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ABSTRACT

We analyze the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, play a major role in achieving differentiated quality-of-service in integrated-services networks. We prove that, in certain scenarios, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. This phenomenon contrasts with previous results which show that, under certain conditions, an individual flow with long-tailed traffic characteristics is effectively served at a constant rate. In particular, the flow is then essentially immune from excessive activity of flows with 'heavier'-tailed traffic characteristics. The sharp dichotomy in qualitative behavior illustrates the crucial importance of the weight parameters in protecting individual flows.

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1 Introduction

Statistical data analysis has provided convincing evidence of long-tailed (subexponential) traffic characteristics in high-speed communication networks. Early indications of the long-range dependence of Ethernet traffic, attributed to long-tailed file size distributions, were reported in Leland *et al.* [21]. Long-tailed characteristics of the scene length distribution of MPEG video streams were explored in Heyman & Lakshman [15] and Jelenković *et al.* [18].

These empirical findings have encouraged active theoretical developments in the modeling and queueing analysis of long-tailed traffic phenomena. We refer to Boxma & Dumas [9] for a comprehensive survey on fluid queues with long-tailed arrival processes. See also Jelenković [16] for an extensive list of references on subexponential queueing models.

Despite significant progress, however, the practical implications are not yet thoroughly understood, in particular issues relating to control and priority mechanisms in the network. To gain a better understanding of those issues, the present paper analyzes the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. As a design paradigm, GPS is at the heart of commonly-used scheduling algorithms for high-speed switches, such as Weighted Fair Queueing, see for instance Parekh & Gallager [25, 26].

The impact of priority and scheduling mechanisms on long-tailed traffic phenomena has received relatively little attention. Some recent studies have investigated the effect of the scheduling discipline on the waiting-time distribution in the classical M/G/1 queue, see for instance Anantharam [2]. For FCFS, it is well-known (see Cohen [13]) that the waiting-time tail is regularly varying of index $1 - \nu$ iff the service time tail is regularly varying of index $-\nu$. For LCFS preemptive resume as well as for Processor Sharing, the waiting-time tail turns out to be regularly varying of the *same* index as the service time tail, see Boxma & Cohen [8], and Zwart & Boxma [32], although with different pre-factors. In the case of Processor Sharing with several customer classes, Zwart [29] showed that the sojourn time distribution of a class-i customer is regularly varying of index $-\nu_i$ iff the service time distribution of that class is regularly varying of index $-\nu_i$, *regardless* of the service time distributions of the other classes. In contrast, for two customer classes with ordinary non-preemptive priority, the tail behavior of the waiting- and sojourn time distributions is determined by the *heaviest* of the (regularly-varying) service time distributions, see Abate & Whitt [1].

In the present paper, we consider the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, play a major role in achieving differentiated quality-of-service in integrated-services networks. The queueing analysis of GPS is extremely difficult. Interesting partial results for exponential traffic models were obtained in Bertsimas *et al.* [3], Dupuis & Ramanan [14], Massoulié [22], Zhang [27], and Zhang *et al.* [28]. Here, we focus on non-exponential traffic models. We show that, in certain scenarios, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. This complements results previously obtained in [6] which show that, under certain conditions, an individual flow with long-tailed traffic characteristics is effectively served at a constant rate. The latter rate only depends on the traffic characteristics of other flows through their average rate. In particular, the flow is then essentially immune to excessive activity of flows with 'heavier'-tailed traffic characteristics. The sharp dichotomy in qualitative behavior illustrates the crucial importance of the weight parameters in protecting individual flows.

The remainder of the paper is organized as follows. In Section 2, we present a detailed model description. In Section 3, we derive generic lower and upper bounds for the workload distribution. We then show, in Section 4, that for long-tailed traffic characteristics, the lower and upper bounds have the same asymptotic behavior, yielding exact asymptotics for the tail distribution of the workload. In Section 5, we make some concluding remarks.

2 Model description

We consider two traffic flows sharing a link of unit rate. Traffic from the flows is served in accordance with the Generalized Processor Sharing (GPS) discipline, which operates as follows. There are weights ϕ_i, $i = 1, 2$, associated with each of the flows, with $\phi_1 + \phi_2 = 1$. As long as both flows are backlogged, flow i is served at rate ϕ_i, $i = 1, 2$. If one of the flows is not backlogged, however, then the capacity is reallocated to the other flow, which is then served at the full link rate (if backlogged).

