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Universal Algebraic Convergence in Time of Pulled Fronts: the Common Mechanism for Difference-differential and Partial Differential Equations

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Abstract. We analyze the front structures evolving under the difference-differential equation $\partial_t C_j = -C_j + C_{j-1}^2$ from initial conditions $0 \le C_j(0) \le 1$ such that $C_j(0) \to 1$ as $j \to \infty$ sufficiently fast. We show that the velocity $v(t)$ of the front converges to a constant value v^* according to $v(t) = v^* - 3/(2\lambda^* t) + (3\sqrt{\pi}/2)\ D\lambda^*/(\lambda^{*2} D t)^{3/2} + \mathcal{O}(1/t^2)$. Here v^*, λ^* and D are determined by the properties of the equation linearized around $C_j = 1$. Ebert and Van Saarloos recently derived the same asymptotic expression for fronts in the nonlinear diffusion equation where the values of the parameters λ^*, v^* and D are specific to the equation. The identity of methods and results for both equations is due to a common propagation mechanism of pulled fronts. This gives reasons to believe, that this universal algebraic convergence actually occurs in an even larger class of equations.

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1 Introduction

We consider the invasion of one homogeneous state by another in a one-dimensional system, creating a propagating front between them. Most familiar are fronts in bistable systems, where the invading as well as the invaded state are dynamically stable against small perturbations. If, on the other hand, the invaded state is unstable, one can identify two basically distinct mechanisms of propagation that depend on further properties of the dynamical equation and apply to the evolution of all initial conditions decaying sufficiently rapidly into the unstable state. The two mechanisms are conveniently distinguished by the notion of the asymptotic front speed, v_{as}. For differential equations, this speed is defined as the large time limit of the slope $v(t)$ of level curves of the front-type solution in the (x,t)-plane. The *linear spreading velocity* v^* is defined as the asymptotic speed for the evolution equation linearized around the unstable invaded state. Since a nonlinear front never can move slower than the linear spreading speed, since otherwise the leading edge would outrun the nonlinear profile, it is clear that $v_{as} \geq v^*$. The distinction between the two types of fronts lies in whether v_{as} is larger or equal to v^*. Fronts for which $v_{as} = v^*$ are sometimes referred to as *pulled*, while those for which $v_{as} > v^*$ are then called *pushed* [1, 2]. Here we will focus on pulled fronts.

Although these ideas are often phrased in different languages within different communities, they are all illustrated by the properties of solutions of the celebrated nonlinear diffusion equation

$$\partial_t \phi = \partial_x^2 \phi + f(\phi) \;, \qquad f(\phi) = \phi \Leftrightarrow \phi^3, \tag{1}$$

which goes back to the work of Fisher [3] and Kolmogoroff *et al.* [4]. It is well known [3, 4, 5] that sufficiently rapidly decaying initial conditions $\phi(x,0)$ such that $\lim_{x\to\infty} \phi(x,0)\, e^x = 0$ lead to fronts with $v_{as} = v^* = 2$, so that fronts are indeed *pulled*.

It is the main purpose of this paper to draw attention to the fact that the general mechanism underlying the formation of pulled fronts extends far beyond the simple statement $v_{as} = v^*$ and is shared by a large variety of dynamical systems. In particular, we will focus on the universality of the convergence towards the asymptotic front speed and shape, caused by the general dynamical mechanism of pulled front propagation.

We illustrate this observation by considering front propagation in which a stable state invades an unstable state in a differential-difference equation. The equation concerned arises in kinetic theory [6] and is given by

$$dC_j(t)/dt = -C_j(t) + C_{j-1}^2(t) \ . \qquad (2)$$

The unstable state is here $C_j \equiv 1$ and the stable state is $C_j \equiv 0$. We consider initial conditions such that there exists a $\lambda_0 > \lambda^*$ such that

$$0 \le C_j(0) \le 1 \ \text{ for all } j \quad \text{and} \quad \lim_{j\to\infty} [1 - C_j(0)]\, e^{\lambda_0 j} = 0 \ . \qquad (3)$$

We will refer to these initial conditions as “sufficiently steep” [7]. Since for difference equations level curves are not easily introduced, we define the front position $x_f(t)$ as

$$x_f(t) = \sum_{j=0}^{\infty} [1 - C_j(t)] \ . \qquad (4)$$

The central result of this paper is that for initial values which satisfy (3), the front velocity $v(t) = \dot{x}_f(t)$ is asymptotically given by

$$v(t) = v^* + \dot{X}(t) \ , \qquad (5)$$

$$\dot{X}(t) = -\frac{3}{2\lambda^* t}\left(1 - \frac{\sqrt{\pi}}{\lambda^* \sqrt{Dt}}\right) + \mathcal{O}\left(\frac{1}{t^2}\right), \qquad t \to \infty. \qquad (6)$$

Here λ^* is the solution of

$$2e^{\lambda^*} = \frac{2e^{\lambda^*} - 1}{\lambda^*} \qquad \Rightarrow \qquad \lambda^* = 0.768039 \ , \qquad (7)$$

and

$$v^* = \frac{2e^{\lambda^*} - 1}{\lambda^*} = 4.31107 \ , \qquad D = e^{\lambda^*} = 2.155535 \ . \qquad (8)$$

The asymptotic expression for $v(t)$ presented in (6) is exactly the same as the expression that was recently derived [7, 8] for the velocity relaxation in (1) and in higher order evolution equations that admit uniformly translating pulled fronts; for (1) we obtain $\lambda^* = 1$, $D = 1$ and $v^* = 2$.

Thus, while the behavior of kinks or fronts between two (meta)stable states can change drastically when the nonlinear diffusion equation is replaced by a finite difference approximation (one possibility being propagation failure [9, 10, 11]), the dynamical mechanism that leads to pulled fronts is completely the same in both types of equations. In fact, by a combination of analytical and numerical methods, we have argued [7], that (6) holds for all equations that admit uniformly translating pulled fronts. The parameters v^*, λ^* and D in (6) can generally be expressed in terms of the dispersion relation $\omega(k)$ of the evolution equation linearized about the unstable state. This is discussed in Appendix A.

