both X_1 and X_2 are Markov with respect to $\sim$, $p_i(\cdot)$, $i \in \{1, 2\}$, can be written as a product of clique interaction functions which we will denote by $\phi_i(\cdot)$.

Using formula (3.1),

$$p_s(\emptyset) = e^{-\mu(A)}\, \phi_1(\emptyset)\, \phi_2(\emptyset),$$

$$p_s(\{\xi\}) = e^{-\mu(A)} \left[\phi_1(\emptyset)\, \phi_1(\{\xi\})\, \phi_2(\emptyset) + \phi_1(\emptyset)\, \phi_2(\emptyset)\, \phi_2(\{\xi\})\right] = p_s(\emptyset) \left[\phi_1(\{\xi\}) + \phi_2(\{\xi\})\right],$$

and

$$\begin{aligned} p_s(\{\xi, \eta\}) &= e^{-\mu(A)} [\phi_1(\emptyset)\, \phi_1(\{\xi\})\, \phi_1(\{\eta\})\, \phi_1(\{\xi, \eta\})\, \phi_2(\emptyset) \\ &+ \phi_1(\emptyset)\, \phi_1(\{\xi\})\, \phi_2(\emptyset)\, \phi_2(\{\eta\}) \\ &+ \phi_1(\emptyset)\, \phi_1(\{\eta\})\, \phi_2(\emptyset)\, \phi_2(\{\xi\}) \\ &+ \phi_1(\emptyset)\, \phi_2(\emptyset)\, \phi_2(\{\xi\})\, \phi_2(\{\eta\})\, \phi_2(\{\xi, \eta\})] \end{aligned} \tag{3.3}$$

for any $\xi, \eta \in A$. Substitution in (3.2) yields

$$\phi_s(\emptyset) = e^{-\mu(A)}\, \phi_1(\emptyset)\, \phi_2(\emptyset) \tag{3.4}$$

and

$$\phi_s(\{\xi\}) = \phi_1(\{\xi\}) + \phi_2(\{\xi\}), \tag{3.5}$$

using that $p_s(\emptyset) = \phi_s(\emptyset) > 0$ for the second equation. To obtain the pair interaction function, rewrite (3.3) as

$$\begin{aligned} p_s(\{\xi,\eta\}) &= \phi_s(\emptyset)\,\phi_s(\{\xi\})\,\phi_s(\{\eta\}) + \phi_s(\emptyset)\,\phi_1(\{\xi\})\,\phi_1(\{\eta\})\,(\phi_1(\{\xi,\eta\}) - 1) \\ &+ \phi_s(\emptyset)\,\phi_2(\{\xi\})\,\phi_2(\{\eta\})\,(\phi_2(\{\xi,\eta\}) - 1). \end{aligned}$$

First, consider the case that $\phi_s(\{\xi\}) = 0$, or equivalently $\phi_1(\{\xi\}) = \phi_2(\{\xi\}) = 0$. Then, $p_s(\{\xi,\eta\}) = 0$ and, by definition (3.2), $\phi_s(\{\xi,\eta\}) = 1\{\xi \not\sim \eta\}$. The same arguments apply when $\phi_s(\{\eta\}) = 0$.

Finally, turn to the case where $\phi_s(\{\xi\})$ and $\phi(\{\eta\})$ are both strictly positive. Then

$$\phi_s(\{\xi,\eta\}) = 1 + \frac{\phi_1(\{\xi\})\phi_1(\{\eta\})}{\phi_s(\xi)\phi_s(\{\eta\})}(\phi_1(\{\xi,\eta\}) - 1) + \frac{\phi_2(\{\xi\})\phi_2(\{\eta\})}{\phi_s(\xi)\phi_s(\{\eta\})}(\phi_2(\{\xi,\eta\}) - 1). \tag{3.6}$$

If $\xi \not\sim \eta$, $\phi_i(\{\xi,\eta\}) = 1$ for both $i = 1$ and $i = 2$. Consequently, $\phi_s(\{\xi,\eta\})$ reduces to 1 as well whenever $\{\xi,\eta\}$ is not a clique. □

A counterexample for triples is obtained by considering the superposition of two independent Strauss processes [8] with *different* interaction parameters. The example should be compared to Counterexample 1.

