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Integral Representations of Affine Transformations in Phase Space with an Application to Energy Localization Problems

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ABSTRACT
Applying the fractional Fourier transform and the Wigner distribution on a signal in a cascade fashion is equivalent with a rotation of the time and frequency parameters of the Wigner distribution. This report presents a formula for all unitary operators that are related to energy preserving transformations on the parameters of the Wigner distribution by means of such a cascade of operators. Furthermore, such operators are used to solve certain type of energy localization problems via the Weyl correspondence.

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1. Introduction

For analysing signals $f \in L^2(\mathbb{R})$ one may use the Fourier transform. This transform maps a function f to a function $\hat{f}$. For a function of time (a signal), $\hat{f}$ represents the intensity of the fluctuations (frequencies) in the signal f. Analysing a signal in this way is called spectral analysis. Besides the representation in time f and the representation in frequency $\hat{f}$, there exists transformations $f \mapsto \tilde{f}$ to represent a signal both in time and in frequency. Amongst others a well-known time-frequency transformation is the Wigner distribution.

The Wigner distribution is defined by

$$\mathcal{WV}[f](x,\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x+t/2)\overline{f(x-t/2)} e^{-it\omega}\, dt, \quad \forall_{x,\omega\in\mathbb{R}} \tag{1.1}$$

The Wigner distribution is in fact the Fourier transform of the function

$$R_{f,x}(t) = f(x+t/2)\,\overline{f(x-t/2)}/\sqrt{2\pi}.$$

Consequently, the Wigner distribution is non-linear and it also represents a signal redundantly in time and frequency. Therefore, a signal can be reconstructed from its Wigner distribution, but this cannot be done in a unique way. The Wigner distribution is discussed extensively in Section 2.

A representation of a signal in a domain different from the time or frequency domain is given by the fractional Fourier transform (FRFT). This transform is given by

$$\mathcal{F}_\alpha[f](x) = \frac{C_\alpha}{\sqrt{2\pi\,|\sin\alpha|}} \int_{\mathbb{R}} f(u)\, e^{i\,((u^2+x^2)\cdot(\cot\alpha)/2\,-\,ux\csc\alpha)}\, du, \tag{1.2}$$

for some parameter $\alpha \neq k\pi$, $k \in \mathbb{Z}$ and a constant C_α, with $|C_\alpha| = 1$. This transform may seem a bit peculiar, however representation (1.2) can be derived by defining

$$\mathcal{F}_\alpha = \mathcal{F}^\alpha, \tag{1.3}$$

where $\mathcal{F}$ denotes the Fourier transform. In Section 3 we discuss the definition and properties of the FRFT. There we also show that taking the Wigner distribution of $\mathcal{F}_\alpha f$ corresponds to the Wigner distribution of the function f followed by a rotation over an angle α in the Wigner plane.

The rotation property of the FRFT inspired mathematicians in the past to study also other transformations in the Wigner plane, that correspond to linear operators on $L^2(\mathbb{R})$. However, already before the introduction of the FRFT De Bruijn proposed in [2] a class of operators that are related to linear operators in the Wigner plane. In Section 4 we study this problem for the n-dimensional Wigner distribution. Furthermore, we show that the FRFT is a special element of this class, since it is the only transformation that corresponds to an orthogonal symplectic transformation in the one-dimensional case.

We derive a classification of all unitary operators on $L^2(\mathbb{R}^n)$ that correspond to linear energy preserving transformations in the $2n$-dimensional Wigner plane. Using this classification we present a representation formula for these unitary operators. This is done in Section 5.

Sections 6 and 7 are devoted to a celebrated problem in signal analysis, namely energy localization in time and frequency. In Section 6 two well-known problems are discussed rigorously, namely maximalization of the energy of time-limited signal within a compact frequency interval and maximalization of a signal's energy within a disc in the Wigner plane. In Section 7 we show how a generalization of the FRFT can be used to solve a class of localization problems in the phase plane, if the solution of one problem in such a class is known. This procedure is illustrated by using it for the classical localization problems of Section 6.

The sequel of this introductory section is devoted to mathematical preliminaries, namely the Fourier transform and Lie group theory.

1.1 The Fourier transform

To obtain information on the frequency behaviour of a function we may consider its Fourier transform. This transform computes the frequency spectrum of a given function. We discuss the Fourier transform first for functions in $L^1(\mathbb{R})$ and subsequently for functions in $L^2(\mathbb{R})$.

For $f \in L^1(\mathbb{R})$ its Fourier transform $\hat{f}$ is given by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx. \tag{1.4}$$

Formally, an inverse Fourier transform exists and is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\omega) e^{ix\omega}\, d\omega. \tag{1.5}$$

However, convergence of the integral in (1.5) is not guaranteed. Indeed, the following example shows that $\hat{f}$ is not necessarily in $L^1(\mathbb{R})$ if $f \in L^1(\mathbb{R})$.

Example 1.1 Let $f \in L^1(\mathbb{R})$ be given by

$$f(x) = \begin{cases} \sqrt{2\pi}\, e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then its Fourier transform is given by

$$\hat{f}(\omega) = \frac{1}{1 + i\omega},$$

which is not in $L^1(\mathbb{R})$.

Following [3, 11], we present additional conditions on f and $\hat{f}$, that are necessary for a well-defined inversion formula.

Theorem 1.2 *Let $f \in L^1(I\!R)$ and $\hat{f} \in L^1(I\!R)$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{I\!R} \hat{f}(\omega) e^{ix\omega} \, d\omega \;\; a.e., \;\; x \in I\!R.$$

Moreover, the latter results holds for every $x \in I\!R$ if also $f \in C(I\!R)$.

A useful property of the Fourier transform is given by the following lemma.

Lemma 1.3 *Let $f \in L^1(I\!R)$, then $\hat{f} \in C(I\!R)$ and $\|\hat{f}\|_\infty \le \|f\|_1/\sqrt{2\pi}$.*

Proof
Let $f \in L^1(I\!R)$. Then

$$\begin{aligned} |\hat{f}(\omega_1) - \hat{f}(\omega_2)| &= \frac{1}{\sqrt{2\pi}} \left| \int\limits_{I\!R} f(x)(e^{-i\omega_1 x} - e^{-i\omega_2 x}) \, dx \right| \\ &\le \frac{1}{\sqrt{2\pi}} \int\limits_{I\!R} |f(x)| \cdot |1 - e^{i(\omega_1 - \omega_2)x}| \, dx \\ &= \sqrt{\frac{2}{\pi}} \int\limits_{I\!R} |f(x)| \cdot |\sin((\omega_1 - \omega_2)x/2)| \, dx. \end{aligned}$$

Applying the dominated convergence theorem on the latter result yields

$$|\hat{f}(\omega_1) - \hat{f}(\omega_2)| \;\; \to 0 \;\; (\omega_1 \to \omega_2),$$

which shows that $\hat{f}$ is continuous. Furthermore, we have

$$|\hat{f}(\omega)| \le \frac{1}{\sqrt{2\pi}} \int\limits_{I\!R} |f(x) e^{-i\omega x}| \, dx = \frac{1}{\sqrt{2\pi}} \int\limits_{I\!R} |f(x)| \, dx = \|f\|_1/\sqrt{2\pi}.$$

Taking the supremum over ω establishes the proof. □

In the sequel of this report we will focus ourselves on functions in $L^2(I\!R)$. Starting form the definition of the Fourier transform on $L^1(I\!R)$ the Fourier transform on $L^2(I\!R)$ can only be defined if $f \in L^1(I\!R) \cap L^2(I\!R)$. To come to a definition of the Fourier transform on $L^2(I\!R)$ we will define the Fourier transform first on a dense subspace of both $L^1(I\!R)$ and $L^2(I\!R)$ and then extend it uniquely to $L^2(I\!R)$.

A dense subspace of both $L^1(I\!R)$ and $L^2(I\!R)$ is given by the Schwartz class $S(I\!R)$, see [32, 33].

Definition 1.4 *The Schwartz class $S(I\!R^n)$ is the space of rapidly decreasing C^∞-functions on $I\!R^n$, i.e., for each $k, l \in I\!N$*

$$\sup_{|\alpha| \le k, |\beta| \le l, x \in I\!R^n} |x_1^{\beta_1} \cdots x_n^{\beta_n} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(x)| < \infty \;\; \forall_{f \in S(I\!R^n)}.$$

It can be shown that the Fourier transform $\mathcal{F}$, when restricted to $S(I\!R)$, is a bounded linear mapping on $S(I\!R)$ as a subspace of $L^2(I\!R)$. Moreover, $\mathcal{F}$ is an isometry on $S(I\!R)$, with respect to the inner product in $L^2(I\!R)$, see [11, 36]. So, we have Parseval's formula

$$(f, g)_2 = (\mathcal{F}f, \mathcal{F}g)_2,$$

with $(\cdot,\cdot)_2$ the inner product in $L^2(I\!R)$.

Since $S(I\!R)$ is dense in $L^2(I\!R)$, $\mathcal{F}$ can be uniquely extended to a Hilbert space isometry of $L^2(I\!R)$. It can be shown that this definition is equivalent with the following expression, which we shall refer to as the definition of the Fourier transform on $L^2(I\!R)$.

Definition 1.5 *Let $f \in L^2(I\!R)$. Then its Fourier transform $\hat{f} = \mathcal{F}f$ is given by*

$$\mathcal{F}[f](\omega) = \textit{l.i.m.}_{N\to\infty} \; \frac{1}{\sqrt{2\pi}} \int\limits_{-N}^{N} f(x)e^{-i\omega x}\, dx, \tag{1.6}$$

where l.i.m. stands for limit in L^2 mean.

Remark, that this definition coincides with (1.4) if $f \in L^1(I\!R) \cap L^2(I\!R)$. Also we observe that by this definition $\mathcal{F}f$ is a function, defined almost everywhere on $I\!R$ and belonging to $L^2(I\!R)$. Moreover, with this construction Parseval's formula can be extended to $L^2(I\!R)$.

$$\int\limits_{I\!R} f(x)\overline{g(x)}\, dx = \int\limits_{I\!R} \hat{f}(\omega)\overline{\hat{g}(\omega)}\, d\omega, \tag{1.7}$$

for all $f, g \in L^2(I\!R)$. As a result we also have Plancherel's formula

$$\int\limits_{I\!R} |f(x)|^2\, dx = \int\limits_{I\!R} |\hat{f}(\omega)|^2\, d\omega, \tag{1.8}$$

for all $f \in L^2(I\!R)$. The two equal sides of (1.8) give the energy of $f \in L^2(I\!R)$.

Since $\hat{f} \in L^2(I\!R)$ for $f \in I\!R$, we can derive an inversion formula using the same construction as for (1.6), i.e.,

$$f(x) = \text{l.i.m.}_{N\to\infty} \; \frac{1}{\sqrt{2\pi}} \int\limits_{-N}^{N} \hat{f}(\omega)e^{ix\omega}\, d\omega. \tag{1.9}$$

Another result on the Fourier transform that is used in the sequel of this report deals with convolution products. The following lemma presents two relations between convolution products and the Fourier transform. For a proof we refer to [46].

Lemma 1.6 *Convolution products and the Fourier transform are related by*

1. $(f*g)\hat{}\,(\omega) = \sqrt{2\pi}\,\hat{f}(\omega)\cdot\hat{g}(\omega)$, *for* $f \in L^1(I\!R) \cup L^2(I\!R)$ *and* $g \in L^1(I\!R)$,
2. $\sqrt{2\pi}\,(f\cdot g)\hat{}\,(\omega) = (\hat{f}*\hat{g})(\omega)$, *for* $f, g \in L^2(I\!R)$.

A subspace of $L^2(I\!R)$, which is of special interest in signal analysis is $L^2_{\text{comp}}(I\!R)$, i.e., the space of all functions in $L^2(I\!R)$ with compact support. Related to this space we can define two types of signals.

Definition 1.7 *A signal $f \in L^2(I\!R)$ is called time-limited if $f \in L^2_{\text{comp}}(I\!R)$. If $\hat{f} \in L^2_{\text{comp}}(I\!R)$, then f is called band-limited.*

Another special class of functions in $L^2(I\!R)$ is the class of functions of exponential type.

Definition 1.8 *A function $f \in L^2(\mathbb{R})$ is called of exponential type if it extends to a holomorphic function on $\mathbb{C}$ and if there are two positive constants C and Ω such that*

$$|f(z)| < Ce^{\Omega|\mathrm{Im}\, z|}, \;\; \forall z \in \mathbb{C}.$$

Functions of exponential type can be related to band-limited functions by means of the Paley-Wiener theorem; for a proof, see [45].

Theorem 1.9 (Paley-Wiener) *If $f \in L^2(\mathbb{R})$ is holomorphic and of exponential type, then f is band-limited. Conversely, if f is band-limited, then f is holomorphic and of exponential type.*

Since a holomorphic function $f \in L^2(\mathbb{R})$, vanishing at a certain interval, has to be identically zero, the Paley-Wiener theorem immediately yields

Corollary 1.10 *If $f \in L^2(\mathbb{R})$ is both time-limited and band-limited, then $f = 0$.*

The previous corollary states that there does not exist a non-trivial time-limited signal f, whose energy is contained within a finite interval in the frequency domain, say $[-\omega_0, \omega_0]$. In Section 6.1 we will deal with this phenomenon. There, we will consider the problem of maximizing the energy of a band-limited signal within a finite interval $[-\omega_0, \omega]$ in the frequency domain.

1.2 Lie Group Theory and the Heisenberg Group

In this section we will discuss Lie groups. In particular the Heisenberg group will be studied. In the following section we will see that this group can be related to time-frequency operators.

We start with some standard definitions on Lie group theory, that can be found in e.g. [38, 41].

Definition 1.11 *A set G having both a topological and a group structure is called a topological group if the mapping*

$$(x, y) \mapsto xy^{-1} \tag{1.10}$$

is a continuous mapping from $G \times G$ onto G. A topological group G is called a Lie group if there is a differentiable structure on G, compatible with its topology, such that G converts into a C^∞-manifold and for which the mapping (1.10) is C^∞.

Related to a Lie group G we can also look for a Lie subgroup G' defined as a Lie group that is a subgroup of the group G and a C^∞-submanifold of the C^∞-manifold G. In the following example we shall consider a well-known Lie group and some of its Lie subgroups.

Example 1.12 Consider the group $GL(n) = \{M \in \mathbb{R}^{n\times n} \mid \det M \neq 0\}$. It can be verified rather easily that $GL(n)$ is a Lie group using the fact that the mapping $M \mapsto \det M$ is continuous. Some well-known Lie subgroups of $GL(n)$ are given by

1. $SL(n) = \{M \in GL(n) \mid \det M = 1\}$,
2. $O(n) = \{M \in GL(n) \mid M^T M = I\}$,
3. $SO(n) = \{M \in O(n) \mid \det M = 1\}$.

Another example of a well-known Lie group is the Heisenberg group, which is defined as follows.

Definition 1.13 *The $2n + 1$-dimensional Heisenberg group H_n is identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the multiplication law*

$$(p_1, q_1, t_1)\,(p_2, q_2, t_2) = (p_1 + p_2, q_1 + q_2, t_1 + t_2 + ((q_1, p_2) - (p_1, q_2))/2). \tag{1.11}$$

To relate a topological group to an operator on a separable Hilbert space, we use the concept of topological group representations.

Definition 1.14 *Let G be a topological group, H be a Hilbert space and $B(H)$ be the space of all bounded operators on H. Then a representation of G in H is a mapping $\mu : G \to B(H)$ for which*

1. $\mu(x)\mu(y) = \mu(xy)$, *for all* $x, y \in G$,
2. $\mu(e) = \mathcal{I}$, *with* e *the identity of* G *and* $\mathcal{I}$ *the identity operator on* $\mathcal{H}$,
3. $x \mapsto \mu(x)f$ *is a continuous mapping from* G *to* H, *for all* $f \in H$.

Note, that Definition 1.14 yields that μ is a group homomorphism, which is continuous in the strong operator topology of $B(H)$.

Topological group representations may satisfy several important properties. A first desirable property of a representation is that it is unitary, i.e. $\mu(x) \in U(H)$, for all $x \in G$, where $U(H)$ denotes the space of all unitary operators on H. Furthermore, μ is said to be irreducible if $\{0\}$ and H are the only closed subspaces of H that are invariant under the action of $\mu(x)$, for all $x \in G$. A last property concerns the equivalence of two representations. A representation μ is said to be equivalent with a representation $\rho : G \to B(H)$ if there exists an operator $\mathcal{V} \in U(H)$, such that

$$\rho(x) = \mathcal{V}^* \mu(x) \mathcal{V} \;\; \forall_{x \in G}. \tag{1.12}$$

Note that a unitary representation μ is a group homomorphism, which is continuous in the strong operator topology of $U(H)$. Also we observe, that for unitary representations it can be proved, see e.g. [16], that μ is irreducible if and only if for $\rho = \mu$, (1.12) only holds for $V = C\mathcal{I}$, with $|C| = 1$.

An irreducible unitary representation of H_n in the space $L^2(\mathbb{R}^n)$ is given by the Schrödinger representation

$$\mu(p, q, t)[f](x) = e^{i(p,x)} e^{i(t+(p,q)/2)} f(x + q). \tag{1.13}$$

In the sequel of this section the representation μ will denote the Schrödinger representation.

2. The Wigner Distribution

A well-known representation of a signal f in both time and frequency is the Wigner distribution. This is a quadratic time-frequency representation given by

$$\mathcal{WV}[f](x, \omega) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} f(x + t/2)\overline{f(x - t/2)} e^{-it\omega}\, dt, \tag{2.1}$$

for all $f \in L^2(\mathbb{R})$. In the sequel we will refer to the domain of the Wigner distribution as the Wigner plane.

This representation was already introduced in 1932 by Wigner in his paper [43]. He presented this representation in the field of quantum mechanics. In 1948, Ville introduced the representation in the fields of signal analysis in [39]. Therefore, this representation is also known in the literature as the Wigner-Ville distribution.

Later in this report we will also use the mixed Wigner distribution given by

$$\mathcal{WV}[f, g](x, \omega) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} f(x + t/2)\overline{g(x - t/2)} e^{-it\omega}\, dt, \tag{2.2}$$

for all $f, g \in L^2(\mathbb{R})$. Obviously, this representation coincides with the Wigner distribution if $f = g$.

The n-dimensional Wigner distribution is defined from a straightforward generalization of (2.1) by

$$\mathcal{WV}[f](x,\omega) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x+t/2)\overline{f(x-t/2)}e^{-i(t,\omega)}\,dt, \tag{2.3}$$

for all $f \in L^2(\mathbb{R}^n)$ and with $(\cdot,\cdot)$ the inner product in $\mathbb{R}^n$. For simplicity we only discuss properties of the Wigner distribution for $f \in L^2(\mathbb{R})$. Generalizations of these results for functions in $L^2(\mathbb{R}^n)$ can be made in a rather direct way.

The Wigner distribution is invariant under the action of both translation $\mathcal{T}_b$ and frequency modulation $\mathcal{M}_{\omega_0}$, given by

$$\mathcal{T}_b[f](x) = f(x-b) \tag{2.4}$$
$$\mathcal{M}_{\omega_0}[f](x) = e^{i\omega_0 x}f(x), \tag{2.5}$$

for $b \in \mathbb{R}$ and $\omega_0 \in \mathbb{R}$. A straightforward calculation shows

$$\mathcal{WV}[\mathcal{T}_b f](x,\omega) = \mathcal{WV}[f](x-b,\omega) \;\text{ and }\; \mathcal{WV}[\mathcal{M}_{\omega_0} f](x,\omega) = \mathcal{WV}[f](x,\omega-\omega_0).$$

Furthermore, by a change of variables in (2.1) it follows immediately that the Wigner distribution is real-valued, i.e., $\overline{\mathcal{WV}[f]} = \mathcal{WV}[f]$, and that

$$\mathcal{WV}[\overline{f}](x,\omega) = \mathcal{WV}[f](x,-\omega), \tag{2.6}$$

for all $f \in L^2(\mathbb{R})$. In particular Relation (2.6) yields $\mathcal{WV}[f](x,\omega) = \mathcal{WV}[f](x,-\omega)$ for all real-valued $f \in L^2(\mathbb{R})$. Rewriting (2.1) enables us to derive more useful properties of the Wigner distribution.

By defining $h_{x,\omega}(t) = f(x+t/2)e^{-it\omega/2}/\sqrt{2\pi}$, for $f \in L^2(\mathbb{R})$, we can also write (2.1) as

$$\mathcal{WV}[f](x,\omega) = \int_{\mathbb{R}} h_{x,\omega}(t)\,\overline{h_{x,\omega}(-t)}\,dt.$$

Now, Parseval's formula (1.7) yields

$$\mathcal{WV}[f](x,\omega) = \int_{\mathbb{R}} \hat{h}_{x,\omega}(\theta)\,\overline{\hat{h}_{x,\omega}(-\theta)}\,d\theta = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega+\theta/2)\,\overline{\hat{f}(\omega-\theta/2)}e^{i\theta x}\,d\theta, \tag{2.7}$$

for all $f \in L^2(\mathbb{R})$. Relation (2.7) shows that $\mathcal{WV}[f](\cdot,\omega)$ is the Fourier transform of a function in $L^1(\mathbb{R})$. Consequently, Lemma 1.3 can be applied. This yields that $\mathcal{WV}[f](\cdot,\omega)$ is bounded and continuous for fixed $\omega \in \mathbb{R}$. In the same manner it follows from (2.1) that $\mathcal{WV}[f](x,\cdot)$ is bounded and continuous for fixed $x \in \mathbb{R}$. Moreover, we can show that $\mathcal{WV}[f] \in C(\mathbb{R}^2)$, for all $f \in L^2(\mathbb{R})$, see e.g. [24]. Concluding, we have $\mathcal{WV}[f] \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, for all $f \in L^2(\mathbb{R})$.

