The Stable Central Limit Theorem for Local Martingales with Bounded Jumps via Skorohod Embedding

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ABSTRACT

The stable central limit theorem for properly normalized local martingales with bounded jumps is proved. Instead of the usual characteristic function-type methods we use an embedding technique in combination with a result on nested Brownian motions. In this approach, the stability of the CLT is explained by the fact that nested Brownian motions are asymptotically independent of any other random element. As was previously shown in the special case of continuous local martingales, the embedding technique leads to short and transparent arguments. In the conclusion we discuss the direction in which further research is needed to make the embedding method applicable in an even larger number of situations.

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1 Introduction

In this paper we study the weak convergence of a normalized local martingale M with bounded jumps as 'time' tends to infinity. We suppose that there exist positive numbers c_t increasing to infinity such that $\langle M \rangle_t / c_t \to \eta$ in probability as $t \to \infty$, where η is some nonnegative random variable. We prove that for any random element X on the same probability space as M and with values in an arbitrary Polish space $\mathcal{X}$, we have the weak convergence $(M_t/\sqrt{c_t}, X) \Rightarrow (\sqrt{\eta}Z, X)$ in $\mathbb{R} \times \mathcal{X}$, where Z is a standard normal random variable that is independent of (η, X). In the terminology of stable convergence (see [1]) this amounts to saying that $M_t/\sqrt{c_t} \Rightarrow V$ (stably), where V is a random variable with characteristic function $u \mapsto \mathrm{E} \exp(-\frac{1}{2}\eta u^2)$.

It is well-known that many martingale limit theorems are stable. See for example [1, 8] for stable central limit theorems in discrete time and [7, 11] for continuous-time results. The main result of this paper, theorem 4.1 below, could also be deduced from [11, theorem 5.5.3]. Rather than on the result itself however, our focus is on the methods that give us stable limit theorems. The usual approach uses characteristic function-type methods or rather the 'method of stochastic exponentials' (see [8, 11, 7, 9]). This method is quite powerful and has lead to very general stable limit theorems. A drawback is however the high technical level of the method, which may somewhat obscure intuition.

In the paper [15] we proved the (multivariate) stable central limit theorem for continuous local martingales by using the Dambis-Dubins-Schwarz time-change theorem (see e.g. [13]), which states that every continuous local martingale is in fact a time-changed Brownian motion. In this approach, the stability of the central limit theorem turned out to be a consequence of the fact that 'nested' Brownian motions are asymptotically independent of any other random element (see [15, theorem 3.1]). The purpose of this paper is to present a first extension of this method, namely to the case of local martingales that have bounded jumps. The Dambis-Dubins-Schwarz theorem can not be used in this situation, since it holds only for martingales with continuous paths. Instead we use a general version of Skorohod's embedding theorem, developed in the papers [12, 10, 5]. In the literature on central limit theory the Skorohod embedding theorem has primarily been used to investigate the rate of convergence of the (functional) central limit theorem for martingales (see [8, 10, 5, 2]). In the present paper we use it to prove a version of the stable central limit theorem. This approach leads to short and transparent arguments.

The result that we present deals 'only' with one-dimensional local martingales that have bounded jumps. Of course, we would also like to consider for instance martingales with arbitrarily large jumps and higher dimensional martingales. In the concluding section we indicate in which direction further research is needed in order to make the embedding technique also applicable those situations.

2 Nested Brownian motions

One advantage of the embedding approach is that it gives us a nice explanation of the stability of the martingale central limit theorem. We will get this as a consequence of the following result that states that nested Brownian motions are asymptotically independent of any other random element. The theorem is in fact a special case of a more general result on nested Brownian motions that was presented in [15].

Theorem 2.1. *Let W be a Brownian motion and let $(c_t)_{t\geq 0}$ be a collection of positive numbers that increase to infinity. For every $t \geq 0$, define the Brownian motion W^t by putting*

$$W_s^t = \tfrac{1}{\sqrt{c_t}} W_{c_t s}, \quad s \geq 0. \tag{2.1}$$

Then for any random element X defined on the same probability space as W and with values in an arbitrary Polish space $\mathcal{X}$ we have the weak convergence $(W^t, X) \Rightarrow (B, X)$ in $C[0,\infty) \times \mathcal{X}$, where B is a Brownian motion that is independent of X.