Counterexample 2 Let $\sim$ be a symmetric neighbourhood relation on A. Let X_1 and X_2 be independent Strauss processes [8], defined by their densities

$$p_i(\mathbf{x}) = \alpha_i \gamma_i^{s(\mathbf{x}_i)} \qquad i = 1, 2$$

with different interaction parameters $\gamma_1 \neq \gamma_2 \in (0,1)$. The exponent $s(\mathbf{x}_i)$ denotes the number of neighbour pairs in $\mathbf{x}_i$ $(i = 1, 2)$. Suppose A is sufficiently large to allow for a configuration $\{\xi,\eta,\zeta\}$ for which $\xi \sim \eta \sim \zeta$ but $\xi \not\sim \zeta$. Then

$$p_s(\{\xi,\eta,\zeta\}) = \phi_s(\emptyset)\left((\gamma_1+1)^2 + (\gamma_2+1)^2\right),$$

which implies

$$\phi_s(\{\xi,\eta,\zeta\}) = 2\frac{(\gamma_1+1)^2 + (\gamma_2+1)^2}{(\gamma_1+\gamma_2+2)^2}.$$

Hence $\phi_s(\{\xi,\eta,\zeta\}) \neq 1$, unless $\gamma_1 = \gamma_2$.

When the two component processes are identically distributed – in addition to interactions between pairs of points – the third order interactions vanish as well.

Theorem 2 *Let X_1 and X_2 be independent and identically distributed Markov point processes with respect to the same symmetric relation $\sim$ on A. Then, for configurations $\mathbf{x} \in \mathcal{C}$ containing at most three points, the Hammersley–Clifford factorisation (2.2) holds, that is*

$$p_s(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi_s(\mathbf{y})$$

where $\phi_s(\mathbf{x}) = 1$ whenever $\mathbf{x}$ is not a $\sim$-clique.

Proof: Write $p(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{y})$ for the factorisation of $p(\cdot)$ over cliques. The interaction functions up to second order were derived in the proof of Theorem 1. In the current context their expressions can be simplified, and we obtain

$$\begin{aligned}
\phi_s(\emptyset) &= e^{-\mu(A)}\, \phi(\emptyset)^2 \\
\phi_s(\{\xi\}) &= 2\phi(\{\xi\}) \\
\phi_s(\{\xi, \eta\}) &= \begin{cases} 1\{\xi \not\sim \eta\} & \text{if } \phi(\xi) = 0 \text{ or } \phi(\eta) = 0 \\ 1 + \frac{1}{2}(\phi(\{\xi, \eta\}) - 1) & \text{else} \end{cases}
\end{aligned}$$

for $\xi \neq \eta \in A$.

In order to compute the third order interaction function, note that for distinct $\xi, \eta, \zeta \in A$,

$$p_s(\{\xi, \eta, \zeta\}) = 2\phi_s(\emptyset)\, \phi(\{\xi\})\, \phi(\{\eta\})\, \phi(\{\zeta\})[$$

$$\phi(\{\xi, \eta\})\, \phi(\{\xi, \zeta\})\, \phi(\{\eta, \zeta\})\, \phi(\{\xi, \eta, \zeta\}) + \phi(\{\xi, \eta\} + \phi(\{\xi, \zeta\}) + \phi(\{\eta, \zeta\})].$$

Assuming $\prod_{\mathbf{y} \subset \{\xi, \eta, \zeta\}} \phi_s(\mathbf{y}) > 0$,

$$\begin{aligned}
\phi_s(\{\xi, \eta, \zeta\}) &= \frac{1}{4} \frac{\phi(\{\xi, \eta\})\, \phi(\{\xi, \zeta\})\, \phi(\{\eta, \zeta\})}{\phi_s(\{\xi, \eta\})\, \phi_s(\{\xi, \zeta\})\, \phi_s(\{\eta, \zeta\})} \phi(\{\xi, \eta, \zeta\}) \\
&+ \frac{1}{4} \frac{\phi(\{\xi, \eta\}) + \phi(\{\xi, \zeta\}) + \phi(\{\eta, \zeta\})}{\phi_s(\{\xi, \eta\})\, \phi_s(\{\xi, \zeta\})\, \phi_s(\{\eta, \zeta\})} \\
&= 1 + \frac{1}{4} \frac{\phi(\{\xi, \eta\})\, \phi(\{\xi, \zeta\})\, \phi(\{\eta, \zeta\})}{\phi_s(\{\xi, \eta\})\, \phi_s(\{\xi, \zeta\})\, \phi_s(\{\eta, \zeta\})} (\phi(\{\xi, \eta, \zeta\}) - 1) \\
&+ \frac{1}{8} \frac{(\phi(\{\xi, \eta\}) - 1)\, (\phi(\{\xi, \zeta\}) - 1)\, (\phi(\{\eta, \zeta\}) - 1)}{\phi_s(\{\xi, \eta\})\, \phi_s(\{\xi, \zeta\})\, \phi_s(\{\eta, \zeta\})}.
\end{aligned} \tag{3.7}$$