We ignore some technicalities here which may arise for general arrival processes when the inflow rate r_i of flow i may be smaller than the weight ϕ_i. In that case, only the *excess* capacity, i.e., $\phi_i - r_i$, is reallocated to the other flow. These subtleties however will not arise for the arrival processes that we consider. We refer to Dupuis & Ramanan [14] for a more thorough discussion of these issues.

Denote by $A_i(s,t)$ the amount of traffic generated by flow i during the time interval $(s,t]$. We assume that the process $A_i(s,t)$ is stationary.

Denote by $V_i(t)$ the workload of flow i at time t. Let $\mathbf{V}_i$ be a stochastic variable with as distribution the limiting distribution of $V_i(t)$ for $t \to \infty$ (assuming it exists).

Define $B_i(s,t)$ as the amount of service received by flow i during the time interval $(s,t]$. Then the following identity relation holds,

$$V_i(t) = V_i(s) + A_i(s,t) - B_i(s,t) \tag{1}$$

for all $0 \leq s \leq t$.

For any $c \geq 0$, denote by $V_i^c(t) := \sup_{0 \leq s \leq t} \{A_i(s,t) - c(t-s)\}$ the workload at time t in a queue of capacity c fed by flow i only. Denote by ρ_i the traffic intensity of flow i as will be defined in

detail below for the two traffic scenarios that we consider. For $c > \rho_i$, let $\mathbf{V}_i^c$ be a stochastic variable with as distribution the limiting distribution of $V_i^c(t)$ for $t \to \infty$.
Denote by $\mathbf{P}_i^c$ the busy period associated with the workload process $\mathbf{V}_i^c$. For conciseness, we occasionally suppress the superscript c when the capacity is clear from the context.
Similarly to the identity relation above,

$$V_i^c(t) = V_i^c(s) + A_i(s,t) - B_i^c(s,t) \tag{2}$$

for all $0 \leq s \leq t$, with

$$B_i^c(s,t) = c \int_s^t \mathrm{I}_{\{V_i^c(u)>0\}} \mathrm{d}u. \tag{3}$$

Before describing the traffic model, we first introduce some additional notation. For any two real functions $g(\cdot)$ and $h(\cdot)$, we use the notational convention $g(x) \sim h(x)$ to denote $\lim_{x\to\infty} g(x)/h(x) = 1$, or equivalently, $g(x) = h(x)(1 + o(1))$ as $x \to \infty$. For any stochastic variable $\mathbf{X}$ with distribution function $F(\cdot)$, $\mathbb{E}\mathbf{X} < \infty$, denote by $F^r(\cdot)$ the distribution function of the residual lifetime of $\mathbf{X}$, i.e., $F^r(x) = \frac{1}{\mathbb{E}\mathbf{X}} \int_0^x (1 - F(y))\mathrm{d}y$, and by $\mathbf{X}^r$ a stochastic variable with that distribution.
The classes of *long-tailed*, *subexponential*, *regularly varying*, *intermediately regularly varying*, and *dominatedly varying* distributions are denoted with the symbols $\mathcal{L}$, $\mathcal{S}$, $\mathcal{R}$, $\mathcal{IR}$, and $\mathcal{DR}$, respectively. The definitions of these classes may be found in Appendix A.

We now describe the two traffic scenarios that we consider.

2.1 Instantaneous bursts

Here, a flow generates instantaneous traffic bursts according to a renewal processes. The interarrival times between bursts of flow i have distribution function $U_i(\cdot)$ with mean $1/\lambda_i$. The burst sizes of flow i have distribution $S_i(\cdot)$ with mean $\sigma_i < \infty$. Thus, the traffic intensity of flow i is $\rho_i = \lambda_i \sigma_i$.

We now state some results which will play a crucial role in the analysis.

