Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

Nearest Neighbor Estimation of the Intensity Function of a Cyclic Poisson Process

I W. Mangku

Probability, Networks and Algorithms (PNA)

PNA-R9914 December 31, 1999

Report PNA-R9914
ISSN 1386-3711

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

Nearest Neighbor Estimation of the Intensity Function of a Cyclic Poisson Process

I Wayan Mangku
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

and
Department of Mathematics, Bogor Agricultural University
Jl. Raya Pajajaran Bogor, 16144 Indonesia

ABSTRACT
We consider the problem of estimating the intensity function of a cyclic Poisson point process. We suppose that only a single realization of the cyclic Poisson point process is observed within a bounded 'window', and our aim is to estimate consistently the intensity function at a given point. A nearest neighbor estimator of the intensity function is proposed, and we show that our estimator is weakly and strongly consistent, as the window expands.

1991 Mathematics Subject Classification: 60G55, 62G05, 62G20.
Keywords and Phrases: cyclic Poisson point process, cyclic intensity function, nonparametric estimation, nearest neighbor estimator, period, weak consistency, strong consistency.
Note: Work carried out under project PNA 3.2, and supported by a cooperation project between The Netherlands and Indonesia on "Applied Mathematics and Computational Methods". This paper is submitted for publication.

1. Introduction

Let X be a cyclic Poisson point process X in $\boldsymbol{R}$ with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν, and with (unknown) locally integrable intensity function $\lambda : \boldsymbol{R} \to \boldsymbol{R}^+ \cup \{0\}$. In addition, λ is assumed to be cyclic with (unknown) period $\tau \in \boldsymbol{R}^+$, i.e.

$$\lambda(s + k\tau) = \lambda(s) \tag{1.1}$$

for all $s \in \boldsymbol{R}$ and $k \in \mathbb{Z}$.

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and let us suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the cyclic Poisson point process X is observed, though only within a bounded interval, called 'window', $W \subset \boldsymbol{R}$. The aim of this paper is to estimate consistently the intensity function λ at a given point s using an estimator based on nearest neighbor distances, from a single realization $X(\omega)$ of the Poisson process X observed in $W = W_n$, in such a way that

$$|W_n| \to \infty, \tag{1.2}$$

as $n \to \infty$, where $|W_n| = \nu(W_n)$ denotes the size (or the Lebesgue measure) of the window W_n.

Let $\hat{\tau}$ be an estimator of the period τ, e.g. the one proposed and studied in Helmers and Mangku [4], or perhaps the estimator of τ investigated by Vere-Jones [13]. We assume that the estimator $\hat{\tau} = \hat{\tau}_n$ satisfies the condition

$$|W_n|\,|\hat{\tau}_n - \tau| = o_p\left(\frac{k_n}{|W_n|}\right), \tag{1.3}$$

as $n \to \infty$, with k_n as in (1.6) and (1.7).

Let $s_i,\ i = 1, \ldots, X(W_n, \omega)$, denote the locations of the points in the realization $X(\omega)$ of the Poisson process X, observed in window W_n. Here $X(W_n, \omega)$ is nothing but the cardinality of the data set $\{s_i\}$.

It is well-known (see, e.g. Cressie [2], p. 651) that, conditionally given $X(W_n) = m$, $(s_1, \ldots, s_m)$ can be viewed as a random sample of size m from a distribution with density f, which is given by

$$f(u) = \frac{\lambda(u)}{\int_{W_n} \lambda(v)dv} \mathbf{I}(u \in W_n), \tag{1.4}$$

while the simultaneous density $f(s_1, \ldots, s_m)$, of $(s_1, \ldots, s_m)$ is given by

$$f(s_1, \ldots, s_m) = \frac{\prod_{i=1}^m \lambda(s_i)}{\left(\int_{W_n} \lambda(v)dv\right)^m} \mathbf{I}\left((s_1, \ldots, s_m) \in W_n^m\right). \tag{1.5}$$

Let $\hat{s}_i,\ i = 1, \ldots, m$, denote the location of the point s_i $(i = 1, \ldots, m)$, after translation by a multiple of $\hat{\tau}_n$ such that $\hat{s}_i \in B_{\hat{\tau}_n}(s)$, for all $i = 1, \ldots, m$, where $B_{\hat{\tau}_n}(s) = [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$. The translation can be described more precisely as follows. We cover the window W_n by $N_{n,\hat{\tau}_n}$ adjacent disjoint intervals $B_{\hat{\tau}_n}(s + j\hat{\tau}_n)$, for some integer j, and let $N_{n,\hat{\tau}_n}$ denote the number of such intervals, provided $B_{\hat{\tau}_n}(s + j\hat{\tau}_n) \cap W_n \neq \emptyset$. Then, for each j, we shift the interval $B_{\hat{\tau}_n}(s + j\hat{\tau}_n)$ (together with the data points of $X(\omega)$ contained in this interval) by the amount $j\hat{\tau}_n$ such that after translation the interval coincide with $B_{\hat{\tau}_n}(s)$.

Let $k = k_n$ be a sequence of positive integers such that

$$k_n \to \infty, \tag{1.6}$$

and

$$\frac{k_n}{|W_n|} \downarrow 0, \tag{1.7}$$

as $n \to \infty$.