Also Relation (2.7) yields immediately

$$\mathcal{WV}[\mathcal{F}f](x,\omega) = \mathcal{WV}[f](-\omega,x), \tag{2.8}$$

for all $f \in L^2(\mathbb{R})$.

By rewriting the integrand in (2.7) we get

$$\hat{f}(\omega+\theta/2)\,\overline{\hat{f}(\omega-\theta/2)} = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\,\overline{f(y)}\,e^{-ix(\omega+\theta/2)}\,e^{iy(\omega-\theta/2)}\,dx\,dy =$$
$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u+t/2)\,\overline{f(u-t/2)}\,e^{-iu\theta}\,e^{-it\omega}\,du\,dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} M[f](-\theta,t)e^{-it\omega}\,dt,$$

with

$$M[f](\theta,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u+t/2)\,\overline{f(u-t/2)} e^{iu\theta}\,du. \tag{2.9}$$

The function $M[f]$ is called the characteristic function of the Wigner distribution. Note that $M[f](-\cdot,t)$ is the Fourier transform of $f(\cdot+t/2)\,\overline{f(\cdot-t/2)}$, which is in $L^1(\mathbb{R})$ for all $t \in \mathbb{R}$. Using this characteristic function we obtain

$$\mathcal{WV}[f](x,\omega) = (2\pi)^{-3/2} \int_{\mathbb{R}} \int_{\mathbb{R}} M[f](\theta,t) e^{-i\theta x} e^{-it\omega}\,d\theta\,dt. \tag{2.10}$$

Introducing the function $R_{f,x}$ of $f \in L^2(\mathbb{R})$ by

$$R_{f,x}(t) = f(x+t/2)\,\overline{f(x-t/2)}/\sqrt{2\pi}$$

gives the last representation of the Wigner distribution which we discuss in this report. We have

$$\mathcal{WV}[f](x,\omega) = \mathcal{F}[R_{f,x}](\omega). \tag{2.11}$$

We proceed our discussion of the Wigner distribution with a counterpart of Plancherel's formula. To deduce such a formula for the Wigner distribution we use relation (2.11).

Lemma 2.1 *Let $f \in L^2(\mathbb{R})$. Then*

$$\left(\int_{\mathbb{R}} |f(x)|^2\,dx\right)^2 = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{WV}[f](x,\omega)|^2\,d\omega\,dx.$$

Proof
We derive

$$\begin{aligned}
\left(\int_{\mathbb{R}} |f(x)|^2\,dx\right)^2 &= \int_{\mathbb{R}} |f(u)|^2\,du \int_{\mathbb{R}} |f(v)|^2\,dv \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+t/2)|^2\,|f(x-t/2)|^2\,dt\,dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+t/2)\,f(x-t/2)|^2\,dt\,dx \\
&= 2\pi \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |R_{f,x}(t)|^2\,dt\right)\,dx.
\end{aligned}$$

Applying Plancherel's formula on the inner integral of the latter result yields

$$\begin{aligned}
\left(\int_{\mathbb{R}} |f(x)|^2\,dx\right)^2 &= 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}[R_{f,x}](\omega)|^2\,d\omega\,dx \\
&= 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{WV}[f](x,\omega)|^2\,d\omega\,dx,
\end{aligned}$$

which follows from (2.11). $\square$

We observe that this lemma also yields $\mathcal{WV}[f] \in L^2(\mathbb{R}^2)$ for all $f \in L^2(\mathbb{R})$. A counterpart of Parseval's formula also exists. This is given by Moyal's formula, which we present in the following theorem.

Theorem 2.2 (Moyal) *Let $f, g \in L^2(\mathbb{R})$. Then*

$$|(f,g)|^2 = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{WV}[f](x,\omega)\, \mathcal{WV}[g](x,\omega)\, d\omega\, dx.$$

Proof
First we observe that $\mathcal{WV}[f](x,\omega)\, \mathcal{WV}[g](x,\omega) \in L^1(\mathbb{R}^2)$. This follows from Schwarz's inequality and Lemma 2.1

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{WV}[f](x,\omega)\, \mathcal{WV}[g](x,\omega)|\, d\omega\, dx &\leq \|\mathcal{WV}[f]\|_2\, \|\mathcal{WV}[g]\|_2 \\ &= \|f\|_2^2\, \|g\|_2^2/2\pi. \end{aligned}$$

Using Parseval's formula we derive as a corollary of Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{WV}[f](x,\omega)\, \mathcal{WV}[g](x,\omega)\, d\omega &= \int_{\mathbb{R}} \mathcal{F}[R_{f,x}](\omega) \overline{\mathcal{F}[R_{g,x}](\omega)}\, d\omega \\ &= \int_{\mathbb{R}} R_{f,x}(t) \overline{R_{g,x}(t)}\, dt. \end{aligned}$$

Integrating the latter result over x yields

$$\begin{aligned} &2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{WV}[f](x,\omega)\, \mathcal{WV}[g](x,\omega)\, d\omega\, dx = \\ &\int_{\mathbb{R}} \int_{\mathbb{R}} f(x+t/2) \overline{g(x+t/2)}\ \overline{f(x-t/2)} g(x-t/2)\, dt\, dx = \\ &\int_{\mathbb{R}} \int_{\mathbb{R}} f(u) \overline{g(u)}\ \overline{f(v) \overline{g(v)}}\, du\, dv = |(f,g)|^2. \end{aligned}$$

□

For signal analysis a further desirable property of the Wigner distribution is given in the following theorem.

Theorem 2.3 *Let $f \in L^2(\mathbb{R})$. Then*

$$|f(x)|^2 = \int_{\mathbb{R}} \mathcal{WV}[f](x,\omega)\, d\omega, \quad \text{if } \hat{f} \in L^1(\mathbb{R}) \tag{2.12}$$

$$|\hat{f}(\omega)|^2 = \int_{\mathbb{R}} \mathcal{WV}[f](x,\omega)\, dx, \quad \text{if } f \in L^1(\mathbb{R}). \tag{2.13}$$

Proof
We derive from (2.7)

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{WV}[f](x,\omega)|\, d\omega &\leq \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{f}(\omega+\theta/2)|\, |\hat{f}(\omega-\theta/2)|\, d\theta\, d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{f}(u)|\, |\hat{f}(v)|\, du\, dv = \|\hat{f}\|_1^2/2\pi. \end{aligned}$$

Fix $x \in \mathbb{R}$. Then $\mathcal{WV}[f](x,\cdot) \in L^1(\mathbb{R})$ if $\hat{f} \in L^1(\mathbb{R})$. Equivalently, $\mathcal{F}R_{f,x} \in L^1(\mathbb{R})$ if $\hat{f} \in L^1(\mathbb{R})$, cf. (2.11). Also $R_{f,x} \in C(\mathbb{R})$, since f is continuous. This follows from applying Theorem 1.3 on $\hat{f}$. Finally we have $R_{f,x} \in L^1(\mathbb{R})$ since $f \in L^2(\mathbb{R})$. Now, Theorem 1.2 can be applied. This yields

$$|f(x)|^2 = \sqrt{2\pi} R_{f,x}(0) = \int_{\mathbb{R}} \mathcal{F}[R_{f,x}](\omega) d\omega = \int_{\mathbb{R}} \mathcal{WV}[f](x,\omega)\, d\omega.$$

This proves (2.12). Relation (2.13) is proved in the same manner by replacing $\hat{f}$ by f. $\square$

Relations (2.12) and (2.13) are called the time-frequency marginals, see also [5].

A last result on the energy density of the Wigner distribution is obtained from integrating (2.13) over ω. This yields

$$\|f\|_2^2 = \int_{I\!R} \mathcal{WV}[f](x,\omega)\, dx\, d\omega, \tag{2.14}$$

for $f \in L^1(I\!R) \cap L^2(I\!R)$ or $\hat{f} \in L^1(I\!R) \cap L^2(I\!R)$.

For a comprehensive list of other properties of the one dimensional Wigner distribution we refer to $[4,\ 13]$.

In the sequel of this report we will use a group theoretical approach for the Wigner distribution. This approach uses a relation between the Heisenberg group and the Wigner distribution. This relation can be derived using the characteristic function $M[f]$ for the n-dimensional Wigner distribution. We derive

$$\begin{aligned} M[f](p,q) &= (2\pi)^{-n/2} \int_{I\!R^n} f(u+q/2)\, \overline{f(u-q/2)} e^{i(p,u)}\, du \\ &= (2\pi)^{-n/2} e^{i(p,q)/2} \int_{I\!R^n} f(u+q)\, \overline{f(u)} e^{i(p,u)}\, du \\ &= (2\pi)^{-n/2} (\mu(p,q,0)f, f)_2. \end{aligned} \tag{2.15}$$

This yields

$$\mathcal{WV}[f](x,\omega) = (2\pi)^{-n/2} \mathcal{F}[M[f]](x,\omega) = (2\pi)^{-n} \mathcal{F}[(\mu(\cdot,\cdot,0)f,f)_2](x,\omega), \tag{2.16}$$

with $\mathcal{WV}$ the n-dimensional Wigner distribution and $\mathcal{F}$ the n-dimensional Fourier transform. By polarization, we see that (2.16) also holds for the mixed Wigner distribution, i.e.,

$$\mathcal{WV}[f,g](x,\omega) = (2\pi)^{-n} \mathcal{F}[(\mu(\cdot,\cdot,0)f,g)_2](x,\omega). \tag{2.17}$$

Since μ is irreducible, we have for unitary operators $\mathcal{V}$, as a corollary of (2.19),

$$\mathcal{WV}[f] = \mathcal{WV}[\mathcal{V}f] \iff \mathcal{V} = C\mathcal{I},\ \ |C| = 1. \tag{2.18}$$

We have seen that the Wigner distribution is related to the Schrödinger representation by means of the characteristic function. Now, assume that there exists a unitary representation ρ of H_n in $U(L^2(I\!R^n))$, for which $\mu = \mathcal{V}^*\rho\mathcal{V}$, for some $\mathcal{V} \in U(L^2(I\!R^n))$. Then

$$\begin{aligned} \mathcal{WV}[\mathcal{V}f](x,\omega) &= (2\pi)^{-n} \mathcal{F}[(\mu(\cdot,\cdot,0)\mathcal{V}f, \mathcal{V}f)_2](x,\omega) \\ &= (2\pi)^{-n} \mathcal{F}[(\mathcal{V}^*\mu(\cdot,\cdot,0)\mathcal{V}f, f)_2](x,\omega) \\ &= (2\pi)^{-n} \mathcal{F}[(\rho(\cdot,\cdot,0)f, f)_2](x,\omega),. \end{aligned}$$

for all $f \in L^2(I\!R^n)$. This yields

$$\mathcal{WV}[\mathcal{V}f](x,\omega) = (2\pi)^{-2n} \int_{I\!R^n} \int_{I\!R^n} (\rho(p,q,0)f,f)_2\, e^{-i(p,x)} e^{-i(q,\omega)}\, d\theta\, dv. \tag{2.19}$$

We will return to the latter relation in Section affiensect.

3. The Fractional Fourier Transform

The fractional Fourier transform (FRFT) was introduced by Namias in [23] as a Fourier transform of fractional order. This was done starting from fractional powers of the eigenvalues of the Fourier transform and their corresponding eigenvalues. With this formalism he derived in a heuristic manner an integral representation of this operator. In [15, 19], McBride and Kerr provided a rigorous mathematical framework in which the formal work of Namias could be situated. We discuss this mathematical framework and Namias formal work in the first part of this section.

Recently, the FRFT turned out to be an interesting transformation for time-frequency signal processing and optical engineering. This growing interest for the FRFT is the consequence of a series of papers that deal with the relation of the FRFT to time-frequency representations of a signal, like the Wigner distribution, see e.g. [1, 22, 25, 26]. This relation is discussed in the second part of this section.

3.1 Definition and Properties

We start with the definition of the FRFT for functions in $L^2(\mathbb{R})$.

Definition 3.1 *Take $f \in L^2(\mathbb{R})$. Its fractional Fourier transform of order $\alpha \in (-\pi, \pi]$ is given by*

$$\mathcal{F}_\alpha[f](x) = \frac{C_\alpha}{\sqrt{2\pi\,|\sin\alpha|}} \int_{\mathbb{R}} f(u)\, e^{i\,((u^2+x^2)\cdot(\cot\alpha)/2\,-\,ux\csc\alpha)}\, du, \tag{3.1}$$

for $0 < |\alpha| < \pi$, with

$$C_\alpha = e^{i\,(\frac{\pi}{4}\operatorname{sgn}\alpha - \alpha/2)}. \tag{3.2}$$

Furthermore, for $\alpha = 0$ and $\alpha = \pi$ the FRFT is defined by

$$\mathcal{F}_0[f](x) = f(x) \;\; \textit{and} \;\; \mathcal{F}_\pi[f](x) = f(-x).$$

For $\alpha \notin (\pi, \pi]$ the FRFT is defined by periodicity $\mathcal{F}_{\alpha+2\pi} = \mathcal{F}_\alpha$.

Particularly, we have from this definition

$$\mathcal{F}_{\pi/2} = \mathcal{F} \;\text{ and }\; \mathcal{F}_{n\pi/2} = \mathcal{F}^n \;\forall_{n\in\mathbb{Z}},$$

with $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R})$.

The factor C_α in (3.2) is chosen to guarantee that $\mathcal{F}_\alpha$ is properly normalized and that $\mathcal{F}_\alpha$ is continuous in α. Indeed, it can be shown that

$$\lim_{\beta\to\alpha} \|\mathcal{F}_\beta f - \mathcal{F}_\alpha f\|_2 = 0, \tag{3.3}$$

for all $f \in L^2(\mathbb{R})$ and for this particular choice of C_α.

This result is obtained by combining two properties of the FRFT. The first property of the FRFT is known as the index law, i.e.,

$$\mathcal{F}_\alpha \mathcal{F}_\beta f = \mathcal{F}_{\alpha+\beta} f, \tag{3.4}$$

for all $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. A rigorous proof of this property for functions in the Schwartz space $S(\mathbb{R})$ is given in [19]. Consequently, this result can be extended to functions in $L^2(\mathbb{R})$.

The second property we need for proving the continuity of $\mathcal{F}_\alpha$ is the continuity of the FRFT either in $\alpha = 0$ or $\alpha = \pi$. In [15], it is proven that

$$\lim_{\alpha\to 0} \|\mathcal{F}_\alpha f - f\|_2 = 0, \tag{3.5}$$

for all $f \in L^2(I\!R)$. Result (3.3) can now be obtained in a straightforward way by combining (3.4) and (3.5). We observe, that (3.3) also holds for other choices of C_α, see e.g. [1].

Considering again Relation (3.4) we have in particular $\mathcal{F}_\alpha \mathcal{F}_{-\alpha} = \mathcal{I}$ and $\mathcal{F}_{-\alpha}\mathcal{F}_\alpha = \mathcal{I}$. Consequently, the inverse of $\mathcal{F}_\alpha$ is given by $\mathcal{F}_{-\alpha}$, for all $\alpha \in I\!R$.

For a better understanding of the action of the FRFT we introduce two unitary operators on $L^2(I\!R)$. For $t \in I\!R$, we define the operator $\mathcal{C}_t$ on $L^2(I\!R)$ by

$$\mathcal{C}_t[f](x) = e^{itx^2/2} f(x). \tag{3.6}$$

Obviously, $\mathcal{C}_t$ multiplies a given function $f \in L^2(I\!R)$ with a quadratic chirp, i.e., a Fourier mode with a quadratic argument. Furthermore, we introduce for $a \neq 0$ the dilation operator $\mathcal{D}_a$ on $L^2(I\!R)$ by

$$\mathcal{D}_a[f](x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x}{a}\right). \tag{3.7}$$

Using the operator $\mathcal{C}_{\cot\alpha}$ and $\mathcal{D}_{\sin\alpha}$, we can write $\mathcal{F}_\alpha$, $\alpha \in (-\pi,\pi)$, also as

$$\mathcal{F}_\alpha f = C_\alpha\, \mathcal{C}_{\cot\alpha} \mathcal{D}_{\sin\alpha} \mathcal{F} \mathcal{C}_{\cot\alpha}. \tag{3.8}$$

The fact that all operators in the right-hand side of (3.8) are unitary operators on $L^2(I\!R)$ and that $|C_\alpha| = 1$ yields that $\mathcal{F}_\alpha$ is a unitary operator on $L^2(I\!R)$, for all $\alpha \in I\!R$. Note, that $\mathcal{F}_0$ and $\mathcal{F}_\pi$ are also unitary, which follows directly from Definition 3.1. As a consequence we also have Parseval's formula for the FRFT

$$\int\limits_{I\!R} f(x)\overline{g(x)}\, dx = \int\limits_{I\!R} \mathcal{F}_\alpha[f](x)\overline{\mathcal{F}_\alpha[g](x)}\, dx, \tag{3.9}$$

for all $\alpha \in I\!R$ and $f, g \in L^2(I\!R)$. Furthermore, as a result we have Plancherel's formula for the FRFT

$$\int\limits_{I\!R} |f(x)|^2\, dx = \int\limits_{I\!R} |\mathcal{F}_\alpha[f](x)|^2\, dx, \tag{3.10}$$

for all $\alpha \in I\!R$ and $f \in L^2(I\!R)$.

From the preceding derivations and the definition of $\mathcal{F}_0$ it follows that $G_{fr} = \{\mathcal{F}_\alpha \mid \alpha \in I\!R\}$ is a strongly continuous subgroup of unitary operators on $L^2(I\!R)$. A cyclic subgroup of order 4 is given by the integer powers of the Fourier transform $\{\mathcal{F}^n \mid n = 0,1,2,3\}$. Consequently, the discrete cyclic group with generating element $\mathcal{F}$ is embedded in the continuous group G_{fr}.

A further relation with the classical Fourier transform on $L^2(I\!R)$ can be obtained by considering the formal derivation of the FRFT by Namias in [23]. His starting point was to consider the eigenvalues and eigenfunctions of the Fourier transform.

It is known, see e.g. [9], that the eigenfunctions of the Fourier transform are given by the Hermite functions

$$h_k(x) = \left(2^k k! \sqrt{\pi}\right)^{-1/2} e^{-x^2/2} H_k(x), \tag{3.11}$$

where H_k are the Hermite polynomials given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k e^{-x^2}. \tag{3.12}$$

The Hermite functions form an orthonormal basis for $L^2(I\!R)$ and they satisfy $\mathcal{F}h_k = e^{ik\pi/2} h_k$. The first idea of an FRFT was to define an operator $\mathcal{F}_\alpha$, satisfying

$$\mathcal{F}_\alpha h_k = e^{ik\alpha} h_k, \tag{3.13}$$

for $\alpha \in \mathbb{R}$. For $\alpha = m\pi/2$, with $m \in \mathbb{Z}$, we have $\mathcal{F}_{m\pi/2} = \mathcal{F}^m$. Particularly, if m mod $4 = 0$, then $\mathcal{F}^m = \mathcal{I}$. For a formal representation of $\mathcal{F}_\alpha$, with $0 < \alpha < \pi/2$, we follow Namias in [23].

We write $f \in L^2(\mathbb{R})$ as $f = \sum_{k=0}^{\infty} (f, h_k)_2 h_k$. Consequently, we have

$$\begin{aligned}
\mathcal{F}_\alpha[f](x) &= \sum_{k=0}^{\infty}(f,h_k)_2 \mathcal{F}_\alpha[h_k](x) = \sum_{k=0}^{\infty}(f,h_k)_2\, e^{ik\alpha} h_k(x) \\
&= \int_{\mathbb{R}} f(u) \left(\sum_{k=0}^{\infty} e^{ik\alpha} h_k(u) h_k(x) \right) du \\
&= \int_{\mathbb{R}} f(u) \left(\sum_{k=0}^{\infty} \frac{e^{ik\alpha}}{2^k k! \sqrt{\pi}} H_k(u) H_k(x) e^{-u^2/2 - x^2/2} \right) du.
\end{aligned}$$

The latter expression can be rewritten using Mehler's formula, see [20],

$$\sum_{k=0}^{\infty} \frac{z^k}{2^k k! \sqrt{\pi}} H_k(u) H_k(x) = \frac{1}{\sqrt{\pi(1-z^2)}} \exp\left(\frac{2xuz - z^2(x^2+u^2)}{1-z^2} \right). \tag{3.14}$$

Here $1/(1-z^2)$ lies in the right half plane and the square root in $1/(1-z^2)$ is the branch that is positive for $z > 0$. Furthermore, we observe that the series converges in L^2 with respect to u, for all x and z, see [9]. Using Mehler's formula in the previous result yields

$$\begin{aligned}
\mathcal{F}_\alpha[f](x) &= \frac{1}{\sqrt{\pi e^{i\alpha}} \cdot \sqrt{e^{-i\alpha} - e^{i\alpha}}} \int_{\mathbb{R}} f(u) \exp\left(i \frac{2ixu - i(e^{i\alpha} + e^{-i\alpha})(x^2+u^2)/2}{e^{i\alpha} - e^{-i\alpha}} \right) \\
&= \frac{e^{i\pi/4 - i\alpha/2}}{\sqrt{2\pi \sin\alpha}} \int_{\mathbb{R}} f(u)\, e^{i\,((u^2+x^2)\cdot(\cot\alpha)/2 \,-\, ux \csc\alpha)}\, du.
\end{aligned}$$

For a rigorous framework in which this formal work of Namias can be studied we refer to [15, 19].

