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ABSTRACT
This survey paper is concerned with nonparametric estimation of the global and local intensity of a cyclic Poisson point process X. We assume that only a single realization of X is observed, though only within a bounded 'window', and our aim is to estimate consistently the global intensity and the intensity function at a given point. A simple nonparametric estimator for the global intensity is proposed and studied. We also give a preview of recent work by the present authors, in part joint with R. Zitikis (Winnipeg), on nonparametric estimation of the local intensity at a given point. These results will also be contained in I W. Mangku's forthcoming Ph.D. thesis.

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1. Introduction

Let X denote an inhomogeneous Poisson point process on the real line $\boldsymbol{R}$ with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν and with (unknown) locally integrable intensity function $\lambda : \boldsymbol{R} \rightarrow \boldsymbol{R}^+ \cup \{0\}$, i.e., for any bounded Borel set B, we have $\mu(B) = \int_B \lambda(s)ds < \infty$. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and let us suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson point process X is observed, though only in a bounded interval (called window) $W \subset \boldsymbol{R}$. For any set $B \subset \boldsymbol{R}$, $X(B)$ denotes the number of points of X in B; $\mu(B) = \mathbf{E}X(B)$, for any Borel set B, where $\mathbf{E}$ denotes the expectation. The Poisson process X can be characterized by the following two properties:

(a) $\mathbf{P}\left(X(B) = k\right) = \frac{\mu(B)^k}{k!} e^{-\mu(B)}$, $k = 0, 1, \ldots$, for each Borel set B with $\mu(B) < \infty$.

(b) For each positive integer m and pairwise disjoint Borel sets $B_1, B_2, \ldots B_m$ with $\mu(B_j) < \infty,\ j = 1, \ldots, m$ the random variables $X(B_1), X(B_2), \ldots, X(B_m)$ are independent.

We refer to Kingman [6] for an excellent account of the theory of Poisson processes.

The aim of this paper is to study the statistical problem of estimating the 'global' and 'local' intensity, from a single realization $X(\omega)$ of the Poisson point process X observed only in W. The intensity function $\lambda(s)$ at a given location $s \in \boldsymbol{R}$, i.e. the local intensity, can also be expressed as

$$\lambda(s) = \lim_{B \downarrow \{s\}} \mathbf{P}(X(B) = 1)/|B| \tag{1.1}$$

provided λ is continuous at s. Here $B \downarrow \{s\}$ means that the Borel set B shrinks to $\{s\}$; $|B|$ denotes the Lebesgue measure of a Borel set B and $\{s\}$ is the singleton set, which consists of the point s only. The global intensity θ of the process X, whenever well-defined, can be given by

$$\theta = \lim_{W \uparrow \boldsymbol{R}} \frac{\mathbf{E}X(W)}{|W|} \tag{1.2}$$

where $W \uparrow \boldsymbol{R}$ means that the window $W \subset \boldsymbol{R}$ expands to the real line $\boldsymbol{R}$; this limit is certainly well-defined when X is a cyclic (periodic) Poisson point process; in the latter case θ can also be expressed as in (2.3). We will assume throughout this paper that $\theta > 0$.

Since λ is locally integrable, the Poisson point process X always places only a finite number of points in any bounded subset of $\boldsymbol{R}$. Hence, in order to make consistent estimation possible, one must accumulate the necessary empirical information. One way to achieve this is by adopting a framework called 'increasing domain asymptotics' (see, e.g., Cressie [1], p. 480). We let the window W depend on $n = 1, 2, \ldots$, in such a way that $W_n \uparrow \boldsymbol{R}$, so that

$$|W_n| \to \infty, \tag{1.3}$$

as $n \to \infty$, where $|W_n| = \nu(W_n)$ denotes the size (or the Lebesgue measure) of the window W_n. In this set up, a necessary condition for the existence of a consistent estimator is that (cf. Lemma 1)

$$\int_{\boldsymbol{R}} \lambda(s)ds = \infty, \tag{1.4}$$

i.e. $\mu(\boldsymbol{R}) = \mathbf{E}X(\boldsymbol{R}) = \infty$, which implies that there are, almost surely, infinite number of points in $\boldsymbol{R}$ placed there by the point process X. If, on the other hand, $\mathbf{E}X(\boldsymbol{R}) < \infty$, then there are, almost surely, only a finite number of points in $\boldsymbol{R}$ placed there by the point process X, and consistent estimation is clearly impossible. If X is cyclic (cf.(2.2)) and $\theta > 0$, (1.4) is automatically satisfied.

The condition (1.4) also shows up in [9], [3], and [4], as a necessary condition for consistency.