Let now $|\hat{s}_{(k_n)} - s|$ denote the k_n-th order statistics of $|\hat{s}_1 - s|, \ldots, |\hat{s}_m - s|$, given $X(W_n) = m$. A nearest neighbor estimator for λ at the point s, is given by

$$\hat{\lambda}_n(s) = \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|}. \tag{1.8}$$

Note that $\hat{\lambda}_n(s)$ is well-defined provided $k_n \leq X(W_n)$. Since

$$\mathbf{P}(k_n \leq X(W_n)) = \mathbf{P}(k_n/|W_n| \leq X(W_n)/|W_n|) \ \to \ 1,$$

as $|W_n| \to \infty$, (because of (1.7) and the fact that $X(W_n)/|W_n| \xrightarrow{p} \theta$, with $\theta > 0$, where $\theta = \tau^{-1} \int_0^\tau \lambda(s)ds$, the 'global intensity' of X), we can conclude no matter how we define $\hat{\lambda}_n(s)$ in case $k_n > X(W_n)$, Theorem 1.1 remains valid. To check that the above conclusion also holds for Theorem 1.2, we need to show that

$$\textstyle\sum_{n=1}^\infty \mathbf{P}(k_n > X(W_n)) < \infty.$$

But, by (1.7), the exponential bound for Poisson probabilities (e.g., see Reiss [11], p. 222), and (1.10), it is easy to show that $\mathbf{P}(k_n > X(W_n))$ is summable.

Theorem 1.1 *Suppose that λ is periodic and locally integrable. If, in addition (1.6), (1.7) and (1.3) hold, then*

$$\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s), \tag{1.9}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

Throughout the paper, for any random variables Y_n and Y, we write $Y_n \xrightarrow{c} Y$ to denote that Y_n *converges completely* to Y, as $n \to \infty$.

Theorem 1.2 *Suppose that λ is periodic and locally integrable. If, in addition*

$$\sum_{n=1}^{\infty} \exp(-\epsilon k_n) < \infty, \tag{1.10}$$

for each $\epsilon > 0$, (1.7) holds, and

$$\frac{|W_n|^2}{k_n} |\hat{\tau}_n - \tau| \xrightarrow{c} 0, \tag{1.11}$$

then

$$\hat{\lambda}_n(s) \xrightarrow{c} \lambda(s), \tag{1.12}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

We remark that nearest neighbor estimators for estimating density functions, was studied by [7], [14], [8], [9], and some others. The condition (1.10) also appears in Wagner [14]. In the construction of our nearest neighbor estimator (1.8) we employ the periodicity of λ (cf. (1.1)) to combine different pieces from our data set, in order to mimic the 'infill asymptotic' framework.

Kernel type estimators for the intensity function λ at a given point s, are proposed and studied by Helmers and Zitikis [3] and Helmers, Mangku, and Zitikis [5]. Helmers and Zitikis [3] show that their estimator is L_2-consistent, provided λ has a parametric form, while Helmers, Mangku, and Zitikis [5] consider a cyclic Poisson process and prove that their estimator is weakly and strongly consistent, provided s is a Lebesgue point of λ.

2. The case τ is known

We first consider the situation where we know the period τ. Let $\bar{s}_i$, $i = 1, \dots, X(W_n, \omega)$, denotes the location of the points s_i ($i = 1, \dots, X(W_n, \omega)$), after translation by a multiple of τ such that $\bar{s}_i \in B_\tau(s)$, for all $i = 1, \dots, X(W_n, \omega)$, where $B_\tau(s) = [s - \frac{\tau}{2}, s + \frac{\tau}{2})$. By periodicity of λ, we have that $\lambda(\bar{s}_i) = \lambda(s_i)$, for each $i = 1, \dots, X(W_n, \omega)$. For any $A \subset B_\tau(s)$, let $\bar{X}_n(A)$ denotes the number of points $\bar{s}_i$ in A. Then, of course, $\bar{X}_n(B_\tau(s)) = X(W_n)$, where $\bar{X}_n$ is a Poisson process with intensity function

$$\lambda_n(u) = \lambda(u) \sum_{j=-\infty}^{\infty} \mathbf{I}(u + j\tau \in W_n)$$

(cf. Kingman [6], Superposition Theorem and Restriction Theorem, p. 16-17). As a result, (cf. (1.4) and (1.5)), conditionally given $\bar{X}_n(B_\tau(s)) = m$, $(\bar{s}_1, \dots, \bar{s}_m)$ can be viewed as a random sample of size m from a distribution with density $\bar{f}$, which is given by

$$\bar{f}(u) = \frac{\lambda_n(u)}{\int_{W_n} \lambda(v) dv} \mathbf{I}(u \in B_\tau(s)) = \frac{\lambda_n(u)}{\int_{B_\tau(s)} \lambda_n(v) dv} \mathbf{I}(u \in B_\tau(s)), \tag{2.1}$$

while the simultaneous density $\bar{f}(\bar{s}_1,\dots,\bar{s}_m)$, of $(\bar{s}_1,\dots,\bar{s}_m)$ is given by

$$\bar{f}(\bar{s}_1,\dots,\bar{s}_m) = \frac{\prod_{i=1}^{m}\lambda_n(\bar{s}_i)}{\left(\int_{W_n}\lambda(v)dv\right)^m}\mathbf{I}\left((\bar{s}_1,\dots,\bar{s}_m)\in B_\tau(s)^m\right). \tag{2.2}$$

