Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

Scale invariance and contingent claim pricing

J.K. Hoogland, C.D.D. Neumann

Modelling, Analysis and Simulation (MAS)

MAS-R9914 June 30, 1999

Report MAS-R9914
ISSN 1386-3703

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

Scaling Invariance and Contingent Claim Pricing

J.K. Hoogland and C.D.D. Neumann
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
jiri@cwi.nl *neumann@cwi.nl*

ABSTRACT

Prices of tradables can only be expressed relative to each other at any instant of time. This fundamental fact should therefore also hold for contingent claims, i.e. tradable instruments, whose prices depend on the prices of other tradables. We show that this property induces local scale-invariance in the problem of pricing contingent claims. Due to this symmetry we do *not* require any martingale techniques to arrive at the price of a claim. If the tradables are driven by Brownian motion, we find, in a natural way, that this price satisfies a PDE. Both possess a manifest gauge-invariance. A unique solution can only be given when we impose restrictions on the drifts and volatilities of the tradables, i.e. the underlying market structure. We give some examples of the application of this PDE to the pricing of claims. In the Black-Scholes world we show the equivalence of our formulation with the standard approach. It is stressed that the formulation in terms of tradables leads to a significant conceptual simplification of the pricing-problem.

1991 Mathematics Subject Classification: 90A12, 60H10, 58G11, 58G35.
Keywords and Phrases: contingent claim pricing, scale-invariance, homogeneity, partial differential equation.
Note: Work carried out under project MAS3.1 "Mathematical finance". The research of C.D.D. Neumann was partially supported by the SWON-program "Financial derivatives".

1. Introduction

The essence of trading is the exchange of goods. Every transaction sets a ratio between the value of the two goods. This means that there is no such thing as the absolute value of an object, it can only be defined relative to the value of another object. If we only have one good, we cannot assign a price to the good. We need at least two goods. Then after choosing one of these two goods, the other good can be assigned a price relative to the first one. If we have $n+1$ tradable goods we can choose any of these $n+1$ tradables to assign a price to the other ones. The tradable that is chosen to set the prices of the other tradable is often called a numeraire. In fact, we have even more freedom. We can choose any positive-definitive function of the tradables as a numeraire and express every tradable price in terms of it, e.g. money.

Thus a price is always given in terms of some unit of measurement. It is a measure-stick which is used to relate different objects. As long as everything is expressed in terms of this one unit prices can be compared. Whether we scale the unit does not matter, prices will scale accordingly. This scale-invariance is of great importance. Not only the prices of tradables which are used to set up the basic economy should scale with a change in numeraire, but any derived tradable like contingent claims, depending on other tradables, should act in the same way. This leads in a natural way to the constraint that the price of a claim as a function of the underlying tradables should be homogeneous[1] of degree 1. Otherwise the economy is not well posed.

Although Merton [Mer73] already noticed the homogeneity property for the case of a simple European warrant, it was apparently not recognized that this property should be an intrinsic property of any economy in which tradables and derivatives on these tradables have prices relative to some numeraire. More recently, Jamshidian [Jam97] discussed interest-rate models and showed that if a payoff is a homogeneous function of degree 1 in the tradables, it leads naturally to self-financing trading strategies for interest-rate contingent claims. But again it is not appreciated that the homogeneity

[1] A function $f(x_0,\dots,x_n)$ is called homogeneous of degree r if $f(ax_0,\dots,ax_n) = a^r f(x_0,\dots,x_n)$. Homogeneous functions of degree r satisfy the following property (Euler): $\sum_{\mu=0}^{n} x_\mu \frac{\partial}{\partial x_\mu} f(x_0,\dots,x_n) = r f(x_0,\dots,x_n)$

is a fundamental property, which any economy should possess to be properly defined.

To compute the price of a contingent claim [HP81] one normally starts with the definition of the stochastic dynamics of the underlying tradables. The next step is to find a self-financing trading strategy which replicates the payoff of the claim at the maturity of the contract. If the economy does not allow for arbitrage and is complete, this self-financing trading strategy gives a unique price for the claim price. To arrive at this result, one has to find a measure under which the tradables, discounted by a numeraire, are martingales. This requires a change of measure. When this change of measure exists, we have to show that the discounted payoff of the claim is a martingale under this new measure too. Then the martingale representation theorem is invoked to link the discounted payoff martingale to the underlying discounted tradables. This then gives a self-financing trading strategy using underlying tradables, which replicates the claim at all times and thus yields a price for the claim. The invariance of the choice of numeraire is reflected in the fact that the price of the claim is indeed invariant under changes of measure, which are associated with different numeraires. Geman *et.al.* [HJ95] used this invariance to show that, depending on the pricing problem at hand, it is useful to select a numeraire, which most naturally fits the payoff of the claim.

In this paper we start our discussion with the scale-invariance of a frictionless economy of tradables with prices expressed in an arbitrary numeraire. We assume the economy to be complete. Our next step is to define the stochastic dynamics of the prices of tradables. Itô then leads to a SDE for a claim-price. If the claim-price solves a certain PDE then together with the homogeneity property this leads automatically to a self-financing trading strategy replicating the claim price. If no-arbitrage constraints are imposed on the drifts and volatilities of the stochastic prices, this price is unique. The invariance under changes of numeraire becomes very transparent due to the homogeneity-property. We do not have to apply changes of measure and this leads in our view to a conceptually more satisfying and transparent contingent claim pricing argument. Finally the scale-invariance property should be satisfied also in economies which do have friction. The symmetry invokes constraints which may be useful in model-building, e.g. more general stochastic processes. We will discuss this in a forthcoming publication [HN99]. Also a more rigorous exposition of these results will be presented in this publication. In the present paper, we want to focus on the main ideas and defer the mathematical details to a later time. To the best of our knowledge this is the first time that the consequences of the scale-invariant economy for contingent-claim pricing have been outlined and discussed.