Lemma 1: *For any Poisson point process X with mean measure μ, if $\mu(\boldsymbol{R}) = \mathbf{E}X(\boldsymbol{R}) = \infty$ then for P-almost all ω the point pattern $X(\omega)$ contains infinite many points, i.e. $X(\boldsymbol{R}) = \infty$. On the other hand, if $\mu(\boldsymbol{R}) = \mathbf{E}X(\boldsymbol{R}) < \infty$, then the probability that $X(\omega)$ contains only finitely many points is equal to 1.*

Proof: A complete proof of this lemma can be found in Helmers and Mangku [4]. A brief sketch of the proof is as follows. Suppose $A_1, A_2, \ldots$ are disjoint measurable subsets of $\boldsymbol{R}$ such that $\bigcup_{i=1}^{\infty} A_i = \boldsymbol{R}$. Then, we can write

$$X(\boldsymbol{R}) = \sum_{i=1}^{\infty} X(A_i). \tag{1.5}$$

By Fubini's theorem for nonnegative functions, we have that

$$\mathbf{E}X(\boldsymbol{R}) = \sum_{i=1}^{\infty} \mathbf{E}X(A_i) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1.6}$$

Next, it can be shown that $\mathbf{E}X(\boldsymbol{R}) = \infty$ is equivalent to $\sum_{i=1}^{\infty} \mathbf{P}(X(A_i) \geq 1) = \infty$. Because A_i and A_j are disjoint for all $i \neq j$, we have that $X(A_i)$ and $X(A_j)$ are independent, for all $i \neq j$. Therefore, by the Borel-Cantelli lemma, we have that $\mathbf{E}X(\boldsymbol{R}) = \infty$ is equivalent to $\mathbf{P}(X(\boldsymbol{R}) = \infty) = 1$. $\square$

In section 2 we discuss estimation of the global intensity θ, while in section 3 we give a preview of recent work by the present authors, in part joint with R. Zitikis (Winnipeg), on the asymptotic behaviour of two nonparametric estimators for the (local) intensity function at a given point, namely: a kernel type estimator and a nearest neighbor estimator.

2. Estimation of the global intensity

If the Poisson process X is homogeneous, $\mu(B) = \lambda_0 \nu(B) = \lambda_0 |B|$, for some constant $\lambda_0 > 0$ and all Borel sets B, the local intensity is constant, i.e. $\lambda(s) = \lambda_0$ for all $s \in \boldsymbol{R}$. The global intensity θ is precisely equal to λ_0 in this very special case, and the maximum likelihood method can be applied to estimate θ. Let $s_i,\ i = 1, \ldots, X(W_n)$ denote the locations of the points in the realization $X(\omega)$ of the Poisson process, observed in W_n. Then, the likelihood of $(s_1, \ldots, s_{X(W_n)})$ is equal to

$$L_n = e^{-\int_{W_n} \lambda(s)ds} \prod_{i=1}^{X(W_n)} \lambda(s_i) = e^{-\lambda_0 |W_n|} \lambda_0^{X(W_n)},$$

where $X(W_n)$ denotes the observed number of points in W_n (cf. Cressie [1], p. 655). Maximizing $\ln L_n$ gives us:

$$\frac{d \ln L_n}{d\lambda_0} = \frac{d}{d\lambda_0} \left(-\lambda_0 |W_n| + X(W_n) \ln \lambda_0\right) = -|W_n| + \frac{X(W_n)}{\lambda_0} = 0,$$

which directly yields the MLE

$$\hat{\theta}_n = X(W_n)/|W_n| \tag{2.1}$$

of λ_0 and hence of θ as well.

Let us now investigate the asymptotic behaviour of $\hat{\theta}_n$ for the case of a *cyclic Poisson process* X, that is λ, the intensity function, is assumed to be periodic. In other words, we consider a cyclic intensity function λ with (unknown) period $\tau \in \boldsymbol{R}^+$:

$$\lambda(s + k\tau) = \lambda(s) \tag{2.2}$$

for all $s \in \boldsymbol{R}$ and $k \in \mathbb{Z}$. For this case we prove in Lemma 2.1 that θ is well-defined by (1.2) and can now also be written as

$$\theta = \frac{1}{\tau} \int_{U_\tau} \lambda(s) ds, \tag{2.3}$$

where U_τ denote any interval of length τ. In Lemma 2.2 we will show that $\hat{\theta}_n$ is a consistent estimator of the global intensity θ of X. Complete convergence (implying strong consistency) of $\hat{\theta}_n$ is established in Lemma 2.3, while the asymptotic normality $\hat{\theta}_n - \theta$, properly normalized, is derived in Theorem 2.4. A bootstrap CLT for $\hat{\theta}_n - \theta$ is established in Theorem 2.5.

Lemma 2.1 *If λ is periodic (with period τ) and locally integrable, then*

$$\theta_n = \frac{EX(W_n)}{|W_n|} = \frac{1}{|W_n|} \int_{W_n} \lambda(s) ds \to \theta, \tag{2.4}$$

as $n \to \infty$, with θ as in (2.3). Hence $\hat{\theta}_n$ is asymptotically unbiased in estimating θ.