For any real number $x \geq 0$, define

$$\begin{aligned} H_n(x) &= \mathbf{P}\left(|\bar{s}_i - s| \leq x \ \ |X(W_n)=m\right) = \mathbf{P}\left(s-x \leq \bar{s}_i \leq s+x \ \ |X(W_n)=m\right) \\ &= \int_{s-x}^{s+x}\frac{\lambda_n(u)}{\int_{W_n}\lambda(v)dv}\mathbf{I}(u\in B_\tau(s))du. \end{aligned} \tag{2.3}$$

Now we consider the order statistics of the random sample $|\bar{s}_1 - s|,\dots,|\bar{s}_m - s|$ of size m from H_n. Let $|\bar{s}_{(k)} - s|$ denote the k-th order statistics of the sample $|\bar{s}_1 - s|,\dots,|\bar{s}_m - s|$. Define

$$\bar{\lambda}_n(s) = \frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|}. \tag{2.4}$$

Note that, if we replace τ and $\bar{s}_{(k_n)}$ in $\bar{\lambda}_n(s)$ by $\hat{\tau}_n$ and $\hat{s}_{(k_n)}$ respectively, then $\bar{\lambda}_n(s)$ reduces to the estimator $\hat{\lambda}_n(s)$ given in (1.8). We will now first prove that our Theorems are true, when $\hat{\lambda}_n(s)$ is replaced by $\bar{\lambda}_n(s)$. In section 3 we will show that our Theorems are valid for $\hat{\lambda}_n(s)$ as well.

Lemma 2.1 *Suppose that λ is periodic (with period τ), and locally integrable. If, in addition (1.6) and (1.7) hold, then*

$$\bar{\lambda}_n(s) \stackrel{p}{\to} \lambda(s), \tag{2.5}$$

as $n\to\infty$, for each s at which λ is continuous and positive.

Lemma 2.2 *Suppose that λ is periodic (with period τ), and locally integrable. If, in addition (1.10) and (1.7) hold, then*

$$\bar{\lambda}_n(s) \stackrel{c}{\to} \lambda(s), \tag{2.6}$$

as $n\to\infty$, for each s at which λ is continuous and positive.

Proof of Lemma 2.1

In view of the remark following (1.8), we may assume, without loss of generality, that $k_n \leq X(W_n)$.

To prove (2.5), we must show that,

$$\lim_{n\to\infty}\mathbf{P}\left(\left|\frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|} - \lambda(s)\right| \geq \epsilon\right) = 0, \tag{2.7}$$

for each sufficiently small $\epsilon > 0$. Choose $\epsilon < \lambda(s)$. Then, a simple calculation shows that, the probability on the l.h.s. of (2.7) is equal to

$$\begin{aligned} &\mathbf{P}\left(\frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} \leq |\bar{s}_{(k_n)} - s| \quad \text{or} \quad \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)} \geq |\bar{s}_{(k_n)} - s|\right) \\ &\leq \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) + \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right). \end{aligned} \tag{2.8}$$

Then, to prove (2.7), it suffices to check that

$$\lim_{n\to\infty}\mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) = 0, \tag{2.9}$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \le \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right) = 0, \tag{2.10}$$

for each $\epsilon > 0$. Here we only give proof of (2.9), because the proof of (2.10) is similar.

Recall $X(W_n)$ is a Poisson with $\mathbf{E}X(W_n) = Var(X(W_n)) = \int_{W_n} \lambda(s)ds$. Since λ is periodic (with period τ), a simple calculation shows that $\int_{W_n} \lambda(s)ds = \theta|W_n| + O(1)$, as $n \to \infty$. Let

$$C_{1,n} = [\theta|W_n| - (\theta|W_n|)^{1/2}a_n] \;\text{ and }\; C_{2,n} = [\theta|W_n| + (\theta|W_n|)^{1/2}a_n],$$

where a_n is an arbitrary sequence such that $a_n \to \infty$ and $a_n = o(|W_n|^{1/2})$, as $n \to \infty$. Then, we can write the probability on the l.h.s. of (2.9) as

$$\begin{aligned}
&\sum_{m=k_n}^{\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \ge \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\;|X(W_n)=m\right)\mathbf{P}\left(X(W_n)=m\right) \\
\le\; & \sum_{m=k_n}^{C_{1,n}-1} \mathbf{P}\left(X(W_n)=m\right) + \sum_{m=C_{2,n}+1}^{\infty} \mathbf{P}\left(X(W_n)=m\right) \\
+\; & \max_{C_{1,n}\le m\le C_{2,n}} \mathbf{P}\left(X(W_n)=m\right) \sum_{m=C_{1,n}}^{C_{2,n}} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \ge \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}|X(W_n)=m\right).
\end{aligned} \tag{2.11}$$

It suffices now to show that each term on the r.h.s. of (2.11) converges to zero, as $n \to \infty$.