The outline of the article is as follows. In section 2 we introduce some standard notions used to price contingent claims in an economy with stochastic tradables. In subsection 2.1 we show that for an economy to be properly defined it is required to be scale-invariant. The scaling-symmetry restricts the contingent claim price: it should be a homogeneous function of the underlying tradables of degree 1. In subsection 2.2 we introduce the dynamics of the prices of tradables and introduce the notion of deterministic constraints on the dynamics, which may follow from certain choices for the drifts and volatilities of the tradables. In subsection 2.3 we use the homogeneity together with Itô to derive a PDE for the contingent claim value. The homogeneity automatically insures the existence of a self-financing trading strategy for the contingent-claim. In subsection 2.4 we show that the claim price will be unique if the constraints on the dynamics can be written as self-financing portfolios. Finally in subsection 2.5 it is shown that the symmetry is inherited by the PDE for the claim value. This allows us to pick an appropriate numeraire (fix a gauge) and solve the PDE. Section 3 gives various applications of the PDE and the scale-invariance in pricing of contingent claims. In subsection 3.1 we give the explicit formula for a European claim with log-normal prices for the underlying tradables. In subsection 3.2 it is shown that the Black-Scholes PDE is contained in our approach. In subsection 3.3 the pricing of quantos is discussed. In our formulation the pricing becomes trivial. In subsection 3.4 we show that term-structure models fit naturally into our approach and give as an example the price of a log-normal stock in a gaussian HJM model. Another example of the simple formulae is given in subsection 3.5, where we consider a trigger-swap. Finally we give our conclusions and outlook in section 4.

2. Contingent claim pricing

In the following subsections we will discuss some general properties of contingent claim pricing using dimensional analysis.

First let us recall the basic principles. We consider a frictionless market with $n+1$ tradables[2] with prices x_μ, where $\mu = 0, \ldots, n$. The prices $x \equiv \{x_\mu\}_{\mu=0}^n$ follow stochastic processes, driven by Brownian motions[3]. Time is continuous. Transaction costs are zero. Dividends are zero. Short positions in tradables are allowed. We want to value a European claim at time t promising a payoff $f(x)$ at maturity $T > t$. To attach a rational price to the claim at time t we have to find a dynamic portfolio or trading strategy $\phi \equiv \{\phi_\mu(x,t)\}_{\mu=0}^n$ of underlying tradables x with value

$$V(x,t) = \phi_\mu(x,t)x_\mu$$

which replicates the payoff of the claim at maturity, $V(x,T) = f(x)$. Let us apply Itô to the trading strategy:

$$dV = \phi_\mu dx_\mu + x_\mu d\phi_\mu + d[\phi_\mu, x_\mu]$$

Here $[\phi_\mu, x_\mu]$ stands for the covariation of the two processes. We assume that the ϕ are adapted to x, predictable, i.e. given the values of x up to time t we know the ϕ, and of bounded variation. This implies

$$d[\phi_\mu, x_\mu] = 0$$

Furthermore the trading-strategy has to be self-financing, i.e. we set up a portfolio for a certain amount of money today such that no further external cash-flows are required during the life-time of the contract to finance the payoff of the claim at maturity. All changes in the positions $\phi_\mu(x,t)$ at any given instant are financed by exchanging part of the tradables at current market prices for others such that the total cost is null:

$$x_\mu d\phi_\mu = 0$$

If we can find such a trading-strategy, then the rational value of the claim today equals the value of the trading portfolio today. If there is a non self-financing trading-strategy, the claim value at time t will not be unique. Hence arbitrage opportunities exist. Uniqueness of the claim value only follows in special cases, i.e. for specific choices of stochastic dynamics and drifts and volatilities. This will be discussed in more detail in Sec. 2.4. The self-financing property of the trading-strategy is expressed as follows.

$$dV = \phi_\mu dx_\mu$$

Finally we also have to impose the following restriction on the allowed tradingstrategies ϕ to be admissible: the value of a self-financing replicating portfolio is either deterministically zero at any time during the life of the contract or never. Otherwise arbitrage is possible. We come back to this point in Sec. 2.4.

2.1 Homogeneity

For a market to exist we need at least two tradables. Prices are always expressed in terms of a numeraire. The numeraire may be any positive-definite, possibly stochastic, function. The freedom to choose an arbitrary numeraire implies the existence of a scaling-symmetry for prices. The symmetry automatically implies the existence of a delta-hedging strategy for any tradable which depends on other underlying tradables.

---

[2] We will always use Greek symbols for indices running from 0 to n and Latin symbols for indices running from 1 to n. Furthermore, we use Einstein's summation convention: repeated indices in products are summed over.

[3] More general processes will be discussed in Ref. [HN99]

Let us consider again a market with $n+1$ basic tradables with prices x at time t. These prices are in units U of the numeraire. We say that the x have dimension U, or symbolically $[x_\mu] = U$. For the moment we leave the dynamics unspecified. What can be said about the price of a claim today, again in units of U, when expressed in terms of the tradables x? Let us denote the price of the claim by $V(x,t)$. Just on the basis of dimensional analysis we can write down the following form for the price

$$V(x,t) = \phi_\mu(x,t)x_\mu \tag{2.1}$$

Since $[V] = U$ and $[x_\mu] = U$, the functions ϕ_μ are dimensionless, $[\phi_\mu] = 1$. This implies that they can only be functions of ratios of different tradables, which are again dimensionless.

The same arguments apply to any payoff function, for else it is ill-specified. For example, the payoff-function of a vanilla call with maturity T does not seem to have this form at first sight

$$(S(T) - K)^+$$

But what is meant is the following function of a stock $S(t)$ and a discount bond $P(t,T)$, which pays 1 unit of U at time T

$$(S(T) - KP(T,T))^+$$

and this does have the right form.