Proof. See [15, theorem 3.1]. $\square$

Remark. Throughout the paper, we use the symbols $\Rightarrow$ and $\xrightarrow{\mathrm{P}}$ to denote weak convergence and convergence in probability in a Polish spaces, respectively. See [3, chapter 1] for more information on the basic weak convergence theory that we use.

3 Skorohod embedding

The classical Skorohod embedding theorem asserts that any random walk with steps that have zero mean and finite variance can be embedded in a Brownian motion (see for instance [4, section 37]). In this paper we will use the following extension of that fact.

Theorem 3.1. *On a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ *let a square integrable martingale* M *and a random element* X *that takes values in a Polish space* $\mathcal{X}$ *be given. Then there exists a filtered probability space* $(\Omega', \mathcal{F}', \mathbb{F}', \mathrm{P}')$ *supporting a square integrable martingale* M'*, an* $\mathcal{X}$*-valued random element* X' *and a càdlàg, adapted, increasing process* $\tau = (\tau_t)_{t\geq 0}$ *with the following properties:*

(i) $\mathcal{L}((M, [M], X) \,|\, \mathrm{P}) = \mathcal{L}((M', [M'], X') \,|\, \mathrm{P}')$.

(ii) $M'_t = W_{\tau_t}$*, where* W *is a process on* $(\Omega', \mathcal{F}', \mathrm{P}')$ *that is a Brownian motion with respect to its natural filtration.*

(iii) $[M'] - \tau$ *is a martingale.*

(iv) If $\mathrm{E}\, \sup_{s\leq t} |M_s|^4 < \infty$ *for all* $t \geq 0$*, then the process*

$$\left(\sum_{s\leq t} \left(4(\Delta M'_s)^4 - (\Delta \tau_s)^2\right)\right)_{t\geq 0}$$

is a submartingale.

We refer to Monroe [12], Kubilius [10] and Coquet et al. [5] for the explicit construction and the proof of the properties (i)-(iv). A difference between theorem 3.1 and the embedding theorem presented in [5] is that we also 'copy' a random element X to the new probability space. This calls for a straightforward adaptation of the construction given in the cited papers and presents no difficulties.

Let us remark that there exists a filtration $\mathbb{G}$ on $(\Omega', \mathcal{F}', \mathrm{P}')$, that is in general larger than the natural filtration of the Brownian motion W, such that W is still a Brownian motion with respect to $\mathbb{G}$ and that each τ_t is a $\mathbb{G}$-stopping time (see [12, theorem 11 and the remark thereafter]). We also note that statement (iv) has higher order analogues where the powers 4 and 2 are replaced by $2p$ and p, for arbitrary $p \geq 1$ (see [5]). These facts are however not important for our purpose.

4 Main result

The following theorem is the main result of the paper.

Theorem 4.1. *Let M be a local martingale with bounded jumps. Suppose that $(c_t)_{t\geq 0}$ is a collection of positive numbers increasing to infinity and such that for some nonnegative random variable η*

$$\frac{\langle M\rangle_t}{c_t} \xrightarrow{\mathrm{P}} \eta \tag{4.1}$$

as $t \to \infty$. Then for any random element X defined on the same probability space as M and with values in an arbitrary Polish space $\mathcal{X}$ we have the weak convergence $(M_t/\sqrt{c_t}, X) \Rightarrow (\sqrt{\eta}Z, X)$ in $\mathbb{R}\times\mathcal{X}$, where Z is a standard normal random variable that is independent of (η, X).

The proof of the theorem is split in two parts, see subsections 4.1 and 4.2 below. In the first part, we work under the additional assumption that M is a martingale satisfying condition (4.2). This allows us to use statement (iv) of theorem 3.1. In the second part we provide the arguments that allow the extra condition (4.2) to be removed.

4.1 Proof of theorem 4.1, part 1

In addition to the conditions of the theorem, we assume for the moment that

$$\mathrm{E}\sup_{s\leq t}|M_s|^4 < \infty \tag{4.2}$$

for all $t \geq 0$. This extra moment condition will be removed in the next subsection. Let a be a bound for the jumps of M, thus $|\Delta M| \leq a$. Denote by (M', η', X') the 'copy' of (M, η, X) on the filtered probability space $(\Omega', \mathcal{F}', \mathbb{F}', \mathrm{P}')$ constructed in theorem 3.1. Part (ii) of theorem 3.1 states that $M'_t = W_{\tau_t}$, where W is a Brownian motion, and therefore

$$\tfrac{1}{\sqrt{c_t}}M'_t = W^t_{\tau_t/c_t},$$

where W^t is given by formula (2.1). So we see that for $M_t/\sqrt{c_t}$ to have a weak limit, we need to control the behaviour of τ_t/c_t as $t \to \infty$.