3.2 The FRFT and the Wigner Plane

For time-frequency analysis it is interesting to consider the relation of the FRFT with time-frequency operators like the Wigner distribution. Therefore, we compute the Wigner distribution of the FRFT. This will give us insight in how the FRFT acts in the Wigner plane, i.e., the phase space related to the Wigner distribution.

For this computation we need the following lemma.

Lemma 3.2 *Let $\mathcal{T}_b$ and $\mathcal{M}_\omega$, $b, \omega \in \mathbb{R}$, denote respectively the shift operator and frequency modulation on $L^2(\mathbb{R})$ as given in (2.4) and (2.5) respectively. Furthermore, let $\mathcal{F}_\alpha$, $\alpha \in \mathbb{R}$, the fractional Fourier transform on $L^2(\mathbb{R})$ as given in Definition 3.1. Then*

$$\mathcal{F}_\alpha \mathcal{T}_b = e^{ib^2(\sin 2\alpha)/4}\, \mathcal{M}_{-b\sin\alpha}\, \mathcal{T}_{b\cos\alpha} \mathcal{F}_\alpha, \tag{3.15}$$

$$\mathcal{F}_\alpha \mathcal{M}_\omega = e^{-i\omega^2(\sin 2\alpha)/4}\, \mathcal{M}_{\omega\cos\alpha}\, \mathcal{T}_{\omega\sin\alpha} \mathcal{F}_\alpha. \tag{3.16}$$

Proof

For $\alpha = 0$ both results are trivial, since $\mathcal{F}_0 = \mathcal{I}$. For $\alpha = \pi$ both results follow directly from Definition 3.1. Furthermore, equation (3.16) follows from (3.15) by observing that $\mathcal{F}\mathcal{M}_\omega = \mathcal{T}_\omega \mathcal{F}$, with $\mathcal{F}$ the Fourier transform. Indeed, if (3.15) holds, this observation yields

$$\begin{aligned}
\mathcal{F}_\alpha \mathcal{M}_\omega &= \mathcal{F}_\alpha \mathcal{F}^* \mathcal{T}_\omega \mathcal{F} = \mathcal{F}_{\alpha-\pi/2} \mathcal{T}_\omega \mathcal{F}_{\pi/2} \\
&= e^{i\omega^2(\sin(2\alpha-\pi))/4}\, \mathcal{M}_{-\omega\sin(\alpha-\pi/2)}\, \mathcal{T}_{\omega\cos(\alpha-\pi/2)} \mathcal{F}_{\alpha-\pi/2} \mathcal{F}_{\pi/2} \\
&= e^{-i\omega^2(\sin 2\alpha)/4}\, \mathcal{M}_{\omega\cos\alpha}\, \mathcal{T}_{\omega\sin\alpha} \mathcal{F}_\alpha,
\end{aligned}$$

using (3.4). Consequently, the proof is established by showing that (3.15) holds for $0 < |\alpha| < \pi$. We derive for $f \in L^2(I\!R)$, $b \in I\!R$ and $0 < |\alpha| < \pi$

$$\begin{aligned}
\mathcal{F}_\alpha \mathcal{T}_b[f](x) &= \frac{C_\alpha}{\sqrt{2\pi\,|\sin\alpha|}} \int_{I\!R} f(u-b)\, e^{i\,((u^2+x^2)\cdot(\cot\alpha)/2 \,-\, ux\csc\alpha)}\, du \\
&= \frac{C_\alpha}{\sqrt{2\pi\,|\sin\alpha|}} \int_{I\!R} f(u)\, e^{i\,((u^2+x^2+b^2+2ub)\cdot(\cot\alpha)/2 \,-\, (u+b)x\csc\alpha)}\, du \\
&= \frac{C_\alpha}{\sqrt{2\pi\,|\sin\alpha|}}\, e^{i(b^2\cdot(\cos\alpha)/2 - bx)(1-\cos^2\alpha)\csc\alpha} \times \\
&\qquad \int_{I\!R} f(u)\, e^{i\,((u^2+(x-b\cos\alpha)^2)(\cot\alpha)/2 \,-\, (u(x-b\cos\alpha))\csc\alpha)}\, du \\
&= e^{i(b^2(\sin 2\alpha)/4 - bx\sin\alpha)}\, \mathcal{F}_\alpha[f](x - b\cos\alpha) \\
&= e^{ib^2(\sin 2\alpha)/4}\, \mathcal{M}_{-b\sin\alpha}\, \mathcal{T}_{b\cos\alpha} \mathcal{F}_\alpha[f](x).
\end{aligned}$$

□

Using this lemma, we can compute the action of the FRFT in phase space by means of the Wigner distribution. For this we write

$$\begin{aligned}
\mathcal{WV}[f](x,\omega) &= \frac{1}{2\pi} \int_{I\!R} f(x+t/2)\overline{f(x-t/2)} e^{-it\omega}\, dt \\
&= \frac{1}{\pi} \int_{I\!R} f(t+x)\overline{f(x-t)} e^{-2it\omega}\, dt = (\mathcal{M}_{-\omega}\mathcal{T}_{-x} f, \mathcal{M}_\omega \mathcal{T}_x \mathcal{F}_\pi f)/\pi.
\end{aligned}$$

Using Lemma 3.2 we derive

$$\begin{aligned}
\mathcal{F}_{-\alpha}\mathcal{M}_\omega \mathcal{T}_x &= e^{i(\omega^2 - x^2)\cdot(\sin 2\alpha)/4}\, \mathcal{M}_{\omega\cos\alpha}\, \mathcal{T}_{-\omega\sin\alpha}\, \mathcal{M}_{x\sin\alpha}\, \mathcal{T}_{x\cos\alpha} \mathcal{F}_{-\alpha} \\
&= e^{i(\omega^2 - x^2)\cdot(\sin 2\alpha)/4}\, e^{ix\omega\sin^2\alpha} \mathcal{M}_{x\sin\alpha + \omega\cos\alpha}\, \mathcal{T}_{x\cos\alpha - \omega\sin\alpha} \mathcal{F}_{-\alpha}.
\end{aligned}$$

Combining these two results yields

$$\begin{aligned}
&\mathcal{WV}[\mathcal{F}_\alpha f](x,\omega) = (\mathcal{M}_{-\omega}\mathcal{T}_{-x}\mathcal{F}_\alpha f, \mathcal{M}_\omega \mathcal{T}_x \mathcal{F}_\pi \mathcal{F}_\alpha f)/\pi = \\
&(\mathcal{F}_{-\alpha}\mathcal{M}_{-\omega}\mathcal{T}_{-x}\mathcal{F}_\alpha f, \mathcal{F}_{-\alpha}\mathcal{M}_\omega \mathcal{T}_x \mathcal{F}_\pi \mathcal{F}_\alpha f)_2/\pi = \\
&(\mathcal{M}_{-x\sin\alpha - \omega\cos\alpha}\, \mathcal{T}_{-x\cos\alpha + \omega\sin\alpha} f, \mathcal{M}_{x\sin\alpha+\omega\cos\alpha}\, \mathcal{T}_{x\cos\alpha - \omega\sin\alpha}\mathcal{F}_\pi f)/\pi = \\
&\mathcal{WV}[f](x\cos\alpha - \omega\sin\alpha, x\sin\alpha + \omega\cos\alpha) = \mathcal{WV}[f](R_\alpha(x,\omega)),
\end{aligned} \tag{3.17}$$

where $R_\alpha(x,\omega)$ represents the matrix vector product with matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.18}$$

We conclude from this derivation that the FRFT of order α acts like a rotation by α in the Wigner plane. In particular, we have a rotation by $\pi/2$ in the Wigner plane for $\mathcal{F}_{\pi/2}$, which is a result that coincides with (2.8).

The action of the FRFT in the Wigner plane leads us in a natural way to the question which operators on $L^2(I\!R)$ act like a linear transformation in the Wigner plane. The following section is devoted to this question. However, instead of operators on $L^2(I\!R)$ we consider operators acting on $L^2(I\!R^n)$. It will turn out that finding a solution for the n-dimensional problem does not follow straightforwardly from the solution for the one-dimensional case.

Since we want to give an answer to our problem for operators on $L^2(\mathbb{R}^n)$, we introduce the fractional Fourier transform on $L^2(\mathbb{R}^n)$ by

$$\mathcal{F}_{\alpha_1,\ldots,\alpha_n} = \widetilde{\mathcal{F}}_{\alpha_1} \cdots \widetilde{\mathcal{F}}_{\alpha_n}, \tag{3.19}$$

for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Here $\widetilde{\mathcal{F}}_{\alpha_i}$ is given by

$$\widetilde{\mathcal{F}}_{\alpha_i}[f](x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) = \mathcal{F}_{\alpha_i}[g_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}](y),$$

with $g_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}(y) = f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$, for fixed $x_1, .., x_{i-1}, x_{i+1}, .., x_n \in \mathbb{R}$. Computing the n-dimensional Wigner distribution of this FRFT yields

$$\mathcal{WV}[\mathcal{F}_{\alpha_1,\ldots,\alpha_n} f](x, \omega) = \mathcal{WV}[f](R_{\alpha_1,\cdots,\alpha_n}(x, \omega)), \tag{3.20}$$

with

$$R_{\alpha_1,\cdots,\alpha_n} = \begin{pmatrix} \cos\alpha_1 & & 0 & -\sin\alpha_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \cos\alpha_n & 0 & & -\sin\alpha_n \\ \sin\alpha_1 & & 0 & \cos\alpha_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \sin\alpha_n & 0 & & \cos\alpha_n \end{pmatrix}. \tag{3.21}$$

This result follows in a straightforward way from (3.17).

4. Affine Transformations in the Wigner Plane

Inspired by the fractional Fourier transform and its action in the Wigner plane, we search for linear operators $\mathcal{V}$ on $L^2(\mathbb{R}^n)$ such that there exist a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$ for which

$$\mathcal{WV}[\mathcal{V}f](x, \omega) = \mathcal{WV}[f](A(x, \omega) + b), \tag{4.1}$$

holds for all $f \in L^2(\mathbb{R}^n)$. We observe, that De Bruijn already considered this problem in [2] where he dealt with a new class of generalized functions. Here we will follow an approach based on group theory, see [30, 31, 40]. These results will be placed within the concept of the FRFT in order to embed this transform in a larger class of unitary transformations. Also new results will be added.

We restrict ourselves to matrices A for which $\det A = \pm 1$. For these matrices we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{WV}[f](A[x, \omega] + b)\, d\omega\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{WV}[f](x, \omega)\, d\omega\, dx.$$

We shall refer to such affine transformations in the Wigner plane as energy preserving affine transformations. Indeed, for these transformations the corresponding operators $\mathcal{V}$ on $L^2(\mathbb{R}^n)$ satisfy

$$\begin{aligned} (\mathcal{V}f, \mathcal{V}f) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{WV}[\mathcal{V}f](x, \omega)\, d\omega\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{WV}[f](A(x, \omega) + b)\, d\omega\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{WV}[f](x, \omega)\, d\omega\, dx = (f, f), \end{aligned}$$

for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ or $\hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ which follows from (2.14). We observe that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a dense subspace of $L^2(\mathbb{R}^n)$. Concluding, an operator on $L^2(\mathbb{R}^n)$ that yields an energy preserving affine transformation in the Wigner plane has to be an isometry on $L^2(\mathbb{R}^n)$. On the other hand, Equation (4.1) follows directly from applying (2.14) on both sides of the equation $(\mathcal{V}f, \mathcal{V}f) = (f, f)$, for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ or $\hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Before dealing with a classification of all unitary operators that satisfy (4.1), we present some well-known operators for which (4.1) holds.

Multiplication
We start our set of unitary operators on $L^2(\mathbb{R}^n)$ with a trivial one, namely multiplication by a constant C with $|C| = 1$. Result (2.18) already showed that $\mathcal{WV}[f] = \mathcal{WV}[Cf]$, for all $|C| = 1$. Consequently, this multiplication operator satisfies (4.1) with $A = I_{2n}$, the $(2n \times 2n)$ identity matrix, and $b = 0$.

Complex conjugation
Besides linear operators there also exists a non-linear operator for which (4.1) holds, namely the operator $f \mapsto \overline{f}$. For the one-dimensional case we have already seen in (2.6) that

$$\mathcal{WV}[\overline{f}](x, \omega) = \mathcal{WV}[f](x, -\omega).$$

For $f \in L^2(\mathbb{R}^n)$ this result also holds. This follows from a straightforward generalization of (2.6). We conclude, that taking the complex conjugate also satisfies (4.1) with

$$A = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \text{ and } b = 0.$$

We observe that we have $\det A = (-1)^n$ for the complex conjugation. Later in this section it will turn out that a necessary condition on a linear operator $\mathcal{V}$, such that (4.1) holds, is given by $\det A = 1$.

Space and frequency shift
For $x_0, \omega_0 \in \mathbb{R}^n$ we introduce the shift operator and the frequency shift operator on $L^2(\mathbb{R}^n)$ by

$$T_{x_0}[f](x) = f(x - x_0) \text{ and } M_{\omega_0}[f](x) = e^{i(\omega_0, x)} f(x)$$

respectively, with $f \in L^2(\mathbb{R}^n)$. Remark, that these operators coincide with the shift and frequency shift operators (2.4) and (2.5) in the one-dimensional case.

We combine the introduced unitary operators T_{x_0} and M_{ω_0} into one unitary operator on $L^2(\mathbb{R}^n)$, given by

$$\mathcal{N}_{(x_0,\omega_0)}[f](x) = T_{x_0} M_{\omega_0}[f](x) = e^{i(\omega_0, x)} f(x - x_0). \tag{4.2}$$

Computing the Wigner transform of this operator yields

$$\mathcal{WV}[\mathcal{N}_{(x_0,\omega_0)} f](x, \omega) = \mathcal{WV}[f](x - x_0, \omega - \omega_0),$$

which is a result we have seen before in discussing the one-dimensional Wigner distribution. From this result we conclude, that (4.1) holds for $\mathcal{N}_{(x_0,\omega_0)}$, namely by taking $A = 0$ and $b = (x_0, \omega_0)$.

We observe that all possible translations $b \in \mathbb{R}^n$ in (4.1) can be obtained from $\mathcal{N}_b$. This means, that if we are looking for a unitary operator $\mathcal{V}$ on $L^2(\mathbb{R}^n)$ such that (4.1) holds, then we only have to find a linear operator $\mathcal{U}$ on $L^2(\mathbb{R}^n)$ such that

$$\mathcal{WV}[\mathcal{U}f](x, \omega) = \mathcal{WV}[f](A(x, \omega)), \tag{4.3}$$

for all $f \in \mathbb{R}^n$. The operator $\mathcal{V}$ we are looking for is then given by $\mathcal{V} = \mathcal{N}_b \mathcal{U}$. Therefore, we will restrict ourselves from now on to operators $\mathcal{U}$ that satisfy (4.3) with $\det A = \pm 1$.

The Fourier transform
In Section 2 we already derived for the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R})$

$$\mathcal{WV}[\mathcal{F}f](x, \omega) = \mathcal{WV}[f](-\omega, x). \tag{4.4}$$

For $f \in L^2(\mathbb{R}^n)$ and the n-dimensional Fourier transform $\mathcal{F}$ this relation remains the same, which follows directly from a generalization of Relation (2.7) for the n-dimensional Wigner distribution. Consequently, the Fourier transform on $L^2(\mathbb{R}^n)$ satisfies (4.3) with $A = J_n^T$. Here J_n denotes the $(2n \times 2n)$ matrix given by

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{4.5}$$

In the sequel of this section this matrix will play an important role in classifying all unitary operators $\mathcal{U}$ that satisfy (4.3).

The dilation operator

For $B \in \mathbb{R}^{n\times n}$, with $\det B \neq 0$, the dilation operator $\mathcal{D}_B$ on $L^2(\mathbb{R}^n)$ is defined by

$$\mathcal{D}_B[f](x) = \frac{1}{\sqrt{|\det B|}} f(B^{-1}x), \tag{4.6}$$

with inverse

$$\mathcal{D}_B^{-1}[f](x) = \sqrt{|\det B|} f(Bx).$$

We use the definition of the Wigner distribution to derive the action of $\mathcal{D}_B$ in the Wigner plane. We compute

$$\begin{aligned} \mathcal{WV}[\mathcal{D}_B f](x,\omega) &= \frac{1}{(2\pi)^n |\det B|} \int_{\mathbb{R}^n} f(B^{-1}(x+\tau/2))\overline{f(B^{-1}(x-\tau/2))} e^{-i(\tau,\omega)}\, d\tau \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(B^{-1}x+\tau/2)\overline{f(B^{-1}x-\tau/2)} e^{-i(\tau,B^T\omega)}\, d\tau \\ &= \mathcal{WV}[f](B^{-1}x, B^T\omega). \end{aligned} \tag{4.7}$$

Concluding, also $\mathcal{D}_B$ corresponds to a linear transformation in the Wigner plane. For $\mathcal{D}_B$ Relation (4.3) holds with

$$A = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix}.$$

Multiplication with a chirp

The last example of a unitary operator that satisfies (4.3) is the operator that multiplies a function in $L^2(\mathbb{R}^n)$ with a quadratic chirp. This operator is given by

$$\mathcal{C}_S[f](x) = e^{i(Sx,x)/2} f(x), \tag{4.8}$$

with $S \in \mathbb{R}^{n\times n}$ symmetric. Remark, that we have seen this operator already for the one-dimensional case in (3.6), which coincides with (4.8) for $n = 1$. Obviously its inverse is given by

$$\mathcal{C}_S^{-1}[f](x) = \mathcal{C}_S^*[f](x) = e^{-i(Sx,x)/2} f(x).$$

We use (2.19) to derive the action of $\mathcal{C}_S$ in the Wigner plane

$$\mathcal{WV}[C_S f](x,\omega) = (2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} ((C_S^*\mu(p,q,0)C_S)f, f)_2\, e^{-i(p,x)} e^{-i(q,\omega)}\, dp\, dq.$$

In a direct way we get

$$\begin{aligned} (\mathcal{C}_S^*\mu(p,q,0)\mathcal{C}_S)[f](x) &= e^{-i(Sx,x)/2} e^{i(p,x)} e^{i(p,q)/2} e^{i(S(x+q),x+q)/2} f(x+q) \\ &= e^{i(p+Sq,x)} e^{i(p+S(q,q))/2} f(x+q) \\ &= \mu(p+Sq,q,0)[f](x), \end{aligned}$$

which yields

$$\begin{aligned} \mathcal{WV}[C_S f](x,\omega) &= (2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mu(p+Sq,q,0)f,f)_2\, e^{-i(p,x)} e^{-i(q,\omega)}\, dp\, dq \\ &= (2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mu(p,q,0)f,f)_2\, e^{-i((p,q),A(x,w))}\, dp\, dq \\ &= \mathcal{WV}[f](A(x,\omega)), \end{aligned} \tag{4.9}$$

with $A = \begin{pmatrix} I_n & 0 \\ -S & I_n \end{pmatrix}$. Consequently, also $\mathcal{C}_S$ satisfies (4.3) with A as given before.

4.1 A Group Theoretical Approach

In the latter example we have already seen that the relation between a unitary operator on $L^2(\mathbb{R}^n)$ and its affine action in the Wigner plane can be given by using (2.19). This relation can also be used to translate our problem in terms of group theory. This can be done in the following way.

Given a unitary operator $\mathcal{V}$ on $L^2(\mathbb{R}^n)$, we define a unitary representation ρ of the Heisenberg group H_n by $\rho(g) = \mathcal{V}^*\mu(g)\mathcal{V}$, for all $g \in H_n$ and μ the Schrödinger representation. Then by (2.19) we have for such ρ and $\mathcal{V}$

$$\begin{aligned} \mathcal{WV}[\mathcal{V}f](x,\omega) &= (2\pi)^{-2n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} ((\mathcal{V}^*\mu(p,q,0)\mathcal{V})f,f)_2\, e^{-i(p,x)}e^{-i(q,\omega)}\,dp\,dq. \\ &= (2\pi)^{-2n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} (\rho(p,q,0)f,f)_2\, e^{-i(p,x)}e^{-i(q,\omega)}\,dp\,dq. \end{aligned}$$

Consequently, if there exists a linear transformation A such that $\mu(g,0) = \rho(A^Tg,0)$ for all $g \in H_n'$, with

$$H_n' = \{g \in \mathbb{R}^{2n} \mid \forall_{t\in\mathbb{R}}(g,t) \in H_n\},$$

then

$$\begin{aligned} \mathcal{WV}[\mathcal{V}f](x,\omega) &= (2\pi)^{-2n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} (\mu(A^{-T}(p,q),0)f,f)_2\, e^{-i(p,x)}e^{-i(q,\omega)}\,dp\,dq. \\ &= |\det A| \cdot \mathcal{WV}[f](A(x,\omega)), \end{aligned} \tag{4.10}$$

using the notation $A^{-T} = (A^{-1})^T$.