First we show that the first term on the r.h.s. of (2.11) is $o(1)$, as $n \to \infty$. Since $|\mathbf{E}X(W_n) - \theta|W_n|| = O(1)$, as $n \to \infty$, this quantity is equal to

$$\begin{aligned}
&\mathbf{P}\left(X(W_n) \le C_{1,n} - 1\right) \le \mathbf{P}\left(X(W_n) \le \theta|W_n| - (\theta|W_n|)^{1/2}a_n\right) \\
&\le \mathbf{P}\left(|X(W_n) - \mathbf{E}X(W_n)| \ge (\theta|W_n|)^{1/2}a_n - |\mathbf{E}X(W_n) - \theta|W_n||\right) \\
&= \mathbf{P}\left((\mathbf{E}X(W_n))^{-1/2}|X(W_n) - \mathbf{E}X(W_n)| \ge O(1)a_n\right) \le O(1)\exp\left(-\frac{a_n^2}{2+o(1)}\right),
\end{aligned} \tag{2.12}$$

which is $o(1)$, since $a_n \to \infty$, as $n \to \infty$. Here we used an exponential bound for Poisson probabilities (e.g., see Reiss [11], p. 222). A similar argument also shows that the second term on the r.h.s. of (2.11) is $o(1)$, as $n \to \infty$.

Next we prove that the third term on the r.h.s. of (2.11) is $o(1)$, as $n \to \infty$. Let $m = m_n$ be a positive integer, such that $C_{1,n} \le m_n \le C_{2,n}$. Then $m_n \sim \theta|W_n|$, which implies that $k_n/m_n = o(1)$, as $n \to \infty$ (by (1.7)). Recall that $X(W_n)$ has a Poisson distribution with parameter $\mu(W_n) = \int_{W_n} \lambda(s)ds$. A simple calculation, using Stirling's formula, shows that

$$\max_{m_n,\; C_{1,n}\le m_n\le C_{2,n}} \mathbf{P}\left(X(W_n) = m_n\right) = O(|W_n|^{-1/2}),$$

as $n \to \infty$. It is well-known (see, e.g. Reiss [10], p. 15) that, conditionally given $\bar{X}_n(B_\tau(s)) = X(W_n) = m_n$, $|\bar{s}_{(k_n)} - s|$ has exactly the same distribution as $H_n^{-1}(Z_{k_n:m_n})$, where $Z_{k_n:m_n}$ is the k_n-th order statistics of a sample $Z_1, \dots, Z_{m_n}$ of size m_n from the uniform $(0,1)$ distribution. (We remark in passing that $k_n \le m_n$ for all n sufficiently large). Note that the same device was employed by Ralescu [9] in his analysis of multivariate nearest neighbor density estimators. As a result, the third term on the r.h.s. of (2.11) is equal to

$$O(|W_n|^{-1/2}) \sum_{m_n=C_{1,n}}^{C_{2,n}} \mathbf{P}\left(H_n^{-1}(Z_{k_n:m_n}) \ge \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right). \tag{2.13}$$

First note that, by choosing $\epsilon < \lambda(s)$, we have

$$\frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)} = \frac{\tau k_n}{2\lambda(s)|W_n|\left(1-\frac{\epsilon}{\lambda(s)}\right)} \geq \frac{\tau k_n}{2\lambda(s)|W_n|}\left(1+\frac{\epsilon}{\lambda(s)}\right)$$
$$= \frac{\tau k_n}{2\lambda(s)|W_n|} + \frac{\tau\epsilon k_n}{2\lambda^2(s)|W_n|}. \tag{2.14}$$

We know that, for each m_n,

$$\mathbf{E}Z_{k_n:m_n} = k_n/(m_n+1)$$

and

$$Var(Z_{k_n:m_n}) = O(k_n/(m_n^2)).$$

We now need a stochastic expansion for $H_n^{-1}(Z_{k_n:m_n})$. First we simplify the r.h.s. of (2.3) to get

$$\begin{aligned} H_n(x) &= \frac{(|W_n|/\tau + O(1))}{(\theta|W_n| + O(1))}\int_{s-x}^{s+x}\lambda(u)\mathbf{I}(u\in B_\tau(s))du \\ &= \left(\frac{1}{\theta\tau}+O(|W_n|^{-1})\right)\int_{s-x}^{s+x}\lambda(u)\mathbf{I}(u\in B_\tau(s))du \\ &= \frac{1}{\theta\tau}\int_{s-x}^{s+x}\lambda(u)\mathbf{I}(u\in B_\tau(s))du + O(|W_n|^{-1}), \end{aligned} \tag{2.15}$$

as $n\to\infty$, uniformly in x. This because $\int_{s-x}^{s+x}\lambda(u)\mathbf{I}(u\in B_\tau(s))du \leq \theta\tau$. Let $H(x)$ denote the first term on the r.h.s. of (2.15). The density of $H(x)$ is given by

$$h(x) = \frac{\lambda(s+x)\mathbf{I}(s+x\in B_\tau(s)) + \lambda(s-x)\mathbf{I}(s-x\in B_\tau(s))}{\theta\tau}\mathbf{I}(x\geq 0). \tag{2.16}$$