Theorem 2.1 *(Pakes [24]) If $S_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c$, then*

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \sim \frac{\rho_i}{c - \rho_i} \mathbb{P}\{\mathbf{S}_i^r > x\}.$$

Theorem 2.2 *(Zwart [30]) If $U_i(\cdot)$ is an exponential distribution, i.e., the arrival process is Poisson, $S_i(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then*

$$\mathbb{P}\{\mathbf{P}_i > x\} \sim \frac{c}{c - \rho_i} \mathbb{P}\{\mathbf{S}_i > x(c - \rho_i)\}.$$

In fact, the preceding theorem can be extended to non-Poisson arrival processes, see Zwart [30]. In the analysis we will need a slight modification:

Theorem 2.3 *If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then*

$$\mathbb{P}\{\mathbf{P}_i^r > x\} \sim \frac{c}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i^r > x(c-\rho_i)\}.$$

Remark 2.1 *Although Theorem 2.3 is only a minor extension of Theorem 2.2, the proof is new and might be of independent interest. It directly uses Theorem 2.1 to derive the asymptotic behavior of the residual busy period. Note that if $S_i(\cdot) \in \mathcal{IR}$, then Theorem 2.2 implies Theorem 2.3. However, if we only assume $S_i^r(\cdot) \in \mathcal{IR}$, then we cannot directly use Theorem 2.2, since $S_i^r(\cdot) \in \mathcal{IR}$ does not necessarily imply $S_i(\cdot) \in \mathcal{IR}$.*

Proof of Theorem 2.3

For compactness, we suppress the subscript i, e.g., $V^c(t) \equiv V_i^c(t)$, $\rho \equiv \rho_i$, etc.

For $0 < \delta < c - \rho$, define

$$L^\delta(t) := \sup_{0\le s\le t}\{B^c(s,t) - (c-\delta)(t-s)\},$$

with $B^c(s,t)$ as in (3).

Observe that $L^\delta(t)$ and $V^c(t)$ represent the workload processes in a priority queue with service rate c and arrival processes $\delta(t-s)$ and $A(s,t)$, respectively, with $L^\delta(t)$ having lower priority. Since the total workload does not depend on the priority mechanism, the sum of the workloads equals

$$L^\delta(t) + V^c(t) = V^{c-\delta}(t) = \sup_{0\le s\le t}\{A(s,t) - (c-\delta)(t-s)\}. \tag{4}$$

(Upper bound) By the previous equality and Theorem 2.1, in steady state,

$$\begin{aligned}\mathbb{P}\left\{\mathbf{L}^\delta > \delta x\right\} &\le \mathbb{P}\left\{\mathbf{V}^{c-\delta} > \delta x\right\} \\ &\sim \frac{\rho}{c-\delta-\rho}\mathbb{P}\{\mathbf{S}^r > \delta x\}.\end{aligned} \tag{5}$$

Let $\mathbf{P}^{b,r}$ be the past lifetime of the busy period of $V^c(t)$ in steady state. By symmetry, $\mathbf{P}^{b,r}$ is equal in distribution to $\mathbf{P}^r$. Hence,

$$\mathbb{P}\left\{\mathbf{L}^\delta > x\right\} \ge \mathbb{P}\left\{\mathbf{V}^c > 0, \mathbf{P}^{b,r} > x/\delta\right\} = \mathbb{P}\{\mathbf{V}^c > 0\}\,\mathbb{P}\{\mathbf{P}^r > x/\delta\}, \tag{6}$$

where we use the fact that in steady state the event $\{\mathbf{V}^c > 0\}$ is independent of $\mathbf{P}^{b,r}$.

Since the busy period $\mathbf{P}$ is larger than the time $\mathbf{S}/c$ it takes to serve a single service request, it easily follows that

$$\mathbb{P}\{\mathbf{P}^r > x\} \ge \frac{\mathbb{E}\mathbf{S}}{c\mathbb{E}\mathbf{P}}\mathbb{P}\{\mathbf{S}^r > xc\}, \tag{7}$$

which, in conjunction with (5), (6) and $S^r(\cdot) \in \mathcal{IR}$, implies that $P^r(\cdot) \in \mathcal{DR}$, and therefore $P^r(\cdot) \in \mathcal{S}$.

Now observe that $L^\delta(t)$ may also be interpreted as the workload at time t in a queue with constant service rate $c - \delta$ fed by an On-Off process with On- and Off-periods equal to the busy and idle periods associated with the workload process $V^c(t)$, respectively. During the On-periods, traffic is produced at constant rate c. The fraction Off-time is $1 - \rho/c$. The On- and Off-periods are independent because $U(\cdot)$ is an exponential distribution. Hence, by Theorem 2.4,

$$\mathbb{P}\left\{\mathbf{L}^\delta > \delta x\right\} \sim \frac{c-\rho}{c}\frac{\rho}{c-\delta-\rho}\mathbb{P}\left\{\mathbf{P}^r > x\right\}. \tag{8}$$

Now, (5) and (8) yield

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left\{\mathbf{P}^r > x\right\}}{\mathbb{P}\left\{\mathbf{S}^r > \delta x\right\}} \leq \frac{c}{c-\rho},$$

and the upper bound follows by letting $\delta \uparrow c - \rho$.