Since the jumps of M are bounded, it follows from Lenglart's inequality that (4.1) also holds with $[M]$ in the place of $\langle M\rangle$, i.e.

$$\frac{[M]_t}{c_t} \xrightarrow{\mathrm{P}} \eta. \tag{4.3}$$

By part (i) of theorem 3.1 we have $\mathcal{L}(([M'], \eta') \,|\, \mathrm{P}') = \mathcal{L}(([M], \eta) \,|\, \mathrm{P})$ so it follows from (4.3) that

$$\frac{[M']_t}{c_t} \xrightarrow{\mathrm{P}'} \eta'. \tag{4.4}$$

Let us show that (4.4) implies the convergence

$$\frac{\tau_t}{c_t} \xrightarrow{\mathrm{P}'} \eta'. \tag{4.5}$$

Consider the martingale $Z = [M'] - \tau$ (cf. part (iii) of theorem 3.1). Clearly, it suffices to show that

$$\frac{Z_t}{c_t} \xrightarrow{\mathrm{P}'} 0. \tag{4.6}$$

Observe that

$$\Delta Z_t = \Delta[M']_t - \Delta\tau_t = (\Delta M'_t)^2 - \Delta\tau_t.$$

So for the quadratic variation of Z we have

$$[Z]_t = \sum_{s\le t}(\Delta Z_s)^2 = \sum_{s\le t}((\Delta M'_s)^2 - \Delta\tau_s)^2 \le 2\sum_{s\le t}((\Delta M'_s)^4 + (\Delta\tau_s)^2).$$

Using the fact that $Z^2 - [Z]$ is a martingale we thus find that Z^2 is Lenglart dominated by the process $2\sum_{s\le\cdot}((\Delta M'_s)^4 + (\Delta\tau_s)^2)$. In view of assumption (4.2) we may conclude from part (iv) of theorem 3.1 that the process $\sum_{s\le\cdot}(\Delta\tau_s)^2$ is Lenglart dominated by $4\sum_{s\le\cdot}(\Delta M'_s)^4$. It follows that Z^2 is Lenglart dominated by

$$10\sum_{s\le\cdot}(\Delta M'_s)^4 \le 10a^2[M'].$$

So by Lenglart's inequality (see e.g. [9, lemma I.3.30]) and the fact that $|\Delta M| \le a$ we have

$$\mathrm{P}'\left(Z_t^2 \ge \varepsilon\right) \le \frac{\delta}{\varepsilon} + \frac{10a^4}{\varepsilon} + \mathrm{P}'\left(10a^2[M']_t \ge \delta\right) \tag{4.7}$$

for all $\varepsilon, \delta, t > 0$. For given $\lambda, t > 0$ take $\delta = \lambda c_t\sqrt{c_t}$ and $\varepsilon = \lambda c_t^2$ in (4.7) to find that

$$\mathrm{P}'\left(\frac{Z_t^2}{c_t^2} \ge \lambda\right) \le \frac{1}{\sqrt{c_t}} + \frac{10a^4}{\lambda c_t^2} + \mathrm{P}'\left(10a^2\frac{[M']_t}{c_t\sqrt{c_t}} \ge \lambda\right).$$

Then it follows from (4.4) that we indeed have (4.6), which completes the proof of (4.5).

Now that (4.5) has been established, we can finish this part of the proof of theorem 4.1. It follows from theorem 2.1 and (4.5) that we have the weak convergence

$$(W^t, \tau_t/c_t, X') \Rightarrow (B, \eta', X') \tag{4.8}$$

in $C[0,\infty) \times \mathbb{R} \times \mathcal{X}$, where B is a Brownian motion that is independent of (η', X'). Define the map $\phi : C[0,\infty) \times \mathbb{R} \times \mathcal{X} \to \mathbb{R} \times \mathcal{X}$ by $\phi(f, t, x) = (f(t), x)$. This map is continuous, so by part (ii) of theorem 3.1 and the continuous mapping theorem

$$(M'_t/\sqrt{c_t}, X') = \phi(W^t, \tau_t/c_t, X') \Rightarrow \phi(B, \eta', X') = (B_{\eta'}, X').$$

in $\mathbb{R}\times\mathcal{X}$. Using part (i) of theorem 3.1 and noting that $(B_{\eta'}, X')$ has the same law as $(\sqrt{\eta}Z, X)$, where Z is a standard normal random variable that is independent of (η, X) we find that we indeed have the desired weak convergence of $(M/\sqrt{c_t}, X)$. So under the additional assumption that (4.2) holds for all $t \ge 0$, we have proved theorem 4.1.