Since $H_n(x) = H(x) + O(|W_n|^{-1})$, as $n\to\infty$, we can write

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n}) &= \inf\{x : H_n(x)\geq Z_{k_n:m_n}\} = \inf\{x : H(x)\geq Z_{k_n:m_n} + O(|W_n|^{-1})\} \\ &= H^{-1}\left(Z_{k_n:m_n} + O(|W_n|^{-1})\right), \end{aligned} \tag{2.17}$$

as $n\to\infty$. Now we compute $H^{-1}(0)$. Since $\lambda(s)>0$ and λ is continuous at s, we see from the first term on the r.h.s. of (2.15) that $H(x)>0$, while $x>0$. In other words, the first term on the r.h.s. of (2.15) is equal to zero, if and only if, $x=0$. Hence $H^{-1}(0)=0$. Since h is right-continuous at 0, the first derivative of H^{-1} at 0 can be computed as

$$H^{-1'}(0) = \frac{1}{h\left(H^{-1}(0)\right)} = \frac{1}{h(0)} = \frac{\theta\tau}{2\lambda(s)}. \tag{2.18}$$

Since $H^{-1'}(0)$ is finite, by Young's form for Taylor's theorem (Serfling [12], p. 45), we can write

$$\begin{aligned} H^{-1}\left(\frac{k_n}{m_n+1}+O(|W_n|^{-1})\right) &= H^{-1}(0) + \left(\frac{k_n}{m_n+1}+O(|W_n|^{-1})\right)H^{-1'}(0)(1+o(1)) \\ &= \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right), \end{aligned} \tag{2.19}$$

as $n\to\infty$. Because λ is continuous at s, we can compute $H^{-1'}(k_n/(m_n+1)+O(|W_n|^{-1}))$ as follows.

$$H^{-1'}\left(\frac{k_n}{m_n+1}+O(|W_n|^{-1})\right) = \frac{1}{h\left(H^{-1}\left(\frac{k_n}{m_n+1}+O(|W_n|^{-1})\right)\right)} = \frac{1}{h\left(|o(1)|\right)}$$
$$= \frac{\theta\tau}{2\lambda(s+|o(1)|)} = \frac{\theta\tau}{2\lambda(s)} + o(1), \tag{2.20}$$

as $n \to \infty$. Because $H^{-1'}(k_n/(m_n+1)+O(|W_n|^{-1})) = O(1)$, as $n \to \infty$, by Young's form for Taylor's theorem, we can write $H_n^{-1}(Z_{k_n:m_n})$ as (cf. (2.17))

$$\begin{aligned} H_n^{-1}(Z_{k_n:m_n}) &= H^{-1}\left(Z_{k_n:m_n} + O(|W_n|^{-1})\right) = H^{-1}\left(\frac{k_n}{m_n+1} + O(|W_n|^{-1})\right) \\ &+ \left(Z_{k_n:m_n} - \frac{k_n}{m_n+1} + O(|W_n|^{-1})\right) H^{-1'}\left(\frac{k_n}{m_n+1} + O(|W_n|^{-1})\right)(1+o(1)) \\ &= \frac{\theta\tau k_n}{2\lambda(s)(m_n+1)} + o\left(\frac{k_n}{|W_n|}\right) + \left(Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right)\left(\frac{\theta\tau}{2\lambda(s)} + o(1)\right), \end{aligned} \tag{2.21}$$

as $n \to \infty$. Since $m_n \geq C_{1,n}$, the first term on the r.h.s. of (2.21) does not exceed

$$\begin{aligned} &\frac{\theta\tau k_n}{2\lambda(s)\left([\theta|W_n| - (\theta|W_n|)^{1/2}a_n] + 1\right)} \leq \frac{\theta\tau k_n}{2\lambda(s)\left(\theta|W_n| - (\theta|W_n|)^{1/2}a_n\right)} \\ &= \frac{\theta\tau k_n}{2\theta\lambda(s)|W_n|\left(1 - (\theta|W_n|)^{-1/2}a_n\right)} = \frac{\tau k_n}{2\lambda(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right). \end{aligned} \tag{2.22}$$

Combining (2.21), (2.22), and (2.14), and by noting also that the first term on the r.h.s. of (2.22) cancels with the first term on the r.h.s. of (2.14), we then found that, for sufficiently large n, the quantity in (2.13) does not exceed

$$\begin{aligned} &O(|W_n|^{-1/2})\,(C_{2,n} - C_{1,n} + 1)\,\mathbf{P}\left(o(\frac{k_n}{|W_n|}) + \frac{\theta\tau}{\lambda(s)}\left|Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right| \geq \frac{\tau\epsilon k_n}{2\lambda^2(s)|W_n|}\right) \\ &\leq O(1)a_n\mathbf{P}\left(\frac{\theta\tau}{\lambda(s)}\left|Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right| \geq \frac{\tau\epsilon k_n}{2\lambda^2(s)|W_n|}(1+o(1))\right) \\ &\leq O(1)a_n\mathbf{P}\left(\left|Z_{k_n:m_n} - \frac{k_n}{m_n+1}\right| \geq \frac{\epsilon k_n}{4\theta\lambda(s)|W_n|}\right), \end{aligned} \tag{2.23}$$

as $n \to \infty$. By Chebyshev's inequality, we found that the probability on the r.h.s. of (2.23) is of order $O(k_n^{-1})$, as $n \to \infty$. By (1.6) and choosing now $a_n = o(k_n)$, as $n \to \infty$, we have that the r.h.s. of (2.23) is $o(1)$ as $n \to \infty$. Hence (2.9) is proved. This completes the proof. □

