Centrum voor Wiskunde en Informatica
REPORT*RAPPORT*

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Software Engineering (SEN)

SEN-R9923 November 30, 1999

Report SEN-R9923
ISSN 1386-369X

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ABSTRACT

Building on the work of L. Moss on *coalgebraic logic*, I study in a general setting a class of infinitary modal logics for F-coalgebras, designed to capture simulation and bisimulation. For a notion of coalgebraic simulation, I use the work of A. Thijs on modelling simulation in terms of *relators* Γ (extensions of SET-functors along some family of preoders): simulation is the analogue for relators of the notion of bisimulation for functors. I prove the logics introduced here can indeed *capture coalgebraic simulation and bisimulation.* Moreover, one can *characterize any given coalgebra up to simulation* (and, in certain conditions, up to bisimulation) by a *single sentence.* An interesting feature of this logics is that their notion of *truth* or satisfaction can be understood as *a simulation relation* itself, but with respect to a (relator associated to a) richer functor $\overline{F}$; moreover, truth is the *the largest simulation*, i.e. the *similarity relation* between states of the coalgebra and elements of the language. This sheds a new perspective on the classical preservation and characterizability results, and also on *logic games.* The two kinds of games normally used in logic ("truth games" to define the semantics dynamically, and "similarity games" between two structures) are seen to be *the same kind of game* at the level of coalgebras: *simulation games.*

1991 Mathematics Subject Classification: 03G30, 03G99, 03B45, 03B70, 03C05, 18A15
1991 ACM Computing Classification System: F.4.1, F.3.2, I.6.1
Keywords and Phrases: coalgebras, modal logic, simulation, bisimulation, relator, semantics, infinitary logic, coinduction
Note: This work has been carried out under the project SEN3.4 "PROMACS". I thank Jan Rutten for useful discussion and comments.

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1. Introduction

It has been said that *modal logic is the logic of dynamics*, as the natural language describing bisimulation-invariant properties of transition systems. On the other hand, a current trend in the semantics of programming languages is based on taking *coalgebras* as standard models for dynamical systems, generalizing (various notions of) transition systems. A natural problem that arises from this work is to *generalize modal logics to coalgebraic logics for various functors.* There are at least two different directions to explore: first, extending modal logic concepts, methods and ideas to the coalgebraic context; second, applying coalgebraic methods and concepts back to the study of modal logics and of logic in general. Some work along the first line has been done by L. Moss, M. Rossiger, A. Kurz and B. Jacobs, but there still remains a whole area of open problems concerning finding the coalgebraic analogues of classical constructions in model theory of modal logic. The second line of research has not even been touched until now, as far as I know.

Coalgebras are generalizations of transition systems, which can be seen to be *dual* to algebras, from a category-theoretical point of view. See [Rut] and [JacRut] for a general introduction to the theory of coalgebras. Informally, one could say that algebraic operations are ways to *build up* complex objects from simpler ones, in a "bottom-up" fashion, while coalgebraic operations are ways to *analyse, unfold or observe* the *dynamical behaviour of objects*, in a "top-down" manner. They have been used to model computational features of infinite objects (streams, automata, processes, infinite data types, probabilistic transition systems etc.), in the semantics of concurent and object-oriented languages and (more recently) in understanding the logico-algebraical-computational aspects of some basic proof methods used in clasical analysis. In universal algebra one is primarily concerned with *initial algebras*, which contain *all the objects that can be built recursively* using some given operations, and which give us *recursion* and *induction principles.* Dually, in the theory of coalgebras, the important objects are the *final coalgebras*, which describe *all possible behaviours* and give us *corecursion and coinduction principles.* Coinduction is usually formulated in terms of *bisimulation*, a notion originally coming from concurent programming languages (but studied earlier by set-theorists under different names) and later seen to be the categorial dual of the notion of congruence. It has been first defined for transition systems and non-wellfounded sets, and generalized to coalgebras by Aczel and Mendler [AM]. It later turned out this definition has an equivalent, but much more transparent and natural, formulation in terms of *relators* (i.e. functors in the category *Rel*, having *sets* as objects and *binary relations* as arrows). Moreover, if this notion of relator is weakened in a natural way, the same definition gives a concept of *coalgebraic simulation.* The relation of *similarity* is defined as the largest simulation relation between two coalgebras. This is a weaker, "asymmetric" version of bisimilarity, proposed by A. Thijs [Thijs], as a generalization of the standard notions of simulation for transition systems and for deterministic automata.

This paper builds on, generalizes and simplifies the work done by Larry Moss on *coalgebraic logics.* He introduced, for every suitable functor F on sets, an infinitary language that generalizes a fragment of standard modal logic, strong enough to characterize coalgebras up to $\mathbf{F}$-bisimilarity (generalizing the Hennessey-Milner property). He uses the standard Aczel-Milner definition of coalgebraic bisimulation. We are using the relator definition, which we find to be simpler, more natural and easier to use in proofs. In fact, we use Thijs's above-mentioned generalization of this notion, which allows uso introduce, for every suitable relator Γ on sets, *an infinitary "modal" logic that captures coalgebraic* Γ-*simulation* (and bisimulation, as a particular case). Among these logics, there are many natural logics for various types of transition systems (including probabilistic transition systems), as well as their existential and universal sufragments. We prove analogues of classical modal characterization-up-to-(bi)simulation (Scott and Hennessey-Milner properties), definability, preservation and interpolation theorems for these logics. We obtain in particular (simpler proofs for) all the results of Larry Moss.

We also study the associated logical-semantical (pebble) games, under the name of *simulation games.* These generalize the "pebble" versions of Ehrenfeucht-Fraisse games for modal logics, designed to decide whether two structures are "similar" (i.e. related by a simulation/bisimulation). An interesting observation is that the truth-condition for the universal modality $\Box$ resembles the definition of simulation (and the condition for $\Diamond$ - the one of converse simulation). This resemblance is the basis of my current work. The language can be naturally endowed with a coalgebraic structure, but with respect to a more complex functor. Truth (the satisfaction relation) for this logic can be seen as the largest simulation relation between the model (the given coalgebra) and the language. So one could say that, for coalgebraic logic, *truth is similarity*.

This leads to a better understanding of some of the proofs of classical results, which become more transparent in this general setting: for instance, preservation of existential sentences under embeddings (simulations) becomes a simple consequence of the transitivity of similarity. As a project for future work, we plan to pursue in general this *coinductive definition of truth*, which can be seen as a *dual of Tarski's inductive definition*. Such a notion can be meaningfully applied to self-referential, fixed-point or, more generally, non-inductive languages. These usually come with an underlying coalgebra structure, in the same way that classical inductive languages come with an algebraic (Lindenbaum) structure. Also, one can see some non-compositional logical semantics, namely some of the so-called *game semantics* (used by some authors to define truth for both classical logic and for more problematic fragments of natural languages), as instances of this coinductive notion. This is natural: games in general are coinductive objects. But what is interesting is that, for coalgebraic logics, the *truth-defining games can be understood as simulation games*.