By Theorem 1, $p_s(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi_s(\mathbf{y})$ for any $\mathbf{x} \in \mathcal{C}$ with $n(\mathbf{x}) \leq 2$, and moreover the product can be restricted to $\sim$-cliques. It remains to prove a similar factorisation for $\mathbf{x} = \{\xi, \eta, \zeta\}$ where ξ, η and ζ are distinct points in A. Now, if $\{\xi, \eta, \zeta\})$ is not a clique, $\phi(\{\xi, \eta, \zeta\}) = 1$. Furthermore, there must be a pair in $\{\xi, \eta, \zeta\})$ that are not neighbours, hence at least one of $\phi(\{\xi, \eta\})$, $\phi(\{\xi, \zeta\})$, $\phi(\{\eta, \zeta\}$ must be 1. Thus, by (3.7), $\phi_s(\mathbf{x}) = 1$.

Finally, suppose $\prod_{\mathbf{x} \neq \mathbf{y} \subseteq \mathbf{x}} \phi_s(\mathbf{y}) = 0$. If a first order term equals zero, the superposition density also vanishes. If this is not the case, a second order term, say $\phi_s(\{\xi, \eta\})$ must be equal to 0. But then $\phi(\{\xi, \eta\}) = -1$, contradicting the fact that interaction functions are non-negative. In summary, the Hammersley–Clifford factorisation holds for all configurations consisting of three points, which completes the proof. □

4. Asymptotics

In this section, let us consider what happens when a large number of independent realisations X_i of a Markov point process X defined by a density $p(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \phi(\mathbf{x})$ are superposed. Write ϕ_n for the interaction function of $\cup_{i=1}^n X_i$ and assume that the first order interaction function of X is strictly positive. Then, by iterating (3.4)–(3.6), one obtains

$$\phi_n(\{\xi\}) = n\phi(\{\xi\}); \tag{4.1}$$

$$\phi_n(\{\xi,\eta\}) = 1 + \frac{1}{n}(\phi(\{\xi,\eta\}) - 1). \tag{4.2}$$

Therefore, as n tends to infinity, $\phi_n(\{\xi,\eta\})$ tends to 1. Intuitively this means that each time a new X_{n+1} is added, the intensity increases while the inter-point interactions grow weaker. The remainder of the section is devoted to making this claim more precise.

Construct a sequence $\tilde{X}^{(n)}$ of point process obtained as the superposition of n independent, identically distributed Markov point processes $\tilde{X}_{n1}, \ldots, \tilde{X}_{nn}$ with density

$$\tilde{p}_n(\mathbf{x}) = \frac{\tilde{\alpha_n}}{\alpha} \left(\frac{1}{n}\right)^{n(\mathbf{x})} p(\mathbf{x}).$$

Hence $\tilde{X}_{ni}$ has the same second and higher order interaction functions as X, but the first order interaction terms are standardised to avoid explosion (cf. 4.1). Write $N_{\tilde{X}_{n1}}(B)$ for the random variable that counts the number of points of $\tilde{X}_{n1}$ in the Borel set $B \subseteq A$. Then

$$EN_{\tilde{X}_{n1}}(B) = \int_B E\tilde{\lambda}_n(\xi \mid \tilde{X}_{n1})\, d\mu(\xi) = \frac{1}{n}\int_B E\lambda(\xi \mid X)\, d\mu(\xi) \tag{4.3}$$

where $\tilde{\lambda}_n(\xi \mid \tilde{X}_{n1})$ denotes the Papangelou conditional intensity of $\tilde{X}_{n1}$ at ξ. Thus in the limit $\tilde{X}^{(n)}$ has intensity function $E\lambda(\cdot \mid X)$.

Lemma 2 *The random variable $N_{\tilde{X}_{n1}}(A)$ is asymptotically negligable, that is*

$$\lim_{n\to\infty} P\left(N_{\tilde{X}_{n1}}(A) > 0\right) = 0.$$

Proof: Since

$$\tilde{\alpha}_n e^{-\mu(A)} = \frac{e^{-\mu(A)}}{\sum_{k=0}^{\infty} \frac{e^{-\mu(A)}}{k!}\left(\frac{1}{n}\right)^k \int \frac{p(\mathbf{x})}{\alpha}\, d\mu(\mathbf{x})} \to \frac{e^{-\mu(A)}}{e^{-\mu(A)}} = 1,$$

$P\left(N_{\tilde{X}_{n1}}(A) > 0\right) = 1 - \tilde{\alpha}_n e^{-\mu(A)}$ tends to 0 as $n \to \infty$. □

Lemma 3 *Let X be a locally stable Markov point process. Then*

$$\lim_{n\to\infty} n\,P\left(N_{\tilde{X}_{n1}}(B) \geq 2\right) = 0; \qquad \lim_{n\to\infty} n\,P\left(N_{\tilde{X}_{n1}}(B) \geq 1\right) = \int_B E\lambda(\xi \mid X)\, d\mu(\xi)$$

for every Borel set $B \subseteq A$.