In our view, the common features of pulled fronts expressed by (6) suggest that many of the methods developed in the mathematical literature for (1) (See, e.g., [3, 4, 5, 12, 13]) may be generalized to much larger classes of equations.

2 Derivation of the main results

We now turn to the derivation of these results for Eq. (2). We shall do this through the series of steps (i)–(vii). below. To facilitate the comparison with (1), we first transform to the variables $\phi_j = 1 \Leftrightarrow C_j$. In these, the dynamical equation reads

$$d\phi_j/dt = 2\phi_{j-1} \Leftrightarrow \phi_j \Leftrightarrow \phi_{j-1}^2 \ , \tag{9}$$

and the initial condition (3) becomes

$$0 \leq \phi_j(0) \leq 1 \quad \text{for all } j \ , \quad \text{and} \ \lim_{j\to\infty} \phi_j(0)\, e^{\lambda_0 j} = 0 \quad \text{for some} \quad \lambda_0 > \lambda^* \ . \tag{10}$$

The invaded unstable state is now $\phi_j \equiv 0$ and the invading state is $\phi_j \equiv 1$. From (2) and (3), we see that $C_j(t) \geq 0$, so that $\phi_j(t) \leq 1$ for all j and $t \geq 0$, and an elementary comparison argument shows that $\phi_j(t) \geq 0$ for all j and $t \geq 0$. Thus

$$0 \leq \phi_j(t) \leq 1 \quad \text{for} \quad j \in \mathbf{Z},\ t \geq 0. \tag{11}$$

(i) *Instability and dispersion relation.* That the state $\phi_j = 0$ is unstable, can easily be seen as follows. We linearize the dynamical equation about $\phi = 0$ to get

$$d\phi_j/dt = 2\phi_{j-1} \Leftrightarrow \phi_j \tag{12}$$

and substitute a Fourier mode $\phi(x,t) = Ae^{-i(\omega t - kj)}$ with k in the "Brillouin zone" $\Leftrightarrow \pi < k \leq \pi$. This yields the dispersion relation

$$\Leftrightarrow i\omega(k) = 2e^{-ik} \Leftrightarrow 1 \ . \tag{13}$$

As the growth rate is given by $\mathrm{Re}(\Leftrightarrow i\omega) = \mathrm{Im}\omega = 2\cos k \Leftrightarrow 1$, modes with $|k| < \pi/3$ grow in time, so that the state $\phi = 0$ is unstable.

(*ii*) *Nonlinear versus linear dynamics.* For a given initial condition $\phi_j(0)$, the dynamics resulting from the linearized equation (12) is an upper bound for the dynamics resulting from the nonlinear equation (9). To see this suppose that ϕ_j is the solution of the nonlinear equation and $\hat{\phi}_j$ the solution of the linear equation, and that ϕ_j and $\hat{\phi}_j$ have the same initial values. Then the difference $z_j = \hat{\phi}_j \Leftrightarrow \phi_j$ satisfies the equation

$$dz_j/dt = 2z_{j-1} \Leftrightarrow z_j + \phi_{j-1}^2, \qquad z_j(0) = 0. \tag{14}$$

Because the inhomogeneous term is ϕ_{j-1}^2 is nonnegative it follows from standard theory [14] that $z_j(t) \geq 0$, and hence that $\phi_j(t) \leq \hat{\phi}_j(t)$ for all j and all $t \geq 0$. Thus solutions of the linear equation provide an upper bound to solutions of the nonlinear equation.

(*iii*) v^* *as upper bound for sufficiently steep initial conditions, and pulling.* We now introduce the family of comparison functions

$$\eta_j(t;\lambda, v) = e^{-\lambda(j-vt)}, \qquad \lambda \in \mathbf{R}^+,\ v \in \mathbf{R}. \tag{15}$$

Substitution shows that η_j is a solution of the linear equation (12) if λ and v are related by

$$v = v(\lambda) = \frac{2e^{\lambda} \Leftrightarrow 1}{\lambda}. \tag{16}$$

We choose λ such that the speed $v(\lambda)$ defined by (16) takes on its smallest possible value. This is the case for $\lambda = \lambda^*$, as defined in (7). The corresponding velocity is denoted by v^*. Then, for any $A > 0$, we have

$$\phi_j(t) \leq A e^{-\lambda^*(j-v^*t)} \quad \text{if} \quad \phi_j(0) \leq A e^{-\lambda^* j} \quad \text{for all} \quad j. \tag{17}$$

Thus, for initial data which decay sufficiently fast as defined by (10), the nonlinearity in (9) cannot push the front in the large time limit to a velocity higher than v^*, which is determined by the linear equation. Rather, the leading edge, i.e., the region defined through $\phi_j \approx 0$, will pull the front along. This creates the particular mode of pulled front propagation, that is unlike the nonlinear mechanisms dominating pushed and bistable fronts.

The upper bound (17) is not strong enough for our subsequent analysis. Using a comparison function with decay rate $\lambda = \lambda_0 > \lambda^*$ that bounds the initial data (10), it follows immediately that for every fixed and finite $t \in \mathbf{R}^+$, the sequence $\phi_j(t)$ can also be bounded by

$$\phi_j(t) \le A\, e^{\lambda_0 v(\lambda_0)\, t} \;\cdot\; e^{-\lambda_0\, j} \quad \text{for } \lambda_0 > \lambda^* \; . \tag{18}$$

(*iv*) *Leading edge representation.* Let us now turn to the systematic calculation of the wave speed. To understand the convergence towards an asymptotic nonlinear front profile in the pulled regime, we introduce, what has been called the *leading edge representation* ψ [7]. This involves the transformation to a coordinate frame $\xi_X = x - v^* t - X(t)$ which moves with a speed $v^* + \dot{X}(t)$. Thus $X(t)$ is a — as yet undetermined — time dependent shift. On the basis of an asymptotic analysis (cf. Appendix A and [7]) we make the Ansatz that $\dot{X}(t) = c_1/t + \mathcal{O}(1/t^{3/2})$ as $t \to \infty$, where $c_1 < 0$. Plainly then,

$$X(t) = \int_0^t \{v(s) - v^*\}\, ds \approx c_1 \ln t \to -\infty \quad \text{as} \quad t \to \infty \; . \tag{19}$$

The importance of using the logarithmically shifted time frame ξ_X for calculating the long time asymptotic behavior of pulled fronts is illustrated in Fig. 1 for the nonlinear diffusion equation (1). The solid lines show different steps of the temporal evolution of a front, that has started from a sufficiently steep initial condition. The dashed lines in this figure are the asymptotic uniformly translating front solutions $\Phi^*(x - v^* t)$ with $v^* = 2$. As the data show, the actual front shape $\phi(x,t)$ is quite similar to Φ^* for times $t \ge 5$, but as the fat solid line illustrates, the distance between the actual transient front $\phi(x,t)$ and the uniformly translating solution $\Phi^*(x - v^* t)$ increases without bound, in accordance with (19). Although it is numerically less easy to visualize this for our difference equation (9), the same logarithmic shift occurs here or for any other equation admitting pulled fronts as well.