Proof: Let $N_{n\tau} = [\frac{|W_n|}{\tau}]$, $W_{N_{n\tau}}$ denotes an interval of length $\tau N_{n\tau}$ contained in W_n, and $R_{n\tau} = W_n \setminus W_{N_{n\tau}}$. Then we can write

$$\theta_n = \frac{|W_{N_{n\tau}}|}{|W_n|} \frac{1}{|W_{N_{n\tau}}|} \int_{W_{N_{n\tau}}} \lambda(s) ds + \frac{1}{|W_n|} \int_{R_{n\tau}} \lambda(s) ds. \tag{2.5}$$

First note that

$$\frac{1}{|W_{N_{n\tau}}|} \int_{W_{N_{n\tau}}} \lambda(s) ds = \theta \tag{2.6}$$

because λ is periodic with period τ. Since $|R_{n\tau}| < \tau$ for all n, we have that

$$\frac{|W_{N_{n\tau}}|}{|W_n|} = \frac{|W_n| - |R_{n\tau}|}{|W_n|} \ \rightarrow \ 1, \tag{2.7}$$

as $n \to \infty$. Because λ is locally integrable and $|R_{n\tau}| = \mathcal{O}(1)$, as $n \to \infty$, we also know that

$$\int_{R_{n\tau}} \lambda(s) ds = \mathcal{O}(1), \text{ as } n \to \infty.$$

Hence, the first term on the r.h.s. of (2.5) converges to θ, while its second term (by (1.3)) converges to zero, as $n \to \infty$. This completes the proof. $\Box$

Lemma 2.2 *If λ is periodic (with period τ) and locally integrable, then*

$$\hat{\theta}_n \xrightarrow{p} \theta, \tag{2.8}$$

as $n \to \infty$.

Proof: To prove (2.8) we must show, for each $\epsilon > 0$,

$$\mathbf{P}(|\hat{\theta}_n - \theta| \geq \epsilon) \to 0, \tag{2.9}$$

as $n \to \infty$. Since $X(W_n)$ has Poisson distribution with parameter $\mu(W_n) = \int_{W_n} \lambda(s) ds$, we know that $\mathbf{E}X(W_n) = Var(X(W_n)) = \int_{W_n} \lambda(s) ds$. Then we have

$$\mathbf{E}(\hat{\theta}_n) = \frac{1}{|W_n|} \int_{W_n} \lambda(s) ds, \quad \text{and} \quad Var(\hat{\theta}_n) = \frac{1}{|W_n|^2} \int_{W_n} \lambda(s) ds.$$

Now we write

$$\mathbf{P}\left(|\hat{\theta}_n - \theta| \geq \epsilon\right) \leq \mathbf{P}\left(|\hat{\theta}_n - \mathbf{E}\hat{\theta}_n| + |\mathbf{E}\hat{\theta}_n - \theta| \geq \epsilon\right).$$

By Lemma 2.1, for sufficiently large n, we have $|\mathbf{E}\hat{\theta}_n - \theta| \leq \epsilon/2$. Then, for sufficiently large n, we have

$$\mathbf{P}\left(|\hat{\theta}_n - \theta| \geq \epsilon\right) \leq \mathbf{P}\left(|\hat{\theta}_n - \mathbf{E}\hat{\theta}_n| \geq \frac{\epsilon}{2}\right). \tag{2.10}$$

By Chebyshev's inequality and Lemma 2.1, the r.h.s. of (2.10) does not exceed

$$\frac{4Var(\hat{\theta}_n)}{\epsilon^2} = \frac{4}{\epsilon^2 |W_n|^2} \int_{W_n} \lambda(s) ds = \frac{4}{\epsilon^2 |W_n|}(\theta + o(1)), \tag{2.11}$$

as $n \to \infty$. By (1.3), the r.h.s. of (2.11) is $o(1)$, as $n \to \infty$. This completes the proof. $\Box$

Throughout the paper, for any random variables Y_n and Y on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, we write $Y_n \xrightarrow{c} Y$ to denote that Y_n *converges completely* to Y, as $n \to \infty$. We say that Y_n converges completely to Y if $\sum_{n=1}^{\infty} \mathbf{P}(|Y_n - Y| > \epsilon) < \infty$, for every $\epsilon > 0$.