Proof: For $c \in \{1,2\}$ write

$$n\,P\left(N_{\tilde{X}_{n1}}(B) \geq c\right) = n \sum_{l=c}^{\infty} \frac{1}{l!} \int_{B^l} \left\{ \sum_{k=0}^{\infty} \frac{e^{-\mu(A)}}{k!} \int_{A^k} \tilde{p}_n(\mathbf{x} \cup \mathbf{y})\, d\mu(\mathbf{y}) \right\} d\mu(\mathbf{x}).$$

Now, since X is locally stable, its Papangelou conditional intensity $\lambda(\cdot \mid \cdot)$ is uniformly bounded from above by some $L > 0$. Therefore

$$\tilde{p}_n(\mathbf{x} \cup \mathbf{y}) = \tilde{\lambda}_n(x_1 \mid (\mathbf{x} \cup \mathbf{y}) \setminus \{x_1\}) \cdots \tilde{\lambda}_n(x_l \mid \mathbf{y})\, \tilde{p}_n(\mathbf{y}) \leq \left(\frac{L}{n}\right)^l \tilde{p}_n(\mathbf{y}).$$

Hence the terms with $l \geq 2$ vanish as $n \to \infty$. Consequently $n\,P\left(N_{\tilde{X}_{n1}}(B) \geq 2\right)$ tends to 0 as $n \to \infty$. Finally,

$$\lim_{n \to \infty} n\,P\left(N_{\tilde{X}_{n1}}(B) \geq 1\right) = \int_B E\lambda(\xi \mid X)\, d\mu(\xi).$$

□

Combining the above Lemmata with theorem 9.2.V in Daley and Vere–Jones [4] yields the following limit theorem. The result is in accordance with (4.2) and (4.3).

Theorem 3 *Let X be a locally stable Markov point process with density $p(\mathbf{x})$ and $\tilde{X}_{ni}$, $i = 1, \ldots, n$, be independent, identically distributed Markov point processes with density $\tilde{p}_n(\mathbf{x}) \propto \left(\frac{1}{n}\right)^{n(\mathbf{x})} p(\mathbf{x})$. Then the superposition $\tilde{X}^{(n)} = \cup_{i=1}^{n} \tilde{X}_{ni}$ converges weakly to an (inhomogeneous) Poisson process on A with intensity $E\lambda(\xi|X)$.*

5. Conclusion

In this paper we have shown that the independent superposition of Markov point processes in general is not a Markov point process with respect to the same neighbourhood relation $\sim$. Indeed, higher order correlation is introduced, as can be seen from the fact that if X_1 and X_2 are independent identically distributed Markov pairwise interaction processes with density $p(\mathbf{x}) = \alpha\beta^{n(\mathbf{x})} \prod_{i<j} \gamma(x_i, x_j)$, the superposition interaction function for a triple $\{x_1, x_2, x_3\}$

$$\phi_s(\{x_1, x_2, x_3\}) = 1 + \frac{1}{8} \frac{(\gamma(x_1,x_2) - 1)(\gamma(x_1,x_3) - 1)(\gamma(x_2,x_3) - 1)}{(\gamma_s(x_1,x_2) - 1)(\gamma_s(x_1,x_3) - 1)(\gamma_s(x_2,x_3) - 1)}$$

is not necessarily identically 1 for cliques. Here $\gamma_s(\xi,\eta) = 1 + \frac{1}{2}(\gamma(\xi,\eta) - 1)$.

On the other hand, lower order interactions are not introduced. More specifically, suppose that a Markov density is of the form

$$p(\mathbf{x}) = \alpha \prod_{\mathbf{y} \subseteq \mathbf{x};\, n(\mathbf{y}) > k} \phi(\mathbf{y}).$$

Then $\phi_s \equiv 1$ on $\mathcal{C} \cap \{\mathbf{y} \in \mathcal{C} : 1 \leq n(\mathbf{y}) \leq k\}$.

Finally, asymptotic results were discussed, showing that in the limit all interpoint interactions disappear.

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