(Lower bound) From (4), in steady state, for any $\epsilon > 0$,

$$\begin{aligned} \mathbb{P}\left\{\mathbf{L}^\delta > \delta x\right\} &= \mathbb{P}\left\{\mathbf{V}^{c-\delta} - \mathbf{V}^c > \delta x\right\} \\ &\geq \mathbb{P}\left\{\mathbf{V}^{c-\delta} > (1+\epsilon)\delta x, \mathbf{V}^c \leq \epsilon\delta x\right\} \\ &\geq \mathbb{P}\left\{\mathbf{V}^{c-\delta} > (1+\epsilon)\delta x\right\} - \mathbb{P}\left\{\mathbf{V}^c > \epsilon\delta x\right\}. \end{aligned} \tag{9}$$

Hence, by (8), (9), and Theorem 2.1,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left\{\mathbf{P}^r > x\right\}}{\mathbb{P}\left\{\mathbf{S}^r > (1+\epsilon)\delta x\right\}} \geq \frac{c}{c-\rho} - \frac{c(c-\delta-\rho)}{(c-\rho)^2}\limsup_{x\to\infty}\frac{\mathbb{P}\left\{\mathbf{S}^r > \epsilon\delta x/(r-c)\right\}}{\mathbb{P}\left\{\mathbf{S}^r > (1+\epsilon)\delta x\right\}},$$

which, by letting first $\delta \uparrow c - \rho$ and then $\epsilon \downarrow 0$ completes the proof of the lower bound.

□

2.2 On-Off processes

Here, a flow generates traffic according to an On-Off process, alternating between On- and Off-periods. The Off-periods of flow i have distribution function $U_i(\cdot)$ with mean $1/\lambda_i$. The On-periods of flow i have distribution $S_i(\cdot)$ with mean $\sigma_i < \infty$. While On, flow i produces traffic at a constant rate r_i, so the mean burst size is $\sigma_i r_i$. The fraction of time that flow i is Off is

$$p_i = \frac{1/\lambda_i}{1/\lambda_i + \sigma_i} = \frac{1}{1+\lambda_i\sigma_i}.$$

The traffic intensity of flow i is

$$\rho_i = (1-p_i)r_i = \frac{\lambda_i\sigma_i r_i}{1+\lambda_i\sigma_i}.$$

We now state the analogues of Theorems 2.1, 2.2, and 2.3 for the case of On-Off processes.

Theorem 2.4 *(Jelenković & Lazar [17]) If $S_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c < r_i$, then*

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \sim p_i \frac{\rho_i}{c-\rho_i} \mathbb{P}\{\mathbf{S}_i^r > x/(r_i - c)\}.$$

Theorem 2.5 *(Boxma & Dumas [10], Zwart [30]) If $U_i(\cdot)$ is an exponential distribution, i.e., the Off-periods are exponentially distributed, $S_i(\cdot) \in \mathcal{IR}$, and $\rho_i < c < r_i$, then*

$$\mathbb{P}\{\mathbf{P}_i > x\} \sim p_i \frac{c}{c-\rho_i} \mathbb{P}\{\mathbf{S}_i > x(c-\rho_i)/(r_i - \rho_i)\}.$$

In addition, the following minor extension of the preceding theorem holds:

Theorem 2.6 *If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c < r_i$, then*

$$\mathbb{P}\{\mathbf{P}_i^r > x\} \sim p_i \frac{c}{c-\rho_i} \mathbb{P}\{\mathbf{S}_i^r > x(c-\rho_i)/(r_i - \rho_i)\}.$$