Lemma 2.3 *Suppose that λ is periodic (with period τ) and locally integrable. If, in addition, for each $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} \exp\{-\epsilon |W_n|\} < \infty, \tag{2.12}$$

then, as $n \to \infty$,

$$\hat{\theta}_n \overset{c}{\to} \theta. \tag{2.13}$$

Proof: To establish (2.13) we must show

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\hat{\theta}_n - \theta| > \epsilon\right) < \infty, \tag{2.14}$$

for each $\epsilon > 0$. Now, recall from the proof of Lemma 2.2 that, for sufficiently large n, the probability on the l.h.s. of (2.14) does not exceed that on the r.h.s. of (2.10). Then, to prove (2.14), it suffices to check that the probability on the r.h.s. of (2.10) is summable. To do this, we first recall an exponential bound for Poisson probabilities (e.g., see Reiss [10], p. 222), which states that, for any Poisson random variable X and for any $\epsilon > 0$, we have

$$\mathbf{P}\left((\mathbf{E}X)^{-1/2}|X - \mathbf{E}X| \geq \epsilon\right) \leq 2\exp\left\{-\frac{\epsilon^2}{2 + \epsilon(\mathbf{E}X)^{-1/2}}\right\}. \tag{2.15}$$

Then, by an application of (2.15) and with θ_n as in (2.4), the probability on the r.h.s. of (2.10) can be bounded above as follows.

$$\begin{aligned}
&\mathbf{P}\left(|\hat{\theta}_n - \mathbf{E}\hat{\theta}_n| \geq \frac{\epsilon}{2}\right) = \mathbf{P}\left(|W_n|^{-1}|X(W_n) - \mathbf{E}X(W_n)| \geq \frac{\epsilon}{2}\right) \\
&= \mathbf{P}\left((\mathbf{E}X(W_n))^{-1/2}|X(W_n) - \mathbf{E}X(W_n)| \geq \frac{\epsilon|W_n|}{2(\mathbf{E}X(W_n))^{1/2}}\right) \\
&\leq 2\exp\left\{-\frac{\epsilon^2 4^{-1}|W_n|^2(\mathbf{E}X(W_n))^{-1}}{2 + \epsilon 2^{-1}|W_n|(\mathbf{E}X(W_n))^{-1}}\right\} = 2\exp\left\{-\frac{\epsilon^2|W_n|}{8\theta_n + 2\epsilon}\right\}.
\end{aligned} \tag{2.16}$$

For sufficiently large n, since by Lemma 2.1 we have $\theta_n = \theta + o(1)$, as $n \to \infty$, the r.h.s. of (2.16) does not exceed $2\exp\{-(\epsilon^2|W_n|)(16\theta + 2\epsilon)^{-1}\}$. By assumption (2.12), we can conclude that the quantity on the r.h.s. of (2.16) is summable. This completes the proof. □

Asymptotic normality of $\hat{\theta}_n$, properly normalized, is established in the following theorem:

Theorem 2.4 *If λ is periodic (with period τ) and locally integrable, then*

$$|W_n|^{1/2}\left(\hat{\theta}_n - \theta\right) \overset{d}{\to} N(0, \theta), \tag{2.17}$$

as $n \to \infty$.

Proof: First we write

$$|W_n|^{1/2}\left(\hat{\theta}_n - \theta\right) = |W_n|^{1/2}\left(\hat{\theta}_n - \theta_n\right) + |W_n|^{1/2}(\theta_n - \theta), \tag{2.18}$$

where θ_n is given by the l.h.s. of (2.4). Then, to prove (2.17), it suffices to check

$$|W_n|^{1/2}(\hat{\theta}_n - \theta_n) \overset{d}{\to} N(0, \theta), \tag{2.19}$$

and

$$|W_n|^{1/2}(\theta_n - \theta) \to 0, \tag{2.20}$$

as $n \to \infty$.

First we prove (2.19). The l.h.s. of (2.19) can be written as

$$|W_n|^{1/2}\left(\frac{X(W_n)}{|W_n|} - \frac{\int_{W_n}\lambda(s)ds}{|W_n|}\right) = \frac{(\int_{W_n}\lambda(s)ds)^{1/2}}{|W_n|^{1/2}}\left(\frac{X(W_n) - \int_{W_n}\lambda(s)ds}{(\int_{W_n}\lambda(s)ds)^{1/2}}\right). \tag{2.21}$$

By Lemma 2.1, (1.3), and the normal approximation to the Poisson distribution, the r.h.s. of (2.21) can be written as $(\theta^{1/2} + o(1))(N(0,1) + o_p(1))$, which converges in distribution to $N(0,\theta)$ as $n \to \infty$.

Next we prove (2.20). Combining (2.5), (2.6), and by writing $|W_{N_{n\tau}}|$ as $(|W_n| - |R_{n\tau}|)$, we can simplify the r.h.s. of (2.5) to get

$$\theta_n = \theta - \frac{\theta|R_{n\tau}|}{|W_n|} + \frac{1}{|W_n|}\int_{R_{n\tau}}\lambda(s)ds. \tag{2.22}$$

The l.h.s. of (2.20) now reduces to

$$|W_n|^{1/2}\left(-\frac{\theta|R_{n\tau}|}{|W_n|} + \frac{1}{|W_n|}\int_{R_{n\tau}}\lambda(s)ds\right) = \left(-\frac{\theta|R_{n\tau}|}{|W_n|^{1/2}} + \frac{\int_{R_{n\tau}}\lambda(s)ds}{|W_n|^{1/2}}\right). \tag{2.23}$$