Now suppose that we change our unit of measurement. If we scale the unit by a, such that $U \to U/a$, then the prices of the tradables will scale accordingly, $x_\mu \to a x_\mu$. Using the dimensional analysis result above we then find the following property for the price of the claim

$$V(ax,t) = \phi_\mu(ax,t)ax_\mu = a\phi_\mu(x,t)x_\mu = aV(x,t) \tag{2.2}$$

The price of the claim is a homogeneous function of degree 1. Note the scaling factor a may be local, $a = a(x,t)$. Differentiating Eq. 2.2 with respect to a, this immediately yields the following relation, valid for any homogeneous function[4] of degree 1,

$$V(x,t) = \frac{\partial V(x,t)}{\partial x_\mu} x_\mu \equiv V_{x_\mu}(x,t)x_\mu \tag{2.3}$$

This result is independent of the choice of dynamics. Even if we relax the frictionless market assumptions, this scaling-symmetry should not be broken.

As already mentioned various authors [Mer73, Jam97] already touched upon the homogeneity-property of certain claim prices, but they always inferred this property as a consequence of the no-arbitrage conditions they imposed on the drift and volatilities of the tradables. Furthermore their claim is that this property only holds in certain cases. In fact Jamshidian [Jam97] gives a theorem which is very similar to what we discuss in subsection 2.3, except that he doesn't recognize the fact that the required homogeneity should always be satisfied. This should be contrasted with our presentation above, where we show that this homogeneity property is one of the most fundamental properties any market model must posses to be well-posed. The homogeneity property just expresses the fact that one needs a proper coordinate-system. It could be termed: 'the relativity principle of finance'.

2.2 Dynamics: the market model

The prices of tradables, relative to a numeraire, change over time. Let us assume that the dynamics of the tradables is given by the following stochastic differential equation:

$$dx_\mu(t) = \alpha_\mu(x,t)x_\mu(t)dt + \sigma_\mu(x,t)x_\mu(t) \cdot dW(t) \tag{2.4}$$

[4] We allow generalized functions.

where we have k independent Brownian motions driving the n tradables and initial conditions[5] $x_\mu(t)$. The Brownian motion is defined under the measure with respect to the numeraire. This is often called the real-world measure in the literature. To determine a price for the claim we will *always* work under this measure. This should be contrasted with the usual approach, where one first applies a change of measure to make the tradables martingales under the new measure. Then one invokes the martingale representation theorem to determine the claim price. This change of measure is not required, as we will show later, for the determination of a rational price. In fact we do not even have to require the tradables to be strictly positive. If one of the tradables would become zero, this is allowed as long as it hits zero in a non-deterministic way. The tradable should not be used as a numeraire.

For the properties of the drift and volatilities we refer to Appendix 1. Both the LHS and RHS have dimension 1. It is convenient to extract a unit of x_μ from the drift and volatilities in Eq. 2.4 to make them dimensionless[6] Thus the only allowed form for the drift and volatility-structure are functions of the ratios of the tradables. This is a fundamental requirement for any viable and properly posed market model.

A priori it could well be that deterministic relations exist between the tradables. These relations should satisfy certain constraints in order to attach a unique rational price to a claim. If these constraints are satisfied, arbitrage is not possible. We will come back to this point in section 2.4.

2.3 Deriving the basic PDE

The results of the previous sections are precisely what is needed to obtain a PDE for the price of a contingent claim. It will be shown that the homogeneity-property, together with this PDE, is all that is necessary to obtain a unique self-financing trading-strategy in an arbitrage-free market. We do not have to make a detour using martingale techniques to prove this fact. This is a substantial conceptual simplification of the standard theory.

Let us consider the evolution of the contingent claim price $V(x,t)$ in time. Using Itô we arrive at the following SDE

$$dV = \left(V_t + \frac{1}{2}\sigma_\mu \cdot \sigma_\nu x_\mu x_\nu V_{x_\mu x_\nu}\right) dt + V_{x_\mu} dx_\mu$$

At this point the homogeneity property of $V(x,t)$ is used. Since

$$V = V_{x_\mu} x_\mu$$

we see that if the claim value solves the PDE

$$V_t + \frac{1}{2}\sigma_\mu \cdot \sigma_\nu x_\mu x_\nu V_{x_\mu x_\nu} \equiv \mathcal{L}V = 0 \tag{2.5}$$

a replicating portfolio, containing V_{x_μ} of tradable x_μ, is indeed self-financing.

$$dV = V_{x_\mu} dx_\mu$$

As usual, the payoff of the claim is specified as the boundary condition of the PDE.

Note that the drift terms did not enter the derivation of the PDE at all. We did not have to apply a change of measure to obtain an equivalent martingale measure and use the martingale representation theorem. All that is needed is the homogeneity of the contingent claim price as a function of the underlying tradables.

The PDE in Eq. 2.5 provides, in our view, the most natural formulation of the valuation of claims on tradables in a Brownian motion setting. It allows us to easily derive the classical result of Black, Scholes, and Merton (subsection 3.2), but also the results of Heath-Jarrow-Morton (subsection 3.4). Although we considered European claims up till now, it is not too difficult to include path-dependent properties. This will be discussed in Ref. [HN99].

[5] Here σ_μ and dW should be understood as k-dimensional vectors. We denote the inner product by a dot.

[6] In the literature the α_μ and σ_μ are often called *relative* drift and volatilities.

2.4 Uniqueness: No arbitrage revisited
In the previous subsection we showed that if the claim-value solves Eq. 2.5 then the replicating portfolio for the claim is self-financing. If deterministic relations between tradables exist, this is too strong a condition. In that case the constraints introduce a redundancy (gauge-freedom) in the space of tradables. This implies that we only have to solve $\mathcal{L}V = 0$ modulo the constraints. The deterministic relations between tradables allow the construction of deterministic portfolios with zero value for all times. We will call them null-portfolios. Suppose that there exist m deterministic relations

$$P_i(t) = \psi_{i,\mu}(x,t)x_\mu = 0$$

with $i = 1, \ldots, m$. We will assume for the moment that these relations are independent such that they span the null-space $\mathcal{P}$. Otherwise we can find a smaller set of independent constraints to span the null-space. We also assume that the dimension of the null-space is constant over time. Thus we can write the null-space $\mathcal{P}$ as follows.

$$\mathcal{P} = \{f_i(x,t)P_i(t)|\text{arbitrary} f_i(x,t)\}$$

where the f_i are predictable homogeneous functions of degree 0 of the prices. Taking into account the constraints we require

$$\mathcal{L}V \approx 0$$

Here we use the notation ≈ 0 to write $\mathcal{L}V = 0$ modulo elements in the null-space $\mathcal{P}$.