Remark 2.2 *Theorems 2.5 and 2.6 follow directly from Theorems 2.2 and 2.3 because of a beautiful equivalence relation observed by Boxma & Dumas [10] and Zwart [31]. The busy period in a fluid queue is equal in distribution to the busy period in a corresponding $G/G/1$ queue scaled by a factor $r_i/(r_i - c_i)$. The interarrival times in the $G/G/1$ queue are exactly the Off-periods in the fluid queue, and the service times correspond to the* net input *during the On-periods. Thus, with some minor abuse of notation, $\mathbb{P}\{\mathbf{P}_i > x\} = \mathbb{P}\left\{\mathbf{P}_i^{G/G/1} > x(r_i - c)/r_i\right\}$ for all values of x, with $U_i^{G/G/1}(\cdot) = U_i(\cdot)$ and $\mathbf{S}_i^{G/G/1} := (r_i - c)\mathbf{S}_i$.*
From Theorem 2.2, noting that $c - \rho_i^{M/G/1} = (c-\rho_i)/p_i$ and $p_i r_i = r_i - \rho_i$,

$$\begin{aligned}\mathbb{P}\left\{\mathbf{P}_i^{M/G/1} > x(r_i - c)/r_i\right\} &\sim \frac{c}{c-\rho_i^{M/G/1}} \mathbb{P}\left\{\mathbf{S}_i^{M/G/1} > x(c-\rho_i^{M/G/1})(r_i - c)/r_i\right\} \\ &= p_i \frac{c}{c-\rho_i} \mathbb{P}\{\mathbf{S}_i > x(c-\rho_i)/(r_i-\rho_i)\},\end{aligned}$$

yielding Theorem 2.5.
In Boxma & Dumas [10], Theorem 2.6 was essentially obtained in this manner from a weaker version of Theorem 2.2 in De Meyer & Teugels [23] for the case $S_i(\cdot) \in \mathcal{R}$. Similarly, Theorem 2.6 for the residual busy period can be directly obtained from Theorem 2.3.
Alternatively, Theorem 2.6 can be proved by mimicking the proof of Theorem 2.3. The only difference is that in Equations (5) and (9), one uses Theorem 2.4 instead of Theorem 2.1, and replaces c in (7) by c/r.

3 Bounds

We now derive some generic bounds for the workload distribution which we will use in the next section to analyze the tail behavior. Without loss of generality we focus on flow 1. The bounds apply for the scenario of instantaneous bursts as well as On-Off processes as described in Subsections 2.1 and 2.2, respectively.

We first introduce some notation. Define

$$Q^{\epsilon}(t) := \sup_{0 \le s \le t} \{B_2^{\phi_2}(s,t) - (1 - \rho_1 - \epsilon)(t - s)\},$$

with $B_2^{\phi_2}(s,t)$ as in (3). For $\epsilon < 1 - \rho_1 - \rho_2$, let $\mathbf{Q}^{\epsilon}$ be a stochastic variable with as distribution the limiting distribution of $Q^{\epsilon}(t)$ for $t \to \infty$.
Define $Z_1^c(s) := \sup_{u \ge s}\{c(u-s) - A_1(s,u)\}$. For $c < \rho_1$, let $\mathbf{Z}_1^c$ be a stochastic variable with as distribution the distribution of $Z_1^c(s)$ (which does not depend on s because the process $A_1(s,t)$ is stationary).

We first present a lower bound for the workload distribution of flow 1.

Lemma 3.1 *(Lower bound) If $\rho_1 > \phi_1$, then for any $\epsilon > \rho_1 + \rho_2 - 1$ sufficiently small and y,*

$$\mathbb{P}\{\mathbf{V}_1 > x\} \ge \mathbb{P}\left\{\mathbf{Q}^{-\epsilon} > x + y\right\} \mathbb{P}\left\{\mathbf{Z}_1^{\rho_1 - \epsilon} \le y\right\}.$$

Proof
Define

$$s^* := \arg \sup_{0 \le s \le t} \{B_2^{\phi_2}(s,t) - (1 - \rho_1 + \epsilon)(t - s)\}, \tag{10}$$

so that

$$Q^{-\epsilon}(t) = B_2^{\phi_2}(s^*,t) - (1 - \rho_1 + \epsilon)(t - s^*). \tag{11}$$