The coalgebraic perspective might hence give a unified view on these various uses of games in logic: both the analogues of pebble (Ehrenfeucht-Fraisse) games and the "truth-defining" games for coalgebras turned out to be simulation (and bisimulation) games. So the pebble games can be seen (at least in the case of coalgebras) to be games *of the same type* as the ones used in game semantics.

2. Coalgebras, Relators and Simulation

In the following, *Set* will be the category of *sets* and *functions*. For convenience, we also move one-step beyond sets and consider the category SET of *classes* and *set-continuous functions* (i.e. functions f between classes s.t. $f(C) = \bigcup\{f(c) : c \subseteq C$ and C is a set$\}$). (This can be avoided, but it is only necessary in order to talk about "the final coalgebra" and "the largest fixed-point" of some functors like the powerset $\mathcal{P}$.) Similarly, *Rel* (and REL) is the category of having as objects *sets* (and *classes*, respectively), and as arrows *relations* (and *set-continuous relations* between classes, respectively, i.e. relations R s.t. ARB iff there exist some sets $a \subseteq A, b \subseteq B$ with aRb). We identify relations and functions with their graphs, writing sometimes afb or $(a,b) \in f$ for $f(a) = b$, and similarly $(a,b) \in R$ for aRb. We denote by$\circ$ *relational composition*, defined by $A(R \circ R')B$ iff there exists C s.t. $ARCR'B$, and (because of identification of functions with their graphs) we use the same relational composition for functions (instead of the usual functional composition, which goes the other way around).

Let F be a functor on SET. The following definitions are taken from [Moss]:

Definition 2.1 F is *set-based* if for all classes C and all $a \in F(C)$, there exists some set $C_0 \subseteq C$ and some $a_0 \in F(C_0)$ s.t. $a = (Fi)a_0$, where i is the inclusion of C_0 in C. F is said to be *standard* if the image of every inclusion $i; A \hookrightarrow B$ is also an inclusion $Fi : FA \hookrightarrow FB$.

Every standard functor is *monotone*: if $A \subseteq B$ then $FA \subseteq FB$. Every monotone and set-based operator F has a least fixed point $F_\star$ and a largest fixed point $F^\star$. All functors considered in this paper are assumed to be standard and set-based. All the natural functors on SET are like this (including powerset, identity, constant functors and all the functors obtained from these by applying compositions $F \circ G$, products $F \times G$, sums $F + G$ and exponentials F^C with constant exponent-set).

Definition 2.2 A *coalgebra* for an endofunctor $F : SET \to SET$ is a pair (A, α) s.t. A is a set or a class, and $\alpha : A \to FA$ is a map called the *coalgebra map*. An *algebra* is the dual notion: a pair (A, α) with $\alpha : FA \to A$.

Observe that any fixed point of F, considered with its identity map, is both an F-algebra and an F-coalgebra.
Examples of coalgebras are: Kripke structures, transition systems, automata, streams etc. In particular, a Kripke structure $(A, \rightarrow)$ (where $\rightarrow$ is a binary relation on A, called the *accessibility* relation) can be understood as a coalgebra for the powerset-functor (A, α), with $\alpha : A \rightarrow \mathcal{P}(A)$, $\alpha(a) = \{b \in A : a \rightarrow b\}$. Similarly, a nondeterminist transition system with action labels in a set Act is a coalgebra for the functor $X \mapsto \mathcal{P}(Act \times X)$.

Definition 2.3 A *coalgebraic morphism* between F-coalgebras (A, α), (B, β) is a function $f : A \longrightarrow B$ such that $\alpha \circ (Ff) = f \circ \beta$. (Usually, this is stated the other way around, but remember we decided to use *relational composition* even for functions, where this is the dual of functional composition.)

The *category* SET_F *of F-coalgebras* has F-coalgebras as objects and coalgebraic morphisms as arrows.

Definition 2.4 (Aczel-Mendler) A *coalgebraic bisimulation* between F-coalgebras $(A, \alpha), (B, \beta)$ is an F-coalgebra (R, γ), s.t.:

R is a binary relation $R \subseteq A \times B$

the projections $\pi_1^R : R \rightarrow A, \pi_2^R \rightarrow B$ are (coalgebraic) morphisms, i.e.: $\pi_1^R \circ \alpha = \gamma \circ (F\pi_1^R)$ and $\gamma \circ (F\pi_2^R) = \pi_2^R \circ \beta$. (Remember that we use relational composition!)

It can be shown that the relevant part of a bisimulation is the underlying binary relation R. In the particular case of Kripke structures (or transition systems) this is equivalent to the standard definition: $R \subseteq A \times B$ is a bisimulation if

$$aRb \Longrightarrow [\forall a' \leftarrow_A a \exists b' \leftarrow_B b \; a'Rb' \;\&\; \forall b' \leftarrow_B b \, \exists a' \leftarrow a \; a'Rb'].$$

As mentioned in the Introduction, an alternative definition of the same notion can be given in terms of *relators*.

Definition 2.5 A *strong relator* Γ is a functor in the category REL; i.e. Γ maps classes (sets) to classes (sets) and binary relations to binary relations s.t.:

1. if $R \subseteq_A \times B$ then $\Gamma R \subseteq \Gamma A \times \Gamma B$
2. $\Gamma(R \circ R') = \Gamma R \circ \Gamma R'$
3. $\Gamma(\Delta_A) = \Delta_{\Gamma A}$, where Δ_A is the identity relation on A.

   The notion of a *weak relator* is obtained by dropping condition 3, and requiring only 1 and 2. A weak relator Γ is said to *extend a functor* F if we have:

   - $\Gamma A = FA$ for every set A
   - $\Gamma(f) \supseteq Ff$ for every function f (identified with its graph-relation)

   It is easy to see that a (weak) relator Γ can extend a functor only if we have (at least) half of the "strong-relator"-condition, i.e. $\Gamma(\Delta_A) \supseteq \Delta_A$ for every A (-this is because Δ_A is a *function* on A). Also, one can show that a *strong* relator extending F will have the stronger property that: $\Gamma(f) = Ff$ for every function f. Also, one can check that a strong relator has the following properties:

4. If $R \subseteq R'$ then $\Gamma R \subseteq \Gamma R'$

5. For $R \subseteq A \times B, f : A \to A', g : B \to B'$, we have

$$(\Gamma f \times \Gamma g)[\Gamma R] \subseteq \Gamma((f \times g)[R]).$$

(Here $f[A] =: \{f(a) : a \in A\}$ is the image of A through f.)

This can be made into a definition (due to A. Thijs), that captures (the only) properties of strong relators which will be relevant in this paper:

A weak relator is *monotonic* if it satisfies conditions 4 and 5 (besides conditions 1 and 2, which define weak relators).

All relators used in this paper are monotonic. From now on we drop the adjectives "weak" and "monotonic" and we shall speak of *F-relators* to designate monotonic weak relators extending some functor F.

There is a theorem by Carboni, Kelly and Wood that says a functor F on SET can be extended to a strong relator iff F has a certain categorial property (preservation of weak pullbacks). More importantly for our purposes, they also prove that:

Proposition 2.6 *If F has an extension to a strong relator Γ_F then this extension is unique, and is given by:*

$$\Gamma_F R =: (F\pi_2^R) \circ (F\pi_1^R)^{-1},$$

where R is an arbitrary relation $R \subseteq A \times B$, with projections $\pi_1^R : R \to A, \pi_2^R : R \to B$.