The null-portfolios are either self-financing or not. In the first case, the price of the claim is unique up to arbitrary null-portfolios for all times. No external cash-flows are required to keep the null-portfolio null. In the second case we can find two portfolios which replicate the payoff at maturity but whose values diverge as one moves away from maturity. There will be no unique price and arbitrage is possible.

A market will have self-financing null-portfolios if the drift and volatilities satisfy certain constraints. A null-portfolio $P = \psi_\mu x_\mu \in \mathcal{P}$ satisfies by definition

$$dP \approx 0 \tag{2.6}$$

Since the null-portfolio is by definition deterministic, this leads automatically to the following constraints on the volatilities

$$\frac{\partial P}{\partial x_\mu}\sigma_\mu x_\mu = \psi_\mu \sigma_\mu x_\mu + \frac{\partial \psi_\nu}{\partial x_\mu}\sigma_\mu x_\mu x_\nu \approx 0 \tag{2.7}$$

If a null-portfolio is self-financing, we have

$$dP = \psi_\mu dx_\mu$$

But Eq. 2.6 immediately gives

$$\psi_\mu dx_\mu \approx 0 \tag{2.8}$$

which implies

$$\begin{aligned} \psi_\mu \alpha_\mu x_\mu &\approx 0 \\ \psi_\mu \sigma_\mu x_\mu &\approx 0 \end{aligned}$$

If these constraints are satisfied for all null-portfolios, then the null-portfolios will be self-financing and hence no arbitrage is possible.

As a simple example of such constraints, let us consider two tradables $x_{1,2}$ with one Brownian motion

$$\frac{dx_{1,2}}{x_{1,2}} = \alpha_{1,2}dt + \sigma_{1,2}dW(t)$$

and constant drifts $\alpha_{1,2}$ and volatilities $\sigma_{1,2}$ and initial values $x_{1,2}(0) = 1$. Note that this is the usual setting of Black-Scholes. The SDE for the ratio x_2/x_1 then becomes

$$\frac{dx_2/x_1}{x_2/x_1} = (\alpha_2 - \alpha_1 - \sigma_1(\sigma_2 - \sigma_1))dt + (\sigma_2 - \sigma_1)dW$$

If the tradables satisfy a deterministic relation, we see that this is only possible if the volatilities are equal, $\sigma_1 = \sigma_2 \equiv \sigma$. In that case the above SDE reduces to an ODE

$$\frac{dx_2/x_1}{x_2/x_1} = (\alpha_2 - \alpha_1)dt$$

Solving the ODE, we find the following deterministic relation

$$x_2(t) = x_1(t)e^{(\alpha_2-\alpha_1)t} \tag{2.9}$$

The existence of this relation allows us to construct a null-portfolio with zero value and previsible coefficients for all times. Indeed

$$P(t) = x_2(t) - x_1(t)e^{(\alpha_2-\alpha_1)t}$$

is trivially zero. Two cases can be distinguished. The portfolio P is self-financing or it is not. Consider the evolution of P

$$dP = dx_1 - e^{(\alpha_2-\alpha_1)t}dx_2 + (\alpha_2 - \alpha_1)e^{(\alpha_2-\alpha_1)t}x_1 dt$$

It should be clear that only if $\alpha_1 = \alpha_2$ the portfolio P will be self-financing and x_1 can be hedged using x_2. Otherwise arbitrage is possible. Intuitively this should be obvious, two tradables with equal risk σ should yield the same return α.

Let us consider the consequences for the price V of a claim if $\alpha_1 \neq \alpha_2$. We construct a portfolio with constant coefficients $\psi_{1,2}$ and price process

$$P(t) = \psi_1 x_1(t) + \psi_2 x_2(t)$$

If we set

$$\psi_2 = -\psi_1 e^{(a_1-\alpha_2)T}$$

then the value of the portfolio at time T is $P(T) = 0$. However at $t < T$ we have

$$P(t) = \psi_1 x_1(t)\left(1 - e^{(a_1-\alpha_2)(T-t)}\right)$$

Since ψ_1 can take any value, the value of the contract which pays zero at time T can have any value. But this implies that we can ask any price $V(t)+P(t)$ for a claim paying $V(T)$ by adding an arbitrary portfolio with $P(T) = 0$.

2.5 Gauge invariance of the PDE

It was shown that a fundamental property of any viable market-model is the scale-invariance of the prices of tradables as expressed through the freedom of choice of the numeraire. It leads automatically to the requirement that the claim-price should be a homogeneous function of degree 1 in terms of prices of tradables. This invariance should be inherited by the dynamical equations governing the price-process for the claim. Indeed, by differentiating Eq. 2.3 again we obtain

$$x_\mu V_{x_\mu x_\nu} = 0 \tag{2.10}$$

Using this result it is a simple exercise to show that $\mathcal{L}V$ is invariant under the (simultaneous) substitutions

$$\sigma_\mu(x,t) \to \sigma_\mu(x,t) - \lambda(x,t)$$

This invariance-property represents the fact that volatility is a relative concept. It can only be measured with respect to some numeraire. Prices should not depend on this[7]. We can exploit this freedom to reduce the dimension of the problem. For example, choosing x_0 as a numeraire corresponds to taking $\lambda(x,t) = \sigma_0(x,t)$. Then

$$V_t + \frac{1}{2}(\sigma_i(x,t) - \sigma_0(x,t)) \cdot (\sigma_j(x,t) - \sigma_0(x,t)) x_i x_j V_{x_i x_j} = 0 \tag{2.11}$$

Now one can introduce

$$V(x_0, \ldots, x_n, t) = x_0 E\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}, t\right) \tag{2.12}$$

Then $E(x_1, \ldots, x_n, t)$ again satisfies Eq. 2.11. Interesting things happen when V is independent of x_0. In that case, E is homogeneous again, the $\sigma_0(x,t)$ dependence drops out, and the game can be repeated. Furthermore it should be noted, that the numeraire does not have to be a tradable. As stated earlier it may be be any positive-definite stochastic function. This freedom can be exploited to simplify calculations. Finally recall Eqs. 2.3 and 2.10. These relations give some interesting relations between the various greeks. This can be of use in numerical schemes to solve the PDE.