It is easily verified that for any ϵ sufficiently small,

$$V_2^{\phi_2}(s^*) = 0, \tag{12}$$

because otherwise $B_2^{\phi_2}(s^* - \Delta, t) = B_2^{\phi_2}(s^*, t) + \Delta\phi_2$, contradicting the optimality of s^*, as $\phi_2 = 1 - \phi_1 > 1 - \rho_1 + \epsilon$.
The GPS discipline ensures that

$$V_2(t) \le V_2^{\phi_2}(t), \tag{13}$$

since each flow i is guaranteed to receive a minimum service rate ϕ_i whenever backlogged.
Combining (1), (2), (12), and (13),

$$B_2(s^*,t) \ge B_2^{\phi_2}(s^*,t). \tag{14}$$

By definition, $B_1(s,t) + B_2(s,t) \le t - s$ for all $0 \le s \le t$, so that

$$B_1(s^*,t) \le t - s^* - B_2(s^*,t). \tag{15}$$

Substituting (14), (15) into (2), with $i = 1$, using (11),

$$\begin{aligned} V_1(t) &\geq A_1(s^*,t) + B_2^{\phi_2}(s^*,t) - (t - s^*) \\ &= Q^{-\epsilon}(t) + A_1(s^*,t) - (\rho_1 - \epsilon)(t - s^*) \\ &\geq Q^{-\epsilon}(t) + \inf_{u \geq s^*} \{A_1(s^*,u) - (\rho_1 - \epsilon)(u - s^*)\} \\ &= Q^{-\epsilon}(t) - \sup_{u \geq s^*} \{(\rho_1 - \epsilon)(u - s^*) - A_1(s^*,u)\} \\ &= Q^{-\epsilon}(t) - Z_1^{\rho_1 - \epsilon}(s^*). \end{aligned}$$

Note that, by (10), (11), s^*, $Q^{-\epsilon}(t)$ depend only on $A_2(s,t)$, $0 \leq s \leq t$, and are independent of $Z_1^{\rho_1-\epsilon}(s)$, $s \geq 0$ (fixed). Hence,

$$\begin{aligned} \mathbb{P}\{V_1(t) > x | s^*\} &\geq \mathbb{P}\left\{Q^{-\epsilon}(t) - Z_1^{\rho_1-\epsilon}(s^*) > x | s^*\right\} \\ &\geq \mathbb{P}\left\{Q^{-\epsilon}(t) > x - y, Z_1^{\rho_1-\epsilon}(s^*) \leq y | s^*\right\} \\ &= \mathbb{P}\left\{Q^{-\epsilon}(t) > x - y | s^*\right\} \mathbb{P}\left\{\mathbf{Z}_1^{\rho_1-\epsilon} \leq y\right\}, \end{aligned}$$

which immediately yields the statement of the lemma.

□

We now present an upper bound for the workload distribution of flow 1.

Lemma 3.2 *(Upper bound) For any $\epsilon < 1 - \rho_1 - \rho_2$ and δ,*

$$\mathbb{P}\{\mathbf{V}_1 > x\} \leq \mathbb{P}\{\mathbf{Q}^\epsilon > (1 - \delta)x\} + \mathbb{P}\left\{\mathbf{V}_1^{\rho_1+\epsilon} > \delta x\right\}.$$

Proof

The GPS discipline implies that

$$V_1(t) = \sup_{0 \leq s \leq t} \{A_1(s,t) - C_1(s,t)\}, \tag{16}$$

(assuming $V_1(0) = 0$), with

$$C_1(s,t) = \int_s^t (1 - \phi_2 \mathrm{I}_{\{V_2(u) > 0\}}) \mathrm{d}u. \tag{17}$$

From (3), (13),

$$C_1(s,t) = \int_s^t (1 - \phi_2 \mathrm{I}_{\{V_2(u) > 0\}}) \mathrm{d}u \geq \int_s^t (1 - \phi_2 \mathrm{I}_{\{V_2^{\phi_2}(u) > 0\}}) \mathrm{d}u = t - s - B_2^{\phi_2}(s,t). \tag{18}$$

Substituting (17), (18) into (16), for any $\epsilon > 0$,

$$\begin{aligned} V_1(t) &\leq \sup_{0 \leq s \leq t} \{A_1(s,t) - (t - s) + B_2^{\phi_2}(s,t)\} \\ &\leq \sup_{0 \leq s \leq t} \{A_1(s,t) - (\rho_1 + \epsilon)(t - s)\} + \sup_{0 \leq s \leq t} \{B_2^{\phi_2}(s,t) - (1 - \rho_1 - \epsilon)(t - s)\} \\ &= V_1^{\rho_1+\epsilon}(t) + Q^\epsilon(t), \end{aligned}$$

from which the statement of the lemma directly follows.