In addition to the moving frame ξ_X, we introduce an exponential factor $e^{-\lambda^* \xi_X}$ that is motivated by (17). Thus, we put

$$\phi_j(t) = e^{-\lambda^* \xi_X}\; \psi(\xi_X, t) \; , \qquad \xi_X = j - v^* t - X(t) \; . \tag{20}$$

Of course, at any fixed time t, the variable ξ_X is only defined on discrete points, whose position varies linearly with t. However, the transformation from $\phi_j(t)$ to $\psi(\xi_X, t)$ in (20) anticipates that for large t and ξ_X, the solution $\psi(\xi_X, t)$ will

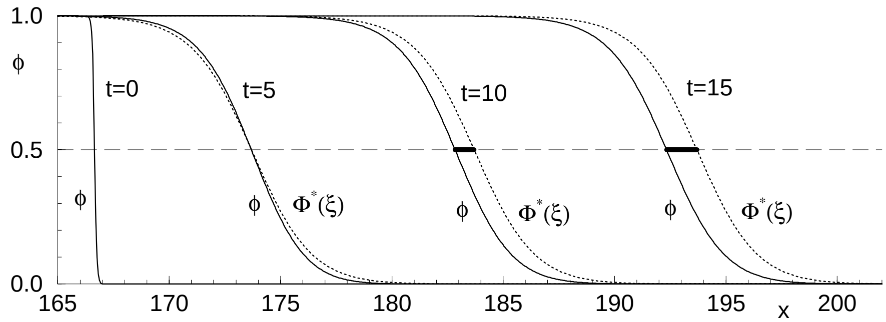

Figure 1: Illustration of the fact, that even though the shape of a front profile is quite close to $\phi(x,t) = \Phi^*(x - v^*t)$, the position of a front is shifted logarithmically in time relative to the uniformly translating profile Φ^*. Solid lines: evolution of some initial condition of the form $\phi(x,0) = 1/\left[\,1 + e^{10(x-x_0)}\,\right]$ under $\partial_t\phi = \partial_x^2\phi + \phi - \phi^3$ at times $t = 0, 5, 10, 15$. Dotted lines: $\Phi^*(x - v^*t)$ at times $t = 5, 10, 15$. The initial position of Φ^* is chosen in such a way that the amplitude $\Phi^* = 1/2$ coincides with $\phi(x,t) = 1/2$ at time $t = 5$. The logarithmic temporal shift is indicated by the fat line.

be arbitrarily slowly varying in time and space, so that discretization effects become unimportant. Transforming the equation (9) for $\phi_j(t)$ into an equation for $\psi(\xi_X, t)$, we find

$$\begin{aligned} \frac{\partial\psi(\xi_X,t)}{\partial t} &= v^* \left[\psi(\xi_X \Leftrightarrow 1, t) \Leftrightarrow \psi(\xi_X, t) + \frac{\partial\psi(\xi_X,t)}{\partial\xi_X}\right] \\ &\quad + \dot{X}(t)\left[\frac{\partial}{\partial\xi_X} \Leftrightarrow \lambda^*\right]\psi(\xi_X,t) \Leftrightarrow \frac{v^*}{2}\, e^{-\lambda^*(\xi_X-1)}\, \psi(\xi_X \Leftrightarrow 1, t)^2. \end{aligned} \tag{21}$$

In deriving (21), we divided out a common factor $e^{-\lambda^*\xi_X}$, and used the identities (7) and (8) for v^* and λ^*. The idea is now to determine $X(t)$ such that $\psi(\xi_X, t)$ converges to a time independent limit $\Psi(\xi_X)$ as $t \to \infty$, i.e.,

$$|\psi(\xi_X,t) \Leftrightarrow \Psi(\xi_X)| \to 0 \quad \text{as} \quad t \to \infty, \tag{22}$$

uniformly on intervals of the form $(\Leftrightarrow\infty, L]$, for any $L \in \mathbf{R}$. We shall refer to this limit as the *asymptotic shape* of the front. In Fig. 1, we already illustrated that the convergence (22) occurs only in the properly shifted frame ξ_X.

(v) *The asymptotic shape of the leading edge.* The profile $\Psi(\xi)$ in the long time asymptotics is the solution of the equation

$$\Psi(\xi \Leftrightarrow 1) \Leftrightarrow \Psi(\xi) + \Psi'(\xi) = \frac{1}{2}\, e^{-\lambda^*(\xi-1)}\, \Psi^2(\xi \Leftrightarrow 1)\ , \tag{23}$$

which we have obtained from (21) by suppressing the t-dependence, and setting $\dot{X}(t) = 0$. Since in the leading edge transformation (20) we already divided out the dominant exponential factor, the relevant function $\Psi(\xi)$ does not diverge exponentially to infinity, so that the right hand side of (23) decays exponentially as $\xi \to \infty$. Hence, by an elementary argument involving the Laplace transform, $\Psi(\xi)$ behaves asymptotically as[1]

$$\Psi(\xi) \sim \alpha\xi + \beta \quad \text{as} \quad \xi \to \infty \quad (\alpha, \beta \in \mathbf{R})\ . \tag{24}$$

Since ϕ_j approaches 1 behind the front, the transformation (20) implies that

$$\Psi(\xi) = \mathcal{O}\left(e^{\lambda^*\xi}\right) \quad \text{as} \quad \xi \to \Leftrightarrow\infty\ . \tag{25}$$

By integrating (23) over $(\Leftrightarrow\infty, b)$, letting $b \to \infty$ and using (24) and (25) we obtain

$$\alpha = \int_{-\infty}^{\infty} e^{-\lambda^*\xi}\, \Psi^2(\xi)\, d\xi > 0. \tag{26}$$

The fact that α is positive stems from the nonlinearity in the equation as is clear from Eq. (23). As is discussed in Appendix A, this is why the linearized equation fails to give the correct long time convergence of the solution, even though the linear spreading velocity v^* is a property of the linear equations.