In this case, Thijs proved that Γ_F is *the minimal F-relator*:

Proposition 2.7 *If F has an extension Γ_F then $\Gamma_F R \subseteq \Gamma R$, for every F-relator Γ.*

Given an F-relator Γ we define, for each class (set) A:

$$\rhd_{\Gamma,A} =: \Gamma(\Delta_A) \text{ (the image of the identity).}$$

Then Thijs has proved that:

- $\rhd_{\Gamma,A}$ is always a preorder on A
- if $A \subseteq B, x, x' \in FA$ then we have: $x \rhd_{\Gamma,A} x'$ iff $x \rhd_{\Gamma,B} x'$. This is a "mutual consistency" requirement which ensures this preorder is a "universal" relation, which does not actually depend on the underlying set (similarly to the fact that identity Δ is also such a "universal" relation). This justifies dropping the subscript A and introducing the notation

  $$\rhd_\Gamma =: \Gamma(\Delta).$$

- Γ is uniquely determined by (F and) the preorder it induces; more concretely,

  $$\Gamma R = \rhd_\Gamma \circ (\Gamma_F R) \circ \rhd_\Gamma,$$

  where Γ_F is the (unique) (strong) minimal F-relator.

He also gave necessary and sufficient conditions in which a given "universal preorder" (i.e. family of preorders which are mutually consistent as above) induces an F-relator. (The condition is the generalization of the above-mentioned categorial condition of "preservation of weak pullbacks").

So a relator can be uniquely determined by specifying the pair of the underlying functor F and the underlying preorder $\rhd_\Gamma$. We shall do this by identifying the relator with this pair:

$$\Gamma = (F, \rhd_\Gamma).$$

Examples of relators extending functors: Let $F = \mathcal{P}$ be the powerset functor (defined by $FA = \mathcal{P}A$, and for functions $f : A \to B$, $Ff : FA \to FB$ is given by $(Ff)(A') =: f[A'] = \{f(a') : a' \in A'\}$ for every $A' \in \mathcal{P}A$). Then:

1. the minimal (strong) $\mathcal{P}$-relator $\Gamma_{\mathcal{P}}$ is given by defining, for classes A, B and relations $R \subseteq A \times B$, the relation $\Gamma_{\mathcal{P}} R \subseteq \mathcal{P}A \times \mathcal{P}B$ by:

$$A'(\Gamma_{\mathcal{P}} R)B' \qquad \text{iff} \qquad \forall a' \in A \exists b' \in B a' R b' \;\&\; \forall b' \in B \exists a' \in A a' R b'$$

(for all $A' \subseteq A, B' \subseteq B$). One can easily recognize the well-known back-and-forth conditions present in the standard notion of bisimulation for transition systems. Of course, the underlying preorder is (as for any strong relator) the identity Δ.

2. another $\mathcal{P}$-relator $\Gamma_{\subseteq} = (\mathcal{P}, \subseteq)$ is obtained by taking *set-inclusion* $\subseteq$ as its underlying preorder: $\rhd_{\Gamma} =: \subseteq$. This arrives to taking only the "forth" condition above:

$$A'(\Gamma_{\subseteq} R)B' \qquad \text{iff} \qquad \forall a' \in A \exists b' \in B a' R b'.$$

3. yet another $\mathcal{P}$-relator $\Gamma_{\supseteq}$ is obtained by taking the converse inclusion $\supseteq$ as the underlying preorder. This arrives to taking only the "back" condition in the above definition of $\Gamma_{\mathcal{P}}$.

In general for an F-relator Γ, one can define *the transposed F-relator* $\Gamma^{\sim}$, given by: $\Gamma^{\sim} R =: (\Gamma(R^{\sim}))^{\sim}$, where in general $R^{\sim}$ is the converse of the relation R. For example, $\Gamma_{\supseteq} = \Gamma_{\subseteq}^{\sim}$.

Definition 2.8 (A. Thijs) Given an F-relator Γ, a Γ-*simulation* between two F-coalgebras $(A, \alpha), (B, \beta)$ is a relation $R \subseteq A \times B$ s.t., for all $a \in A, b \in B$:

$$\text{if } aRb \text{ then } \alpha(a)(\Gamma R)\beta(b).$$

Define the relation of Γ-*similarity* $\preceq_{\Gamma}$ between the two coalgebras as the greatest Γ-simulation between them. For $a \in A, b \in B,$, we write $R : a \stackrel{\preceq}{\rightarrow}_{\Gamma} b$ if R is some Γ-simulation relating a with b. A Γ-simulation equivalence is a relation R such that both R and $R^{\sim}$ are Γ-simulations (or equivalently, R is both a Γ-simulation and $\Gamma^{\sim}$-simulation). We write $a \simeq_{\Gamma} b$ if a and b are related by some Γ-simulation equivalence.

Then we have the following:

- $\Gamma^{\sim}$-similarity is the converse of Γ-similarity.
- The relation of Γ-similarity satisfies:

$$a \preceq_{\Gamma} b \qquad \Longleftrightarrow \qquad \alpha(a)(\Gamma \preceq_{\Gamma})\beta(b).$$

- Γ_F-similarity is equivalent to F-bisimilarity (in the Aczel-Mendler sense). This is the alternative definition we announced: bisimilarity is just similarity with respect to the strong relator.
- F-bisimilarity implies Γ-equivalence, which in its turn implies both Γ-similarity and $\Gamma^{\sim}$-similarity (or converse similarity). In general, all the converses fail.
- If $\rhd_{\Gamma}$ is a partial order (i.e. is antisymmetric) then Γ-equivalence is equivalent to F-bsimulation.
- More generally, the following generalization of Aczel-Mendler's definition

  $R \subseteq A \times B$ is a Γ-simulation between (A, α) and (B, β) iff there exists an F-coalgebra (R, γ), such that the projections π_1^R, π_2^R satisfy: $\pi_1^R \circ \alpha \rhd_{\Gamma} \gamma \circ (F\pi_1^R)$ and $\gamma \circ (F\pi_2^R) \rhd_{\Gamma} \pi_2^R \circ \beta$.

Examples: In the particular case of *Kripke structures* (or, more generally, non-deterministic transition systems), $\Gamma_{\mathcal{P}}$-similarity coincides of course with bisimulation. But $\Gamma_{\subseteq}$-similarity coincides with the standard notion of simulation (i.e. "half" of a bisimulation: if aRb then $\forall a' \leftarrow a \exists b' \leftarrow b\ a'Rb'$). One can also see that the usual notion of simulation for *deterministic automata* can be otained in this frame b y taking an appropriate preorder.