Since $|R_{n\tau}| < \tau$ for all n and $\int_{R_{n\tau}}\lambda(s)ds = \mathcal{O}(1)$, as $n \to \infty$, then by (1.3), the r.h.s. of (2.23) is $o(1)$, as $n \to \infty$. This completes the proof. $\square$

To conclude this section we derive a bootstrap CLT, parallel to Theorem 2.4. Conditionally given $X(W_n)$, let $X^*(W_n)$ denote a realization from a Poisson distribution with parameter $X(W_n)$. If $X(W_n)$ happens to be equal to zero, we set $X^*(W_n) = 0$. Define

$$\hat{\theta}_n^* = \frac{X^*(W_n)}{|W_n|}. \tag{2.24}$$

To obtain a bootstrap counterpart of (2.17), we replace $\hat{\theta}_n - \theta$ by $\hat{\theta}_n^* - \hat{\theta}_n$, with $\hat{\theta}_n$ as in (2.1), and establish bootstrap consistency, i.e. we shall prove that $|W_n|^{\frac{1}{2}}(\hat{\theta}_n^* - \hat{\theta}_n)$ has - in P-probability - the same limit distribution as $|W_n|^{\frac{1}{2}}(\hat{\theta}_n - \theta)$, that is a normal $(0,\theta)$ distribution. Note that we have employed a 'parametric bootstrap' here. There is no use for Efron's bootstrap, instead our bootstrap is based on a parametric model, namely a Poisson distribution with estimated parameter. We refer to Helmers and Putter [2] for a general introduction to bootstrap resampling.

Theorem 2.5 *If λ is periodic (with period τ) and locally integrable, then*

$$|W_n|^{1/2}\left(\hat{\theta}_n^* - \hat{\theta}_n\right) \xrightarrow{d} N(0,\theta), \tag{2.25}$$

as $n \to \infty$, in P-probability. Hence our parametric bootstrap works.

Proof: Since $X^*(W_n)$ has Poisson distribution with parameter $X(W_n)$, it suffices to write the l.h.s. of (2.25) as

$$|W_n|^{1/2}\left(\frac{X^*(W_n)}{|W_n|} - \frac{X(W_n)}{|W_n|}\right) = \left(\frac{X(W_n)}{|W_n|}\right)^{1/2}\left(\frac{X^*(W_n) - X(W_n)}{(X(W_n))^{1/2}}\right). \tag{2.26}$$

By Lemma 2.2, (1.3), and the normal approximation to the Poisson distribution, the r.h.s. of (2.26) can be written as

$$\left(\theta^{1/2} + o_p(1)\right)\left(N(0,1) + o_p(1)\right), \tag{2.27}$$

since $X(W_n) \to \infty$, in P-probability, as $\int_{W_n} \lambda(s)ds \to \infty$, which is implied by $|W_n| \to \infty$ (cf.(1.3)), because $\theta > 0$. Hence, by Slutsky (cf. Serfling [11], p. 19), the quantity in (2.27) converges in distribution to $N(0, \theta)$, as $n \to \infty$, in P-probability. This completes the proof. □

3. Estimation of the intensity function

In this section we give a preview of recent work by the present authors, in part joint with R. Zitikis (Winnipeg), on two nonparametric methods for estimating a cyclic intensity function at a given point, namely: a 'kernel type' estimator proposed and studied in Helmers, Mangku, and Zitikis [5], and a 'nearest neighbor' estimator investigated by Mangku [7]. A complete account of all this will be contained in Mangku's forthcoming Ph.D. thesis. As in the previous section, we will assume throughout that $\theta > 0$ and the intensity function λ is cyclic with period τ, that is λ satisfies (2.2). Note that in this case the condition $\int_{\boldsymbol{R}} \lambda(s)ds = \infty$ (cf. Lemma 1), which is really necessary to make consistent estimation possible, is automatically satisfied.

3.1 Kernel type estimator

The basic idea behind the construction of our estimator can be described as follows: Recall that our aim is to estimate the local intensity (cf.(1.1)), i.e. the intensity function λ at a given point s. Since λ is periodic (with period τ), to estimate λ at s, we not only can use the information in a neighborhood of s, but also the information in a neighborhood of $\{s + k\tau\} \cap W_n$, for any integer k. But, in order to be able to employ this idea when we do not know τ, we have to estimate the period τ.

Let $\hat{\tau}_n$ be a consistent estimator of the period τ, and let h_n be a sequence of positive real numbers such that

$$h_n \downarrow 0, \tag{3.1}$$

as $n \to \infty$. An estimator $\hat{\lambda}_n$ of λ at a given point s is given by

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap W_n\right), \tag{3.2}$$

where $B_{h_n}(x)$ denotes the interval $[x-h, x+h]$. The estimator $\hat{\lambda}_n(s)$ can be viewed as a kernel type estimator, using uniform kernel $K(s) = \frac{1}{2}\mathbf{I}_{(-1,1)}(s)$. An example is given in Figure 1.