3. Applications

In this section we give several examples, which show the simplicity and clarity with which one derives results for contingent claim prices using the scale-invariance of the PDE.

3.1 General solution for the log-normal case

We compute the claim price for a path-independent European claim with an arbitrary number of underlying tradables, when the prices of the tradables are log-normally distributed,

$$\frac{dx_\mu}{x_\mu} = \alpha_\mu(t)dt + \sigma_\mu(t) \cdot dW(t)$$

It is easy to write the general solution for a path-independent European claim in this case. First we perform a change of variables

$$x_\mu = \exp(y_\mu)$$

such that the PDE becomes

$$V_t + \frac{1}{2}\sigma_\mu(t) \cdot \sigma_\nu(t)(V_{y_\mu y_\nu} - \delta_{\mu\nu} V_{y_\mu}) = 0$$

[7] This is called a gauge-invariance in physics' parlance and change of numeraire in finance parlance.

A Fourier transformation yields an ODE in t

$$\tilde{V}_t - \frac{1}{2}\sigma_\mu(t)\cdot\sigma_\nu(t)(\tilde{y}_\mu\tilde{y}_\nu - i\delta_{\mu\nu}\tilde{y}_\mu)\tilde{V} = 0$$

where i denotes the imaginary unit. The ODE has the solution

$$\tilde{V}(t) = \tilde{V}(T)\exp\left(-\frac{1}{2}\Sigma_{\mu\nu}(\tilde{y}_\mu\tilde{y}_\nu - i\delta_{\mu\nu}\tilde{y}_\mu)\right)$$

with

$$\Sigma_{\mu\nu} \equiv \int_t^T \sigma_\mu(u)\cdot\sigma_\nu(u)du$$

Clearly Σ is a non-negative symmetric matrix. If we perform a singular value decomposition, we get

$$\Sigma_{\mu\nu} = A_{\mu\sigma}A_{\nu\rho}B_{\sigma\rho}, \qquad B = \mathrm{diag}(\lambda_0,\ldots,\lambda_{m-1},0,\ldots)$$

where A is an orthogonal matrix and m equals the rank of Σ (so $\lambda_i > 0$ for $0 \leq i < m$). It will turn out to be convenient to introduce the matrix

$$\Theta_{\mu\nu} = \begin{cases} A_{\mu\nu}\sqrt{\lambda_\nu} & \text{for } \nu < m \\ A_{\mu\nu} & \text{otherwise} \end{cases}$$

Clearly, this matrix is invertible, $\det\Theta = \sqrt{\lambda_0\cdots\lambda_{m-1}}$, and it satisfies

$$\Sigma_{\mu\nu} = \Theta_{\mu\sigma}\Theta_{\nu\rho}\Lambda_{\sigma\rho}, \qquad \Lambda = \mathrm{diag}(\underbrace{1,\ldots,1}_{m},0,\ldots)$$

We now perform an inverse Fourier transformation on the solution of the ODE, and find

$$\begin{aligned} V(x_0,\ldots,x_n,t) &= \frac{1}{(2\pi)^{n+1}}\iint V(\exp(y_0),\ldots,T) \\ &\quad\times\exp\left(-\frac{1}{2}\Sigma_{\mu\nu}(\tilde{y}_\mu\tilde{y}_\nu - i\delta_{\mu\nu}\tilde{y}_\mu) + i\tilde{y}_\mu(y_\mu - \ln x_\mu)\right)dyd\tilde{y} \\ &= \frac{1}{(2\pi)^{n+1}}\iint V(x_0\exp(y_0 - \frac{1}{2}\Sigma_{00}),\ldots,T) \\ &\quad\times\exp\left(-\frac{1}{2}\Sigma_{\mu\nu}\tilde{y}_\mu\tilde{y}_\nu + i\tilde{y}_\mu y_\mu\right)dyd\tilde{y} \end{aligned}$$

Next we introduce new variables as follows

$$y_\mu = \Theta_{\mu\nu}z_\nu, \qquad \Theta_{\mu\nu}\tilde{y}_\mu = \tilde{z}_\nu$$

In terms of these variables, the integral becomes (note that the Jacobian of this transformation exactly equals one)

$$\frac{1}{(2\pi)^{n+1}}\iint V(x_0\exp(\Theta_{0\nu}z_\nu - \frac{1}{2}\Sigma_{00}),\ldots,T)\exp\left(-\frac{1}{2}\Lambda_{\mu\nu}\tilde{z}_\mu\tilde{z}_\nu + i\tilde{z}_\mu z_\mu\right)dzd\tilde{z}$$

The integral over the $\tilde{z}_\mu$ can be calculated explicitly. It gives rise to an m-dimensional standard normal PDF, multiplied by some δ-functions

$$\frac{1}{(2\pi)^{n+1}}\int\exp\left(-\frac{1}{2}\Lambda_{\mu\nu}\tilde{z}_\mu\tilde{z}_\nu + i\tilde{z}_\mu z_\mu\right)d\tilde{z} = \phi(z)\delta(z_m)\cdots\delta(z_n)$$

$$\phi(z) = \frac{1}{\left(\sqrt{2\pi}\right)^m} \exp\left(-\frac{1}{2}\sum_{i=0}^{m-1} z_i^2\right)$$