Proof of Lemma 2.2

To establish (2.6), we must show that

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\left|\frac{\tau k_n}{2|W_n||\bar{s}_{(k_n)} - s|} - \lambda(s)\right| \geq \epsilon\right) < \infty, \tag{2.24}$$

for each $\epsilon > 0$. By (2.8), to prove (2.24) it suffices to show, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)}\right) < \infty, \tag{2.25}$$

and

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s) + \epsilon)}\right) < \infty. \tag{2.26}$$

Here we only give the proof of (2.25), because the proof of (2.26) is similar. To prove (2.25), it suffices clearly to show that, each of the terms on the r.h.s. of (2.11) converges completely to zero, as $n \to \infty$.

Let $C_{1,n}$ and $C_{2,n}$ be as in the proof of Lemma 2.1. In order to deal with the first and second term of (2.11), the sequence a_n will now have to satisfy, in addition to the assumption $a_n = o(|W_n|^{1/2})$ which was already needed in the proof of Lemma 2.1, the additional requirement $\sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty$. The argument given in (2.12) will then imply that these terms converge completely to zero, as $n \to \infty$.

It remains to show that the third term on the r.h.s. of (2.11) also converges completely to zero, as $n \to \infty$. To do this, it is clear from the proof of Lemma 2.1, that it suffices now to check that the r.h.s. of (2.23) is summable, for each $\epsilon > 0$.

Let us now consider the probability appearing on the r.h.s. of (2.23). To obtain an appropriate exponential bound we apply Bernstein's inequality (cf., for instance, Albers, Bickel, and van Zwet [1], p. 149) and obtain that there exists a positive constant C_0 such that the probability on the r.h.s. of (2.23) does not exceed

$$2 \exp\left\{-C_0 t_n^2\right\}, \tag{2.27}$$

where

$$t_n = \left(\frac{m_n}{k_n/(m_n+1)\,(1-k_n/(m_n+1))} \right)^{1/2} \frac{k_n \epsilon}{4\theta\lambda(s)|W_n|} \tag{2.28}$$

which, for sufficiently large n, can be replaced with impunity by $\epsilon/(8\lambda(s))k_n^{\frac{1}{2}}$. Hence, for sufficiently large n, the r.h.s. of (2.23) does not exceed

$$\begin{aligned}
&O(1) a_n \exp\left\{-\frac{C_0\epsilon^2}{64(\lambda(s))^2} k_n\right\} \\
&= O(1) \exp\left\{\log a_n - \frac{C_0\epsilon^2}{128(\lambda(s))^2} k_n\right\} \exp\left\{-\frac{C_0\epsilon^2}{128(\lambda(s))^2} k_n\right\} \\
&= O(1) \exp\left\{-\frac{C_0\epsilon^2}{128(\lambda(s))^2} k_n\right\},
\end{aligned} \tag{2.29}$$

provided we require a_n to satisfy $\log a_n = o(k_n)$, as $n \to \infty$. Note that, e.g. the choice $a_n = 2(\log n)^{1/2}$ satisfies each of the three conditions imposed on a_n, namely $a_n = o(|W_n|^{1/2})$, $\sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty$, and $\log a_n = o(k_n)$, provided (1.7) and (1.10). By assumption (1.10), we have that the r.h.s. of (2.29) is summable. Hence (2.25) is proved. This completes the proof. □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1

To prove (1.9), it suffices to check that

$$\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \xrightarrow{p} \lambda(s), \tag{3.1}$$

and

$$\left| \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|} - \frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \right| \xrightarrow{p} 0, \tag{3.2}$$

as $n \to \infty$, for each s at which λ is continuous and positive.

First, we prove (3.1). To do this, we must show that

$$\lim_{n\to\infty} \mathbf{P}\left(\left| \frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} - \lambda(s) \right| \geq \epsilon \right) = 0, \tag{3.3}$$

for each sufficiently small $\epsilon > 0$. Choose $\epsilon < \lambda(s)$. Then, a simple calculation like the one leading from (2.7) to (2.9) and (2.10), shows that it suffices to check

$$\lim_{n\to\infty} \mathbf{P}\left(|\hat{s}_{(k_n)} - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s)-\epsilon)}\right) = 0, \tag{3.4}$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(|\hat{s}_{(k_n)} - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s)+\epsilon)}\right) = 0, \tag{3.5}$$

for each $\epsilon > 0$. We only prove (3.4), because the proof of (3.5) is similar.