Operations with relators:

(Preordered) Sum. Given relators $\Gamma = (F, \triangleright), \Gamma' = (F', \triangleright')$, their (preordered) sum $\Gamma + \Gamma' = (F + F', \triangleright' + \triangleright'')$ is obtained by taking the *coproduct (sum)* $F + F'$ of two functors (which maps sets S to the disjoint sum $FS + F'S$ and functions f to the direct sum $f + g$) together with the *preordered sum of the two preorders* $\triangleright + \triangleright'$, where, for every S:

$$\begin{array}{lll} i(x)(\triangleright + \triangleright')i(y) & \text{iff} & x \triangleright y\ \text{, for } x, y \in FS \\ i'(x)(\triangleright + \triangleright')i'(y) & \text{iff} & x \triangleright' y\ \text{, for } x, y \in F'S \\ i(x)(\triangleright + \triangleright')i'(y) & \text{iff} & x \in FS, y \in F'S \end{array}$$

(where $i : FS \to FS + F'S$, $i' : F'S \to FS + F'S$ are the canonical injections).

(Preordered) Product. Given relators $\Gamma = (F, \triangleright), \Gamma' = (F', \triangleright')$, their (preordered) product $\Gamma \times \Gamma' = (F = F', \triangleright \times \triangleright')$ is obtained by taking the *product* $F \times F'$ of two functors (which maps sets S to the Cartesian product $FS \times F'S$ and functions f to $Ff \times F'f$) together with the *product of the two preorders*:

$$(x, y)(\triangleright \times \triangleright')(x', y') \text{ iff } x \triangleright y \text{ and } x' \triangleright y'.$$

Natural Relator Transformations:

Given two relators $\Gamma = (F, \triangleright), \Gamma' = (F', \triangleright')$, a *natural relator transformation* $\mu : \Gamma \to \Gamma'$ is a natural transformation $\mu : F \to F'$ (i.e. a family $\{\mu_S : FS \to F'S : S \in Set\}$ s.t. for every function $f : S \to T$ we have $\mu_S \circ F'f = Ff \circ \mu_T$, using relational composition notation), which is *preorder-invariant*, in the sense that:

$$x \triangleright y \text{ iff } \mu_S(x) \triangleright' \mu_S(y),$$

for all sets S and all $x, y \in FS$.

It is easy to see every natural relator transformation is *"relator-invariant"*, in the following sense:

$$x(\Gamma R)y \text{ iff } \mu_S(x)(\Gamma' R)\mu_S(y),$$

for all sets relations $R \subseteq S \times T$ between sets S and T, and all $x \in FS, y \in FT$.

It is known that, given a natural transformation $\mu : F \to F'$, any F-coalgebra (A, α) can be seen as an F'-coalgebra by composing α with μ_A. We restate in our relational notation the following

Theorem 2.9 *(J. Rutten)*

A natural transformation $\mu : F \to F'$ between functors F and $F' : SEt \to Set$ induces a functor between the categories of F-coalgebras and F'-coalgebras $\mu \circ (-) : Set_F \to Set_{F'}$, which maps (A, α) to $(A, \alpha \circ \mu_A)$ and any coalgebraic F-morphism $f : (A, \alpha) \to (B, \beta)$ to the F'-morphism $f : (A, \alpha \circ \mu_A) \to (B, \beta \circ \mu_B)$. Moreover, this functor preserves bisimulations.

We can now stregthen this to the following

Theorem 2.10 *If μ from the above theorem is a natural relator transformation between the functors Γ and Γ' then the induced functor is simulation-invariant, i.e. we have:*

$$R : a \overset{\prec}{\to}_{\Gamma} b \text{ iff } R : a \overset{\prec}{\to}_{\Gamma'} b,$$

for all F-coalgebras A, B, all relations $R \subseteq A \times B$ and all $a \in A, b \in B$. Hence we also have: $a \simeq_{\Gamma} b$ iff $a \simeq_{\Gamma'} b$.

Examples

1. **Add-Bottom-transformation** $1 + (-)$:

   Let $1 = \{\star\}$ be a singleton set and let (by sistematic ambiguity) $1 : Set \to Set$ be the constant functor given by: $1(A) =: 1, 1(f) =: \Delta_1 = (\star, \star)$. Consider on it the obvious relator 1, given by taking identity as the preorder. For a given relator Γ, consider now the (preordered) sum $1 + \Gamma$. This new relator is basically obtained by adding a "bottom" to the preorder $\rhd_\Gamma$. The canonical inclusion (embedding) $\mu_A : FA \to 1 + FA$ of each FA into $1 + FA$ gives a *natural relator transformation* $\mu : \Gamma \to 1 + \Gamma$. We denote this transformation by $1 + (-)$. By the above theorem, it induces a simulation-invariant functor (which is actually an embedding) from Set_F to Set_{1+F}.

2. **Sum-to-Product-transformation**:

   There is a standard natural embedding from a sum-functor $\sum_{i\in I} F_i$ to a product-functor. Namely, take some privileged element $\star$, put as above $1 = \{\star\}$, and then define $\mu : \sum_{i\in I} F_i \to \prod_{i\in I}(1 + F_i)$ in the following way: for a set A, define $\mu_A : \sum_{i\in I} F_i A \to \prod_{i\in I}(1 + F_i A)$, by

   $$\begin{array}{lll} \mu_A(in_i(x))(j) =: & \star, & \text{if } i \neq j, x \in F_i, \\ \mu_A(in_i(x))(i) =: & in_2 x, & \text{if } i = j, x \in F_i, \end{array}$$

   where in_i is the canonical embedding of $F_i A$ into $\sum_{i\in I} F_i A$ and in_2 is the canonical embedding of $F_i A$ into $1 + F_i A$.

   If we consider now a family of relators $\{\Gamma_i = (F_i, \rhd_i) : i \in I\}$, The above-defined natural transformation can be easily seen to be a natural *relator transformation* (embedding) from $\sum_{i\in I} \Gamma_i$ in to $\prod_{i\in I}(1 + F_i A)$.

   Again, by the above theorem, this gives us a natural simulation-invariant functor from the category of $\sum_{i\in I} \Gamma_i$-coalgebras to $\prod_{i\in I}(1 + F_i A)$-coalgebras.

3. Identity-to-Powerset:

   There exists a natural transformation $\mu : I \to \mathcal{P}$ (where I is the identity functor and $\mathcal{P}$ is powerset), given by the *singleton map*: $\mu_A : I(A) = A \to \mathcal{P}A$, $\mu_A(a) =: \{a\}$. This can be easily seen to be a natural relator transformation from the identity relator I to any of the following: strong powerset relator $\Gamma_{\mathcal{P}}$, inclusion-set-relator $\Gamma_{\subseteq}$, converse-inclusion relator $\Gamma_{\supseteq}$. From these, we shall later consider the second, which we call the *identity-to-powerset-inclusion transformation*.

3. Some Classical Modal Logic

I review here some standard results in classical modal logic, taking the opportunity to also introduce some less-standard notation that would later make more transparent the connection with coalgebraic logic. For simplicity, I only treat unimodal logic with Kripke structures as models.