□

4 Asymptotic analysis

We now use the bounds from the previous section to determine the tail distribution of the workload. We consider both the scenario of instantaneous bursts and On-Off processes as described in Subsections 2.1 and 2.2, respectively. We also allow for mixed traffic scenarios, where one flow generates instantaneous bursts, while the other produces traffic according to an On-Off process. Define $c_2 := \phi_2$ if flow 2 generates instantaneous traffic, and $c_2 := \min\{r_2, \phi_2\}$ if flow 2 produces traffic according to an On-Off process, and denote $\mathbf{P}_2^r := (\mathbf{P}_2^{c_2})^r$.

Theorem 4.1 *Assume that $U_2(\cdot)$ is an exponential distribution.*
If $\rho_1 + \rho_2 < 1$, $\rho_1 + c_2 > 1$, $S_2^r(\cdot) \in \mathcal{IR}$, and $\mathbb{P}\{\mathbf{S}_1^r > x\} = o(\mathbb{P}\{\mathbf{S}_2^r > x\})$ as $x \to \infty$, then

$$\mathbb{P}\{\mathbf{V}_1 > x\} \sim \frac{c_2 - \rho_2}{c_2} \frac{\rho_2}{1 - \rho_1 - \rho_2} \mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1 + c_2 - 1)\},$$

with $\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1 + c_2 - 1)\}$ as in Theorems 2.3 and 2.6, respectively.

The above theorem complements the results obtained in [6] which in the context of the present model may be formulated as follows.

Theorem 4.2 *If either (i) $\rho_1 < \phi_1$, and $S_1^r(\cdot) \in \mathcal{S}$ (instantaneous bursts) or $S_1^r(\cdot) \in \mathcal{IR}$ (On-Off process); or (ii) $S_1^r(\cdot) \in \mathcal{IR}$, $\mathbb{P}\{\mathbf{S}_2^r > x\} = o(\mathbb{P}\{\mathbf{S}_1^r > x\})$ as $x \to \infty$, and $r_1 > \gamma_1$, then*

$$\mathbb{P}\{\mathbf{V}_1 > x\} \sim \mathbb{P}\{\mathbf{V}_1^{\gamma_1} > x\},$$

with $\gamma_1 := \max\{\phi_1, 1 - \rho_2\}$ and $\mathbb{P}\{\mathbf{V}_1^{\gamma_1} > x\}$ as in Theorems 2.1 and 2.4, respectively.

Before giving the formal proof of Theorem 4.1, we first provide an intuitive interpretation. When flow 2 is backlogged, flow 1 is only served at rate $1 - c_2$, while it generates traffic at average rate $\rho_1 > 1 - c_2$. Thus the queue of flow 1 has positive drift $\rho_1 + c_2 - 1 > 0$ when flow 2 is backlogged. Now suppose that flow 2 generates a large burst or experiences a long On-period. It will then become backlogged, and because of the positive drift, flow 1 will soon become backlogged too; flow 2 will thus experience a busy period as if it were served at rate c_2; flow 1 will be served at rate $1 - c_2$, and its queue will roughly grow at rate $\rho_1 + c_2 - 1$.
Of course, its queue may also build up when flow 1 itself generates a large burst or experiences a long On-period. However, these effects are dominated by the build-up during the busy periods of flow 2, because the traffic characteristics of flow 2 are heavier.
Thus, the most likely scenario for flow 1 to build a large queue is for flow 2 to generate a large burst, or experience a long On-period, while flow 1 itself shows average behavior. As long as flow 2 is backlogged, the queue of flow 1 will then roughly grow at rate $\rho_1 + c_2 - 1$. When flow 2 is not backlogged, the queue of flow 1 will drain at approximately rate $1 - \rho_1$. Thus, the

queue of flow 1 behaves as a queue with constant service rate $1-\rho_1$ fed by an On-Off process, with On- and -Off periods exactly equal to the busy and idle periods of flow 2 when served at constant rate c_2. This is reflected in Theorem 4.1, if we use Theorem 2.4 to interpret the right-hand side.
In contrast, under the assumptions of Theorem 4.2, the above scenario cannot arise: either (i) $\rho_1 < \phi_1$, so that the queue of flow 1 retains negative drift when flow 2 is backlogged; or (ii) $\mathbb{P}\{\mathbf{S}_2^r > x\} = o(\mathbb{P}\{\mathbf{S}_1^r > x\})$ as $x \to \infty$ so that the congestion effects due to activity of flow 1 itself dominate the build-up during the busy-periods of flow 2. In this case, the most likely scenario for flow 1 to build a large queue is to generate a large burst or experience a long On-period itself, while flow 2 shows average behavior. Flow 1 will then approximately be served at a constant rate γ_1, as confirmed by Theorem 4.2.
In [7], analogues of Theorems 4.1 and 4.2 were obtained for a closely related coupled-processors model using an explicit expression for the workload transforms. In addition, the analysis in [7] covers the theoretically interesting case that $\mathbb{P}\{\mathbf{S}_1 > x\} \sim K\mathbb{P}\{\mathbf{S}_2 > x\}$ with $0 < K < \infty$.