(vi) *The spatial decay of the evolving front.* The initial condition (10) implies that $\psi(\xi_X, 0) \to 0$ as $\xi_X \to \infty$. Equation (18) shows that this stays true for any fixed and finite time $t > 0$, since

$$\psi(\xi_X, t) \;\leq\; A(t) \cdot e^{-\delta\,\xi_X} \quad , \quad A(t) = A\, e^{\lambda_0\,[v(\lambda_0)t - v^*t - X(t)]} \tag{27}$$
$$\text{for} \;\; \xi_X \to \infty \;\; \text{and fixed } t \in \mathbf{R}^+ \;\; \text{and} \;\; \delta = \lambda_0 \Leftrightarrow \lambda^* > 0\ .$$

[1] Another argument, that leads to the same conclusion is to transform Eq. (23) back to the ϕ variable, which yields $\Phi^*(\xi) - v^*\partial_\xi\Phi^*(\xi) = 2\Phi^*(\xi-1) - \Phi^{*2}(\xi-1)$. By investigating the flow near the fixed points $\Phi = 1$ and $\Phi = 0$, such a front can be shown directly to decay towards $\xi \to \infty$ as $\Phi^*(\xi) = (\alpha\xi + \beta)\, e^{-\lambda^*\xi}$. For Ψ this implies the behavior given in (24).

On the other hand, for any fixed $\xi_X \gg 1$, (22) together with (24) gives

$$\psi(\xi_X, t) = \alpha\xi_X + \beta \quad \text{for} \;\; t \to \infty \;\; \text{and fixed} \;\; 1 \ll \xi_X < \infty, \tag{28}$$

if we make the appropriate choice for $X(t)$. Note that according to (27) the large ξ_X limit of ψ vanishes for any finite time, while in the infinite time limit ψ diverges linearly in ξ_X. This illustrates that the limits $t \to \infty$ and $\xi_X \to \infty$ do not commute. The divergence of the linear growth in the limit of (28) for $t \to \infty$ illustrates the buildup in the intermediate asymptotic region.

(*vii*) *The rate of convergence.* We now determine the large time asymptotics of $X(t)$ from the solution of equation (21) with limiting conditions (27) and (28). For large ξ_X, the nonlinearity in (21) can be neglected, since $e^{-\lambda^*\xi_X}\psi^2 \ll \psi$ because of (27) and $e^{-\lambda^*\xi_X} \ll 1$. Anticipating that also the higher order derivatives become small for large values of ξ_X, we expand $\psi(\xi_X \Leftrightarrow 1)$ at ξ_X:

$$\psi(\xi_X \Leftrightarrow 1, t) = \psi(\xi_X, t) + \partial_{\xi_X}\psi(\xi_X, t) + \frac{1}{2}\partial^2_{\xi_X}\psi(\xi_X, t) + \frac{1}{3!}\partial^3_{\xi_X}\psi(\xi_X, t) + \ldots . \tag{29}$$

Explicit solutions of the time-dependent linearized equation show, that this expansion is justified for sufficiently large values of t, when ψ approaches the smooth function $\Psi(\xi_X) = \alpha\xi_X + \beta$. When we substitute this expansion into equation (21) we obtain

$$\partial_t\, \psi = D\, \partial^2_{\xi_X}\psi + D_3\, \partial^3_{\xi_X}\psi + \ldots + \dot{X}(t)\Big[\partial_{\xi_X} \Leftrightarrow \lambda^*\Big]\psi\ , \tag{30}$$

where the function ψ is now everywhere evaluated at (ξ_X, t), and $D_n = (\Leftrightarrow 1)^n v^*/n!$ with $D = D_2$. If we would set $\dot{X} = 0$, and omit the derivatives of order three and higher, we would obtain the classical diffusion equation. This motivates the use of the Gaussian similarity variable

$$z = \frac{\xi_X^2}{4Dt} \tag{31}$$

as a substitution for ξ_X. For $X(t)$ we anticipate an expansion of the type

$$\dot{X}(t) = \frac{c_1}{t} + \frac{c_{3/2}}{t^{3/2}} + \frac{c_2}{t^2} + \cdots, \tag{32}$$

where the leading order $1/t$ is consistent with (19), and the further expansion in powers of $1/\sqrt{t}$ is motivated by the substitution of ξ_X by z. For ψ in the region $\xi_X \leq \mathcal{O}(\sqrt{t})$, we make an Ansatz with the same structure,

$$\psi(\xi_X, t) = \Psi(\xi_X) + \frac{\psi_1(\xi_X)}{t} + \frac{\psi_{3/2}(\xi_X)}{t^{3/2}} + \dots \quad \text{for } \xi_X \leq \mathcal{O}(\sqrt{t}) \ . \tag{33}$$