A *Kripke structure* is a pair $(A, \to)$ consisting of a set (or class) A and a binary relation $\to$ on A. The *(negation-free) infinitary modal language* L^{mod} is defined as the least class of sentences such that: if φ is a sentence in L^{mod} and Φ is a set of sentences in L^{mod}, then $\bigwedge \Phi, \bigvee \Phi, \Box\varphi, \Diamond\varphi$ are also sentences of L^{mod}. For simplicity, we do not consider any atomic sentences (but the "true" sentence $\top$ and the "false" sentence" $\bot$ are in the language, since $\top = \bigwedge \emptyset$ and $\bot = \bigvee \emptyset$). We shall not need negation, so we do not consider it. For *sets of sentences* Φ we introduce the following notations:

$$\Box_\infty \Phi =: \Box \bigvee \Phi$$
$$\Diamond_\infty \Phi =: \bigwedge\{\Diamond\varphi : \varphi \in \Phi\}$$
$$\triangle\Phi =: \Box\Phi \wedge \Diamond\Phi.$$

For each list $Op \subseteq \{\bigwedge, \bigvee, \Box, \Diamond, \Box_\infty, \Diamond_\infty\}$ of logical or modal operators, we denote by L^{mod}_{Op} the subfragment of L^{mod} which can be built using only operators in Op. Sentences of $L^{\text{mod}}_{\bigwedge,\bigvee,\Box}$ are called *universal sentences*, while the sentences of $L^{\text{mod}}_{\bigwedge,\bigvee,\Diamond}$ are called *existential*.

The semantics is the usual one for logical connectives, while for modalities: $a \models \Box\varphi$ iff $a \models \varphi$ for every $a' \leftarrow a$, and $a \models \Diamond\varphi$ iff $a \models \varphi$ for some $a' \leftarrow a$. Observe that our newly introduced infinitary modalities behave very similarly respectively to a simulation, a dual simulation and a bisimulation:

$a \models \Box_\infty\Phi$ iff $\forall a' \leftarrow a\, \exists\varphi \in \Phi$ s.t. $a' \models \varphi$
$a \models \Diamond_\infty\Phi$ iff $\forall\varphi \in \Phi\, \exists a' \leftarrow a$ s.t. $a' \models \varphi$
$a \models \triangle\Phi$ iff $a \models \Box_\infty\Phi$ and $a \models \Diamond_\infty\Phi$

(This observation will be used later.)

Here is a list of classical results:

1. Two (states in two) Kripke structures are (related by a) bisimilar(ity) iff they satisfy the same infinitary modal sentences, which happens iff they satisdfy the same $L^{\text{mod}}_{\bigwedge,\triangle}$. Similarity for Kripke frames is equivalent to preservation of existential sentences and also equivalent to preservation of $L^{\text{mod}}_{\bigwedge\Box_\infty}$. A dual statement holds for dual (converse) simulations.
2. Every state of a Kripke structure K can be characterized up to bisimilarity by a sentence in $L^{\text{mod}}_{\bigwedge,\triangle}$. Similar statements hold for similarity (converse similarity) with respect to $L^{\text{mod}}_{\bigwedge\Box_\infty}$.
3. A modal sentence is preserved under similarity (converse similarity) iff it is equivalent to an existential (universal) formula.

Our purpose is to generalize these facts to arbitrary coalgebras.

4. Coalgebraic Γ-Logic

Let F be some standard, set-based endofunctor on SET, having an extension to a relator Γ.

Syntax. The *language* of our coalgebraic logic is defined as the least class L such that:

if $\Phi \subseteq L$ is set of formulas in L and $X \in F\Phi$, then $\bigwedge\Phi, \bigvee\Phi, \Box X, \Diamond X$ are formulas of X.

Alternatively, one could have defined L as the least fixed-point of the operator $\mathcal{P} + \mathcal{P} + F + F$, and denote by $\bigwedge$ the injection $i_1 : \mathcal{P}L \to L$ of the first component of the sum into L, by $\bigvee$ the injection i_2 of the second component, by $\Box$ the injection i_3 of FL into L etc. We do not introduce negation, as it is not needed for our purposes. We could of course introduce it (with its classical semantics), but one should observe that even so, the modalities are not interdefinable in the usual way via negation: even if $\neg\phi$ makes sense for $\phi \in L$, there is nothing like $\neg X$ for $X \in FL$.

One should think of our coalgebraic operators $\Box$ and $\Diamond$ as generalizations of the infinitary modal operators $\Box_\infty$, $\Diamond_\infty$ (but not of the unary modalities $\Box$ and $\Diamond$). We also introduce the notation

$$\triangle X =: \Box X \wedge \Diamond X.$$

As in the modal case, we also consider sublanguages of the form L_{Op}, for some set of operators, and in particular the *universal fragment* $L_{\bigwedge\bigvee\Box}$ (built without using $\Diamond$) and its dual, the *existential fragment* $L_{\bigwedge\bigvee\Diamond}$, built similarly without using $\Box$.

Semantics: Given a F-coalgebra (A, α), we define the relation of *satisfaction* $\models \subseteq A \times L$ as the *unique* relation satisfying

- the usual inductive clauses for $\bigwedge$ and $\bigvee$
- $a \models \Box X$ iff $\alpha(a)(\Gamma \models)X$
- dually, $a \models \Diamond X$ iff $\alpha(a)(\Gamma^{\sim} \models)X$.

The correctness of the definition is ensured by the fact L is a least fixed point, which ensures the uniqueness of $\models$ with these properties. In other words, this is indeed an inductive definition of truth. On the other hasnd, it can be also understood as a co-inductive definition: it is easy to see that the we have the following

Proposition 4.1 ***("Coinductive" characterization of satisfaction)***. *The above relation of satisfaction can be characterized as* ***the greatest*** *relation between A and L satisfying:*

1. *if $a \models \bigwedge \Phi$ then $a \models \varphi$ for every $\varphi \in \Phi$*
2. *if $a \models \bigvee \Phi$ then $a \models \varphi$ for some $\varphi \in \Phi$*
3. *if $a \models \Box X$ then $\alpha(a)(\Gamma \models)X$*
4. *if $a \models \Diamond X$ then $\alpha(a)(\Gamma^{\sim} \models)X$.*

Similarly, one can easily check that there are such coinductive characterizations of satisfaction for the natural sublanguages of L, among of which we mention the following

> Satisfaction for $L_{\bigwedge\Box}$ can be characterized coinductively as the greatest relation between A and L satisfying conditions 1. and 3. from the above characterization of satisfaction.

Examples There are a lot of examples: for each pair of a "natural" functor F and an F-relator, one obtains a "natural" logic falling under our general definition. I only point here that, if we take $F = \mathcal{P}$ (powerset) and $\Gamma = \Gamma_{\subseteq}$, we obtain exactly the classical modal logic $L^{\mathrm{mod}}_{\bigwedge\bigvee\Box_\infty\Diamond_\infty}$, which is essentially equivalent in expressive power to the full modal logic L^{mod}. On the other hand, if we take the strong $\mathcal{P}$-relator $\Gamma_{\mathcal{P}}$, then all our coalgebraic modalities collapse to the same thing:

$$\Diamond_{\Gamma_{\mathcal{P}}} = \Box_{\Gamma_{\mathcal{P}}} = \triangle_{\Gamma_{\mathcal{P}}} = \triangle.$$

More generally, $\Box, \Diamond, \triangle$ collapse to the same thing for every strong relator. The logic obtained in this particular case of strong relators is exactly *Larry Moss's coalgebraic logic.*

To distinguish the semantics of our logic from the one of the logic of L. Moss, we use the term "Γ-language" when the semantics is given as above, in terms of some arbitrary F-relator Γ, and we reserve the term "F-language" for the Γ_F-language.