Recall that s_i, $(i = 1, \ldots, m)$ denotes the location of the points in the realization $X(\omega)$ of the Poisson process X. Let $\hat{\jmath}_i$ denote the random integer, depending on $\hat{\tau}_n$ and s_i, such that $\hat{s}_i = s_i + \hat{\jmath}_i\hat{\tau}_n$. Similarly, let $\bar{\jmath}_i$ denote an integer, depends on τ and s_i, such that $\bar{s}_i = s_i + \bar{\jmath}_i\tau$. If $s_{(k_n)}$ denotes the point corresponding to $\hat{s}_{(k_n)}$ before translation, then obviously $\hat{s}_{(k_n)} = s_{(k_n)} + \hat{\jmath}_{k_n}\hat{\tau}_n$. Furthermore we have that

$$\begin{aligned} |\hat{s}_{(k_n)} - s| &= |s_{(k_n)} + \hat{\jmath}_{k_n}\hat{\tau}_n - s| \leq |s_{(k_n)} + \bar{\jmath}_{k_n}\tau - s| + |\hat{\jmath}_{k_n}\hat{\tau}_n - \bar{\jmath}_{k_n}\tau| \\ &\leq |\bar{s}_{(k_n)} - s| + |\hat{\jmath}_{k_n}||\hat{\tau}_n - \tau| + \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \end{aligned} \tag{3.6}$$

To prove (3.4), it suffices now to check

$$\lim_{n\to\infty} \mathbf{P}\left(|\bar{s}_{(k_n)} - s| \geq \frac{\tau k_n}{6|W_n|(\lambda(s)-\epsilon)}\right) = 0, \tag{3.7}$$

$$\lim_{n\to\infty} \mathbf{P}\left(|\hat{\jmath}_{k_n}||\hat{\tau}_n - \tau| \geq \frac{\tau k_n}{6|W_n|(\lambda(s)-\epsilon)}\right) = 0, \tag{3.8}$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \geq \frac{k_n}{6|W_n|(\lambda(s)-\epsilon)}\right) = 0, \tag{3.9}$$

for each $\epsilon > 0$. First note that, the proof of (2.9) also yields (3.7). Since $|\hat{\jmath}_{k_n}| = O_p(|W_n|)$, as $n \to \infty$, assumption (1.3) yields that $|\hat{\jmath}_{k_n}||\hat{\tau}_n - \tau| = o_p(k_n/|W_n|)$, as $n \to \infty$, which directly implies (3.8). Hence, it remains to check (3.9).

Here we only give the proof of (3.9) for the case $\hat{\tau}_n \geq \tau$ and $\hat{\jmath}_{k_n}$, $\bar{\jmath}_{k_n}$ are both positive; because the proofs of the other seven cases are similar and therefore omitted. Since $\hat{\tau}_n \geq \tau$, we also know that $\hat{\jmath}_{k_n} \leq \bar{\jmath}_{k_n}$. Hence we have that $\hat{\tau}_n = \tau + |\hat{\tau}_n - \tau|$ and $\hat{\jmath}_{k_n} = \bar{\jmath}_{k_n} - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}|$. Then, we can write

$$\begin{aligned} \hat{s}_{(k_n)} &= s_{k_n} + \hat{\jmath}_{k_n}\hat{\tau}_n = s_{k_n} + \left(\bar{\jmath}_{k_n} - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}|\right)(\tau + |\hat{\tau}_n - \tau|) \\ &= \bar{s}_{k_n} + \bar{\jmath}_{k_n}|\hat{\tau}_n - \tau| - \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}||\hat{\tau}_n - \tau|. \end{aligned} \tag{3.10}$$

Since $\hat{s}_{(k_n)} \in [s - \frac{\hat{\tau}_n}{2}, s + \frac{\hat{\tau}_n}{2})$, it follows now from (3.10) that

$$\begin{aligned} s - \frac{\tau}{2} - \frac{|\hat{\tau}_n - \tau|}{2} &\leq \bar{s}_{k_n} + \bar{\jmath}_{k_n}|\hat{\tau}_n - \tau| - \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}||\hat{\tau}_n - \tau| \\ &< s + \frac{\tau}{2} + \frac{|\hat{\tau}_n - \tau|}{2}. \end{aligned} \tag{3.11}$$

Since we also know that $\bar{s}_{k_n} \in [s - \frac{\tau}{2}, s + \frac{\tau}{2})$, (3.11) directly yields that

$$-\tfrac{|\hat{\tau}_n - \tau|}{2} \leq \bar{\jmath}_{k_n}|\hat{\tau}_n - \tau| - \tau|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| - |\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}||\hat{\tau}_n - \tau| \leq \tfrac{|\hat{\tau}_n - \tau|}{2},$$

which is equivalent to

$$\left(\bar{\jmath}_{k_n} - \frac{1}{2}\right)|\hat{\tau}_n - \tau| < (\tau + o_p(1))\,|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \leq \left(\bar{\jmath}_{k_n} + \frac{1}{2}\right)|\hat{\tau}_n - \tau|. \tag{3.12}$$

Since $\bar{\jmath}_{k_n} = O(|W_n|)$, as $n \to \infty$, together with assumption (1.3), we find that $|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| = o_p(k_n|W_n|^{-1})$, as $n \to \infty$, which implies (3.9). Hence (3.1) is proved.