We note in passing that the estimate $\hat{\lambda}_n(s)$ plotted in Figure 1 is not really a good one. This appears to be due to the slow rate of convergence of $\hat{\tau}_n$. However one can do better, but this is outside the scope of this paper.

The general kernel type estimator of λ at a given point s can be written down as

$$\hat{\lambda}_{n,K}(s) = \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_R K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) X(dx \cap W_n). \tag{3.3}$$

In Helmers, Mangku, and Zitikis [5] it is proved that: *if λ is periodic and locally integrable, the kernel K is a probability density function, bounded, has support in $[-1, 1]$, and has only a finite number of discontinuities, (3.1) holds,*

$$|W_n|h_n \to \infty, \ \textit{and}\ |W_n||\hat{\tau}_n - \tau|/h_n \xrightarrow{p} 0, \tag{3.4}$$

as $n \to \infty$, then

$$\hat{\lambda}_{n,K}(s) \ \xrightarrow{p}\ \lambda(s), \tag{3.5}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

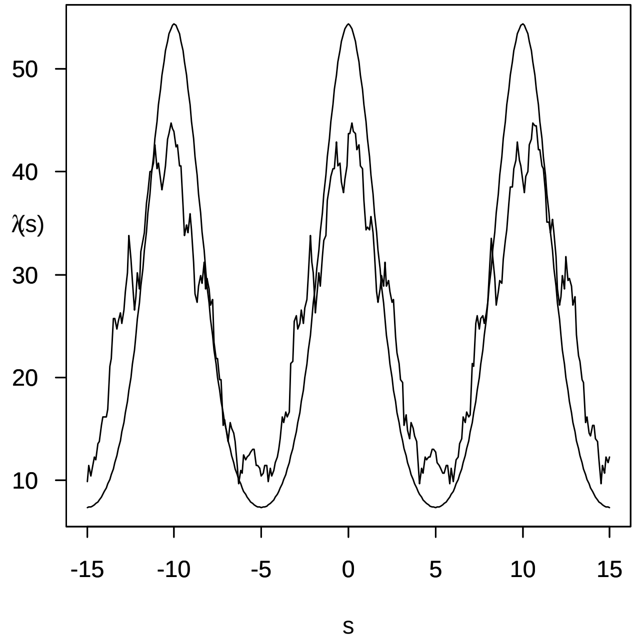

Figure 1: Graph of $\lambda(s) = 20 \exp\{\cos(\pi s/5)\}$ and its estimate $\hat{\lambda}_n(s)$ using a realization observed in window $W_n = [-n, n] = [-50, 50]$, with $h_n = 0.2$ and $\hat{\tau}_n = 10.404$ given by (3.7).

Note that s is a Lebesgue point of λ when we have

$$\lim_{h \to 0} \frac{1}{2h} \int_{s-h}^{s+h} |\lambda(u) - \lambda(s)| du = 0.$$

By local integrability of λ, the set of all Lebesgue points of λ is dense in $\boldsymbol{R}$. Hence, this assumption appears to be a mild one.

Let us give a brief outline of the proof of (3.5). A complete proof can be found in Helmers, Mangku, Zitikis [5]. Here we restrict attention to the estimator given in (3.2), but the general case is similar. Define

$$N_n = \#\{k : \ s + k\tau \in W_n\},$$

i.e., N_n denotes the number of integers k such that $s + k\tau \in W_n$. The basic idea is to establish rigorously in a number of steps the following approximations:

$$\begin{aligned}
\hat{\lambda}_n(s) \quad &= \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap W_n\right) \\
&\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap W_n\right) \\
&\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\tau) \cap W_n\right) \\
&\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \mathbf{E} X\left(B_{h_n}(s + k\tau) \cap W_n\right)
\end{aligned}$$

$$= \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \int_{-h_n}^{h_n} \lambda(s + k\tau + x)\mathbf{I}(s + k\tau + x \in W_n)dx$$
$$\approx \lambda(s). \tag{3.6}$$