This derivation shows that the problem we are considering is equivalent to the problem of finding operators $\mathcal{V} \in U(L^2(\mathbb{R}^n))$ for which there exist matrices $A \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{V}^*\mu(g,t)\mathcal{V} = \mu(A^{-T}g,t), \tag{4.11}$$

for all $g \in H_n'$ and $t \in \mathbb{R}$.

Besides the Lie groups in Example 1.12 we introduce another Lie group for solving this problem, namely the symplectic group $Sp(n)$. This group is defined by

$$Sp(n) = \{M \in GL(2n) \mid J_nM^TJ_n^T = M^{-1}\}, \tag{4.12}$$

with J_n as given in (4.5). Note that by definition $M^T \in Sp(n)$ and $\det M = \pm 1$ for any $M \in Sp(n)$. Moreover, it can be shown that $Sp(n)$ is connected, see [9]. This yields that $\det M = 1$ if $M \in Sp(n)$. Furthermore, we observe, that $Sp(n) \subset SL(2n)$, but $Sp(1) = SL(2)$. It will turn out later in this section, that this property of the symplectic group causes the fact that solutions for the n-dimensional problem do not follow straightforwardly from the solution for the one-dimensional case.

To solve our problem we start with the introduction of G, the subgroup of $U(L^2(\mathbb{R}^n))$ given by

$$G = \{\mathcal{V} \in U(L^2(\mathbb{R}^n)) \mid \forall_{g\in\mathbb{R}^{2n}}\forall_{t\in\mathbb{R}}\exists_{g'\in\mathbb{R}^{2n}} :\ \mathcal{V}^*\mu(g,t)\mathcal{V} = \mu(g',t)\}. \tag{4.13}$$

Obviously, G is a semi-group. Later we will show that every $g \in G$ has an inverse element in G, which yields that G is a group. This group can be equipped with the strong operator topology of $U(L^2(\mathbb{R}^n))$. Furthermore, it is clear from (1.13) that g' in (4.13) is uniquely determined. So a mapping $\nu(\mathcal{V}) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ can be defined, which depends on $\mathcal{V} \in G$. This $\nu(\mathcal{V})$ is given by $\nu(\mathcal{V})g = g'$, with $g, g' \in \mathbb{R}^{2n}$. Also $\nu(\mathcal{V})$

is a homomorphism for all $\mathcal{V} \in G$. This is shown in the following way.

For $\alpha, \beta \in I\!R$ and $p_1, p_2, q_1, q_2 \in I\!R^n$ we have

$$\begin{aligned}&\mathcal{V}^*\mu(\alpha p_1, \alpha q_1, 0)\,\mu(\beta p_2, \beta q_2, 0)\mathcal{V} =\\&\mathcal{V}^*\mu(\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2, (\alpha q_1, \beta p_2)/2 - (\alpha p_1, \beta q_2)/2)\mathcal{V} =\\&\mu(\nu(\mathcal{V})(\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2), (\alpha J_n(p_1, q_1), \beta(p_2, q_2))/2).\end{aligned}$$

On the other hand we also have

$$\begin{aligned}&\mathcal{V}^*\mu(\alpha p_1, \alpha q_1, 0)\,\mu(\beta p_2, \beta q_2, 0)\mathcal{V} =\\&\mu(\alpha\nu(\mathcal{V})(p_1, q_1), 0)\,\mu(\beta\nu(\mathcal{V})(p_2, q_2), 0) =\\&\mu(\alpha\nu(\mathcal{V})(p_1, q_1) + \beta\nu(\mathcal{V})(p_2, q_2), (y_1, x_2)/2 - (x_1, y_2)/2),\end{aligned}$$

with $(x_1, y_1) = \alpha\nu(\mathcal{V})(p_1, q_1)$ and $(x_2, y_2) = \beta\nu(\mathcal{V})(p_2, q_2)$. Taking these results together yields

$$\begin{aligned}&\mu(\nu(\mathcal{V})(\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2), (\alpha J_n(p_1, q_1), \beta(p_2, q_2))/2) =\\&\mu(\alpha\nu(\mathcal{V})(p_1, q_1) + \beta\nu(\mathcal{V})(p_2, q_2), (y_1, x_2)/2 - (x_1, y_2)/2).\end{aligned} \tag{4.14}$$

A necessary condition such that (4.14) holds for all $\alpha, \beta, p_1, p_2, q_1$ and q_2 is given by the linearity of $\nu(\mathcal{V})$ for all $\mathcal{V} \in G$. Consequently, $\nu(\mathcal{V}) : I\!R^{2n} \to I\!R^{2n}$ is a homomorphism, that satisfies

$$\mathcal{V}^*\mu(p, q, t)\mathcal{V} = \mu(\nu(\mathcal{V})(p, q), t). \tag{4.15}$$

Using this relation we can show, that $\nu(\mathcal{V})$ is also injective. To do this, we assume $\nu(\mathcal{V})g = 0$, or equivalently $\mu(g, t)\mathcal{V} = \mu(0, t)$. Then $\mu(g, t) = \mathcal{V}\mu(0, t)\mathcal{V}^* = \mu(0, t)$, which yields $g = 0$.

Furthermore, ν satisfies

$$\mu(\nu(C\mathcal{V})(p, q), t) \quad = \quad (\overline{C}\mathcal{V}^*)\mu(p, q, t)(C\mathcal{V}) = \mu(\nu(\mathcal{V})(p, q), t)$$

and

$$\begin{aligned}\mu(\nu(\mathcal{V}_1\mathcal{V}_2)(p, q), t) &= \mathcal{V}_2^*(\mathcal{V}_1^*\mu(p, q, t)\mathcal{V}_1)\mathcal{V}_2 = \mathcal{V}_2^*\mu(\nu(\mathcal{V}_1)(p, q), t)\mathcal{V}_2\\&= \mu(\nu(\mathcal{V}_2)\nu(\mathcal{V}_1)(p, q), t),\end{aligned}$$

for all $\mathcal{V}_1, \mathcal{V}_2 \in U(L^2(I\!R^n))$ and $|C| = 1$. In the following lemma we deal with some other properties of the mapping ν.

Lemma 4.1 *Let G be the subgroup of $U(L^2(I\!R^n))$ as defined in (4.13) and let ν be the mapping as defined in (4.15). Then ν is a continuous mapping from G onto $Sp(n)$ in the subspace topology of $G \subset U(L^2(I\!R^n))$. The kernel of ν is given by Ker $\nu = \{C\mathcal{I} \mid |C| = 1\}$.*

Proof
Since g' is uniquely determined in (4.13) it follows that $\nu(\mathcal{V})$ is a non-singular mapping on $I\!R^{2n}$, or equivalently $\nu(\mathcal{V}) \in GL(2n)$ for all $\mathcal{V} \in G$. To show that $\nu(\mathcal{V}) \in Sp(n)$, we take $T = \nu(\mathcal{V})$ and $p_1, p_2, q_1, q_2 \in I\!R^n$. Then by (4.14) we get for $\alpha = 1$ and $\beta = 1$

$$\begin{aligned}&\mu(T(p_1 + p_2, q_1 + q_2), (J_n(p_1, q_1), (p_2, q_2))/2) =\\&\mu(T(p_1 + p_2, q_1 + q_2), (J_n(x_1, y_1), (x_2, y_2))/2) =\\&\mu(T(p_1 + p_2, q_1 + q_2), (T^T J_n T(p_1, q_1), (p_2, q_2))/2).\end{aligned}$$

This result must hold for all $p_1, p_2, q_1, q_2 \in I\!R^n$. This implies that $T^T J_n T J_n^T = I$, which is equivalent with the condition in (4.12).

To compute the kernel of ν we take $\mathcal{V}$ such that $\nu(\mathcal{V}) = I$. This yields $\mathcal{V}\mu\mathcal{V}^* = \mu$. Since μ is irreducible this equation yields $\mathcal{V} = CI$, with $|C| = 1$.

To complete this proof we show the continuity of the mapping. Let $\mathcal{V}_1, \mathcal{V}_2 \in G$ and $\mathcal{W} = \mathcal{V}_2 - \mathcal{V}_1$. Then for all $p, q \in I\!\!R^n$

$$\begin{aligned}\mu((\nu(\mathcal{V}_2) - \nu(\mathcal{V}_1))(p,q), t) &= \mu(\nu(\mathcal{V}_2)(p,q), 0)\, \mu(\nu(\mathcal{V}_1)(-p,-q), 0) \\ &= \mathcal{V}_2^*\mu(p,q,0)(\mathcal{W} + \mathcal{V}_1)\mathcal{V}_1^*\mu(-p,-q,0)\mathcal{V}_1 \\ &= \mathcal{I} - \mathcal{V}_2^*\mathcal{W} + \mathcal{V}_2^*\mu(p,q,0)\mathcal{W}\mathcal{V}_1^*\mu(-p,-q,0)\mathcal{V}_1,\end{aligned}$$

with $t = -(\nu(\mathcal{V}_1)^T J_n \nu(\mathcal{V}_2)(p,q), (p,q))$. Consequently,

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{p,q\in I\!\!R^n} \;:\; \|\mathcal{V}_2 - \mathcal{V}_1\|_2 < \delta \implies \|\mu((\nu(\mathcal{V}_2) - \nu(\mathcal{V}_1))(p,q), t) - \mu(0,0,0)\|_2 < \varepsilon.$$

It can be shown, see e.g. [40], that $\|\mu(p,q,t) - \mu(0,0,0)\|_2 \to 0$ implies $(p,q,t) \to (0,0,0)$. Since the latter result must hold for all $p, q \in I\!\!R^n$, we get $\|\nu(\mathcal{V}_2) - \nu(\mathcal{V}_1)\|_2 \to 0$. This condition is not only necessary to obtain $\|\mu(x,y,t) - \mu(0,0,0)\|_2 \to 0$. It is also sufficient, since $t \to -(\nu(\mathcal{V}_1)^T J_n \nu(\mathcal{V}_1)(p,q), (p,q)) = -(J_n(p,q), (p,q)) = 0$, if $\nu(\mathcal{V}_2) \to \nu(\mathcal{V}_1)$. □

For solving our original problem, namely to find unitary operators on $L^2(I\!\!R^n)$ that act like affine transformations in the Wigner plane, we combine (4.10), (4.11) and Lemma 4.1. This results into the following theorem.

Theorem 4.2 *Let $\mathcal{V}$ be a unitary operator on $L^2(I\!\!R^n)$ and A a linear transformation on $I\!\!R^{2n}$. Then*

$$\mathcal{WV}[\mathcal{V}f](x,\omega) = \mathcal{WV}[f](A(x,\omega)). \tag{4.16}$$

if and only if

(i) $\mathcal{V} \in G$, *with* G *as defined in (4.13),*

(ii) $A \in Sp(n)$,

(iii) $A = \nu(\mathcal{V})^{-T}$, *with* ν *the continuous mapping from* G *onto* $Sp(n)$ *as defined in (4.15).*

Theorem 4.2 tells us under which conditions unitary operators on $L^2(I\!\!R^n)$ act like affine transformations in the Wigner plane, namely if they belong to G. However, Theorem 4.2 does not tell us explicitly which unitary operators satisfy (4.16), e.g. by means of a representation formula for such operators. In the following examples we revisit three operators, that have been considered in the beginning of this section. We show that these three operators are elements of G and we compute $\nu(\mathcal{V})$. These three operators will give us some insight in the type of operators, that G consists of. In Section 5 we will present a representation formula that gives us an explicit formula for all operators in G.

Example 4.3 The first unitary operator we consider is the Fourier transform on $L^2(I\!\!R^n)$. We derive

$$\begin{aligned}(\mathcal{F}^*\mu(p,q,t)\mathcal{F})[f](x) &= \int_{I\!\!R^n} \hat{f}(\omega + q) e^{i((p,\omega)+(x,\omega)+(p,q)/2+t)}\, d\omega \\ &= \int_{I\!\!R^n} \hat{f}(\omega) e^{i((p,\omega)+(x,\omega)-(p,q)/2-(q,x)+t)}\, d\omega \\ &= e^{i((-q,x)+(-q,p)/2+t)} \int_{I\!\!R^n} \hat{f}(\omega) e^{i(x+p,\omega)}\, d\omega \\ &= \mu(-q,p,t)[f](x),\end{aligned}$$

for all $f \in L^2(I\!\!R^n)$. Consequently, $\mathcal{F} \in G$ and

$$\nu(\mathcal{F}) = J_n^T. \tag{4.17}$$

According to Theorem 4.2 the symplectic transformation in the Wigner plane corresponding to the Fourier transform is given by

$$A = \nu(\mathcal{F})^{-T} = (J_n^T)^{-T} = J_n^T,$$

which corresponds with (4.4).

Example 4.4 The second unitary operator we consider is the dilation operator $\mathcal{D}_B$ on $L^2(\mathbb{R}^n)$, with $B \in \mathbb{R}^{n\times n}$ and $\det B \neq 0$. We derive

$$\begin{aligned}(\mathcal{D}_B^*\mu(p,q,t)\mathcal{D}_B)[f](x) &= e^{i(p,Bx)}e^{i(t+(p,q)/2)}f(x+B^{-1}q)\\ &= e^{i(B^Tp,x)}e^{i(t+(B^Tp,B^{-1}q)/2)}f(x+B^{-1}q)\\ &= \mu(B^Tp,B^{-1}q,t)[f](x).\end{aligned}$$

this shows that also $\mathcal{D}_B \in G$ for $B \in GL(n)$. Moreover, we have

$$\nu(\mathcal{D}_B) = \begin{pmatrix} B^T & 0 \\ 0 & B^{-1} \end{pmatrix}. \tag{4.18}$$

Now, Theorem 4.2 states that the action of the dilation operator in the Wigner plane is given by

$$A = \nu(\mathcal{D}_B)^{-T} = \begin{pmatrix} B^T & 0 \\ 0 & B^{-1} \end{pmatrix}^{-T} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix}.$$

We observe that this result corresponds to the linear transformation that we derived in (4.7).

Example 4.5 The last unitary operator we consider here is the operator $\mathcal{C}_S$ with $S \in \mathbb{R}^{n\times n}$ symmetric, as defined in (4.8). We have already seen

$$(\mathcal{C}_S^*\mu(p,q,t)\mathcal{C}_S)[f](x) \quad = \quad \mu(p+Sq,q,t)[f](x),$$

for $t = 0$. A straightforward computation shows that this result also holds for $t \neq 0$. This result yields that $\mathcal{C}_S \in G$ for $S \in \mathbb{R}^{n\times n}$ symmetric. Furthermore, we have

$$\nu(\mathcal{C}_S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}. \tag{4.19}$$

Theorem 4.2 can also be applied to this operator. This yields

$$A = \nu(\mathcal{C}_S)^{-T} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-T} = \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix},$$

which is the same result we derived in (4.9).

We observe that the fractional Fourier transform on $L^2(\mathbb{R}^n)$ is a combination of the three unitary operators discussed in the previous examples. We have for $0 < |\alpha_i| < \pi$, $i = 1,\ldots,n$,

$$\mathcal{F}_{\alpha_1,\ldots,\alpha_n} = C_{\alpha_1}\cdots C_{\alpha_n}\,\mathcal{C}_{S(\underline{\alpha})}\mathcal{D}_{B(\underline{\alpha})}\mathcal{F}\mathcal{C}_{S(\underline{\alpha})}, \tag{4.20}$$

with

$$S(\underline{\alpha}) = \text{diag}(\cot\alpha_1,\ldots,\cot\alpha_n) \;\text{ and }\; B(\underline{\alpha}) = \text{diag}(\sin\alpha_1,\ldots,\sin\alpha_n).$$

Starting from (4.20) a limit process determines the FRFT if $\alpha_i = 0$ or $\alpha_i = \pi$ for some $i = 1,\ldots,n$.

The following theorem classifies all possible elements of $Sp(n)$. A proof of this result can be found in [9, 40].

Theorem 4.6 (Bruhat Decomposition) *Let G be the group as defined in (4.13) and let ν be the anti-homomorphism from G onto $Sp(n)$ as defined in (4.15). Then ν is surjective. Moreover, let J_n, $\nu(\mathcal{D}_B)$ and $\nu(\mathcal{C}_S)$ be the real valued $(n \times n)$ matrices as given in (4.5), (4.18) and (4.19) and let*

$$G_1 = \{\nu(\mathcal{C}_S) \mid S \in I\!R^{n\times n},\, S^T = S\}$$

and

$$G_2 = \{\nu(\mathcal{D}_B) \mid B \in I\!R^{n\times n},\, \det B \neq 0\},$$

then $Sp(n)$ is generated by $G_1 \cup G_2 \cup \{J_n\}$.

This result is a corollary of the generalized Bruhat decomposition with respect to a suitable maximal parabolic subgroup [42].

The next corollary combines Theorem 4.2 and Theorem 4.6. It characterizes all unitary operators on $L^2(I\!R^n)$ that correspond to linear transformations in the Wigner plane.

Corollary 4.7 *Let $f, g \in L^2(I\!R^n)$. Then*

$$\mathcal{WV}[g](x,\omega) = \mathcal{WV}[f](T(x,\omega)),$$

for some $T \in Sp(n)$ if and only if

$$g = C\,\mathcal{U}_1 \cdots \mathcal{U}_N f,$$

with $|C| = 1$ and $\mathcal{U}_i = \mathcal{C}_S, \mathcal{U}_i = \mathcal{D}_B$ or $\mathcal{U}_i = \mathcal{F}$, with $S \in I\!R^{n\times n}$ symmetric and $B \in I\!R^{n\times n}$ non-singular, for $i = 1, \ldots, N$, and $N \in I\!N$.

We omit the proof of this corollary since it follows immediately from Theorem 4.2 and Theorem 4.6 by observing that $\nu(\mathcal{F})^{-T} = \nu(\mathcal{F})$, $\nu(\mathcal{D}_B)^{-T} = \nu(\mathcal{D}_{B^{-T}})$ and $\nu(\mathcal{C}_S)^{-T} = J_n^T \nu(\mathcal{C}_S) J_n = \nu(\mathcal{F} C_S \mathcal{F}^*)$.

The classification presented in Corollary 4.7 also holds for the mixed Wigner distribution. For a unitary operator $\mathcal{V}$ on $L^2(I\!R^n)$ that corresponds to a linear transformation A in the Wigner plane we also have by polarization

$$\mathcal{WV}[\mathcal{V}f, \mathcal{V}g](x,\omega) = \mathcal{WV}[f,g](A(x,\omega)), \tag{4.21}$$

with $A \in Sp(n)$ and for $f, g \in L^2(I\!R^n)$.

In Section 5 this relation is used to come to a representation formula for the unitary operators as discussed in Corollary 4.7.

4.2 The FRFT Generalized

As we have seen in (4.20) the fractional Fourier transform on $L^2(I\!R^n)$ can be decomposed into four unitary operators, namely a chirp multiplication, the Fourier transform, a dilation and again a chirp multiplication. Both the chirp multiplications and the dilation depend on a set of parameters $\alpha_1, \ldots, \alpha_n$, that determine the FRFT. Therefore, a natural generalization of the FRFT is given by

$$\mathcal{F}_{\Gamma,\Delta} = C\,\mathcal{C}_\Gamma \mathcal{D}_\Delta \mathcal{F} \mathcal{C}_\Gamma, \tag{4.22}$$

for some $|C| = 1$, $\Gamma, \Delta \in I\!R^{n\times n}$, both symmetric and Δ non-singular. We observe, that Δ is not required to be symmetric in (4.6). Here we require the symmetry of Δ to obtain a symmetrical representation formula for the generalized FRFT.

We observe, that (4.22) generalizes the n-dimensional FRFT, which was introduced in Section 3.2. Indeed, by taking

$$\Gamma = \mathrm{diag}(\cot\alpha_1, \ldots, \cot\alpha_n) \;\text{ and }\; \Delta = \mathrm{diag}(\sin\alpha_1, \ldots, \sin\alpha_n) \tag{4.23}$$

the generalized FRFT with the definition of the n-dimensional FRFT.

As a consequence of Corollary 4.7, we have for all operators $\mathcal{F}_{\Gamma,\Delta}$

$$\mathcal{WV}[\mathcal{F}_{\Gamma,\Delta}f](x,\omega) = \mathcal{WV}[f](A(x,\omega)),$$

for some $A \in Sp(n)$. Using (4.17), (4.18) and (4.19) we compute straightforwardly

$$\begin{aligned} A &= \nu(\mathcal{C}_\Gamma \mathcal{D}_\Delta \mathcal{F} \mathcal{C}_\Gamma)^{-T} = \nu(\mathcal{C}_\Gamma)^{-T}\nu(\mathcal{F})^{-T}\nu(\mathcal{D}_\Delta)^{-T}\nu(\mathcal{C}_\Gamma)^{-T} \\ &= \begin{pmatrix} \Delta\Gamma & -\Delta \\ -\Gamma\Delta\Gamma + \Delta^{-1} & \Gamma\Delta \end{pmatrix}. \end{aligned} \tag{4.24}$$

Taking Γ and Δ as in (4.23) we arrive at the matrix A as given in (3.21).