Next we prove (3.2). The l.h.s. of (3.2) can be written as

$$\frac{\tau k_n}{2|W_n||\hat{s}_{(k_n)} - s|} \frac{1}{\tau} |\hat{\tau}_n - \tau| = O_p(1)\ o_p(k_n|W_n|^{-2}) = o_p(1), \tag{3.13}$$

as $n \to \infty$. Here we have used (3.1) and assumption (1.3). Hence (3.2) is proved. This completes the proof. □

Proof of Theorem 1.2

To establish (1.12), it suffices to check that (3.1) and (3.2) remain valid, when $\xrightarrow{p}$ is replaced by $\xrightarrow{c}$, as $n \to \infty$, for each s at which λ is continuous and positive.

First, we prove that the l.h.s. of (3.1) converges completely to $\lambda(s)$, as $n \to \infty$. Following the structure of the proof of Theorem 1.1, it suffices to check that the probabilities appearing on the l.h.s. of (3.4) and (3.5) are summable, for each $\epsilon > 0$. We shall prove that the probability appearing on the l.h.s. of (3.4) is summable; the proof of the other case is similar.

In view of (3.6), it suffices now to show that the probabilities appearing on the l.h.s. of (3.7), (3.8), and (3.9), are summable, for each $\epsilon > 0$. The proof of the probability on the l.h.s. of (3.7) is summable is exactly the same as the proof of (2.25). Since, by assumption (1.11), we have $|\hat{\jmath}_{k_n}| \le \frac{|W_n|}{\tau}(1+o_c(1))$, as $n \to \infty$, (for any r.v. Y_n we write $Y_n = o_c(1)$ to denote that Y_n converges completely to zero, as $n \to \infty$), then by assumption (1.11) once more, we have that the probability on the l.h.s. of (3.8) is summable, for each $\epsilon > 0$. It remains to prove that the probability on the l.h.s. of (3.9) is summable. We only consider the case that $\hat{\tau}_n \ge \tau$ and $\hat{\jmath}_{k_n}$, $\bar{\jmath}_{k_n}$ are both positive; the proofs for the other seven cases are similar. An application of inequality (3.12), by using now assumption (1.11), yields that

$$|\hat{\jmath}_{k_n} - \bar{\jmath}_{k_n}| \le \left(\bar{\jmath}_{k_n} + \tfrac{1}{2}\right) (\tau + o_c(1))^{-1} |\hat{\tau}_n - \tau| \ ,$$

as $n \to \infty$. Since $\bar{\jmath}_{k_n} = O(|W_n|)$, as $n \to \infty$, by assumption (1.11) once more, we have that the probability on the l.h.s. of (3.9) is summable. Hence we have proved (3.1) with $\xrightarrow{p}$ replaced by $\xrightarrow{c}$.

Next we prove (3.2) with $\xrightarrow{p}$ replaced by $\xrightarrow{c}$. First note that, the l.h.s. of (3.2) is the same as the l.h.s. of (3.13). Because we have that the l.h.s. of (3.1) converges completely to $\lambda(s)$, as $n \to \infty$, by assumption (1.11), we also have that the l.h.s. of (3.13) converges completely to zero, which of course implies that the l.h.s. of (3.2) converges completely to zero, as $n \to \infty$. This completes the proof. □

Acknowledgment

The author is grateful to R. Helmers for suggesting the problem and his excellent help and comments throughout the preparation of this paper.

References

1. Albers, W., Bickel, P. J., and van Zwet, W. R. (1976). Asymptotic expansions for the power of distribution free tests in the one-sample problem. *Ann. Statist.* **4**, 108-156.
2. Cressie, N. A. C. (1993). *Statistics for Spatial Data.* Revised Edition. Wiley, New York.
3. Helmers, R., and Zitikis, R. (1999). On estimation of Poisson intensity functions. *Ann. Inst. Stat. Math.* **51**, 2, 265-280.
4. Helmers, R., and Mangku, I W. (2000). Nonparametric estimation of the period of a cyclic Poisson process (part I): consistency. *CWI report,* in preparation.
5. Helmers, R., Mangku, I W., and Zitikis, R. (1999). Consistent estimation of the intensity function of a cyclic Poisson process. *CWI Report PNA-R9915,* submitted for publication.
6. Kingman, J. F. C. (1993). *Poisson Processes.* Clarendon Press, Oxford.
7. Loftsgaarden, D. O., and Quesenberry, C. P. (1965). A nonparametric estimate of a multivariate density function. *Ann. Math. Statist.* **5**, 1049-1051.
8. Moore, D. S., and Yackel, J. W. (1977). Consistency properties of nearest neighbor density function estimators. *Ann. Statist.* **5**, 143-154.
9. Ralescu, S. S. (1995). The law of the iterated logarithm for the multivariate nearest neighbor density estimators. *J. Multivariate Anal.* **53**, 159-178.
10. Reiss, R. D. (1989). *Approximate Distributions of Order Statistics With Applications to Nonparametric Statistics.* Springer-Verlag, New York.
11. Reiss, R. D. (1993). *A Course on Point Processes.* Springer-Verlag, New York.
12. Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics.* John Wiley & Sons, New York.
13. Vere-Jones, D. (1982). On the estimation of frequency in point-process data. *J. Appl. Prob.* **19A**, 383-394.
14. Wagner, T. J. (1973). Strong consistency of a nonparametric estimate of a density function. *IEEE Trans. Systems Man Cybernet.* **3**, 289-290.