4.2 Proof of theorem 4.1, part 2

In this subsection the moment condition (4.2) is removed. The crucial point is that by Bernstein's inequality, the process $M^* = \sup_{s \leq \cdot} |M_s|$ can only grow at the rate $\sqrt{c_t}$ if $\langle M \rangle$ grows at the rate c_t.

Lemma 4.2. *Let M be a local martingale with bounded jumps. If $(c_t)_{t \geq 0}$ is a collection of positive numbers increasing to infinity such that $\langle M \rangle_t = O_P(c_t)$ then $M_t^* = O_P(\sqrt{c_t})$ as $t \to \infty$.*

Proof. Let a be a bound for the jumps of M, thus $|\Delta M| \leq a$. The Bernstein inequality for local martingales with bounded jumps (see [14, p. 899] and also [6] for further references) states that

$$\mathrm{P}\left(M_t^* \geq x\,,\ \langle M \rangle_t \leq y\right) \leq 2e^{-\frac{1}{2}\frac{x^2}{ax+y}}$$

for all $t, x, y > 0$. It follows that

$$\mathrm{P}\left(\frac{1}{\sqrt{c_t}} M_t^* \geq K\right) \leq 2e^{-\frac{1}{2}\frac{K^2}{aK+L}} + \mathrm{P}\left(\frac{\langle M \rangle_t}{c_t} > L\right) \tag{4.9}$$

for all $K, L > 0$ and t large enough. Let $\varepsilon > 0$ be given. By assumption, we can choose L large enough to ensure that the probability on the right hand side of (4.9) is less than $\varepsilon/2$ for all t large enough. Choosing K so large that $2\exp(-\frac{1}{2}\frac{K^2}{aK+L}) \leq \varepsilon/2$ then implies that $\mathrm{P}\left(\frac{1}{\sqrt{c_t}} M_t^* \geq K\right) \leq \varepsilon$ for t large enough, which proves the lemma. $\square$

With the help of lemma 4.2 we can construct a very useful localizing sequence (T_n) for the local martingale M. We define

$$T_n = \inf\{t : M_t^* > n\sqrt{c_t}\}.$$

Clearly, $T_1 \leq T_2 \leq \cdots$. It follows from lemma 4.2 that under (4.1) it holds that $T_n \to \infty$ in probability (and therefore also almost surely, since (T_n) is increasing) . This implies that we can find a collection $(n_t)_{t \geq 0}$ of positive integers that increase to infinity and such that

$$\mathrm{P}(T_{n_t} \leq t) \to 0 \tag{4.10}$$

as $t \to \infty$. Define the process N and the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ by putting

$$N_t = M_{T_{n_t} \wedge t}, \quad \mathcal{G}_t = \mathcal{F}_{T_{n_t} \wedge t}$$

for all $t \geq 0$. Then it follows from the optional sampling theorem (see e.g. [11, theorem 1.4.1]) that N is a martingale with respect to $\mathbb{G}$ and

$$\langle N \rangle_t = \langle M \rangle_{T_{n_t} \wedge t}\,.$$

Using (4.10) it is easily seen that (4.1) implies that

$$\frac{\langle N\rangle_t}{c_t} \xrightarrow{\mathrm{P}} \eta. \tag{4.11}$$

Indeed, simply note that for all $\varepsilon > 0$

$$\mathrm{P}\left(\left|\frac{\langle M\rangle_t}{c_t} - \frac{\langle N\rangle_t}{c_t}\right| > \varepsilon\right) \leq \mathrm{P}(T_{n_t} \leq t) \to 0$$

as $t \to \infty$, which yields (4.11). Also note that by definition of the stopping times T_n and the fact that $|\Delta M| \leq a$, it holds that

$$N_t^* = M^*_{T_{n_t}\wedge t} \leq n_t\sqrt{c_t} + |\Delta M_{T_{n_t}\wedge t}| \leq n_t\sqrt{c_t} + a.$$