A special property of the FRFT is that for its corresponding transformation in the Wigner plane we have $A \in Sp(n) \cap SO(2n)$, the orthonormal symplectic group . One may ask whether the generalized FRFT is also related to an orthogonal transformation in the Wigner plane. The answer to this question is given in the following lemma.

Lemma 4.8 *Let $\mathcal{F}_{\Gamma,\Delta}$ be the generalized FRFT as defined in (4.22), for certain symmetric real valued $(n \times n)$ matrices Γ and Δ. Then A as given by (4.24) is orthogonal if and only if*

(i) $\Delta^{-2} - \Gamma^2 = I$,

(ii) $\Gamma\Delta^{-1}$ *is symmetric.*

Proof
We compute

$$A^T A = \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix},$$

with

$$\begin{aligned} X &= \Gamma\Delta\Gamma - \Gamma\Delta\Gamma^2\Delta\Gamma + \Delta^{-2} - \Delta^{-1}\Gamma\Delta\Gamma - \Gamma\Delta\Gamma\Delta^{-1}, \\ Y &= \Delta^{-1}\Gamma\Delta - \Gamma\Delta^2 - \Gamma\Delta\Gamma^2\Delta, \\ Z &= \Delta + \Delta\Gamma^2\Delta. \end{aligned}$$

For orthonormal A we should have $X = Z = I$ and $Y = 0$. The condition $Z = I$ yields $\Delta^{-1}Z\Delta^{-1} = \Delta^{-2}$, which equals (i). Obviously, Condition (i) is also sufficient to guarantee $Z = I$. Substituting (i) into the matrix Y yields

$$Y = 0 \iff \Gamma\Delta^{-1} = \Delta^{-1}\Gamma \iff \Gamma\Delta^{-1} = (\Gamma\Delta^{-1})^T.$$

After substituting Condition (i) and (ii) in the matrix X we get $X = I$. So for the equation $X = I$ no further conditions are required. □

We observe that Conditions (i) and (ii) in Lemma 4.8 are equivalent with

$$(\Delta^{-1} + \Gamma)(\Delta^{-1} - \Gamma) = I.$$

It follows from this relation, that we have $n^2/2 + n$ degrees of freedom for choosing symmetric matrices Γ and Δ, such that the matrix A corresponding to $\mathcal{F}_{\Gamma,\Delta}$ is orthogonal. Therefore, for higher dimensional function spaces we may expect more variety in the class of operators $\mathcal{F}_{\Gamma,\Delta}$ that yield orthogonal symplectic transformations in the Wigner plane. For the one-dimensional case the one-parameter family of the FRFT turns out to be the only transformation up to a constant, that is in the class of generalized FRFT and that acts like an orthogonal transform in the Wigner plane.

Lemma 4.9 *Let $\mathcal{F}_{\Gamma,\Delta}$ be the unitary operator on $L^2(I\!R)$ as given in (4.22), with $\Gamma, \Delta \in I\!R$. Then $A = \nu(\mathcal{F}_{\Gamma,\Delta})^{-T}$ is orthonormal if and only if $\mathcal{F}_{\Gamma,\Delta} = C\,\mathcal{F}_\alpha$, for some $\alpha \in I\!R$ and C with $|C| = 1$.*

Proof
In the case that Γ and Δ are scalars, the conditions in Lemma 4.8 reduce to

$$\Delta^{-2} = 1 + \Gamma^2.$$

This equation can be parameterized by taking $\Gamma = \cot\alpha$ and $\Delta = \sin\alpha$, for some $\alpha \in I\!R$. Substituting this parameterization into (4.22) leaves the FRFT $\mathcal{F}_\alpha$ up to a constant of absolute value 1, which does not affect A. $\square$

As we expected from the considerations before Lemma 4.9, this lemma cannot be extended in a canonical way to higher dimensions. This is shown by the following example for $n = 2$. Moreover, by extending the example to higher dimensions in a natural way it follows that the preceding lemma can only hold for $\mathcal{F}_{\Gamma,\Delta} \in U(L^2(I\!R))$.

Example 4.10 We consider $\mathcal{F}_{\Gamma,\Delta}$ on $L^2(I\!R^2)$, with

$$\Gamma = \begin{pmatrix} r_1^2\cos^2\alpha + r_2^2\sin^2\alpha & (r_1 - r_2)\cos\alpha\sin\alpha \\ (r_1 - r_2)\cos\alpha\sin\alpha & r_1^2\sin^2\alpha + r_2^2\cos^2\alpha \end{pmatrix} \text{ and}$$

$$\Delta = \begin{pmatrix} \rho_1^2\cos^2\alpha + \rho_2^2\sin^2\alpha & (\rho_1 - \rho_2)\cos\alpha\sin\alpha \\ (\rho_1 - \rho_2)\cos\alpha\sin\alpha & \rho_1^2\sin^2\alpha + \rho_2^2\cos^2\alpha \end{pmatrix}^{-1},$$

with $\alpha \in I\!R$ and $\rho_i^2 = 1 + r_i^2,\ i = 1, 2$. Then

$$\Delta^{-2} - \Gamma^2 = \begin{pmatrix} \rho_1^2 - r_1^2 & 0 \\ 0 & \rho_2^2 - r_2^2 \end{pmatrix} = I,$$

and

$$\Gamma\Delta^{-1} = \begin{pmatrix} r_1^2\rho_1^2\cos^2\alpha + r_2^2\rho_2^2\sin^2\alpha & (r_1\rho_1 - r_2\rho_2)\cos\alpha\sin\alpha \\ (r_1\rho_1 - r_2\rho_2)\cos\alpha\sin\alpha & r_1^2\rho_1^2\sin^2\alpha + r_2^2\rho_2^2\cos^2\alpha \end{pmatrix} = (\Gamma\Delta^{-1})^T.$$

Consequently, the matrices Γ and Δ satisfy the conditions in Lemma 4.8. The orthogonal symplectic transformation in the Wigner plane, that corresponds to $\mathcal{F}_{\Gamma,\Delta}$ is now given by $A = U(\alpha)^T M U(\alpha)$, with

$$M = \begin{pmatrix} -r_1/\rho_1 & 0 & -1/\rho_1 & 0 \\ 0 & -r_2/\rho_2 & 0 & -1/\rho_2 \\ 1/\rho_1 & 0 & -r_1/\rho_1 & 0 \\ 0 & 1/\rho_2 & 0 & -r_2/\rho_2 \end{pmatrix}$$

and

$$U(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha & -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha & \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

Resuming, we have extended the FRFT to a unitary transformation on $L^2(I\!R^n)$ given by $\mathcal{F}_{\Gamma,\Delta}$, where $\Gamma, \Delta \in I\!R^{n\times n}$, both symmetric and Δ non-singular. So the set of all generalizations of the FRFT on $L^2(I\!R^n)$ of this kind are given by the set

$$V_n = \{\mathcal{F}_{\Gamma,\Delta} \mid \Gamma, \Delta \in I\!R^{n\times n} \text{ symmetric},\ \det\Delta \neq 0\}.$$

Furthermore, a subset of V_n is defined consisting of all $\mathcal{F}_{\Gamma,\Delta} \in V_n$ that act like orthogonal transformations in the Wigner plane. This subset is given by

$$W_n = \{\mathcal{F}_{\Gamma,\Delta} \in V_n \mid \Delta^{-2} - \Gamma^2 = I, \Gamma\Delta = (\Gamma\Delta)^T\}.$$

For the FRFT we have $\mathcal{F}_{\alpha_1,\dots,\alpha_n} \in W_n \subset V_n$. Moreover, for the one-dimensional case we have

$$W_1 = \{C\,\mathcal{F}_\alpha \mid \alpha \in I\!R,\ |C| = 1\}$$

and

$$W_n \backslash \{\mu\mathcal{F}_{\alpha_1,\dots,\alpha_n} \mid \alpha_1, \dots, \alpha_n \in I\!R,\ |\mu| = 1\} \neq \emptyset,$$

for $n \geq 2$.

5. A Representation Formula

In this section we present a representation formula for all unitary operators $\mathcal{V}$ on $L^2(I\!R^n)$ for which there exists a transformation A on $I\!R^{2n}$ such that

$$\mathcal{WV}[\mathcal{V}f, \mathcal{V}g](x,\omega) = \mathcal{WV}[f,g](A(x,\omega)). \tag{5.1}$$

We observe, that for the particular choice $f = g$, (5.1) coincides with (4.3). We have already shown that (5.1) can only be realized for symplectic transformations A. Therefore, we start with some properties of symplectic matrices.

Given a matrix $A \in Sp(n)$, then we can represent A by its 2×2 block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{5.2}$$

Since A is symplectic, it has to satisfy (4.12). This yields for the block decomposition

$$A^{-1} = \begin{pmatrix} A_{22}^T & -A_{12}^T \\ -A_{21}^T & A_{11}^T \end{pmatrix}, \tag{5.3}$$

or equivalently

$$A_{22}^T\, A_{11} - A_{12}^T\, A_{21} = I, \tag{5.4}$$
$$A_{11}^T\, A_{21} - A_{21}^T\, A_{11} = 0, \tag{5.5}$$
$$A_{22}^T\, A_{12} - A_{12}^T\, A_{22} = 0. \tag{5.6}$$

Using these relation we prove the following less known properties of symplectic matrices.

Lemma 5.1 *Let $A \in Sp(n)$ be given by its 2×2 block decomposition (5.2). Then the following relations hold*

(i) $(A_{22}^T))^{\leftarrow}(\mathrm{Ran}(A_{12}^T)) = \mathrm{Ran}(A_{12})$,

(ii) $\dim A_{22}(\mathrm{Ker}(A_{12})) = \dim \mathrm{Ker}(A_{12})$,

(iii) $A_{22}(\mathrm{Ker}(A_{12})) = (\mathrm{Ran}(A_{12}))^{\perp}$,

with $\mathrm{Ker}(B)$ *and* $\mathrm{Ran}(B)$ *denoting respectively the null space and range of a linear transformation B and with $B^{\leftarrow}(W)$ denoting the inverse image of a subspace W under the linear transformation B.*

Proof

Let $v \in (A_{22}^T)^{\leftarrow}(\mathrm{Ran}(A_{12}^T))$. Then there exists an $u \in I\!R^n$ such that $A_{22}^T\, v + A_{12}^T\, u = 0$. Hence,

$$A^T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{11}^T\, u + A_{21}^T\, v \\ 0 \end{pmatrix}.$$

Since A is symplectic, we can apply (5.3). This yields

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \begin{pmatrix} A_{21}^T\, v + A_{11}^T\, u \\ 0 \end{pmatrix}.$$

Consequently, $v = -A_{12}\,(A_{21}^T\, v + A_{11}^T\, u) \in \mathrm{Ran}(A_{12})$. On the other hand, if $v \in \mathrm{Ran}(A_{12})$, then there exists a $w \in I\!R^n$ such that $v = A_{12}\, w$. Using (5.6) we get

$$A_{22}^T\, v = A_{22}^T\, A_{12}\, w = A_{12}^T\, A_{22}\, w \in \mathrm{Ran}(A_{12}^T),$$

which proves Property (i).

In order to prove (ii), it is sufficient to show that, if $A_{22}\, u = 0$, for $u \in \mathrm{Ker}(A_{12})$, then $u = 0$. Using (5.4), this follows from

$$u = I_n\, u = A_{11}^T\, A_{22}\, u - A_{21}^T\, A_{12}\, u = 0.$$

For proving Property (iii), we take $u \in A_{22}(\mathrm{Ker}(A_{12}))$ and $v \in \mathrm{Ran}(A_{12})$. Then there exist vectors $x \in \mathrm{Ker}(A_{12})$ and $w \in I\!R^n$, such that $u = A_{22}x$ and $v = A_{12}w$. For proving (iii) we use the following results.

Given a linear transformation B in $I\!R^n$ and a linear subspace V of $I\!R^n$. Then

$$\dim B^{\leftarrow}(V) \geq \dim V, \tag{5.7}$$

$$B^T(V^{\perp}) \subset (B^{\leftarrow}(V))^{\perp}. \tag{5.8}$$

For proving these relations, we put $W = B^{\leftarrow}(V)$. Then, from

$$\dim W = \dim V \cap \mathrm{Ran}(B) + \dim \mathrm{Ker}(B),$$

it follows that

$$\begin{aligned} &\dim W = \dim V + \dim \mathrm{Ran}(B) - \dim(V + \mathrm{Ran}(B)) + \dim \mathrm{Ker}(B) = \\ &\dim V + n - \dim(V + \mathrm{Ran}(B)) \geq \dim V, \end{aligned}$$

which proves (5.7). Now, let $x \in B^T(V^{\perp})$, and $y \in W$. Then $B\, y \in V$, $x = B^T\, u$ for an $u \in V^{\perp}$ and

$$(x, y) = (B^T\, u, y) = (u, B\, y) = 0.$$

Hence, $B^T(V^{\perp}) \subset W^{\perp}$, which proves (5.8). Our next step is to show that if $\dim B^T(V^{\perp}) = \dim V^{\perp}$ then $B^T(V^{\perp}) = (B^{\rightarrow}(V))^{\perp}$. If this result is established, Property (iii) follows immediately from (i) and (ii) by taking $B = A_{22}^T$ and $V = \mathrm{Ran}(A_{12}^T)$. Due to (5.8) we only have to show that

$$\dim(B^{\rightarrow}(V))^{\perp} \leq \dim B^T(V^{\perp}),$$

if $\dim B^T(V^{\perp}) = \dim V^{\perp}$. From (5.7) it follows that

$$\begin{aligned} \dim(B^{\rightarrow}(V))^{\perp} &= n - \dim(B^{\rightarrow}(V)) \leq n - \dim V \\ &= \dim V^{\perp} = \dim B^T(V^{\perp}), \end{aligned}$$

which completes the proof. □

For deriving a representation formula we also need the following result.

Lemma 5.2 *Let W be a subspace of $I\!R^n$ and let B be a linear transformation on $I\!R^n$, such that*

$$\dim(B(W)) = \dim(W) = d.$$

Then

$$\int\limits_W f(Bx)\, dx = \frac{1}{q_W(B)} \int\limits_{B(W)} f(x)\, dx, \quad \forall_{f \in S(I\!R^n)}, \tag{5.9}$$

with $q_W(B)$ the d-dimensional volume of the simplex generated by $B\, e_1, \ldots, B\, e_d$, with $e_1, \ldots e_d$ an orthonormal basis in W.

The proof of this lemma is omitted, since it is straightforward. We observe, that $q_W(B)$ is positive. Furthermore, if W is the null space and B is non-singular, then by setting $q_W(B) = 1$ the definition of $q_W(B)$ is extended in a consistent way.

The last lemma we need to derive our representation formula is as follows.

Lemma 5.3 *Let $f \in S(\mathbb{R}^n)$ and $A \in Sp(n)$ with block decomposition (5.2). Also let* $\dim \text{Ran}(A_{12}) = d > 0$*. Then,*

$$\int\limits_{\text{Ker}(A_{12})} \int\limits_{\mathbb{R}^n} f(u)\, e^{i\,(v, A_{22}^T\, u)}\, du\, dv = \frac{(2\,\pi)^{n-d}}{q_{\text{Ker}(A_{12})}(A_{22})} \int\limits_{\text{Ran}(A_{12})} f(v)\, dv. \tag{5.10}$$

Proof
Since $\dim A_{22}(\text{Ker}(A_{12})) = \dim \text{Ker}(A_{12}) = n - d$, cf. Property (ii) of Lemma 5.1, we may apply Lemma 5.2. This yields

$$\int\limits_{\text{Ker}(A_{12})} \left(\int\limits_{\mathbb{R}^n} f(u)\, e^{i\,(v, A_{22}^T\, u)}\, du \right) dv = (2\,\pi)^{n/2} \int\limits_{\text{Ker}(A_{12})} \hat{f}(A_{22}\, v)\, dv =$$
$$\frac{(2\,\pi)^{n/2}}{q_{\text{Ker}(A_{12})}(A_{22})} \int\limits_{A_{22}(\text{Ker}(A_{12}))} \hat{f}(v)\, dv. \tag{5.11}$$

From Fourier theory we have as a result

$$(2\pi)^{-\dim(W)/2} \int\limits_W \hat{f}(v)\, dv = (2\pi)^{-(n-\dim(W))/2} \int\limits_{W^\perp} f(v)\, dv,$$

for all $f \in S(\mathbb{R}^n)$ and linear subspaces W of $\mathbb{R}^n$. By taking $W = A_{22}(\text{Ker}(A_{12}))$ this result becomes

$$\int\limits_{A_{22}(\text{Ker}(A_{12}))} \hat{f}(v)\, dv = (2\pi)^{n/2-d} \int\limits_{A_{22}(\text{Ker}(A_{12}))^\perp} f(v)\, dv.$$

Since $A_{22}(\text{Ker}(A_{12}))^\perp = \text{Ran}(A_{12})$, we have, cf. Property (iii) of Lemma 5.1,

$$\int\limits_{A_{22}(\text{Ker}(A_{12}))} \hat{f}(v)\, dv = (2\,\pi)^{n/2-d} \int\limits_{\text{Ran}(A_{12})} f(v)\, dv.$$

In combination with (5.11) the latter result establishes the proof. $\square$

The starting point for the derivation of our representation formula is the characteristic function of the Wigner distribution (2.9). For the n-dimensional mixed Wigner distribution, we can also define a characteristic function by

$$M[f,g](\theta,t) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(u + t/2)\, \overline{g(u - t/2)} e^{i(u,\theta)}\, du,$$

or equivalently

$$M[f,g](\theta,t) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(u + t)\, \overline{g(u)} e^{i(u+t/2,\theta)}\, du, \tag{5.12}$$

with $f, g \in L^2(\mathbb{R}^n)$. By the inverse Fourier transform we have

$$f(x)\overline{g(y)} = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} M[f,g](\theta, x - y) e^{-i(\theta, x+y)/2}\, d\theta. \tag{5.13}$$

For the n-dimensional mixed Wigner distribution we have

$$\mathcal{WV}[f](x,\omega) = (2\pi)^{-3n/2} \int\limits_{I\!R^n} \int\limits_{I\!R^n} M[f](\theta,t) e^{-i(\theta,x)} e^{-i(t,\omega)}\, d\theta\, dt. \tag{5.14}$$

Now, let $\mathcal{V}$ be a unitary operator satisfying (5.1). It follows from (5.14) together with (5.1) that

$$M[\mathcal{V}f, \mathcal{V}g] = M[f,g] \circ (A^{-1})^T. \tag{5.15}$$

Combining (5.15) with (5.3) and (5.13) we arrive at

$$\begin{aligned} \mathcal{V}[f](x)\,\overline{\mathcal{V}[g](y)} &= (2\pi)^{-n/2} \int\limits_{I\!R^n} M[f,g]((A^{-1})^T(\theta, x-y)) e^{-i(\theta,x+y)/2}\, d\theta \\ &= (2\,\pi)^{-n} \int\limits_{I\!R^n} \int\limits_{I\!R^n} f(u - A_{12}\,\theta/2 + A_{11}(x-y)/2) \times \\ &\qquad \overline{g(u + A_{12}\,\theta/2 - A_{11}(x-y)/2)}\, E_0(u,\theta,x,y)\, du\, d\theta, a.e.. \end{aligned}$$

for all f and g in $L^2(I\!R^n)$, with

$$E_0(u,\theta,x,y) = \exp(i\,(A_{22}\,\theta - A_{21}(x-y), u) - i\,(\theta, x+y)/2).$$

This last relation only holds formally for general $f,g \in L^2(I\!R^n)$, but it holds rigorously for $f,g \in S(I\!R^n)$. Therefore, we assume $f,g \in S(I\!R^n)$ from now on. After this derivation, we will show that the representation formula also hold for $f \in L^2(I\!R^n)$.