The integrals over z_μ for $\mu \geq m$ are now trivial. To express the result in a compact form, it is useful to introduce a set of m-dimensional vectors

$$(\theta_\mu)_i = \Theta_{\mu i}, \qquad 0 \leq i < m$$

These vectors in fact define a Cholesky-decomposition of the covariance matrix. Indeed, they satisfy

$$\theta_\mu \cdot \theta_\nu = \Sigma_{\mu\nu}$$

Here the inner product is understood to be m dimensional. Combining all, the solution becomes

$$V(x_0, \ldots, x_n, t) = \int \phi(z) V(x_0 \exp(\theta_0 \cdot z - \frac{1}{2}\theta_0 \cdot \theta_0), \ldots, T) d^m z \tag{3.1}$$

Since V is homogeneous, the result can be expressed in an even more compact form

$$V(x_0, \ldots, x_n, t) = \int V(x_0 \phi(z - \theta_0), \ldots, x_n \phi(z - \theta_n), T) d^m z$$

If the number of tradables is small we may be able to compute Eq. 3.1 analytically. Otherwise we have to use numerical techniques.

At this point let us remind the reader that it is easy to include stocks in the model with known future dividend yields. This can be done as follows. Suppose we want to price a European claim V, whose price depends on a dividend paying stock S. The dividend payments occur at times t_i, $1 \leq i \leq n$ during the lifetime of the claim. These dividends are given as a fraction δ_i of the stock-price $S(t_i)$. The effect of the dividend payments on the price of the claim can be incorporated by making the substitution

$$S(t) \to S(t) \prod_{i=1}^{n} (1 + \delta_i)^{-1}$$

in the price function of a similar claim, but depending on a non dividend paying stock. Indeed, a dividend payment at time t_i has the effect of reducing the stock-price by a factor $(1 + \delta_i)^{-1}$. For dividends paid at a continuous rate q, the substitution simply becomes

$$S(t) \to S(t) e^{-q(T-t)}$$

If dividend payments are known in terms of another tradable, e.g. a bond, the situation becomes more complicated. This is so because a dividend payment of δ_i units of a tradable P at time t_i has the effect of reducing the stock-price by a factor

$$(1 + \delta_i \frac{P(t_i)}{S(t_i)})^{-1}$$

This makes the correction factor on S path-dependent in general. We will return to this problem in Ref. [HN99].

3.2 Recovering Black-Scholes

In subsection 2.3 we derived a very general PDE for the pricing of contingent claims, when the stochastic terms are driven by Brownian motion. In this section we show that it reduces to the standard Black-Scholes equation when the underlying tradables are log-normally distributed with constant drift and volatilities. In the Black-Scholes world, we have a number of stocks S_i with SDE's

$$\frac{dS_i}{S_i} = \alpha_i dt + \sigma_i \cdot dW(t)$$

Furthermore we have a deterministic bond P, satisfying

$$\frac{dP(t,T)}{P(t,T)} = rdt$$

with $P(T,T) = 1$. For simplicity we take the interest rate and volatilities to be time-independent. It is not too difficult to extend the present discussion to the time-dependent case. In fact the solution was already computed in the previous section. Our basic equation, Eq. 2.5, gives for the price of a claim

$$V_t + \frac{1}{2}\sigma_i \cdot \sigma_j S_i S_j V_{S_i S_j} = 0$$

Note that V is explicitly a function of P. In the Black-Scholes formulation it is usually defined implicitly. This can be done by defining

$$\begin{aligned} E(S,t) &= V(P,S,t) \\ V(1,S,t) &= \frac{E(P(t)S,t)}{P(t)} \end{aligned} \tag{3.2}$$

Thus we find, as promised,

$$E_t + rS_i E_{S_i} + \frac{1}{2}\sigma_i \cdot \sigma_j S_i S_j E_{S_i S_j} - rE = 0 \tag{3.3}$$

Let us now consider a simple one-dimensional example, a European call option. The solution can be easily found using the results of the previous section.

$$\begin{aligned} V(P,S,t) &= \int (S\phi(z - \sigma\sqrt{T-t}) - KP\phi(z))^+ dz \\ &= S\Phi(d_1) - KP\Phi(d_2) \end{aligned}$$

with

$$d_{1,2} = \frac{\log \frac{S}{KP} \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

This is the well-known Merton's formula [HJ95]. The homogeneity relation, Eq. 2.3, can be used to derive relations between the greeks. In this case it is given by

$$V = SV_S + PV_P$$

Indeed, using $V_S = \Phi(d_1)$ and $V_P = -K\Phi(d_2)$, the equality follows. Since in the Black-Scholes universe P is a deterministic function of r, we have for $\rho \equiv V_r$

$$\rho = V_P P_r = -(T-t)PV_P = (T-t)(SV_S - V)$$

These type of relations were already observed in a different context in Ref. [Car93]. Furthermore, Eq. 2.10 gives the following relations

$$SV_{SS} + PV_{PS} = SV_{SP} + PV_{PP} = 0$$

Again this is easily checked by substitution of the solution V.

3.3 Quantos

Quantos are instruments which have a payoff specified in one currency and pay out in another currency. The pricing of these instruments becomes trivial, when we consider the problem using only tradables in one economy. This requires the introduction of an exchange-rate to relate the instruments denominated in one currency to ones denominated in another currency. The exchange-rate is assumed to be stochastic and driven by Brownian motion. Let us denote the exchange-rate to convert currency 2 into currency 1 by C_{12}, satisfying

$$\frac{dC_{12}}{C_{12}} = \alpha_{12}dt + \sigma_{12} \cdot dW(t)$$

The exchange-rate $C_{21} = C_{12}^{-1}$ to convert currency 1 into currency 2 then satisfies

$$\frac{dC_{21}}{C_{21}} = (-\alpha_{12} + \sigma_{12}^2)dt - \sigma_{12} \cdot dW(t)$$