In particular we have

$$\mathrm{E}\sup_{s\leq t}|N_s|^4 < \infty$$

for all $t \geq 0$. We may thus conclude from the preceding subsection that $(N_t/\sqrt{c_t}, X) \Rightarrow (\sqrt{\eta}Z, X)$ in $\mathbb{R}\times\mathcal{X}$, where Z is a standard normal random variable that is independent of (η, X). Using (4.10) again we see that the difference $N_t/\sqrt{c_t} - M_t/\sqrt{c_t}$ converges to 0 in probability as $t \to \infty$. Therefore, $(M_t/\sqrt{c_t}, X)$ has the same weak limit as $(N_t/\sqrt{c_t}, X)$. This concludes the proof of theorem 4.1.

5 Conclusion

For one-dimensional local martingales with bounded jumps the embedding method proves to be an elegant and very direct method for obtaining the stable central limit theorem. To be able to handle more general cases, such as martingales with arbitrarily large jumps and higher dimensional martingales, further research is needed. Inspection of the proofs indicates that the embedding method can also be used to prove that the statement of theorem 4.1 holds if M is a locally square integrable martingale such that

$$\frac{1}{\sqrt{c_t}}\sup_{s\leq t}|\Delta M_s| \xrightarrow{L^4} 0 \quad \text{and} \quad \frac{[M]_t}{c_t} \xrightarrow{\mathrm{P}} \eta.$$

To relax the conditions on the jumps of M further and to handle d-dimensional martingales it seems necessary to treat sequences of nested local martingales (triangular array scheme). In the case of continuous local martingales, this has been carried out successfully (see [15]). The key point is to relate the nesting property of a sequence of martingales to a nesting property of the Brownian motions in which they are embedded. In the continuous case this was pretty straightforward because the Dambis-Dubins-Schwarz time-change theorem embeds nested continuous local martingales in nested Brownian motions. For the Skorohod theorem this is not the case, at least not for the construction given in [12, 10, 5]. Further research should make clear whether or not a different construction is possible that allows us to treat nested martingales.

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References

[1] D. J. Aldous and G. K. Eagleson. On mixing and stability of limit theorems. *Ann. Probability*, 6(2):325–331, 1978.

[2] C. Bernard. *Vitesses de convergence pour les martingales dans le théorème central limite fonctionnel.* PhD-thesis, Université de Rennes 1, 1998.

[3] P. Billingsley. *Convergence of probability measures.* John Wiley & Sons Inc., New York, 1968.

[4] P. Billingsley. *Probability and measure.* John Wiley & Sons Inc., New York, third edition, 1995.

[5] F. Coquet, J. Mémin, and L. Vostrikova. Rate of convergence in the functional limit theorem for likelihood ratio processes. *Math. Methods Statist.*, 3(2):89–113, 1994.

[6] K. Dzhaparidze and J. H. van Zanten. On Bernstein-type inequalities for martingales. Preprint, 2000. `http://www.cwi.nl/~hvz`.

[7] P. D. Feigin. Stable convergence of semimartingales. *Stochastic Process. Appl.*, 19(1):125–134, 1985.

[8] P. Hall and C. C. Heyde. *Martingale limit theory and its application.* Academic Press Inc., New York, 1980.

[9] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes.* Springer-Verlag, Berlin, 1987.

[10] K. Kubilius. The rate of convergence in the functional central limit theorem for semimartingales. *Litovsk. Mat. Sb.*, 25(1):84–96, 1985.

[11] R. S. Liptser and A. N. Shiryayev. *Theory of martingales.* Kluwer Academic Publishers Group, Dordrecht, 1989.

[12] I. Monroe. On embedding right continuous martingales in Brownian motion. *Ann. Math. Statist.*, 43:1293–1311, 1972.

[13] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer-Verlag, Berlin, 1991.

[14] G. R. Shorack and J. A. Wellner. *Empirical processes with applications to statistics.* John Wiley & Sons Inc., New York, 1986.

[15] J. H. van Zanten. A multivariate central limit theorem for continuous local martingales. Preprint, to appear in Stat. Prob. Letters, 1998. `http://www.cwi.nl/~hvz`.