Next, we generalize the above-mentioned classical results. For this, we introduce the following notation:

Definition 4.2 For coalgebras (A, α), (B, β), states $a \in A, b \in B$ and some fragment $L_0 \subseteq L$ of our logic, we write $a \preceq_{L_0} b$ iff every sentence of L_0 is true at b only if it is true at a:

$$a \preceq_{L_0} b \text{ iff } \forall\varphi \in L_0(b \models \varphi \Rightarrow a \models \varphi).$$

This relation is the converse of L_0-preservation.

Theorem 4.3 ***(Logical Characterization of Similarity)***: *Given coalgebras $(A, \alpha), (B, \beta)$, the following are equivalent, for all $a \in A, b \in B$:*

(i). $a \preceq_{\Gamma} b$

(ii). $a \preceq_{L_{\bigwedge \Box}} b$

(iii). $b \preceq_{L_{\bigwedge \Diamond}} a$, *i.e. all formulas in* $L_{\bigwedge \Diamond}$ *are preserved when passing from* a *to* b.

To summarize it in words: " Similarity coincides with preservation of (the truth of) Γ*-existential sentences" (and with the converse-preservation of universal sentences).*

Proof: By taking the direct sum $A + B$ of the two coalgebras, we can safely asumme that a, b live in the same coalgebra $A = B, \alpha = \beta$. (Observe that, for any pair of states, none of the relations $\models$, $\preceq_\Gamma$, $\preceq_L$ changes: the relations are "locally the same". We shall only prove the equivalence between (i) and (ii); the others are similar (and they can also be obtained from this by duality).

Claim 1 : $(i) \Rightarrow (ii)$; that is, $L_{\bigwedge \Box}$-formulas are preserved by similarity: if $a \preceq_\Gamma b$ and $b \models \varphi \in L_{\bigwedge \Box}$ then $a \models \varphi$.

Proof of claim: The claim is equivalent to saying that $\preceq_\Gamma \circ \models \subseteq \models$. But to show this we need (by the above coinductive charaterization of $\models_{L_{\bigwedge \Box}}$) to prove that $\preceq_\Gamma \circ \models$ **satisfies conditions 1. and 3. above** (when we replace $\models$ with $\preceq_\Gamma \circ \models$).

Condition 1. is trivial to check, so let us check 2. Suppose that we have $a(\preceq_\Gamma \circ \models)\Box X$. Then there exists some $c \in A$ s.t. $a \preceq_\Gamma c$, $c \models \Box X$. But the similarity relation $\preceq_\Gamma$ is a simulation, so we must have $\alpha(a)(\Gamma \preceq_\Gamma)\alpha(c)$. On the other hand, by truth-condition for $\Box$, we have $\alpha(c)(\Gamma \models)X$. By composition, we have that: $(\alpha(a), \alpha(c)) \in (\Gamma \preceq_\Gamma \circ \Gamma \models) = \Gamma(\preceq_\Gamma \circ \models)$. But this means that $\preceq_\Gamma \circ \models$ satisfies condition 2.

Claim 2 There exists *a set* of formulas $\Phi \subseteq L_{\bigwedge \Box}$ such that for all $a, b \in A$, we have: $a \preceq_\Phi b$ iff $a \preceq_{L_{\bigwedge \Box}} b$.

Proof of claim: This is an easy cardinality argument. For each pair $(a, b) \in A^2$ s.t. $a \not\preceq_{L_{\bigwedge \Box}} b$ choose some φ to witness this. Take Φ to be the set of all such φ's (which is not bigger than A^2) and check it works.

Claim 3 There exists some relation $R \subseteq L_{\bigwedge \Box} \times A$ such that: $\models_A \circ R = \preceq_{L_{\bigwedge \Box}}$.

Proof of Claim: For each $a \in A$, put $\theta_a =: \bigwedge\{\varphi \in \Phi : a \models \varphi\}$. Then take $R = \theta^{-1}$ (the inverse relation of the function $\theta : A \to L$) and check the claim, using the Claim 2 about Φ.

Claim 4 $\preceq_{L_{\bigwedge \Box}}$ is a Γ-simulation. (From this, it follows directly that $(ii) \to (i)$, since by definition $\preceq_\Gamma$ is the largest Γ-bisimulation; so after this we are done.)

Proof of claim: Suppose we have $a \preceq_{L_{\bigwedge \Box}} b$. By Claim3, we have $\Delta_A \subseteq \ L_{\bigwedge \Box} \ = \ \models_A \circ R$, so (since Γ is a relator) we have $(\alpha(b), \alpha(b)) \in \Delta_{FA} = F(\Delta_A) \subseteq \Gamma(\Delta_A) \subseteq \Gamma(\models_A \circ R) = (\Gamma \models_A) \circ (\Gamma R)$. So there exists some $X \in FL$ s. t. $\alpha(b)(\Gamma \models_A)X$ and $X(\Gamma R)\alpha(b)$; but then (by truth-defintion of $\Box$) we have $b \models \Box X$. We supposed that $a \preceq_{L_{\bigwedge \Box}} b$, so we conclude that also $a \models \Box X$. Hence (again by truth-definition) $\alpha(a)(\Gamma \models)X$; but remember that $X(\Gamma R)\alpha(b)$, hence by composition we conclude that $(\alpha(a), \alpha(b)) \in (\Gamma \models_A) \circ (\Gamma R) = \Gamma(\models_A \circ R) = \Gamma(\preceq_L)$ (since by Claim 3, $\models_A \circ R = \preceq_{L_{\bigwedge \Box}}$). Hence $\alpha(a)(\Gamma \preceq_{L_{\bigwedge \Box}})\alpha(b)$. So $\preceq_{L_{\bigwedge \Box}}$ is a Γ-simulation.

■

Corollary 4.4 ***(Moss's Logical Characterization of Bisimilarity)****: Bisimilarity coincides with preservation of (=equivalence with respect to) the* F*- language* $L_{\bigwedge \triangle}$*:*

$$a \sim b \text{ iff } a \preceq_{L_{\bigwedge \triangle}} b$$

Proof: Apply the preceding theorem to the strong relator Γ_F. ■

Corollary 4.5 *Γ-equivalence coincides with equivalence with respect to L: In the same conditions as above, a and b are Γ-equivalent iff they satisfy the same sentences of our logic.*

Proof: As a consequence of the above Theorem, elementary equivalence with respect to our logic is both a simulation and a converse simulation, hence it is a Γ-equivalence. ■

Corollary 4.6 ***(Second Characterization of Bisimulation)****: If $\rhd_\Gamma$ is a partial order then bisimilarity is captured by L-equivalence.*

Our theorem above implies that a state in a coalgebra can be characterized-up-to-similarity by its theory in $L_{\bigwedge \diamond}$ (i.e. by the *class* of all sentences in this language which are satisfied by the state). As in the classical case of transition systems, this result can be stregthened to obtain *characterization of a state by a single sentences* in the language:

Theorem 4.7 ***(Characterization-up-to-Similarity)****: For every coalgebra (A, α) and state $a \in A$, there exists some sentence $\theta_a \in L_{\bigwedge \diamond}$ such that, for every coalgebra $(B, b))$ and state $b \in B$, we have:*

$$b \models_B \theta_a \text{ iff } a \preceq_\Gamma b.$$

I*n other words, Γ-similarity is characterizable by a sentence in the Γ-language $L_{\bigwedge \diamond}$. Similarly, $\Gamma^\sim$-similarity (i.e. the converse of Γ-similarity) is characterizable by a sentence in $L_{\bigwedge \Box}$.*

From this we obtain

Corollary 4.8 ***(L. Moss's Characterization-up-to-bisimulation)*** *For every coalgebra (A, α) and state $a \in A$, there exists some "characteristic sentence" $\theta_a \in L_{\bigwedge \triangle}$ in the F-language, such that, for every coalgebra $(B, b))$ and state $b \in B$, we have:*

$$b \models_B \theta_a \text{ iff there exists an } F\text{-bisimulation relating } a \text{ and } b.$$

Corollary 4.9 *Every state in a coalgebra can be characterized up to Γ-equivalence by a sentence in our Γ-language L. In case that $\rhd_\Gamma$ is a partial order , then bisimilarity is captured by L-equivalence.*

5. Simulation Games and Truth-as-Simulation

Ehrenfeucht-Fraisse games, pebbleg games for finite-variable logics and the simulation/bisimulation game for modal logic are various examples of (not unrelated) games used to "compare two structures" and see how much "alike" are they. I am describing below a concrete example from the folklore, the Simulation Game for Kripke structures.

Example:

The *"Simulation game"* can be played in general on transition systems, but to keep it simple we shall restrict our attention to (unimodal) Kripke structures. Two structures $(A, \rightarrow_A), (B, \rightarrow_B)$. There are two playes, Y (yes) and N (no), and a play goes on like that: player N starts by chosing some $a_0 \in A$, and player Y choose some $b_0 \in B$. Then N has to choose some $a_1 \leftarrow_A a_0$, to which player Y answers with some $b_1 \leftarrow_{A'} b_0$; then player N chooses again some $a_2 \leftarrow_A a_1$, to which Y responds with some $b_2 \leftarrow_{A'} b_1$, etc. If, at anytime, some player cannot his move, he loses. If the play goes on forever, Player Y winns.

One can easily see that:

Proposition 5.1 *a and b are related by a simulation on Kripke structures iff, given the initial position $(a_0, b_0) = (a, b)$, Player Y has a winning strategy in the simulation game.*

There is also version for *bisimulation*, in which, when he has to choose, player N is allowed to choose from either of the two structures some successor of either his last choice or his opponent's last choice.

We can immediately generalize these games to coalgebras:

Coalgebraic Simulation Game

For a given F-relator, the "Γ-*Simulation Game"* is played on two F-coalgebras (A, α), (B, β). Player N chooses some pair $(a_0, b_0) \in A \times B$ (claiming they are not related by a simulation). Player Y has to choose some *binary relation* R_0 s.t. $(\alpha(a_0), \beta(b_0)) \in \Gamma R_0$. Then player N chooses some pair $(a_1, b_1) \in R_0$, to which player Y answers with some $R_1 \subseteq A \times B$ s.t. $(\alpha(a_1), \beta(b_1)) \in \Gamma R_1$, etc. The winning rules are as in the previous game.

Proposition 5.2 *$a \preceq_\Gamma b$ iff, given the initial choices $(a_0, b_0) = (a, b)$, player Y has some winning strategy in the Γ-simulation game.*

One can of course adjust the game to check for Γ-equivalence instead of simulation, by allowing the first player to play not just Γ-moves, but also $\Gamma^\sim$-moves. Games like this have a natural *dynamic* and *coinductive flavor*: it is clear that the "moves" of the players follow the "next-steps" of the coalgebras trying to match them in a (bi)simulation. Such semantic games have been used for avariety of purposes.

An interesting observation is that, in the Γ-semantics, the definition of truth for $\Box$ reproduces the "simulation" clause (and the one for $\Diamond$ the one for "converse simulation"). Moreover, a close look at the proofs of the above-mentioned characterization theorems shows that this coinductive simulation-like character of truth is actually the basis of (the non-trivial part) of these proofs. Both for getting a better understanding of these proofs and for the sake of a general, logico-philosophical interest, it is tempting to pursue this analogy and try to use the insights from the theory of processes and other dynamical systems in a study of the apparently-static-unshakable concept of "logical truth". First, we have to *extend this observation to the whole Γ-language, treat language itself as a coalgebra and show truth is a simulation with respect to the underlying functor*.

First, let us try to do this in a simple, classical case: *Kripke structures*. Given a Kripke structure $(A, \rightarrow)$, we can expand the structure by adding the *idle transitions*, transforming A into a transition system with two kind of actions: $\rightarrow$, corresponding to the old transitions, and the "idle transition" $\rightarrow_0 = \Delta_A$ (the identity relation on A). Then observe that infinitary conjunction is an "infinitary diamond" operator (in the sense defined above) for the idle transition, while infinitary disjunction is a "box" for it, because of our earlier observation about their "simulation"-like character:

$$a \models \Box_\infty \Phi \text{ iff } \forall a' \leftarrow a \exists \varphi \in \Phi \text{ s.t.} a' \models \varphi$$

$$a \models \Diamond_\infty \Phi \text{ iff } \forall \varphi \in \Phi \exists a' \leftarrow a \text{ s.t. } a' \models \varphi.$$

This means that truth behaves like a "local simulation" for $\Box$ and $\bigvee$ and a local converse-simulation for $\Diamond$ and $\bigwedge$.

Now, to generalize this idea to coalgebras, we need to endow every F-coalgebra with more structure in order to match the language. We first need first to add the "idle" transitions to our given F-coalgebra:

From F-coalgebras to $I \times F$-coalgebras. Any F-coalgebra (A, α) can be considered as an $I \times F$-coalgebra $(A, \Delta_A \times \alpha)$, where I is the identity functor, by putting

$$\Delta \times \alpha(a) =: (a, \alpha(a)).$$

This correspondence is a natural way to "add" the iddle transitions (given by the identity map) to our coalgebra. It is easy to see that this correspondence is a functor $\Delta \times (-)$ from Set_F to $Set_{I \times F}$, and that moroever it is bisimulation-invariant: R is an F-bisimulation between two F-coalgebras iff it is a $I \times F$-bisimulation between their images under the above correspondence.

The identity functor has a natural extension to an "identity relator" I, given by taking identity as our preorder: $I = (I, \Delta)$. If one considers now some extension of F to a relator $\Gamma = (F, \triangleright)$, this induces a natural extension of $I \times F$ to the relator $I \times \Gamma$ (where the product of relators has been defined above using the product-preorder). Then the above correspondence can be seen to be *simulation-invariant*:

$$R : a \overset{\preceq}{\to}_{\Gamma} b \text{ iff } R : a \overset{\preceq}{\to}_{I\times\Gamma} b,$$

for all F-coalgebras A, B, all relations $R \subseteq A \times B$ and all $a \in A, b \in B$.