Let us consider two assets, one denominated in currency 1, the other in currency 2, with the following dynamics respectively $(i = 1, 2)$,

$$\frac{dx_i}{x_i} = \alpha_i dt + \sigma_i \cdot dW(t)$$

To be able to price the instrument we need two tradables denominated in one currency. Let us define the converted prices $\tilde{x}_1 = C_{21}x_1$ and $\tilde{x}_2 = C_{12}x_2$. The converted prices give us our pairs of tradables $x_1, \tilde{x}_2$ and $\tilde{x}_1, x_2$ needed to price the instrument. The price is identical whether we work in terms of currency 1 or 2. This is a direct consequence of the scale-invariance of the problem. For consider first the case where everything is denoted in terms of currency 1. Then we arrive at the following two SDE's

$$\begin{aligned}
\frac{dx_1}{x_1} &= \alpha_1 dt + \sigma_1 \cdot dW(t) \\
\frac{d\tilde{x}_2}{\tilde{x}_2} &= (\alpha_2 + \alpha_{12} + \frac{1}{2}\sigma_2\sigma_{12})dt + (\sigma_2 + \sigma_{12}) \cdot dW(t)
\end{aligned}$$

Thus the volatilities entering in the pricing problem are σ_1 and $\tilde{\sigma}_2 \equiv \sigma_2 + \sigma_{12}$. Next consider the case where we denominate everything in terms of currency 2. The SDE's become

$$\begin{aligned}
\frac{d\tilde{x}_1}{\tilde{x}_1} &= (\alpha_1 - \alpha_{12} + \sigma_{12}^2 - \frac{1}{2}\sigma_1\sigma_{12})dt + (\sigma_1 - \sigma_{12}) \cdot dW(t) \\
\frac{dx_2}{x_2} &= \alpha_2 dt + \sigma_2 \cdot dW(t)
\end{aligned}$$

In this case, the volatilities which are relevant for the pricing problem are σ_2 and $\tilde{\sigma}_1 \equiv \sigma_1 - \sigma_{12}$. Therefore we see that the difference between calculations in the two currencies amounts to an overall shift in the volatilities by σ_{12}. But we have already seen that solutions of the PDE, Eq. 2.5, are invariant under such a translation. So we obtain a unique price function.

3.4 Heath-Jarrow-Morton

Let us consider the Heath-Jarrow-Morton framework [DA92]. The common approach is to postulate some forward rate dynamics and from there derive the prices of discount-bonds and other interest-rate instruments. But it is well-known that this model can also be formulated in terms of discount-bond prices [Car95]. Since discount bonds are tradables, this approach fits directly into our pricing formalism. Assume the following price process for the bonds[8]

$$\frac{d_t P(t,T)}{P(t,T)} = \alpha(t,T,P)dt + \sigma(t,T,P) \cdot dW(t)$$

[8] Here d_t denotes the stochastic differential w.r.t. t.

As was mentioned before, the drift and volatility functions should be homogeneous of degree zero in the bond prices in order to have a well-defined model. So they can only be functions of ratios of bond prices. In fact the precise form of the drift-terms is not of any importance in deriving the claim-price.

Let us consider as an example the price of an equity option with stochastic interest rates. We restrict our attention to Gaussian HJM models. In that case we have a bond satisfying

$$\frac{d_t P(t,T)}{P(t,T)} = \alpha(t,T)dt + \sigma(t,T)\cdot dW(t)$$

So the drift and volatility only depend on t and T. Note that this form includes both the Vasicek and the Ho-Lee model. As usual, the stock satisfies

$$\frac{dS}{S} = \alpha dt + \sigma \cdot dW(t)$$

Now choosing $P(t,T)$ as a numeraire, we find the following PDE for the price of a claim (cf. Eq. 2.11)

$$V_t + \frac{1}{2}|\sigma - \sigma(t,T)|^2 S^2 V_{SS} = 0$$

where $|v|$ denotes the length of the vector v. Using the standard techniques, this leads to the following price for a call option with maturity T and strike K

$$V(P,S,t) = S\Phi(d_1) - KP\Phi(d_2)$$

with

$$d_{1,2} = \frac{\log\frac{S}{KP} \pm \frac{1}{2}\Sigma(t,T)}{\sqrt{\Sigma(t,T)}}, \qquad \Sigma(t,T) = \int_t^T |\sigma - \sigma(u,T)|^2 du$$

Remember that both σ and $\sigma(t,T)$ are understood to be vectors. Note that in our model it is not necessary to use discount-bonds as fundamental tradables to model the interest rate market. One could equally well use other tradables such as coupon-bonds or swaps, being linear combinations of discount-bonds, or even caplets and swaptions. In our view, it seems to be less natural to model the LIBOR-rate directly, since this is not a traded object. In fact, δ-LIBOR-rates are dimensionless quantities, defined as a quotient of discount bonds

$$L(t,T) = \frac{P(t,T) - P(t,T+\delta)}{\delta P(t,T+\delta)}$$

In this respect, the name 'LIBOR market-model'[Jam97] seems a contradiction in terms.

3.5 A trigger swap

Let us now consider a somewhat more complicated example, a trigger swap. This contract depends on four tradables S_i, and it is defined by its payoff function at maturity T

$$f(S) = (S_3(T) - S_4(T))\mathbf{1}_{S_1(T)>S_2(T)}$$

Here $\mathbf{1}_A$ is the characteristic function, which is unity if A is true and zero otherwise. Note that both exchange options and binary options are special cases of this trigger swap. The former is found by setting $S_3 = S_1$ and $S_4 = S_2$, the latter by setting $S_3 = P(t,T)$ and $S_4 = 0$. Let us assume that the S_i satisfy

$$\frac{dS_i}{S_i} = \alpha_i(t)dt + \sigma_i(t)\cdot dW(t)$$

For this log-normal model, we can immediately write down the following formula for the price of the claim

$$V = \int_{S_1\phi(z-\theta_1)>S_2\phi(z-\theta_2)} (S_3\phi(z-\theta_3) - S_4\phi(z-\theta_4))dz$$