By taking $v = u - A_{11}(x+y)/2$ in the previous result, we have

$$\begin{aligned} &\mathcal{V}[f](x)\,\overline{\mathcal{V}[g](y)} = \\ &(2\,\pi)^{-n} \int\limits_{I\!R^n} \int\limits_{I\!R^n} f(v - A_{12}\,\theta/2 + A_{11}\,x)\,\overline{g(v + A_{12}\,\theta/2 + A_{11}\,y)} \times \\ &\qquad \exp(i\,E_1(v,\theta,x,y))\, dv\, d\theta, \end{aligned}$$

with $E_1(v,\theta,x,y) = (A_{22}\,\theta - A_{21}\,(x-y), v + A_{11}\,(x+y)/2) - (\theta, x+y)/2$. Using Relations (5.4) - (5.6), we can write E_1 as

$$\begin{aligned} E_1(v,\theta,x,y) &= (A_{22}\,\theta - A_{21}\,(x-y), v) + (A_{12}\,\theta, A_{21}\,(x+y))/2 - \\ &\quad (A_{21}\,x, A_{11}\,x)/2 + (A_{21}\,y, A_{11}\,y)/2. \end{aligned}$$

Hence, $\mathcal{V}[f](x)\,\overline{\mathcal{V}[g](y)}$ can be rewritten as

$$\mathcal{V}[f](x)\,\overline{\mathcal{V}[g](y)} = e^{-i\,(A_{21}\,x, A_{11}\,x)/2}\, e^{i\,(A_{21}\,y, A_{11}\,y)/2}\, \mathcal{H}[f,g](x,y), \tag{5.16}$$

with

$$\begin{aligned} \mathcal{H}[f,g](x,y) &= (2\pi)^{-n} \int\limits_{I\!R^n} \int\limits_{I\!R^n} f(v - A_{12}\,\theta/2 + A_{11}\,x)\,\overline{g(v + A_{12}\,\theta/2 + A_{11}\,y)} \times \\ &\qquad e^{i\,(A_{12}\,\theta, A_{21}\,(x+y))/2}\, e^{i\,(A_{22}\,\theta - A_{21}\,(x-y), v)}\, dv\, d\theta. \end{aligned}$$

Our aim is now to write $\mathcal{H}$ in a possible degenerate form. If this is established, then the representation formula for $\mathcal{V}f$ can be read off from this form. To come to such a form we substitute in the latter expression $\theta = \theta_1 + \theta_2$, with $\theta_1 \in \mathrm{Ran}(A_{12}^T)$ and $\theta_2 \in \mathrm{Ker}(A_{12})$. This yields

$$\begin{aligned} &\mathcal{H}[f,g](x,y) = \\ &(2\pi)^{-n} \int\limits_{\mathrm{Ran}(A_{12}^T)} \int\limits_{\mathrm{Ker}(A_{12})} \int\limits_{I\!R^n} f(v - A_{12}\,\theta_1/2 + A_{11}\,x)\,\overline{g(v + A_{12}\,\theta_1/2 + A_{11}\,y)} \times \\ &\qquad e^{i\,(A_{22}\,\theta_1 - A_{21}\,(x-y), v)}\, e^{i\,((A_{12}\,\theta_1, A_{21}\,(x+y))/2 + (A_{22}\,\theta_2, v))}\, dv\, d\theta_2\, d\theta_1. \end{aligned}$$

We are now in the position to apply Lemma 5.3 with respect to the function

$$v \mapsto f(v - A_{12}\,\theta_1/2 + A_{11}\,x)\,\overline{g(v + A_{12}\,\theta_1/2 + A_{11}\,y)}\,e^{i\,(A_{22}\,\theta_1 - A_{21}\,(x-y), v)}.$$

By applying this lemma, we arrive at

$$\begin{aligned}
&\mathcal{H}[f,g](x,y) = \\
&\frac{(2\pi)^{-d}}{q_{\mathrm{Ker}(A_{12})}(A_{22})} \int\limits_{\mathrm{Ran}(A_{12}^T)} \int\limits_{\mathrm{Ran}(A_{12})} f(v - A_{12}\,\theta_1/2 + A_{11}\,x) \times \\
&\qquad\qquad\qquad\qquad \overline{g(v + A_{12}\,\theta_1/2 + A_{11}\,y)} \times \\
&\qquad\qquad e^{i(\,(A_{12}\,\theta_1, A_{21}\,(x+y))/2 + \,(A_{22}\,\theta_1 - A_{21}\,(x-y), v))}\,dv\,d\theta_1,
\end{aligned}$$

with d =dim Ran(A_{12}). Since $v \in$ Ran(A_{12}), we may substitute $v = A_{12}\,w$ with $w \in$ Ran(A_{12}^T), since A_{12} restricted to Ran(A_{12}^T) is a linear bijection onto Ran(A_{12}). We obtain

$$\begin{aligned}
&\mathcal{H}[f,g](x,y) = \\
&C_A^2 \int\limits_{\mathrm{Ran}(A_{12}^T)} \int\limits_{\mathrm{Ran}(A_{12}^T)} f(A_{12}\,w - A_{12}\,\theta_1/2 + A_{11}\,x) \times \\
&\qquad\qquad\qquad \overline{g(A_{12}\,w + A_{12}\,\theta_1/2 + A_{11}\,y)} \times \\
&\qquad e^{i(\,(A_{12}\,\theta_1, A_{21}\,(x+y))/2 + (A_{22}\,\theta_1 - A_{21}\,(x-y), A_{12}\,w))}\,dw\,d\theta_1,
\end{aligned}$$

with

$$C_A = \sqrt{\frac{s(A_{12})}{(2\pi)^d\,q_{\mathrm{Ker}(A_{12})}(A_{22})}}. \tag{5.17}$$

Here $s(A_{12})$ denotes the product of the nonzero singular values of A_{12}, or equivalently

$$s(A_{12}) = q_{\mathrm{Ran}(A_{12}^T)}(A_{12}).$$

Our next step is to substitute $t_1 = w - \theta_1/2$ and $t_2 = w + \theta_1/2$. Then, by using (5.4) - (5.6) one has

$$\begin{aligned}
&(A_{12}\,\theta_1, A_{21}\,(x+y))/2 + (A_{22}\,\theta_1 - A_{21}\,(x-y), A_{12}\,w) = \\
&(A_{12}\,(t_2 - t_1), A_{21}\,(x+y))/2 + (A_{22}\,(t_2 - t_1), A_{12}\,(t_1 + t_2))/2 - \\
&(A_{21}\,(x-y), A_{12}\,(t_1 + t_2))/2 = \\
&-(A_{22}\,t_1, A_{12}\,t_1)/2 + (A_{22}\,t_2, A_{12}\,t_2)/2 - (A_{12}\,t_1, A_{21}\,x) + (A_{21}\,y, A_{12}\,t_2).
\end{aligned}$$

With this result we can rewrite $\mathcal{H}[f,g](x,y)$ in the degenerate form

$$\mathcal{H}[f,g](x,y) = C_(A)^2\,\mathcal{H}_0[f](x)\,\overline{\mathcal{H}_0[g](y)}, \tag{5.18}$$

with

$$\mathcal{H}_0[f](x) = \int\limits_{\mathrm{Ran}(A_{12}^T)} f(A_{12}\,t + A_{11}\,x)\,e^{-i\,((A_{22}\,t, A_{12}\,t)/2 + \,(A_{12}\,t, A_{21}\,x))}\,dt.$$

Finally, combining (5.16) and (5.18) yields the degenerate form for $\mathcal{V}[f](x)\,\overline{\mathcal{V}[g](y)}$

$$\mathcal{V}[f](x)\,\overline{\mathcal{V}[g](y)} = C_A^2\,\mathcal{H}_0[f](x)\overline{\mathcal{H}_0[g](y)}. \tag{5.19}$$

In a natural way this derivation results into the definition of an operator $\mathcal{F}_A$ that satisfies (5.1). We will define this operator on $L^2(\mathbb{R}^n)$ and show that it indeed corresponds to the unitary operator we have been searching for.

Definition 5.4 *Let $A \in Sp(n)$ with block decomposition (5.2). Then the linear operator $\mathcal{F}_A$ on $L^2(I\!R^n)$ is defined as follows. If $\dim(\text{Ran}(A_{12})) > 0$, then*

$$\begin{aligned}\mathcal{F}_A[f](x) \quad &= \quad C_A\, e^{-i\,(A_{11}^T\, A_{21}\, x,\, x)/2} \times \\ &\int\limits_{\text{Ran}(A_{12}^T)} f(A_{12}\, t + A_{11}\, x)\, e^{-i\,(A_{12}^T\, A_{22}\, t,\, t)/2 - i\,(t, A_{12}^T\, A_{21}\, x)}\, dt,\end{aligned} \tag{5.20}$$

for all $f \in L^2(I\!R^n)$ and with C_A as given in (5.17). Furthermore, if $\dim(\text{Ran}(A_{12})) = 0$ then

$$\mathcal{F}_A[f](x) = \sqrt{|\det A_{11}|}\, e^{-i\,(A_{11}^T\, A_{21}\, x,\, x)/2}\, f(A_{11}\, x), \tag{5.21}$$

for all $f \in L^2(I\!R^n)$.

The main theorem of this section can be stated as follows.

Theorem 5.5 *Let $A \in Sp(n)$ and $\mathcal{F}_A$ be given as in Definition 5.4. Then*

$$\mathcal{WV}[\mathcal{F}_A f, \mathcal{F}_A g](x, \omega) = W[f, g](A(x, \omega)),$$

for all $f, g \in L^2(I\!R^n)$.

Proof
If $\dim(\text{Ran}(A_{12})) > 0$ then we conclude from (5.19) and the definition of $\mathcal{F}_A$ that a unitary operator $\mathcal{V}$, for which $W[\mathcal{V}f, \mathcal{V}g](x, \omega) = W[f, g](A(x, \omega))$ holds for all $f, g \in S(I\!R^n)$, must satisfy

$$\mathcal{V}[f](x)\, \overline{\mathcal{V}[g](y)} = \mathcal{F}_A[f](x)\, \overline{\mathcal{F}_A[g](y)} \;\; a.e. \text{ on } I\!R^n,$$

for all $f, g \in S(I\!R^n)$. Hence, $\mathcal{V}$ defined on $S(I\!R^n)$ is equal to $\mathcal{F}_A$ up to a constant C, with $|C| = 1$. Note, that C may depend on A. Since $S(I\!R^n)$ is dense in $L^2(I\!R^n)$, we obtain

$$\mathcal{V} f = C\, \mathcal{F}_A f,$$

for all $f \in L^2(I\!R^n)$. The proof for $\dim(\text{Ran}(A_{12})) > 0$ is completed by assuming, that $\mathcal{V}$ satisfies (5.1).

If $\dim(\text{Ran}(A_{12})) = 0$, we have $A_{12} = 0$. Then (5.4) and (5.5) yield, that A_{11} is non-singular and that $A_{11}^{-1} = {A_{22}}^T$. Moreover, ${A_{11}}^T\, A_{21}$ is symmetric. Using these observations, we compute the mixed Wigner distribution of $\mathcal{F}_A f$ and $\mathcal{F}_A g$ as follows.

$$\begin{aligned}&W[\mathcal{F}_A f, \mathcal{F}_A g](x, \omega) = \frac{|\det A_{11}|}{(2\,\pi)^n} \int\limits_{I\!R^n} f(A_{11}\, x + A_{11}\, t/2) \times \\ &\overline{g(A_{11}\, x - A_{11}\, t/2)}\, e^{-i\,(A_{11}^T\, A_{21}\, x,\, t)}\, e^{-i\,(t, \omega)}\, dt = \\ &(2\pi)^{-n} \int\limits_{I\!R^n} f(A_{11}\, x + t/2)\, \overline{g(A_{11}\, x - t/2)}\, e^{-i\,((A_{11}^T\, A_{21}\, x, A_{11}^{-1}\, t) + (A_{11}^{-1}\, t, \omega))}\, dt.\end{aligned}$$

Hence,

$$\mathcal{WV}[\mathcal{F}_A f, \mathcal{F}_A g](x, \omega) = \mathcal{WV}[f, g](A_{11}\, x, A_{21}\, x + A_{22}\, \omega).$$

This establishes the proof for $\dim(\text{Ran}(A_{12})) = 0$. □

At the end of this section, we present two well-known examples of unitary operators, that satisfy (5.1).

Example 5.6 We recall, that for a set of parameters $\alpha_1, \dots, \alpha_n \in (0, \pi)$ the n-dimensional fractional Fourier transform is given by

$$\mathcal{F}_{\alpha_1, \dots, \alpha_n}[f](x) = \frac{C_{\underline{\alpha}} e^{i\,(B\,x, x)/2}}{\sqrt{(2\,\pi)^n\, |\sin \alpha_1 \cdots \sin \alpha_n|}} \int\limits_{I\!R^n} f(u) e^{i((Bu, u)/2 - (Cx, u))}\, du, \tag{5.22}$$

with $B = \operatorname{diag}(\cot\alpha_1, \ldots, \cot\alpha_n)$, $C = \operatorname{diag}(\csc\alpha_1, \ldots, \csc\alpha_n)$ and $C_{\underline{\alpha}} = C_{\alpha_1} \cdots C_{\alpha_n}$, where C_{α_k} is given by (3.2). The symplectic matrix, that corresponds to this transform in the Wigner plane is given by the rotation matrix $R_{\alpha_1, \cdots, \alpha_n}$ as given in (3.21). We observe, that in this particular case A_{12} is non-singular. This yields $q_{\mathrm{Ker}(A_{12})}(A_{22}) = 1$ and $s(A_{12}) = \det(A_{12})$. Using these simplifications and the substitution $u = A_{12}t + A_{11}x$, Formula (5.20) simplifies to

$$\mathcal{F}_A[f](x) = \frac{e^{-i\,(A_{12}^{-1}\,A_{11}\,x,\,x)/2}}{(2\,\pi)^{n/2}\,\sqrt{|\det A_{12}|}} \int\limits_{\mathbb{R}^n} f(u)\, e^{-i(\,(A_{22}\,A_{12}^{-1}\,u,\,u)/2-\,(x,\,A_{12}^{-1}\,u))}\, du.$$

Taking $A_{11} = A_{22} = \operatorname{diag}(\cos\alpha_1, \ldots, \cos\alpha_n)$ and $A_{12} = \operatorname{diag}(-\sin\alpha_1, \ldots, -\sin\alpha_n)$, the latter representation formula turns into the n-dimensional FRFT as given in (5.22).

Example 5.7 The second example is the unitary operator on $L^2(\mathbb{R}^2)$, which corresponds in the Wigner plane to the symplectic matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Remark, that all matrices in the block decomposition of A are singular.

It can be verified in a straightforward way, that $q_{\mathrm{Ker}(A_{12})}(A_{22}) = 1$ and $s(A_{12}) = 1$. By substituting the block matrices of A into (5.20), the unitary operator, we are dealing with, reads

$$\mathcal{F}_A[f](x_1, x_2) = \frac{1}{\sqrt{2\,\pi}} \int\limits_{\mathbb{R}} f(x_1, \xi)\, e^{-i\,\xi\,x_2}\, d\xi,$$

which is the one-dimensional Fourier-transform of $f(x_1, \cdot)$. We observe, that this operator can also be derived from (5.22) by taking $\alpha_1 \to 0$ and $\alpha_2 \to \pi/2$.

We observe that in [9] and [10] also a representation formula is presented for unitary operators that correspond to symplectic transformations in the Wigner plane. However, both references do not give a formula that can also handle symplectic transformations with a block decomposition, that consists of four singular block matrices, which is the case in the second example.

6. Localization Problems in Phase Space

A celebrated problem in signal processing is the problem of maximizing energy in both time and frequency. This problem already has received much attention in the literature, see e.g. [6, 8, 12, 18].

In this section we discuss two classical problems. The first problem concerns the maximization of energy of time-limited signals within a frequency band, i.e. finite interval in the Fourier domain. For this problem we revisit a series of papers by Slepian and co-workers, [17, 27, 35]. Furthermore, we give a rigorous proof of a conjecture by Slepian [34]. The second problem concerns the maximization of energy within a disk in the Wigner plane, i.e., the phase space related to the Wigner distribution. Although this problem is discussed in several papers [6, 8, 9, 14], we also present alternative proofs and additional results in this section.

In Section 7 the generalized FRFT will be used to relate several classes of energy maximization problems in phase space to the two classical problems, that are discussed in this section.

6.1 Slepian's Energy Problem

The first problem to be considered in this part of the chapter is the concentration of energy in a certain frequency band of a time-limited signal. So we consider for $f \in L^2([-x_0, x_0])$, for some fixed $x_0 > 0$, the ratio

$$E_f(\omega_0) = \frac{\int\limits_{-\omega_0}^{\omega_0} |\hat{f}(\omega)|^2 \, d\omega}{\int\limits_{I\!R} |\hat{f}(\omega)|^2 \, d\omega}, \tag{6.1}$$

with $[-\omega_0, \omega_0]$ the frequency band we are looking at in this problem. Obviously, $E_f(\omega_0) \geq 0$, for all $f \in L^2(I\!R)$. Moreover, Corollary 1.10 yields $E_f(\omega_0) < 1$.

Since $E_f(\omega_0) < 1$ for all $f \in L^2(I\!R)$, the problem arises of maximizing this energy ratio over all $f \in L^2([-x_0, x_0])$.

For solving this problem we introduce two operators. The first operator we discuss is the integral operator $\mathcal{B}(\omega_0) : L^2(I\!R) \to L^2(I\!R)$. For $\omega_0 > 0$ fixed, this operator is given by

$$\mathcal{B}(\omega_0)[f](x) = \sqrt{\frac{2}{\pi}} \int\limits_{I\!R} \frac{\sin(\omega_0(x-u))}{(x-u)} f(u) du, \tag{6.2}$$

for all $f \in L^2(I\!R)$. We observe that

$$\mathcal{F}^{-1}[\chi_{[-\omega_0,\omega_0]}](x) = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega_0 x)}{x}.$$

According to Lemma 1.6 the latter result yields

$$\mathcal{F}\mathcal{B}(\omega_0) f = \chi_{[-\omega_0,\omega_0]} \cdot \mathcal{F} f \quad a.e. \text{ on } I\!R. \tag{6.3}$$

Hence $\mathcal{B}(\omega_0)$ is a Hermitian projection operator; in fact it is an orthonormal projection.

The second operator we introduce in relation to the energy localization problem is the projection $\mathcal{P}(x_0) : L^2(I\!R) \to L^2(I\!R)$. For $x_0 > 0$ fixed, this operator is defined by

$$\mathcal{P}(x_0)[f](x) = \begin{cases} f(x), & \text{if } |x| \leq x_0, \\ 0, & \text{if } |x| > x_0. \end{cases} \tag{6.4}$$

By combining the introduced operators we arrive at

$$\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)[f](x) = \begin{cases} \sqrt{\frac{2}{\pi}} \int\limits_{-x_0}^{x_0} \frac{\sin(\omega_0(x-u))}{(x-u)} f(u) du, & |x| \leq x_0, \\ 0 & |x| \geq x_0, \end{cases} \tag{6.5}$$

for all $f \in L^2(I\!R)$. Since the integral kernel in (6.5) is in $L^2([-x_0, x_0]^2)$, we have that $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$ is a Hilbert-Schmidt operator. Hence, $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$ is a compact operator. Also $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$ is positive definite on $L^2([-x_0, x_0])$, which is shown as follows. Using (6.3) we derive

$$\begin{aligned} (\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)f, f)_2 &= (\mathcal{B}(\omega_0)\mathcal{P}(x_0)f, \mathcal{P}(x_0)f)_2 \\ &= (\mathcal{F}\mathcal{B}(\omega_0)\mathcal{P}(x_0)f, \mathcal{F}\mathcal{P}(x_0)f)_2 \\ &= (\chi_{[-\omega_0,\omega_0]} \cdot \mathcal{F}\mathcal{P}(x_0)f, \mathcal{F}\mathcal{P}(x_0)f)_2 \\ &= (\chi_{[-\omega_0,\omega_0]} \cdot \mathcal{F}\mathcal{P}(x_0)f, \chi_{[-\omega_0,\omega_0]} \cdot \mathcal{F}\mathcal{P}(x_0)f)_2 \geq 0. \end{aligned}$$

If we have, for some $f \in L^2(I\!R)$,

$$(\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)f, f)_2 = 0,$$

then $\mathcal{F}\mathcal{P}(x_0)[f](\omega) = 0$, for almost all $\omega \in [-\omega_0, \omega_0]$. However, $\mathcal{F}\mathcal{P}(x_0)f$ is holomorphic by Theorem 1.9. This yields in combination with the latter result $\mathcal{F}\mathcal{P}(x_0)f = 0$, or equivalently $f(x) = 0$ for almost all $|x| < x_0$.

Following Pollack and Slepian [27, 34], we consider possible solutions $\mathcal{P}(x_0)f_{\max}$, with $f_{\max} \in L^2(I\!R)$, that maximize (6.1). To find these solutions we derive

$$E_{\mathcal{P}(x_0)f_{\max}}(\omega_0) \cdot (\mathcal{F}\mathcal{P}(x_0)f_{\max}, \mathcal{F}\mathcal{P}(x_0)f_{\max})_2 = (\chi_{[-\omega_0,\omega_0]} \cdot \mathcal{F}\mathcal{P}(x_0)f_{\max}, \mathcal{F}\mathcal{P}(x_0)f_{\max})_2.$$

Equivalently, using Parseval's theorem and (6.3),

$$E_{\mathcal{P}(x_0)f_{\max}}(\omega_0) \cdot (\mathcal{P}(x_0)f_{\max}, \mathcal{P}(x_0)f_{\max})_2 = (\mathcal{B}(\omega_0)\mathcal{P}(x_0)f_{\max}, \mathcal{P}(x_0)f_{\max})_2.$$

Since $f_{\max}$ is a stationary solution of this equation, it must satisfy

$$\mathcal{B}(\omega_0)\mathcal{P}(x_0)f_{\max} = \lambda\mathcal{P}(x_0)f_{\max}, \tag{6.6}$$

a homogeneous Fredholm equation of the first kind. In fact, $P(x_0)f_{\max}$ should be an eigenfunction of $\mathcal{B}(\omega_0)$ and $E_{\mathcal{P}(x_0)f_{\max}}(\omega_0)$ is the largest eigenvalue of $\mathcal{B}(\omega_0)$.