Language as coalgebra: Next, we endow our language L with a coalgebra map, in a natural way. Our language was defined as the least fixed-point of the functor $\mathcal{P} + \mathcal{P} + F + F$. Given our assumptions about F, one can easily see that, if we endow L with the *identity map* $\Delta : L \to L$, we obtain an *initial algebra* for the same functor, which is also a *coalgebra* for the (same) functor.

Similarly, the restricted language $L_{\wedge\Box}$ can be considered as an $\mathcal{P} + F$-coalgebra (with the identity map).

Unifying the types of our coalgebras:

We have on one hand our *models* (F-coalgebras, which by the above correspondence can be also considered as) $I \times F$-coalgebras, and on the other hand the languages L and $L_{\wedge\Box}$, endowed with the above coalgebra structures. The problem for defining a simulation between them is that the languages are *coalgebras of different types* than the models. We shall solve this by *raising the types of both* to obtain a common functor.

We first do it for the fragment $L_{\wedge\Box}$, which (by our characterization theorem above) is the natural language for simulation.

First, we transform the language from a sum-coalgebra to a product coalgebra, using the canonical Sum-to-Product natural relator transformation (defined in Section 2). As a result, the language $L_{\wedge\Box}$ becomes a $(1 + \mathcal{P}) \times (1 + F)$-coalgebra, with map γ given by:

$$\begin{aligned} \gamma(\textstyle\bigwedge \Phi) &=: (\Phi, \star) \\ \gamma(\Box X) &=: (\star, X) \end{aligned}$$

Now, on the side of the models: we have already reconsidered our models as $I \times F$-coalgebras. We would like to have them as $(1 + \mathcal{P}) \times (1 + F)$-coalgebras. But there exists a natural transformation from $I \times F$ to $(1+\mathcal{P}) \times (1+F)$. This can be obtained by considering the natural identity-to-powerset transformation mentioned in the second section (call it T), the natural embedding in_2 of F into $1+F$, and then taking

$$(T \circ in_2) \times in_2$$

as our transformation.

By our theorem in Section 2, this induces a simulation-invariant functor from $I \times F$-coalgebras (having $I \times \Gamma$ as relator) to $(1 + \mathcal{P}) \times (1 + F)$-coalgebras (having $(1 + \Gamma_{\subseteq}) \times (1 + \Gamma)$ as relator). And by composing this with the above simulation-preserving functor $\Delta \times (-) : Set_F \to I \times F$, we obtain a simulation-invariant functor fromF-coalgebras to $(1 + \mathcal{P}) \times (1 + F)$-coalgebras.

This functor can be described directly, by putting $H : Set_F \to (1+\mathcal{P}) \times (1+F)$, $H(A, \alpha) =: (A, \overline{\alpha})$, where:

$$\overline{\alpha}(a) =: (\{a\}, \alpha(a)).$$

So our common relator for both $L_{\wedge\Box}$ and A as coalgebras is $\overline{\Gamma} =: (1+\Gamma_{\subseteq}) \times (1+\Gamma)$. As a consequence of the fact that all these functors are simulation-invariant, we obtain that

Lemma 5.3 *For all F-coalgebras $(A, \alpha), (B, \beta)$, we have:*

$$(A, \alpha) \preceq_{\Gamma} (B, \beta) \textit{ iff } (A, \overline{\alpha}) \preceq_{\overline{\Gamma}} (B, \overline{\beta}).$$

Proposition 5.4 ***("Truth is similarity")*** *Our (earlier defined) notion of truth or satisfaction relation between F-coalgebras (A, α) and the Γ-language $L_{\wedge\Box}$ coincides with the relation of $\overline{\Gamma}$-similarity (i.e. the greatest Γ-simulation) between $(A, \overline{\alpha})$ and $(L_{\wedge\Box}, \gamma)$.*

Note: The same move can be applied to the whole language L. Of course, the relevant notion of simulation is the simulation-equivalence. By raising the type, we consider both L and A as coalgebras for the functor $\mathcal{P} \times \mathcal{P} \times (1 + F) \times (1 + F)$, on which we consider the relator given by the product of the following preorders: $1+ \subseteq, 1+ \supseteq, 1 + \rhd_\Gamma, 1 + \lhd_\Gamma$. Then we can see that *truth for L, as defined above, coincides with similarity with respect to this relator* (which also coincides with simulation-equivalence with respect to the previous relator $\overline{\Gamma}$).

6. Conclusions and Future Work

As the last theorem shows, *in coalgebraic logic truth is similarity*, the largest simulation relation, but only if we consider a richer functor: Γ-satisfaction for F-coalgebras is the $\overline{\Gamma}$-similarity between the $\overline{F}$-coalgebra structure of these F-coalgebras and the Γ-language L. Nevertheless, truth is a simulation, and the "local simulation" induced by it on the F-structure is responsinble for some of its nice logical properties.

We would like to pursue this idea in more generality in further work. We list some reasons for which we think this might be interesting;

1. One can obtain *simpler, more iluminating proofs* of some of the classical results. An example is the fact that preservation of certain formulas under certain relations becomes obvious, by the transitivity of similarity (i.e. the fact that a composition of simulations is a simulation). Another example is our proof of the "characterization result" that generalizes Larry Moss's (and Hennessey-Milner's) results, which is much more simple and direct than the previous ones.

2. *Possible generalizations to other languages:* For instance, self-referential languages (like μ-calculus, other fixed-point logic, languages based on "systems of language equations" (e.g. the ones used by J. Barwise in Situation Theory) come naturally equiped with a coalgebra structure. Also, there exist logical languages, closer to natural language, and for which the intended semantics is not "compositional", so truth cannot be built algebraically bootom-up. There are many proposals in this respect, many of them with a "dynamic" or game-theoretical flavor, stated in terms of *possibilities of action*, ways to "use" the information sentence to update your information or to change the environment. But these notions are very close to the dynamical notions that coalgebras have been "invented" for. The hope would be to capture some of these in the general frame of coalgebras and coinduction, as models for dynamics. After all, the fundamental insight behind these attempts seems to be the same as the dynamic-coinductive semantics: instead of starting at the bottom and working your way up until you reach or construct the truth of the formula, you just start with the formula (and a dynamic model) and you "follow" the formula, going wherever it leads and "doing" what it tells you do, and keep doing it unless you reach a "failure" state (proving the formula false).

3. One could *generalize the above-mentioned game-semantics and unify it with the "pebble games".* Indeed, in the coalgebraic logic, satisfaction (truth) and similitude-of-structure (simulation or bisimulation) turned out to be the kind of thing. Accordingly, one can immediately give an (equivalent) game semantics to our logic: just use the above "simulation game", applying it to the coalgebra structure of the language, on one side. In this sense, one can understand our "truth-as-simulation" proposal as a more abstract way to introduce a game semantics. But it is also interesting in that it reveals the deep relation between these two main logical uses of games: "truth-games" (Hintikka-style, say) on one side, "similitude" or equivalence games on the other (pebble games etc.) At the level of coalgebras, these two notions of logical games can be recognized as being *the same kind of games: simulation games.*

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