Here, the θ_i are given by a Cholesky decomposition of the integrated covariance matrix

$$\Sigma_{ij} = \int_t^T \sigma_i(u)\cdot\sigma_j(u)du = \theta_i\cdot\theta_j$$

We will omit the details of the evaluation of this integral. It is a straightforward application of the procedure described in subsection 3.1. The result can be written as

$$V(S_1,S_2,S_3,S_4,t) = S_3\Phi(d_3) - S_4\Phi(d_4)$$

where

$$d_i = \frac{\log\frac{S_1}{S_2} + \frac{1}{2}(\Sigma_{22}(t,T) - \Sigma_{11}(t,T)) + \Sigma_{1i}(t,T) - \Sigma_{2i}(t,T)}{\sqrt{\Sigma_{11}(t,T) - 2\Sigma_{12}(t,T) + \Sigma_{22}(t,T)}}$$

The reader can check that this result is again independent under gauge-transformations $\sigma_i \to \sigma_i - \lambda$, as it should be. Note that V_{S_1} and V_{S_2} are not in general equal to zero. This means that one needs a portfolio consisting of all four underlyings to hedge this claim. Now let us consider the special case of an exchange option, setting $S_3 = S_1$ and $S_4 = S_2$. In this case, the formulae reduce to

$$V(S_1,S_2,t) = S_1\Phi(d_1) - S_2\Phi(d_2)$$

where

$$d_{1,2} = \frac{\log\frac{S_1}{S_2} \pm \frac{1}{2}(\Sigma_{11}(t,T) - 2\Sigma_{12}(t,T) + \Sigma_{22}(t,T))}{\sqrt{\Sigma_{11}(t,T) - 2\Sigma_{12}(t,T) + \Sigma_{22}(t,T)}}$$

In Ref. [LW99] it is claimed that the value of an option to exchange two stocks has a dependence on the interest-rate term structure, or in other words, a dependence on bond-prices. It should be clear from the discussion above that this is in fact impossible, because neither the payoff, nor the volatility functions make any reference to bonds. Therefore, the price of such an exchange option can be calculated in a market where bonds do not even exist.

4. Conclusions and outlook

In the preceding sections we have clearly shown the advantages of a model formulated in terms of tradables only. In this formulation, the relativity of prices manifests itself as a homogeneity condition on the price of any contingent claim, and this fact can be exploited to bypass the usual martingale construction for the replicating trading-strategy. The result is a transparent general framework for the pricing of derivatives.

In this article we have restricted our attention to the problem of pricing European path-independent claims. The generalization to path-dependent and American options is straightforward and will be dealt with in other publications.

Obviously, the applicability of the scaling laws is not restricted to models with Brownian driving factors. Currently we are considering alternative driving factors such as Poisson and Levy processes. We are also looking at implications for modeling incomplete markets. Finally the scaling-symmetry should also hold in markets with friction. This may serve as an extra guidance in the modeling of transaction-costs and restrictions on short-selling.

1. Appendix: Stochastic differential equations

We use stochastic differential equations to model the dynamics of the prices $x_\mu(t)$ of tradables. The governing equation is given by

$$d_t x_\mu(t) = \alpha_\mu(x,t)x_\mu(t)dt + \sigma_\mu(x,t)x_\mu(t) \cdot dW(t)$$

with initial conditions $x_\mu(t)$ and $dW(t)$ denote k-dimensional Brownian motion with respect to some measure. The drifts $\alpha_\mu(x,t)$ and volatilities $\sigma_\mu(x,t)$ are assumed to be adapted to x and predictable. For this equation to have a unique solution, we have to require some regularity-conditions on the drift $\alpha_\mu(x,t)$ and volatility $\sigma_\mu(x,t)$. These can stated as follows [Gar85, Arn74, BS96].

- Lipschitz condition: there exists a $K > 0$ such that for all x, y and $s \in [t, T]$

$$|\alpha_\mu(x,s) - \alpha_\mu(y,s)| + |\sigma_\mu(x,s) - \sigma_\mu(y,s)| \leq K|x-y|$$

- Growth condition: there exists a K such that for all $s \in [t, T]$

$$|\alpha_\mu(x,s)|^2 + |\sigma_\mu(x,t)|^2 \leq K^2(1+|x|^2)$$

The Lipschitz condition above is global, it can in fact be weakened to a local version. If the growth condition is not satisfied, the solution may still exist up to some time t', where the solution $x_\mu(t)$ has a singularity and thus ‘explodes’.

References

[Arn74] L. Arnold. *Stochastic differential equations*. Wiley, 1974.

[BS96] A.N. Borodin and P. Salminen. *Handbook of Brownian motion – facts and formulae*. Birkhäuser, 1996.

[Car93] P. Carr. Deriving derivatives of derivative securities. *working paper*, 1993.

[Car95] A.P. Carverhill. A simplified exposition of the heath, jarrow, and morton model. *Stochastics Reports*, 53:227–240, 1995.

[DA92] D. Heath,R. Jarrow,and A. Morton. Bond pricing and the term structure of interest: a new methodology for contingent claim valuation. *Econometrica*, 60:77–105, 1992.

[Gar85] C.W. Gardiner. *Handbook of stochastic methods for physics, chemistry, and the natural sciences*. Springer, 2nd edition, 1985.

[HJ95] H. Geman,N. El Karoui, and J.-C. Rochet. Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied Probability*, 32:443–458, 1995.

[HN99] J. Hoogland and D. Neumann. *work in progress*, 1999.

[HP81] J.M. Harrison and S.R. Pliska. Martingales, stochastic integrals and continuous trading. *Stochastic processes and their applications*, 11:215–260, 1981.

[Jam97] F. Jamshidian. Libor and swap market-models and measures. *Finance & Stochastics*, 1:293–330, 1997.

[LW99] C. Liu and D.-F. Wang. Exchange options and spread options with stochastic interest rates. *submitted to IJTAF*, 1999. http://xxx.lanl.gov/cond-mat/9904110.

[Mer73] R.C. Merton. Theory of rational pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.