Note that a term of order $1/t^{1/2}$ is absent here; this is derived formally from the resummation of the interior region of the front in Appendix B. An intuitive explanation for this fact is that the interior is slaved to the evolution of the leading edge which has a leading order correction of order $v(t) \Leftrightarrow v^* = \mathcal{O}(1/t)$. In the region $\xi_X \geq \mathcal{O}(\sqrt{t})$, we make the Ansatz

$$\psi(\xi_X, t) = e^{-z} \left[\sqrt{t}\, g_{-1/2}(z) + g_0(z) + \frac{g_{1/2}(z)}{\sqrt{t}} + \dots \right] \quad \text{for } \xi_X \geq \mathcal{O}(\sqrt{t}) \ , \tag{34}$$

where we have used (31) to write the right hand side in terms of z and t. Here a Gaussian e^{-z} is already factorized out for later convenience. By asymptotic matching, Eqs. (28) and (33) determine the small z expansion of the functions g for $z \downarrow 0$ as

$$g_{-1/2}(z) \;=\; 2\alpha\sqrt{z} \; + \mathcal{O}\left(z^{3/2}\right), \tag{35}$$

$$g_0(z) \;=\; \beta \; + \mathcal{O}\left(z\right). \tag{36}$$

The limit of $z \to \infty$ for fixed $t < \infty$ is determined by (27) as

$$g_{-1/2}(z) \leq A(t)\; t^{-1/2} \;\; e^{z}\; e^{-\delta\sqrt{4Dt}\,\sqrt{z}} \quad , \quad g_0(z) \leq A(t)\; e^{z}\; e^{-\delta\sqrt{4Dt}\,\sqrt{z}} \ . \tag{37}$$

The functions $g_{-1/2}(z)$, $g_0(z)$ and $g_{1/2}(z)$ satisfy linear differential equations which we obtain by equating the coefficients of $t^{1/2}$, t^0 and $t^{-1/2}$ to zero. For $g_{-1/2}(z)$ this yields the equation

$$zg'' + \Big(\frac{1}{2} \Leftrightarrow z\Big) g' \Leftrightarrow (1 + c_1\lambda^*) g = 0 \tag{38}$$

and for g_0, we obtain

$$zg'' + \Big(\frac{1}{2} \Leftrightarrow z\Big) g' \Leftrightarrow \Big(\frac{1}{2} + c_1\lambda^*\Big) g = h(z), \tag{39}$$

where

$$h(z) = e^z \left[c_{3/2}\lambda^* \Leftrightarrow \frac{c_1}{\sqrt{D}} \sqrt{z} \, \frac{d}{dz} \Leftrightarrow \frac{D_3 \sqrt{z}}{D^{3/2}} \left(\frac{3}{2} \, \frac{d^2}{dz^2} + z \, \frac{d^3}{dz^3} \right) \right] e^{-z} \, g_{-1/2}(z) \ . \tag{40}$$

The general solutions of the homogeneous differential equations are confluent hypergeometric functions $M(a, b, z)$, for which we use the notation of [15].

Equation (38) only has a solution obeying the boundary conditions (35) and (37), if $\Leftrightarrow c_1\lambda^* \Leftrightarrow 1/2$ is a positive integer [15, 7]. In this case, the solution is

$$g_{-1/2}(z) = 2\alpha\sqrt{z} \, M\Big(c_1\lambda^* + \frac{3}{2}, \frac{3}{2}, z \Big), \tag{41}$$

which can be identified with a Hermite polynomial. The relevant solution is

$$c_1\lambda^* = \Leftrightarrow 3/2 \ , \quad \text{so that} \quad g_{-1/2}(z) = 2\alpha\sqrt{z}, \tag{42}$$

since it is the only solution consistent with the conserved positivity (11) of the solution.

A particular solution of the inhomogeneous equation (39) for $g_0(z)$ with inhomogeneity $h(z)$ given by (40) and (42) has been constructed in [7]. We there find that a construction of the full solution of (39) with boundary conditions (36) and (37) is only possible, if

$$c_{3/2}\lambda^* = \Leftrightarrow \sqrt{\pi} \, \frac{c_1}{\sqrt{D}}. \tag{43}$$

Note, that the coefficient D_3 of the third spatial derivative in (30) enters (40), but does not influence $c_{3/2}$. It does give a contribution to $g_0(z)$, however, whose explicit analytical form can be found in [7].

Eqs. (32), (42) and (43) yield our explicit prediction (6) for the velocity of the evolving front $v(t) = v^* + \dot{X}(t)$.

3 Numerical verification.

In Fig. 2, we show numerical data for the front velocity, obtained by numerically solving Eq. (2) [or (9) after the transformation $\phi_j(t) = 1 \Leftrightarrow C_j(t)$] with initial condition

$$\phi_j(0) = \begin{cases} e^{-j^2} & \text{for } j \geq 0 \\ 1 & \text{for } j < 0. \end{cases} \ , \tag{44}$$

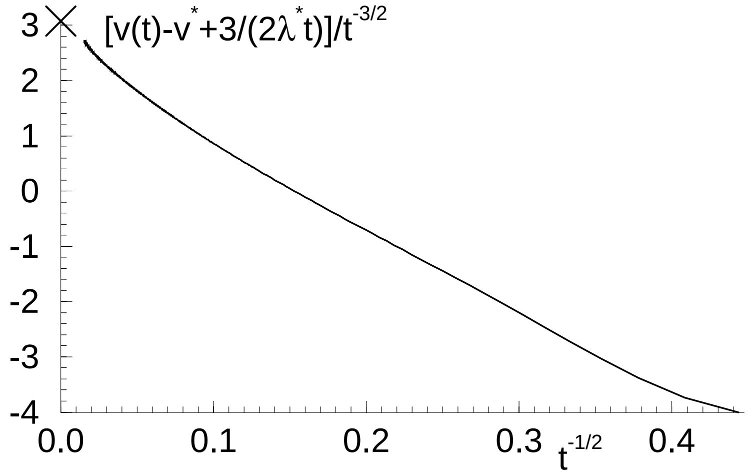

Figure 2: Numerical solution of (2) with initial conditions (44). The velocity $v(t) = \dot{x}_f(t)$ of the front is defined in (4). Plotted is $\big(v(t) - v^* - c_1/t\big)/t^{-3/2}$ as a function of $1/\sqrt{t}$ for times $40 \le t \le 4000$. $c_{3/2}$ is marked by the cross on the axis. The constants are $c_1 = -3/(2\lambda^*)$ (42) and $c_{3/2} = -\sqrt{\pi/D}\; c_1/\lambda^*$ (43) with λ^*, v^*, and D from (7) and (8). The analytical prediction (6) implies that the curve should extrapolate approximately linearly towards the cross. Clearly, the numerics fully confirms this prediction.

which is a sufficiently steep initial condition according to the definition (10). The front velocity is defined in (4) as $v(t) = \dot{x}_f(t) = \sum_{j=0}^{\infty} \dot{\phi}_j(t)$. In order to bring out that all terms up to order $t^{-3/2}$ in our expansion are fully corroborated by our numerical simulations, we plot in Fig. 2 the expression

$$\left[\, v(t) \Leftrightarrow v^* + \frac{3}{2\lambda^* t} \,\right]\; t^{3/2} \tag{45}$$

versus $1/\sqrt{t}$. According to our analysis, this expression should approach the value $3/(2\lambda^{*2})\, \sqrt{\pi/D} = 3.06989$ as $t \to \infty$. This value is indicated with a cross in Fig. 2. Note that at the latest time $t = 4000$, $t^{3/2} = 2.5 \cdot 10^5$, so an error in the sixth decimal place in any of our terms for $v(t)$ would be clearly visible in the figure. The fact that our numerical data approach our analytical value so well, thus confirms our analytical results with extreme precision.