We recall that $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$ is compact. Furthermore, it is positive definite on $L^2([-x_0, x_0])$. These considerations yield that solutions $\mathcal{P}(x_0)f$ for equation (6.6) only exist for a discrete set of real positive values of λ, with the properties that

$$1 > \lambda_0 > \lambda_1 > \lambda_2 > \dots$$

and $\lim_{k\to\infty} \lambda_k = 0$. In general, the eigenvalues of a compact Hermitian operator are not necessarily distinct. However, for this particular Fredholm operator, Pollack and Slepian have shown in [27], that its eigenvalues are distinct. Also Slepian showed, see [34], that the kernel of the integral operator $\mathcal{B}(\omega_0)$ commutes with the second order differential operator

$$\mathcal{D}(x_0\omega_0) = \frac{d}{dx}(1-x^2)\frac{d}{dx} - (x_0\omega_0)^2x^2. \tag{6.7}$$

Since both operators have the same spectrum, they must have the same eigenvectors.

Differential operator (6.7) is a well-known operator. It arises on separating the 3-dimensional scalar wave equation in a prolate spheroidal coordinate system. Its real-valued eigenfunctions $\psi_0, \psi_1, \psi_2, \dots$ are known as prolate spheroidal wave functions (PSWF), see [7]. We observe, that the concentration of energy problem is solved by $\mathcal{P}(x_0)\psi_0$ and that $E_{\mathcal{P}(x_0)f_{\max}}(\omega_0)$ is given by λ_0.

Some useful properties of the PSWF have been derived in the past. We present some of them in the following lemma. For a proof of these properties we refer to [17, 27, 35].

Lemma 6.1 *Let $\psi_0, \psi_1, \psi_2, \dots$ be the eigenfunctions of $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$ and let their corresponding eigenvalues be given by $\lambda_0, \lambda_1, \lambda_2, \dots$. Then*

(i) $\hat{\psi}_k \in L^2([-\omega_0, \omega_0]) \;\; \forall_{k\in I\!N}$,

(ii) $\int_{-x_0}^{x_0} \psi_k(x)\psi_n(x)\,dx = \lambda_k\delta_{k,n}$,

(iii) $\int_{I\!R} \psi_k(x)\psi_n(x)\,dx = \delta_{k,n}$.

Other properties for the PSWF follow from this lemma, e.g. Theorem 1.9 and (i) yield that ψ_k is holomorphic. However, this lemma does not provide us with an explicit expression for ψ_k and consequently for λ_k. More insight in the behaviour of the eigenvalues λ_k is given by a conjecture of Slepian, which can be proven rigorously by using the following classical result, that is due to Landau and Widom, see [18].

Lemma 6.2 *Let $\mathcal{H}(x_0\omega_0) : L^2(I\!R) \to L^2(I\!R)$ be given by*

$$\mathcal{H}(x_0\omega_0) = \mathcal{P}(x_0\omega_0)\mathcal{B}(1)\mathcal{P}(x_0\omega_0).$$

Furthermore, let $N(\mathcal{H}(x_0\omega_0), p)$, $0 < p < 1$, denote the number of eigenvalues of $\mathcal{H}(x_0\omega_0)$ which are greater than or equal to p. Then

$$N(\mathcal{H}(x_0\omega_0), p) = \frac{2x_0\omega_0}{\pi} + \frac{1}{\pi^2}\log(\frac{1-p}{p})\log(x_0\omega_0) + R(x_0\omega_0), \tag{6.8}$$

with $R(x)$ of order $o(\log(x))$ as $x \to \infty$.

In [34] Slepian already noted that this lemma proves his conjecture on the asymptotic behavior of the eigenvalues $\lambda_k, k \in I\!N$. Here we prove Slepian's conjecture in a rigorous way.

Theorem 6.3 (Slepian's conjecture) *Let $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$ be as defined in (6.5) and let $\lambda_k,\ k \in I\!N$, be its eigenvalues. Then for all $\delta, \varepsilon \in (0,1)$ there exists an $M \in I\!N$ such that*

(i) $\lambda_k < \epsilon$, *for* $k \geq (1+\delta)\frac{2x_0\omega_0}{\pi}$, *and* $x_0\omega_0 > M$,

(ii) $1 - \lambda_k < \epsilon$, *and* $1 \leq k \leq (1-\delta)\frac{2x_0\omega_0}{\pi}$, *for* $x_0\omega_0 > M$.

Moreover, for all $\varepsilon > 0$ and $\theta \in I\!R$, there exist $\delta > 0$ and $M \in I\!N$ such that

(iii) $|\lambda_k - (1+e^{\pi\theta})^{-1}| < \epsilon$, *for* $|k - \frac{2x_0\omega_0}{\pi} - \frac{\theta}{\pi}\log(x_0\omega_0)| < \delta\log(x_0\omega_0)$ *and* $x_0\omega_0 > M$.

Proof
We define $\phi_k(x) = \psi_k(x/\omega_0)$. Then, for $|x| < x_0\omega_0$, we derive

$$\begin{aligned} \lambda_k\phi_k(x) &= \sqrt{\frac{2}{\pi}}\int_{-x_0}^{x_0} \frac{\omega_0\sin(x-u\omega_0)}{(x-u\omega_0)}\phi_k(u\omega_0)\,du \\ &= \sqrt{\frac{2}{\pi}}\int_{-x_0\omega_0}^{x_0\omega_0} \frac{\sin(x-v)}{(x-v)}\phi_k(v)\,dv \end{aligned}$$

or equivalently

$$\mathcal{H}(x_0\omega_0)\phi_k = \lambda_k\phi_k \ \ \forall_{k\in I\!N\setminus\{0\}}.$$

Consequently, Lemma 6.2 can also be applied on the eigenvalues of $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$.

Let $0 < \varepsilon < 1$ and $0 < \delta < 1$. We take $M > 0$ such that

$$\delta > \frac{\log\left(\frac{1-\varepsilon}{\varepsilon}\right)\log x}{2\pi x} + \frac{\pi R(x)}{2x}, \tag{6.9}$$

for $x > M$. Then

$$N(\mathcal{H}(x_0\omega_0), \varepsilon) = \frac{2x_0\omega_0}{\pi} + \frac{1}{\pi^2}\log(\frac{1-\varepsilon}{\varepsilon})\log(x_0\omega_0) + R(x_0\omega_0) < (1+\delta)\frac{2x_0\omega_0}{\pi},$$

for $x_0\omega_0 > M$. Consequently, if $k \geq (1+\delta)\frac{2x_0\omega_0}{\pi}$, then $N(\mathcal{H}(x_0\omega_0), \varepsilon) < k$. This result yields $\lambda_k < \varepsilon$.

For proving Property (ii) we also take $M > 0$ such that (6.9) holds. Then

$$N(\mathcal{H}(x_0\omega_0), 1-\varepsilon) = \frac{2x_0\omega_0}{\pi} - \frac{1}{\pi^2}\log(\frac{1-\varepsilon}{\varepsilon})\log(x_0\omega_0) + R(x_0\omega_0) > (1-\delta)\frac{2x_0\omega_0}{\pi},$$

for $x_0\omega_0 > M$. Therefore, if $1 \leq k \leq (1-\delta)\frac{2x_0\omega_0}{\pi}$, then $N(\mathcal{H}(x_0\omega_0), 1-\varepsilon) > k$, which leads to $1 - \lambda_k < \varepsilon$.

Finally, let $\varepsilon > 0$ and $\theta \in I\!R$. Furthermore, take $\delta > 0$ and $M \in I\!N$ such that

$$\delta < \frac{1}{\pi^2} \log\left(\frac{1+\varepsilon+\varepsilon e^{\pi\theta}}{1-\varepsilon-\varepsilon e^{-\pi\theta}}\right) - \frac{R(x)}{\log x},$$

for $x > M$. Then we have

$$\begin{aligned}
&N(\mathcal{H}(x_0\omega_0), (1+e^{\pi\theta})^{-1}+\varepsilon) = \\
&\frac{2x_0\omega_0}{\pi} + \frac{1}{\pi^2}\log\left(e^{\pi\theta}\cdot\frac{1-\varepsilon-\varepsilon e^{-\pi\theta}}{1+\varepsilon+\varepsilon e^{\pi\theta}}\right)\log(x_0\omega_0) + R(x_0\omega_0) = \\
&\frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) - \frac{1}{\pi^2}\log\left(\frac{1+\varepsilon+\varepsilon e^{\pi\theta}}{1-\varepsilon-\varepsilon e^{-\pi\theta}}\right)\log(x_0\omega_0) + R(x_0\omega_0) < \\
&\frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) - \delta\log(x_0\omega_0)
\end{aligned}$$

for $x_0\omega_0 > M$. Consequently, if

$$k > \frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) - \delta\log(x_0\omega_0),$$

or equivalently, if

$$\frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) - k < \delta\log(x_0\omega_0),$$

then $\lambda_k - (1+e^{\pi\theta})^{-1} < \varepsilon$.

In the same way, we derive

$$\begin{aligned}
&N(\mathcal{H}(x_0\omega_0), (1+e^{\pi\theta})^{-1}-\varepsilon) = \\
&\frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) + \frac{1}{\pi^2}\log\left(\frac{1+\varepsilon++\varepsilon e^{\pi\theta}}{1-\varepsilon-\varepsilon e^{-\pi\theta}}\right)\log(x_0\omega_0) + R(x_0\omega_0) > \\
&\frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) + \delta\log(x_0\omega_0)
\end{aligned}$$

for $x_0\omega_0 > M$. Therefore, if

$$k < \frac{2x_0\omega_0}{\pi} + \frac{\theta}{\pi}\log(x_0\omega_0) + \delta\log(x_0\omega_0),$$

or equivalently, if

$$k - \frac{2x_0\omega_0}{\pi} - \frac{\theta}{\pi}\log(x_0\omega_0) < \delta\log(x_0\omega_0),$$

then $\lambda_k - (1+e^{\pi\theta})^{-1} > -\varepsilon$. Combining these two results establishes the proof of Property (iii). □

From this theorem it follows, that for large $x_0\omega_0$ approximately the first $2x_0\omega_0/\pi$ eigenvalues that correspond to the PSWF attain a value close to unity. For index numbers in a region around $2x_0\omega_0/\pi$ the eigenvalues plunge to zero and attain values close to zero afterwards. The number of eigenvalues in the region where the eigenvalues decrease from close to one to close to zero is proportional to $\log x_0\omega_0$. Remark, that the eigenvalues depend on the product $x_0\omega_0$.

In Figure 1 the eigenvalues of $\mathcal{H}(x_0\omega_0)$ are depicted for a) $x_0\omega_0 = 25$ and b) $x_0\omega_0 = 50$ respectively. We observe that in both figures the number of eigenvalues close to unity is given by $2x_0\omega_0/\pi$. For $x_0\omega_0 = 25$, approximately the first 16 eigenvalues are close to unity. For $x_0\omega_0 = 50$, this number is approximately 32. The number of eigenvalues in the plunge region in Figure 1.b is approximately 1.25 times the number of eigenvalues in this region in Figure 1.a. This corresponds with the observation we have made after Theorem 6.3, namely that the multiplication factor is approximately given by $\log 32/\log 16 = 5/4$.

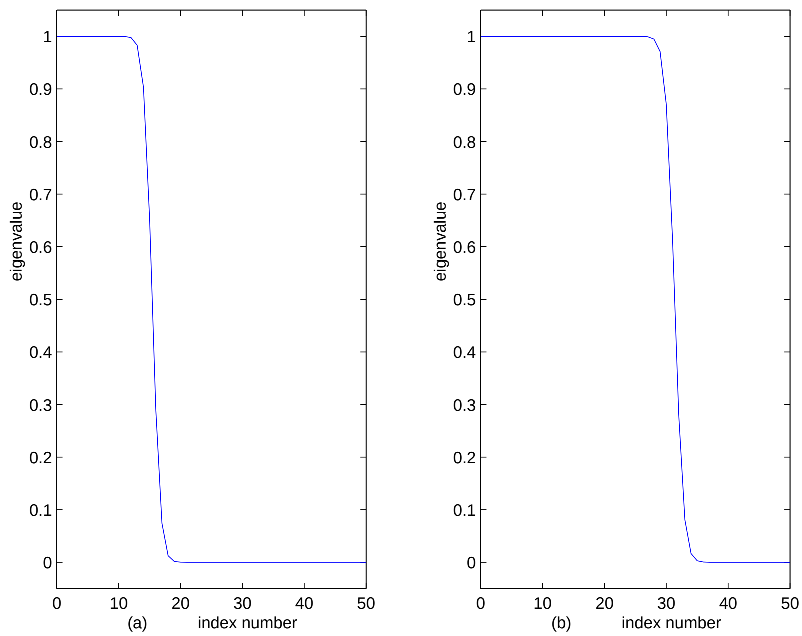

Figure 1: Eigenvalues corresponding to the PSWF for a) $x_0\omega_0 = 25$, b) $x_0\omega_0 = 50$.

6.2 Energy Concentration on a Circle in the Wigner Plane

The second problem to be considered is the concentration of energy in a circular region in the Wigner plane. So we consider a region

$$C_R = \{(x, \omega) \in I\!R^2 \mid x^2 + \omega^2 \leq R\} \tag{6.10}$$

and search for functions $f \in L^2(I\!R)$ for which

$$E_f(R) = \int_{C_R} \mathcal{WV}[f](x, \omega) dx d\omega \,/\, \|f\|_2^2 \tag{6.11}$$

is maximized. An upperbound for $E_f(R)$ follows from an upperbound for $\mathcal{WV}[f]$ which can be derived from (3.16) in the following way

$$|\mathcal{WV}[f](x, \omega)| \quad = \quad |(\mathcal{M}_{-\omega}\mathcal{T}_{-x}f, \mathcal{M}_{\omega}\mathcal{T}_{x}\mathcal{F}f)|/\pi \leq \|f\|_2^2/\pi.$$

This result yields

$$E_f(R) \leq R^2.$$

Of course a better and more natural upperbound for $E_f(R)$ would be given by 1, i.e., if $E_f(R)$ is the total amount of energy of f. A conjecture of Flandrin states that such an upperbound indeed exists, not only for integrals over circular regions, but in general for integrals over convex regions, see [8]. As far as we know, a proof of this conjecture has not been given yet. For non-convex regions this conjecture does not hold, which follows from various examples in [28].

We observe that from (2.14) it follows that

$$E_f(R) \to 1 \ \ (R \to \infty),$$

if also $f \in L^1(I\!R)$ or $\hat{f} \in L^1(I\!R)$. Since the Wigner distribution can attain both positive and negative values, this result is not sufficient to prove Flandrin's conjecture.

In order to solve this energy localization problem, we introduce the localization operator $\mathcal{L}(\sigma)$ on $L^2(I\!R)$, associated with a bounded symbol on $I\!R^2$, by

$$(\mathcal{L}(\sigma)f, g)_2 = \int_{I\!R} \int_{I\!R} \sigma(x,\omega)\, W[f,g](x,\omega)\, dx\, d\omega, \tag{6.12}$$

for all $f, g \in L^2(I\!R)$ and with $\mathcal{WV}[f,g]$ the mixed Wigner distribution of f and g. Then

$$E_f(R) = (\mathcal{L}(\sigma)f, f)_2 \,/ (f,f)_2,$$

with $\sigma = \chi_{C_R}$. Furthermore, we observe that $\mathcal{L}(\sigma)$ is a Weyl transform with symbol $\sigma \in L^2(I\!R^2)$, see [44].

It can be proved, see e.g. [44], that $\mathcal{L}(\sigma)$ is compact for $\sigma \in L^p(I\!R^2)$, $1 \leq p \leq 2$. Moreover, Flandrin showed in [8] that $\mathcal{L}(\sigma)$ is self-adjoint for σ real-valued. This means that $\mathcal{L}(\sigma)$ is a compact Hermitian operator on $L^2(I\!R)$ for real-valued $\sigma \in L^p(I\!R^2)$, $1 \leq p \leq 2$. Consequently, the eigenvectors of $\mathcal{L}(\sigma)$ can be chosen to form an orthonormal basis for $L^2(I\!R)$, the set of real-valued eigenvalues is countable and the only possible accumulation point is 0.

These considerations yield that the function f_{max}, that maximizes $E_f(R)$ is given by the eigenvector ϕ_0 of $\mathcal{L}(\chi_{C_R})$ corresponding to the largest eigenvalue λ_0 of $\mathcal{L}(\chi_{C_R})$. Moreover, $E_{f_{max}}(R)$ is given by λ_0.

The eigenvectors of $\mathcal{L}(\chi_{C_R})$ are given by the Hermite functions $h_k,\ k \in I\!N$, as introduced in (3.11). This result was already given by Janssen in [14]. In the following lemma we come to the same result using a proof based on a property of the fractional Fourier transform.

Lemma 6.4 *Let $C_R = \{(x,\omega) \in I\!R^2 \mid x^2 + \omega^2 \leq R\}$ and $\mathcal{L}(\chi_{C_R})$ as defined in (6.12). Then the eigenvectors of $\mathcal{L}(\chi_{C_R})$ are given by*

$$\{h_k \mid k \in I\!N\}$$

with h_k the Hermite functions as defined in (3.11).

Proof
Since χ_{C_R} is rotation invariant, we have for all $\alpha \in [0, 2\pi)$

$$\begin{aligned}
(\mathcal{L}(\chi_{C_R})\mathcal{F}_\alpha f, \mathcal{F}_\alpha g)_2 &= \int_{C_R} \mathcal{WV}[\mathcal{F}_\alpha f, \mathcal{F}_\alpha g](x,\omega)\, dx\, d\omega \\
&= \int_{C_R} \mathcal{WV}[f,g](R_\alpha(x,\omega))\, dx\, d\omega \\
&= \int_{C_R} \mathcal{WV}[f,g](x,\omega)\, dx\, d\omega = (\mathcal{L}(\chi_{C_R})f, g)_2,
\end{aligned}$$

with R_α the rotation matrix as given in (3.18). Consequently, we have for all $\alpha \in [0, 2\pi)$

$$\mathcal{F}_\alpha \mathcal{L}(\chi_{C_R}) = \mathcal{L}(\chi_{C_R})\mathcal{F}_\alpha.$$

Let now ϕ_k be an eigenvector of $\mathcal{L}(\chi_{C_R})$ and λ_k its corresponding eigenvalue. Then

$$\mathcal{L}(\chi_{C_R})\mathcal{F}_\alpha \phi_k = \mathcal{F}_\alpha \mathcal{L}(\chi_{C_R})\phi_k = \lambda_k \mathcal{F}_\alpha \phi_k.$$

This shows, that if ϕ_k is an eigenvector of $\mathcal{L}(\chi_{C_R})$, then also $\mathcal{F}_\alpha \phi_k$ is an eigenvector of $\mathcal{L}(\chi_{C_R})$ for all $\alpha \in [0, 2\pi)$. Since $\mathcal{L}(\chi_{C_R})$ is compact, the set of eigenvectors

$$\{\mathcal{F}_\alpha \phi_k \mid \alpha \in [0, 2\pi)\}$$

should be finite or countable. This can only be realized if ϕ_k is an eigenvector of $\mathcal{F}_\alpha$ for all $\alpha \in [0, 2\pi)$, i.e., ϕ_k is a Hermite function, following (3.13). □

The eigenvalues λ_k of $\mathcal{L}(\chi_{C_R})$ can be expressed in terms of Laguerre polynomials L_k given by

$$L_k(x) = \frac{1}{k!} e^x \left(\frac{d}{dx}\right)^k \left(e^{-x} x^k\right). \tag{6.13}$$

In the following lemma we present a recurrence relation involving Laguerre polynomials that we shall use to compute the eigenvalues λ_k.

Lemma 6.5 *Define* $I_n(y) = \int\limits_0^y e^{-x/2} L_n(x)\, dx$. *Then*

$$I_{n+1}(y) = -I_n(y) + 2e^{-y/2} \left(L_n(y) - L_{n+1}(y)\right). \tag{6.14}$$

Proof
First we observe that $L_n'(x) = L_{n+1}'(x) + L_n(x)$, which follows from the recurrence relations for Laguerre polynomials, and $L_n(0) = 1$, see e.g. [37]. Integration by parts yields

$$\begin{aligned} I_n(y) &= 2 - 2L_n(y)e^{-y/2} + 2\int\limits_0^y e^{-x/2} L_n'(x)\, dx \\ &= 2 - 2L_n(y)e^{-y/2} + 2I_n(y) + 2\int\limits_0^y e^{-x/2} L_{n+1}'(x)\, dx. \end{aligned}$$

We conclude

$$2\int\limits_0^y e^{-x/2} L_{n+1}'(x)\, dx = -I_n(y) + 2L_n(y)e^{-y/2} - 2.$$

Applying the same procedure on I_{n+1} yields

$$I_{n+1}(y) = 2 - 2L_{n+1}(y)e^{-y/2} + 2\int\limits_0^y e^{-x/2} L_{n+1}'(x)\, dx,$$

or equivalently

$$2\int\limits_0^y e^{-x/2} L_{n+1}'(x)\, dx = I_{n+1}(y) + 2L_{n+1}(y)e^{-y/2} - 2.$$

Combining these two results completes the proof. □

Using this lemma we come to the following recurrence relation for the eigenvalues of $\mathcal{L}(\chi_{C_R})$.