The main problem here is to replace $\hat{\tau}_n$ by τ, that is we replace, for any integer k, the random centre $s + k\hat{\tau}_n$ of $B_{h_n}(s + k\hat{\tau}_n)$ by a deterministic one, namely $s + k\tau$ and obtain $B_{h_n}(s + k\tau)$. To validate this we need a rather fast rate of convergence for $\hat{\tau}_n$ approaching τ. However, the question remains: how do we estimate the period τ? In Helmers and Mangku [4] we propose to estimate τ by

$$\hat{\tau}_n = \arg\min_{\delta \in \Theta} Q_n(\delta), \tag{3.7}$$

where $\Theta \subset \boldsymbol{R}^+$ denotes the parameter space of τ and

$$Q_n(\delta) = \frac{1}{|W_n|} \sum_{i=1}^{N_{n\delta}} \left(X(U_{\delta,i}) - \frac{1}{N_{n\delta}} \sum_{j=1}^{N_{n\delta}} X(U_{\delta,j}) \right)^2,$$

with $N_{n\delta} = \left[\frac{|W_n|}{\delta}\right]$. Note that $N_{n\delta}$ denotes the (maximum) number of adjacent disjoint intervals $U_{\delta,i}$ of length δ in the window W_n. Consistency of $\hat{\tau}_n$ in estimating τ was established. A rate of consistency for $\hat{\tau}_n$ approaching τ was also obtained.

3.2 Nearest neighbor estimator

A simple heuristic explanation of our nearest neighbor estimator for the intensity function λ of a cyclic Poisson process X, at a given point s, will now be given. The complete details of all this can be found in Mangku [7].

To begin with, we describe a shifting procedure to employ the periodicity present, which will enable us to combine different pieces from our dataset, in order to construct an 'infill asymptotic framework', instead of the original 'increasing domain asymptotic' set up (cf (1.3)). In the case that the period τ is known, the shifting procedure can be describe as follows. To do this, we cover the window W_n by adjacent disjoint intervals $\bar{B}_\tau(s + j\tau)$, for all integer j such that $\bar{B}_\tau(s + j\tau) \cap W_n \neq \emptyset$, where $\bar{B}_\tau(s) = [s - \frac{\tau}{2}, s + \frac{\tau}{2})$. Next, for each j, we shift the interval $\bar{B}_\tau(s + j\tau)$ (together with the data points of $X(\omega)$ contained in this interval) by amount $j\tau$, such that after shifting the interval coincide with $\bar{B}_\tau(s)$. We denote the new process after shifting by $\bar{X}_n$. It is easy to see that $\bar{X}_n$ is also a Poisson process with intensity function

$$\lambda_n(u) = \lambda(u) \sum_{j=-\infty}^{\infty} \mathbf{I}(u + j\tau \in W_n) \tag{3.8}$$

for any $u \in \bar{B}_\tau(s)$. To prove this one may rely on the superposition theorem and the restriction theorem for Poisson processes (cf. Kingman [6], p. 16-17). Note that $\bar{X}_n$ is not cyclic anymore; on the contrary, we have exploited the periodicity present in X, to transform X to $\bar{X}_n$, which lives only on $\bar{B}_\tau(s)$, a bounded interval, while $X|W_n$ lives on the expanding window W_n. As a result, the 'increasing domain framework' for 'X observed in W_n' is replaced by the 'infill asymptotic framework' for $\bar{X}_n$ observed in $\bar{B}_\tau(s)$ (cf. Figure 2).

Next suppose that we do not know the period τ, and let $\hat{\tau}_n$ be a consistent estimator of τ. Let $\hat{s}_j,\ j = 1, \dots, m$, with $X(W_n) = m$, denote the 'datapoints' observed in the window W_n, but shifted by a random multiple of $\hat{\tau}_n$, in such a way, that all $\hat{s}_j$'s belong to the interval $\bar{B}_{\hat{\tau}_n}(s) = [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$. We denote the shifted data set by $\hat{X}_n$, which is the same as $\bar{X}_n$ provided τ is replaced by $\hat{\tau}_n$. Now we

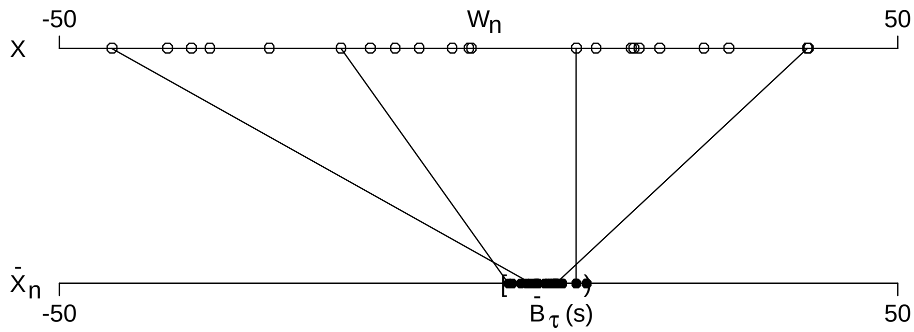

Figure 2: A realization of a Poisson process X with $\lambda(s) = 0.2 \exp\{\cos(\pi s/5)\}$, observed in window $W_n = [-n, n]$, with $n = 50$, is shifted by multiples of $\tau = 10$ to obtain a realization of a Poisson process $\bar{X}_n$ in $\bar{B}_\tau(s) = [3, 13)$, where $s = 8$.

can also write our kernel estimator (3.2) as:

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n}{|W_n|} \frac{\hat{X}_n(B_{h_n}(s))}{2h_n}. \tag{3.9}$$

To obtain our nearest neighbor estimator (3.13), we set the (random) number $\hat{X}_n(B_{h_n}(s))$ in (3.9) equal to a (non-random) positive integer k_n, i.e. $\hat{X}_n(B_{h_n}(s)) = k_n$, which directly yields that $h_n = |\hat{s}_{(k_n)} - s|$, and hence (3.13). Here $|\hat{s}_{(k_n)} - s|$ denotes the k_n-th order statistic of the 'sample' $|\hat{s}_1 - s|, \ldots, |\hat{s}_m - s|$.