4 Conclusion and Outlook.

We finally remark that the prediction (6) for the velocity $v(t)$ of an evolving pulled front is a "universal" result:

(a) It is independent of the precise initial conditions provided they obey the bound (10).

(b) It is independent of the precise nonlinearities, provided they create pulled fronts, where the concept of pulling is explained in (i) – (iv). A different nonlinearity will only affect the value of α in (24), but as long as $\alpha > 0$, the velocity converges to its asymptotic value v^* according to (6).

(c) In the introduction, we already mentioned that the result (6) also holds for the nonlinear diffusion equation (1) with the parameters v^*, λ^* and D depending on the equation through the explicit expressions (49) – (51). As has been stated earlier and is further explained in the Appendices, the asymptotic expression (5), (6) for $v(t)$ holds even more generally for all equations that for $t \to \infty$ generate uniformly translating pulled fronts. The generalization to pulled fronts that generate patterns (such as in the Swift-Hohenberg equation) can be found in [16]. Thus the line of reasoning on which the analysis is built can be viewed as evidence that there is a center manifold governing the convergence of pulled fronts in general.

(d) In this paper, we have focussed on the results for the convergence of the velocity associated with the front position $x_f(t)$ defined in (4). We can actually go much further and analyze the convergence of the front profile as well. Indeed, if we denote with $\Phi_v(\xi)$ the uniformly translating front profile with velocity v, we show in Appendix B along the lines of [7] that $\phi_j(t) = \Phi_{v(t)}(\xi_X) + O(1/t^2)$. This result implies that the velocity $v_\phi(t)$ of a given level $\phi_j(t) = \phi$ is independent of the value of ϕ within the accuracy given in (6). Therefore also the definition of the front velocity through (4) invariantly results in the same prediction (6).

In our view, this universality of pulled front propagation in both the nonlinear diffusion equation (1) and in the difference-differential equation (2) and in other dynamical equations [7, 16] is an indication that many of the methods developed in the mathematical literature [3, 4, 5] for Eq. (1) should be generalizable to much larger classes of equations like higher order $p.d.e.$'s, difference equations, integro-differential equations, sets of coupled equations etc.

A The universal structure of the saddle point approximation of the linearized equation

In these appendices, we summarize some of the essential steps of the more general derivation of the velocity results [7]. The analysis illustrates the mechanism underlying the broad applicability of prediction (6) beyond the equations of motion (1) and (2). We also discuss different definitions of the velocity. In Appendix A, we treat the linearized equation, in Appendix B the nonlinear front region.

The linearized equation (12) for an initial condition $\phi_j(0)$ is solved explicitly by

$$\phi_j(t) = \sum_{j'} G(j \Leftrightarrow j', t)\ \phi_{j'}(0) \qquad ,$$

$$G(j,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\ e^{ikj - i\omega(k)t} \qquad , \qquad \Leftrightarrow i\omega(k) = 2e^{-ik} \Leftrightarrow 1\ , \tag{46}$$

with the dispersion relation $\omega(k)$ as in (13). In a more general context than (12), only the explicit form of $\omega(k)$ depends on the equation considered. Furthermore, the difference structure of the equation restricts the interval of k-integration to $\Leftrightarrow \pi \leq k \leq \pi$. For the nonlinear diffusion equation (1), when linearized about $\phi = 0$, we get for comparison

$$G(j,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ikj - i\omega(k)t} \quad , \quad \Leftrightarrow i\omega(k) = 1 \Leftrightarrow k^2\ . \tag{47}$$

In both cases, the large time asymptotics of G can be extracted from a saddle point or steepest descent evaluation of the integral in (46) or (47) [7, 17]. In a coordinate system $\xi = j \Leftrightarrow vt$ moving with velocity v, we have for fixed ξ and $t \gg 1$

$$G(j,t) \ = \ \int \frac{dk}{2\pi}\ e^{ik\xi - i[\omega(k) - vk]t} \ \approx \ e^{ik_s(v)\xi}\ e^{-i[\omega(k_s(v)) - vk_s(v)]t}\ \frac{e^{-\xi^2/(4Dt)}}{\sqrt{4\pi Dt}}\ , \tag{48}$$

where the saddle point condition

$$\frac{d}{dk}\left[\ \omega(k) \Leftrightarrow vk\ \right]\Big|_{k_s(v)} = 0 \quad \Leftrightarrow \quad v = \frac{d\omega(k)}{dk}\Big|_{k_s(v)} \tag{49}$$

fixes $k_s(v)$ as a function of v, and the diffusion constant $D(v)$ is

$$D(v) = \frac{1}{2} \left. \frac{i d^2\omega}{dk^2} \right|_{k_s(v)} . \tag{50}$$

The requirement that in the moving frame ξ the field ϕ neither grows nor decays in leading order, defines the linear spreading velocity v^* as well as $k^* = q^* + i\lambda^*$ and D:

$$\mathrm{Im} \Big[\, \omega(k^*) \Leftrightarrow v^* k^* \, \Big] = 0 \quad , \quad k^* := k_s(v^*) \quad , \quad D := D(v^*) \; . \tag{51}$$