Theorem 6.6 *Let* $\{\lambda_k \mid k \in \mathbb{N}\}$ *denote the set of eigenvalues of* $\mathcal{L}(\chi_{C_R})$, *with*

$$C_R = \{(x, \omega) \in \mathbb{R}^2 \mid x^2 + \omega^2 \leq R\},$$

with $R > 0$. *Then*

- $\lambda_0 = \left(1 - e^{-R^2}\right)$,
- $\lambda_{k+1} = \lambda_k - (-1)^k\, e^{-R^2} \left(L_k(2R^2) - L_{k+1}(2R^2)\right),\ k \in \mathbb{N}\backslash\{0\}$.

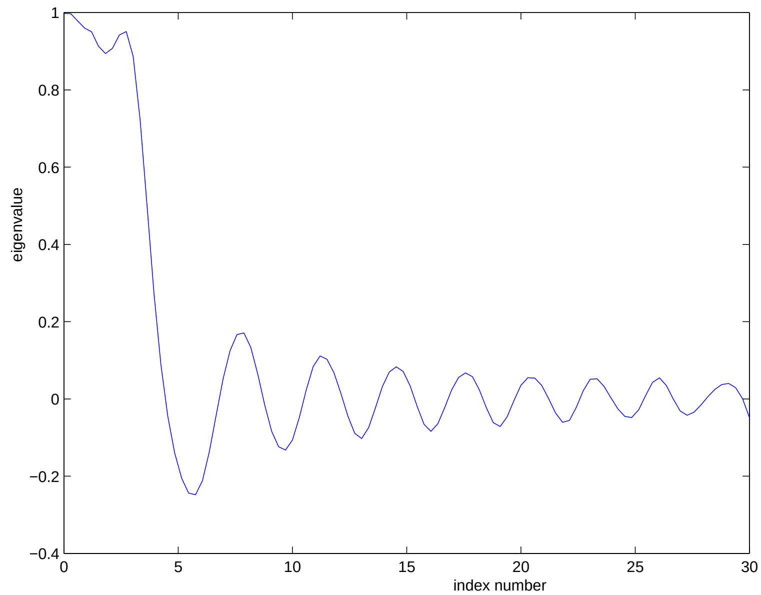

Figure 2: Eigenvalue behavior of the energy localization problem on a disk with radius $R = \sqrt{3}$.

Proof
The Wigner distribution $\mathcal{WV}[h_k](x, \omega)$ can be expressed in terms of Laguerre polynomials, see e.g. [44]. This relation with Laguerre polynomials is given by

$$\mathcal{WV}[h_k](x, \omega) = 2(-1)^k\,(2\pi)^{-1}\,L_k(2(x^2+\omega^2))\,e^{-(x^2+\omega^2)}.$$

Using polar coordinates we get

$$\begin{aligned}\lambda_k &= (\mathcal{L}(\chi_{C_R})h_k, h_k)_2 = \int\limits_{C_R} \mathcal{WV}[h_k](x,\omega)\,dx\,d\omega \\ &= 2\,(-1)^k \int\limits_0^R \rho L_k(2\rho^2)\,e^{-\rho^2}\,d\rho = \frac{(-1)^k}{2}\int\limits_0^{2R^2} e^{-x/2}L_k(x)\,dx \\ &= (-1)^k\,I_k(2R^2)/2.\end{aligned}$$

Consequently, we have

$$\lambda_0 = I_0(2R^2)/2 = 1/2\int\limits_0^{2R^2} e^{-x/2}\,dx = \left(1 - e^{-R^2}\right).$$

Moreover, Lemma 6.5 yields

$$\begin{aligned}\lambda_{k+1} &= (-1)^{k+1}\,I_{k+1}(2R^2)/2 \\ &= (-1)^k\,I_k(2R^2)/2 + (-1)^{k+1}e^{-R^2}\left(L_k(2R^2) - L_{k+1}(2R^2)\right) \\ &= \lambda_k - (-1)^k\,e^{-R^2}\,\left(L_k(2R^2) - L_{k+1}(2R^2)\right).\end{aligned}$$

This gives the recurrence relation for the eigenvalues. □

In Figure 2 the first 30 eigenvalues as given in Theorem 6.6 are depicted for $R = \sqrt{3}$. To emphasize the eigenvalue behavior a spline interpolation function is used in this figure. As we have seen before for the eigenvalues Theorem 6.3, the first eigenvalues are close to λ_0. Later the values plunge down towards zero and remain close to zero for larger index numbers. For the Wigner distribution, the eigenvalues can be negative, which can be observed in Figure 2 as well. Moreover, starting from a certain index number the eigenvalues alternate around zero.

7. Localization Problems and the Generalized FRFT

In this section we return to the fractional Fourier transform as introduced in Section 4.2. This generalized FRFT is used to solve two classes of energy localization problems that are related to the two problems, which we discussed in the previous sections. These two classes of localization problems are related to the discussed problems via the Weyl correspondence.

Although the problems we discuss concern signals in $L^2(\mathbb{R})$ we consider first localization problems for signals in $L^2(\mathbb{R}^n)$. For this we generalize the Weyl correspondence (6.12) to higher dimensions. Then a bounded symbol σ on $\mathbb{R}^{2n}$ is associated with the localization operator $\mathcal{L}(\sigma)$ on $L^2(\mathbb{R}^n)$ by

$$(\mathcal{L}(\sigma)f, g)_2 = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sigma(x,\omega)\, W[f,g](x,\omega)\, dx\, d\omega, \tag{7.1}$$

for all $f, g \in L^2(\mathbb{R}^n)$. Consequently, if $\sigma = \chi_\Omega$, with $\Omega \subset \mathbb{R}^{2n}$, then

$$(\mathcal{L}(\sigma)f, f)_2 = \int\limits_{\Omega} W[f](x,\omega)\, dx\, d\omega$$

represents the energy of f in the Wigner plane within the region Ω.

Using the generalized FRFT $\mathcal{F}_{\Gamma,\Delta}$ as introduced in (4.22) we compute

$$\begin{aligned}
(\mathcal{F}_{\Gamma,\Delta}\mathcal{L}(\sigma)\mathcal{F}^*_{\Gamma,\Delta}f, g)_2 &= (\mathcal{L}(\sigma)\mathcal{F}^*_{\Gamma,\Delta}f, \mathcal{F}^*_{\Gamma,\Delta}g) \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sigma(x,\omega)\, W[\mathcal{F}^*_{\Gamma,\Delta}f, \mathcal{F}^*_{\Gamma,\Delta}g](x,\omega)\, dx\, d\omega \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sigma(x,\omega)\, W[f,g](A^{-1}(x,\omega))\, dx\, d\omega \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sigma(A(x,\omega))\, W[f,g](x,\omega)\, dx\, d\omega \\
&= (\mathcal{L}(\sigma_A)f, g)_2,
\end{aligned}$$

with $\sigma_A(x,\omega) = \sigma(A(x,\omega))$ and A as given in (4.24). Now, assume $\{\phi_k \,|\, k \in \mathbb{N}\}$ is the set of eigenvectors of $\mathcal{L}(\sigma)$ and $\{\lambda_k \mid k \in \mathbb{N}\}$ the set of corresponding eigenvectors. Then

$$\begin{aligned}
\mathcal{L}(\sigma_A)\mathcal{F}_{\Gamma,\Delta}\phi_k &= (\mathcal{F}_{\Gamma,\Delta}\mathcal{L}(\sigma)\mathcal{F}^*_{\Gamma,\Delta})\mathcal{F}_{\Gamma,\Delta}\phi_k \\
&= \mathcal{F}_{\Gamma,\Delta}\mathcal{L}(\sigma)\phi_k = \lambda_k\mathcal{F}_{\Gamma,\Delta}\phi_k.
\end{aligned} \tag{7.2}$$

Consequently, the eigenvectors and eigenvalues of $\mathcal{L}(\sigma_A)$ are given by

$$\{\mathcal{F}_{\Gamma,\Delta}\phi_k \mid k \in \mathbb{N}\} \text{ and } \{\lambda_k \mid k \in \mathbb{N}\}$$

respectively. If $\mathcal{L}(\sigma)$ is a compact operator, both the eigenvectors ϕ_k and $\mathcal{F}_{\Gamma,\Delta}\phi_k$ form an orthonormal set in $L^2(\mathbb{R}^n)$.

7.1 The Rectangle/Parallelogram Case and the Rihaczek Distribution

The first problem we consider is to maximize

$$(\mathcal{L}(\sigma)f, f)_2 \,/\, (f,f)_2 \tag{7.3}$$

for $f \in L^2(I\!R)$, with $\sigma = \chi_{[-x_0,x_0]\times[-\omega_0,\omega_0]}$.

This problem may seem to be similar to Slepian's energy problem in Section 6.1. However, results presented for Slepian's energy problem cannot be related to the problem of localizing the energy on a rectangle in the Wigner plane.

The two problems can only be related to each other if (7.3) is maximized over absolutely integrable $f \in L^2_{\text{comp}}(I\!R)$, with $\text{supp}(f) = [-x_0, x_0]$. Using these constraints (7.3) is equal to (6.1), which follows straightforwardly from Theorem 2.3. If we do not require these constraints on the maximizing function f, we are only provided with some asymptotical results on the eigenvalues of $\mathcal{L}(\sigma)$, see [12, 28].

A less trivial relation with Slepian's energy problem is given for

$$\sigma = \chi_{[-x_0,x_0]\times[-\omega_0,\omega_0]} * \varphi, \tag{7.4}$$

for some $x_0, \omega_0 \in I\!R^+$ and where φ is given by

$$\varphi(x,\omega) = e^{-2ix\omega}.$$

We observe that $\|\sigma\|_\infty \leq 1$, and so $\sigma \in L^\infty(I\!R^2)$.

The following lemma shows that the localization operator $\mathcal{L}(\sigma)$, with σ as in (7.4), can be rewritten as an energy density operator related to the Rihaczek distribution, see [29].

Lemma 7.1 *Let $\mathcal{L}(\sigma)$ be the localization operator as defined in (7.1), with σ the symbol as given in (7.4). Then for all $f, g \in L^2(I\!R)$*

$$(\mathcal{L}(\sigma)f, g)_2 = \int_{I\!R}\int_{I\!R} \chi_{[-x_0,x_0]\times[-\omega_0,\omega_0]}(x,\omega)\mathcal{R}[f,g](x,\omega)\,dx\,d\omega,$$

with $\mathcal{R}[f,g]$ the mixed Rihaczek distribution given by

$$\mathcal{R}[f,g](x,\omega) = f(x)\overline{\hat{g}(\omega)}e^{-i\omega x}/\sqrt{2\pi}. \tag{7.5}$$

Proof
We observe that

$$(\mathcal{L}(\sigma)f, g)_2 = (\sigma_0 * \varphi, \mathcal{WV}[f,g])_2 = (\sigma_0, \varphi * \mathcal{WV}[f,g])_2,$$

with $\sigma_0 = \chi_{[-x_0,x_0]\times[-\omega_0,\omega_0]}$. This expression can be rewritten by

$$\begin{aligned}
&(\varphi * \mathcal{WV}[f,g])(x,\omega) = \\
&\frac{1}{2\pi^2}\int_{I\!R}\int_{I\!R}\int_{I\!R} \varphi(p,q)f(x-p+t)\overline{g(x-p-t)}e^{-2it(\omega-q)}\,dt\,dp\,dq = \\
&\frac{1}{2\pi^2}\int_{I\!R}\int_{I\!R}\int_{I\!R} \varphi(-(u+v)/2,q)f(x+u)\overline{g(x+v)}e^{-i(u-v)(\omega-q)}\,du\,dv\,dq = \\
&\frac{1}{4\pi^2}\int_{I\!R}\int_{I\!R}\int_{I\!R} e^{-iqx}f(u)\overline{g(v)}e^{-iu(\omega-q)}e^{iv\omega}\,du\,dv\,dq = \\
&\frac{1}{2\pi}\int_{I\!R} e^{-iqx}\hat{f}(\omega-q)\overline{\hat{g}(\omega)}\,dq = \frac{1}{2\pi}e^{-i\omega x}\overline{\hat{g}(\omega)}\int_{I\!R} e^{-i\omega x}\hat{f}(q)e^{iqx}\,dq = \\
&f(x)\overline{\hat{g}(\omega)}e^{-i\omega x}/\sqrt{2\pi}.
\end{aligned}$$

This yields $(\mathcal{L}(\sigma)f, g)_2 = \frac{1}{\sqrt{2\pi}}\int_{I\!R}\int_{I\!R} \sigma_0(x,\omega)\, f(x)\overline{\hat{g}(\omega)}e^{-i\omega x}\,dx\,d\omega$ □

Using this lemma we prove the following theorem, that relates $\mathcal{L}(\sigma)$, with σ as in (7.4), with the localization operator of Slepian's energy problem.

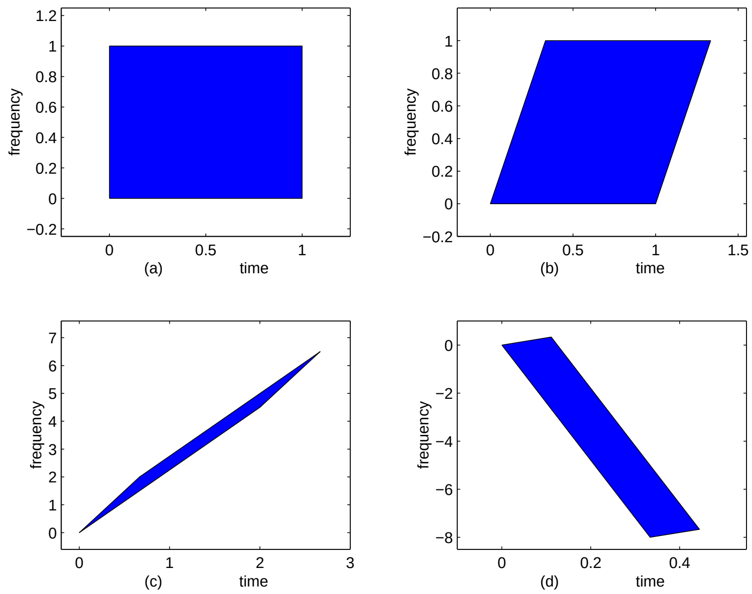

Figure 3: Localisation on a rectangle/parallelogram: fig. a) $\sigma = 1$ on $[0,1] \times [0,1]$ and fig. b, c, d) σ_A with $\Delta = -1/\Gamma, \Delta = -2/\Gamma$ and $\Delta = -1/\Gamma^2$ respectively.

Theorem 7.2 *Let $\mathcal{L}(\sigma)$ be the operator as in (7.1), with σ the symbol as in (7.4). Then*

$$\mathcal{L}(\sigma)^* \mathcal{L}(\sigma) = \mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0),$$

with $\mathcal{B}(\omega_0)$ and $\mathcal{P}(x_0)$ as defined in (6.2) and (6.4) respectively.

Proof
From the preceding lemma it follows immediately that

$$\mathcal{L}(\sigma)^*[g](x) \quad = \quad \chi_{[-x_0,x_0]}(x) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\omega_0}^{\omega_0} \hat{g}(\omega) e^{i\omega x} \, d\omega = \mathcal{P}(x_0)\mathcal{B}(\omega_0).$$

Since both $\mathcal{P}(x_0)$ and $\mathcal{B}(\omega_0)$ are projection operators, we have

$$\mathcal{L}(\sigma)^* \mathcal{L}(\sigma) = \mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0).$$

$\square$

Remark, that although $\sigma \in L^\infty(I\!R)$, $\mathcal{L}(\sigma)$ is compact for σ as in (7.4). This follows from the fact that $\mathcal{L}(\sigma)^*\mathcal{L}(\sigma)$ is compact. Furthermore, we observe that the result of Theorem 7.2 was already given in [8]. However, our aim is not to investigate existing time-frequency distributions, but to consider the generalized FRFT acting on these distributions. In this context, we return to the first part of this section.

We have seen that the eigenvalues of $\mathcal{L}(\sigma)$ and $\mathcal{L}(\sigma_A)$ coincide. In a direct way, we can also show that the eigenvalues of $\mathcal{L}(\sigma)^*\mathcal{L}(\sigma)$ and $\mathcal{L}(\sigma_A)^*\mathcal{L}(\sigma_A)$ coincide. This yields that the singular values of $\mathcal{L}(\sigma)$ and $\mathcal{L}(\sigma_A)$ are the same. These singular values are given by

$$s_k = \sqrt{\lambda_k},$$

where λ_k denote the eigenvalues of the operator $\mathcal{P}(x_0)\mathcal{B}(\omega_0)\mathcal{P}(x_0)$. Since these λ_k satisfy Theorem 6.3, a similar result holds for the singular values. Moreover, the asymptotical behavior of s_k and λ_k is similar.

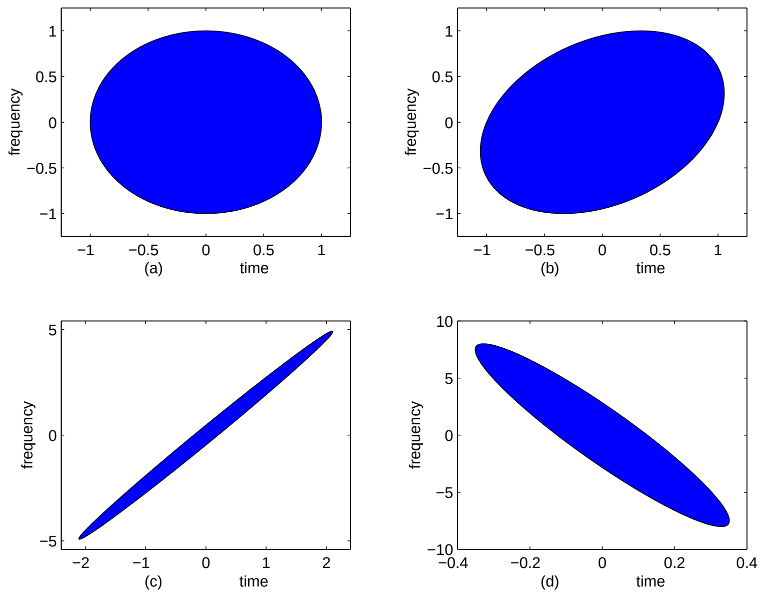

Figure 4: Localisation on a circle/ellipse: fig. a) $\sigma = 1$ on $\{(x, \omega) \in I\!R^2 \mid x^2 + \omega^2 \leq 1\}$, fig. b, c, d) σ_A with $\Delta = -1/\Gamma, \Delta = -2/\Gamma$ and $\Delta = -1/\Gamma^2$ respectively.

The eigenvectors of $\mathcal{L}(\sigma)$ do not follow from Theorem 7.2. Only the eigenvectors of $\mathcal{L}(\sigma)^*\mathcal{L}(\sigma)$ are known, namely the prolate spheroidal wave functions ψ_k. As before we can also show that the eigenvectors of $\mathcal{L}(\sigma_A)^*\mathcal{L}(\sigma_A)$ are then given by $\mathcal{F}_{\Gamma,\Delta}\psi_k$. They can be computed as the eigenvectors of the operator

$$\mathcal{D}'(x_0\omega_0) = \mathcal{F}_{\Gamma,\Delta}\mathcal{D}(x_0\omega_0)\mathcal{F}^*_{\Gamma,\Delta},$$

which is also a second order differential operator that commutes with $\mathcal{L}(\sigma_A)^*\mathcal{L}(\sigma_A)$.

In Figure 3.b,c and d the domain of σ_A is depicted instead of σ, with the substitutions $\Delta = -1/\Gamma$, $\Delta = -2/\Gamma$ and $\Delta = -1/\Gamma^2$ and with $\Gamma = 3$ in (4.24). We observe that with these substitutions $\mathcal{L}(\sigma_A)$ represents the energy of the Rihaczek distribution within differently orientated parallelograms in phase space. The singular values of $\mathcal{L}(\sigma_A)$ for all A related to these parallelograms are the same and are given by $\sqrt{\lambda_k}$, with λ_k as in Theorem 6.3.

7.2 The Circle/Ellipse Case

In Section 6.2 we already discussed the energy localization problem on a circle. Moreover, we studied the operator $\mathcal{L}(\chi_{C_R})$, with C_R a circle in the Wigner plane concentrated around the origin and with radius $R > 0$. It turned out that its eigenvectors are given by the Hermite functions h_k, defined by (3.12), and that the corresponding eigenvalues are given by Theorem 6.6.

It follows from (7.2), that the eigenvectors of $\mathcal{L}(\sigma_A)$, with A as given in (4.24), are given by $\mathcal{F}_{\Gamma,\Delta}h_k$, $k \in I\!N$. The eigenvalues of $\mathcal{L}(\sigma_A)$ are given by the recurrence relation in Theorem 6.6.

In Figure 4.b,c and d the domain of σ_A is depicted with σ the characteristic function of C_R, with the substitutions $\Delta = -1/\Gamma$, $\Delta = -2/\Gamma$ and $\Delta = -1/\Gamma^2$ and with $\Gamma = 3$. With these substitutions $\mathcal{L}(\sigma_A)$ represents the energy in the Wigner plane within differently orientated ellipses. The energy localization problem for each of these ellipsoïdal areas is now solved by the eigenvectors $\mathcal{F}_{\Gamma,\Delta}h_k$, using the corresponding substitutions, and the eigenvalues λ_k.

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