A difficulty which arises is that by carrying out a shift by (random) multiples of $\hat{\tau}_n$, we introduce some dependence in the resulting 'sample': all $\hat{s}_j$'s depend on $\hat{\tau}_n$, our estimate of the period τ, which is based on the whole dataset $X(\omega) \cap W_n$. However, in Mangku [7] it is proved that, by imposing a suitable condition - assumption (3.14) - on the *rate of consistency* of $\hat{\tau}_n$, this problem can be reduced to the much easier case that τ is assumed to be known.

The next step is to analyse our 'transformed problem' in a way similar to the known mathematical analysis for density estimation. The only missing ingredient to do this is a conditioning argument. Conditioned on $\bar{X}_n(\bar{B}_\tau(s)) = X(W_n) = m$, the vector $(\bar{s}_1, \ldots, \bar{s}_m)$ - where $\bar{s}_i$ denotes the location of the ith observed datapoint s_i, after translation by a multiple of τ, such that $\bar{s}_i \in \bar{B}_\tau(s)$, for $i = 1, \ldots, m$ - is distributed as a random sample of size m from a distribution with density

$$\bar{f}_n(u) = \frac{\lambda_n(u)}{\int_{\bar{B}_\tau(s)} \lambda_n(v) dv} \mathbf{I}(u \in \bar{B}_\tau(s)). \tag{3.10}$$

This directly yields (with $\bar{s}_i$ being the ith shifted data point)

$$\begin{aligned} H_n(x) &= \mathbf{P}(|\bar{s}_i - s| \leq x | X(W_n) = m) = \mathbf{P}(s - x \leq \bar{s}_i \leq s + x | X(W_n) = m) \\ &= \int_{s-x}^{s+x} \frac{\lambda_n(u)}{\int_{W_n} \lambda(v) dv} \mathbf{I}(u \in \bar{B}_\tau(s)) du. \end{aligned} \tag{3.11}$$

Now Mangku [7] proceeds by investigating the order statistics of the random sample $|\bar{s}_1 - s|, \ldots, |\bar{s}_m - s|$ of size m from H_n. Let $|\bar{s}_{(k)} - s|$ denote the k-th order statistic of the sample $|\bar{s}_1 - s|, \ldots, |\bar{s}_m - s|$. Define

$$\bar{\lambda}_n(s) = \frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|}, \tag{3.12}$$

which depends on τ, explicitly in the numerator, and implicitly in the denumerator. If we replace τ by $\hat{\tau}_n$ in de definition of $\bar{\lambda}_n(s)$ (i.e. $\hat{\tau}_n$ is substituted for τ and $\bar{s}_{(k_n)}$ is replaced by $\hat{s}_{(k_n)}$) we obtain our nearest neighbor estimator

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \tag{3.13}$$

for $\lambda(s)$. Note that $\hat{\lambda}_n(s)$ is well-defined provided $k_n \leq X(W_n)$.

In Mangku [7], it is proved that: *If λ is periodic and locally integrable, $k_n \to \infty$, $\frac{k_n}{|W_n|} \downarrow 0$, and the rate at which $\hat{\tau}_n$ estimate τ consistently, satisfies*

$$\frac{|W_n|^2}{k_n}|\hat{\tau}_n - \tau| \xrightarrow{p} 0, \tag{3.14}$$

as $n \to \infty$, then

$$\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s), \tag{3.15}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

The almost sure version of this result can also be established, provided the additional conditions $\frac{|W_n|^2}{k_n}|\hat{\tau}_n - \tau| \xrightarrow{c} 0$ and $\sum_{n=1}^{\infty} exp(-\epsilon k_n) < \infty$, for each $\epsilon > 0$, are imposed. So we have to exclude now cases like $k_n \sim \log\log n$.

The final part of Mangku's proof involves some further real analysis. First of all we can transform $|\bar{s}_{(k_n)} - s|$ to uniform order statistics, by applying the classical tranformation $H_n^{-1}(U_{k_n:m}) \stackrel{d}{=} |\bar{s}_{(k_n)} - s|$. Using well-known asymptotic results for uniform order statistics, together with Taylor expansion arguments related to the function(s) H_n^{-1}, the proof can be completed.

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