For both Eqs. (1) and (2), the saddle point equations (49) – (51) yield $q^* = 0$ and Re $\omega(k^*) = 0$. Let us consider only such cases below, as treated in [7]. (Examples where Re $\omega(k^*) \neq 0$ are discussed in [16].) Eq. (48) then becomes

$$G(j,t) \; \approx \; e^{-\lambda^* \xi^*} \; \frac{e^{-\xi^{*2}/(4Dt)}}{\sqrt{4\pi D t}} \; \text{ for } \; \xi^* = j \Leftrightarrow v^* t \text{ fixed and } \; t \gg 1 \; . \tag{52}$$

Furthermore, the saddle point equations for Eq. (2) result in the equations (7) and (8) for v^*, λ^* and D. For Eq. (1), one finds $v^* = 2$, $\lambda^* = 1$, and $D = 1$. Note, that we used a different definition of v^* as $v^* = \min_{0<\lambda<\infty} v(\lambda)$ in Eqs. (16), (17). One easily convinces oneself [7], that the definitions coincide for every dispersion relation $\omega(k)$ that is analytic. For equations with higher spatial derivatives, the dispersion relations are of higher order in k and thus exhibit several saddle points; for a discussion of the choice of the appropriate saddle point, we refer to [7].

The saddle point approximation (52) for $t \gg 1$ and ξ^* fixed shows that G depends on the particular equation only through the saddle point parameters v^*, λ^* and D. If the initial condition $\phi_j(0)$ is sufficiently steep, then the evolution of $\phi_j(t)$ in (46) is essentially given by the long time asymptotics (52) of G. If $\phi_j(0)$ has bounded support, this is straight forward. The more general condition (10) is discussed in [7].

A closer inspection of the structure of (52) motivates the Ansätze from Section 2 and shows their universality:

1) The factor $e^{-\lambda^* \xi^*}/\sqrt{t}$ in (52) can be rewritten as $e^{-\lambda^* \xi_{X_{lin}}}$, where

$$\xi_{X_{lin}} = j \Leftrightarrow v^* t \Leftrightarrow X_{lin}(t) \quad , \quad X_{lin}(t) = \Leftrightarrow \frac{\ln t}{2\lambda^*} \; . \tag{53}$$

Hence the asymptotic rate of convergence towards the linear spreading velocity v^* is

$$v_{lin}(t) = v^* + \dot{X}_{lin}(t) = v^* \Leftrightarrow \frac{1}{2\lambda^* t} + \mathcal{O}\left(\frac{\ln t}{t^2}\right) \qquad \text{(lin.)} \tag{54}$$

for the linearized equation. According to the argument (ii) in the main text, $v_{lin}(t)$ is an upper bound for the evolution of the nonlinear equation. The structure of (53) and (54) motivates the Ansatz for X in (19) and the structure of the leading edge transformation (20).

2) While the $1/t$ rate of approach to the asymptotic linear spreading speed v^* already comes out of the fully linear equation, the prefactor of the $1/t$ term is different from the one for the true front, as (6) shows that

$$v(t) = v^* + \dot{X}(t) = v^* \Leftrightarrow \frac{3}{2\lambda^* t} + \mathcal{O}\left(\frac{1}{t^{3/2}}\right) \qquad \text{(nonlin.)} \tag{55}$$

from our systematic calculation. The different coefficients of the $1/t$ terms in (54) and (55) actually are the fingerprint of the nonlinearity and related to the coefficient α in (24), (26) being nonvanishing. Also the fact that the subdominant term is of order $1/t^{3/2}$ rather than $1/t^2$ only comes out of the full analysis. As is discussed in more detail in [7], the emergence of the coefficient 3/2 of the $1/t$ term can be traced back to the fact that the simple Gaussian $\psi^* \approx e^{-\xi^{*2}/(4Dt)}/t^{1/2}$ does not match to the asymptotic behavior (24) with $\alpha \neq 0$ (26). However, the dipole solution of (57)

$$\psi^* \approx \frac{\xi^* \, e^{-\xi^{*2}/(4Dt)}}{t^{3/2}} \tag{56}$$

does match the asymptotic behavior of (26), and it does give the proper rate of convergence (55). Working with coordinate ξ_X rather than ξ, we recover the solution (35) as the immediate analogue of (56).

3) The large time expansion (52) of G reveals to leading order a Gaussian structure for general $\omega(k)$. The associated equation of motion for $\psi^* = e^{\lambda^* \xi^*} \phi$ in the leading edge and for $t \gg 1$ thus becomes with the same generality

$$\partial_t \psi^* = D \, \partial^2_{\xi^*} \psi^* + D_3 \, \partial^3_{\xi^*} \psi^* + \ldots \tag{57}$$

Expressed in terms of the coordinate ξ_X, rather than ξ^*, Eq. (30) results. It is this universal Gaussian $e^{-\xi_X^2/(4Dt)} = e^{-z}$, that is extracted in (34), and all the further calculations apply equally generally.

B Analysis of the nonlinear front region

Looking at a front as shown, e.g., in Fig. 1, one mainly sees the nonlinear or interior region, where ϕ varies from 0 to 1. This is also where typically the velocity is measured, either by a prescription as in (4) or by tracing the velocity $v_\phi(t)$ of a certain level ϕ. However, in the main text, only the velocity $v(t)$ far ahead in the leading edge (where $\phi \ll 1$) has been determined. In the present Appendix, we will derive that

$$v_\phi(t) = v(t) + \mathcal{O}\left(\frac{g(\phi)}{t^2}\right) \ , \tag{58}$$

which means the velocity $v_\phi(t)$ is independent of ϕ within the accuracy of the velocity prediction (6).

In fact, Equation (58) is an immediate consequence of the general result

$$\phi(x,t) = \Phi_{v(t)}(\xi_X) + \mathcal{O}\left(\frac{1}{t^2}\right) \quad \text{for} \ \ \xi_X \ll \sqrt{4Dt} \tag{59}$$

for the interior region of a pulled uniformly translating front [7], whose derivation we will sketch below. Here $\xi_X = x \Leftrightarrow x_0 \Leftrightarrow \int_0^t d\tau \ \ v(\tau)$ is the comoving coordinate introduced previously and placed at the appropriate position x_0. The functions $\Phi_v(\xi)$ are continuously parametrized by the velocity v; they are front solutions propagating uniformly with *constant* velocity v. In particular, for Equation (9), the functions Φ_v solve the delay equation

$$\Phi_v(\xi) \Leftrightarrow v\partial_\xi\Phi_v(\xi) = 2\Phi_v(\xi \Leftrightarrow 1) \Leftrightarrow \Phi_v^2(\xi \Leftrightarrow 1) \ , \tag{60}$$

while for Equation (1), the functions Φ_v are given by the *o.d.e.*

$$\partial_\xi^2 \ \Phi_v(\xi) + v\partial_\xi \ \Phi_v(\xi) + \Phi_v(\xi) \Leftrightarrow \Phi_v^3(\xi) = 0 \ . \tag{61}$$

Equation (59) expresses that the front interior is "slaved" to the leading edge: the leading edge of the front determines a time dependent velocity $v(t)$; the interior profile of a front with instantaneous velocity $v(t)$ is identical to the interior profile of a front propagating with the same instantaneous, but *time independent* velocity v. The corrections are as small as order $1/t^2$.

The uniformly translating fronts $\Phi_v(\xi)$ in (60) or (61) satisfy the boundary conditions $\lim_{\xi\to-\infty} \Phi_v(\xi) = 1$ and $\lim_{\xi\to\infty} \Phi_v(\xi) = 0$ so that they connect the stable stationary state $\phi = 1$ with the unstable stationary state $\phi = 0$. This defines Φ_v uniquely up to translation in both cases. The position is fixed, e.g., through the requirement $\Phi_v(0) = 1/2$. If $v \geq v^*$, the function $\Phi_v(\xi)$ decreases monotonically to 0 at $\xi \to \infty$. The front $\Phi^* = \Phi_{v^*}$ is the slowest monotone front; for $\xi \gg 1$, it always has the form $\Phi^*(\xi) \approx (\alpha\xi + \beta)\ e^{-\lambda^*\xi}$ with some constants α and β. If $v < v^*$, the function $\Phi_v(\xi)$ oscillates around $\phi = 0$ as $\xi \to \infty$. However, it is just the Φ_v with $v < v^*$ that play a role as a transient in (59), because for sufficiently large times $v(t) < v^*$. This is not in contradiction with the positivity (11) of the solution, because only the positive part of Φ_v up to $\xi_X \ll \sqrt{4Dt}$ contributes in (59).

Let us now define some intermediate expressions and derive (59). If we linearize $\phi(x,t)$ about $\Phi^*(\xi^*)$ in the coordinate system $\xi^* = x \Leftrightarrow v^* t$ moving with the asymptotic velocity v^*, the evolution of the linear perturbation $\eta^* = \phi \Leftrightarrow \Phi^*$

$$\partial_t\, \eta^* = \mathcal{L}^*\, \eta^* + \mathcal{O}(\eta^{*2}) \quad , \quad \phi(x,t) = \Phi^*(\xi^*) + \eta(\xi^*, t) \quad , \quad \xi^* = x \Leftrightarrow v^* t \ , \tag{62}$$

is governed by the linear operator $\mathcal{L}^*$. The explicit form of $\mathcal{L}^*$ depends on the actual evolution equation like (1) or (9) considered, but is not required for the further calculation.

We argued in (iv) in the main text, that we actually should investigate the linear perturbation $\eta(\xi_X, t)$ of the front $\Phi^*(\xi_X)$ in the frame $\xi_X = x \Leftrightarrow v^* t \Leftrightarrow X(t)$. The equation of motion for this $\eta(\xi_X, t)$ is

$$\partial_t\, \eta = \mathcal{L}^*\, \eta + \dot{X}\ \partial_{\xi_X} \Phi^* + \dot{X}\ \partial_{\xi_X} \eta + \mathcal{O}(\eta^2) \quad , \quad \phi = \Phi^*(\xi_X) + \eta(\xi_X, t) \ . \tag{63}$$

As η should decay for large times, it is consistent with the expansion for $\dot{X}$ to write

$$\eta(\xi_X, t) = \frac{\eta_{1/2}(\xi_X)}{t^{1/2}} + \frac{\eta_1(\xi_X)}{t} + \frac{\eta_{3/2}(\xi_X)}{t^{3/2}} + \ldots \tag{64}$$

Ordering in powers of $1/\sqrt{t}$ results in a hierarchy of linear inhomogeneous *o.d.e.*s for the $\eta_{n/2}$:

$$\mathcal{L}^*\ \eta_{1/2} = 0 \tag{65}$$

$$\mathcal{L}^*\ \eta_1 + c_1\ \partial_{\xi_X} \Phi^* = 0 \tag{66}$$

$$\mathcal{L}^*\ \eta_{3/2} + c_{3/2}\ \partial_{\xi_X} \Phi^* = 0 \quad \text{etc.} \tag{67}$$

Evaluated with the appropriate boundary conditions [7], these equations have unique solutions. The homogeneous equation (65) immediately yields $\eta_{1/2} \equiv 0$.

The solutions of equations (66) and (67) can be written as multiples of the so-called shape mode $\eta_{sh} = \delta\Phi_v/\delta v|_{v^*}$, i.e., the variation of Φ_v with respect to v evaluated at $v = v^*$. Upon expanding Eq. (60) or (61) about Φ^*, one derives a linear inhomogeneous equation for η_{sh}:

$$\mathcal{L}^* \, \eta_{sh} + \partial_\xi \Phi^* = 0 \; . \tag{68}$$

Hence one can identify $\eta_1 = c_1 \eta_{sh}$ and $\eta_{3/2} = c_{3/2}\eta_{sh}$. Insertion of the solutions of (65) – (67) into (64) yields $\eta(\xi_X, t) = \dot{X}(t) \; \eta_{sh}(\xi_X) + \mathcal{O}(1/t^2)$. Using the definition of the shape mode η_{sh} for the resummation

$$\Phi_{v(t)}(\xi_X) = \Phi^*(\xi_X) + \dot{X}(t) \; \eta_{sh}(\xi_X) + \mathcal{O}\left(\frac{1}{t^2}\right) \; , \tag{69}$$

the final expression (59) for the interior of a pulled front results.

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