In preparation for the proof of Theorem 4.1 we first state two auxiliary lemmas.

Lemma 4.1 *Assume that $U_2(\cdot)$ is an exponential distribution.*
If $S_2^r(\cdot) \in \mathcal{IR}$, then for any $1-\rho_1-c_2 < \epsilon < 1-\rho_1-\rho_2$,

$$\mathbb{P}\{\mathbf{Q}^\epsilon > x\} \sim \frac{c_2-\rho_2}{c_2}\frac{\rho_2}{1-\rho_1-\rho_2-\epsilon}\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1+c_2+\epsilon-1)\}, \tag{19}$$

with $\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1+c_2+\epsilon-1)\}$ as in Theorems 2.3 and 2.6, respectively.

Proof
Observe that $Q^\epsilon(t) = L^{c+\rho_1+\epsilon-1}(t)$ as defined in the proof of Theorem 2.3 with $c = c_2$. The statement then follows from (8) and the fact that $S_2^r(\cdot) \in \mathcal{IR}$.

□

Lemma 4.2 *If $S_2^r(\cdot) \in \mathcal{IR}$, and $\mathbb{P}\{\mathbf{S}_1^r > x\} = o(\mathbb{P}\{\mathbf{S}_2^r > x\})$ as $x \to \infty$, then for any $c > \rho_1$,*

$$\mathbb{P}\{\mathbf{V}_1^c > x\} = o(\mathbb{P}\{\mathbf{S}_2^r > x\}) \quad \text{as} \quad x \to \infty.$$

Proof
For any $\delta > 0$, construct the stochastic variable $\mathbf{S}_\delta$ with distribution

$$\mathbb{P}\{\mathbf{S}_\delta > x\} = \min\{1, \mathbb{P}\{\mathbf{S}_1 > x\} + \delta\mathbb{P}\{\mathbf{S}_2 > x\}\}.$$

Now consider the workload process $V_\delta^c(t)$ in a queue with service rate c where the stochastic variable $\mathbf{S}_1$ in the arrival process is replaced by $\mathbf{S}_\delta$. For δ sufficiently small, let $\mathbf{V}_\delta^c$ be a stochastic variable with as distribution the limiting distribution of $V_\delta^c(t)$ for $t \to \infty$. (Notice that $\mathbb{E}\mathbf{S}_\delta \leq \mathbb{E}\mathbf{S}_1 + \delta\mathbb{E}\mathbf{S}_2$, so that the queue is stable for δ sufficiently small.)

Clearly, $\mathbf{S}_\delta$ is stochastically larger than $\mathbf{S}_1$, so that

$$\mathbb{P}\{\mathbf{V}_1^c > x\} \le \mathbb{P}\{\mathbf{V}_\delta^c > x\}. \tag{20}$$

Also,

$$\mathbb{P}\{\mathbf{S}_\delta^r > x\} \sim \delta \frac{\mathbb{E}\mathbf{S}_2}{\mathbb{E}\mathbf{S}_\delta} \mathbb{P}\{\mathbf{S}_2^r > x\},$$

which implies that $\mathbb{P}\{\mathbf{S}_\delta^r > x\} \in \mathcal{IR}$. Hence, by Theorems 2.1, 2.4,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_\delta^c > x\}}{\mathbb{P}\{\mathbf{S}_2^r > x\}} \le \delta K \tag{21}$$

for some finite constant K independent of δ.

The lemma follows by combining (20) and (21) and letting $\delta \downarrow 0$.

□

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1

First observe, from Theorems 2.3, 2.6, that $P_2^r(\cdot) \in \mathcal{IR}$, with $P_2^r(x) = \mathbb{P}\{\mathbf{P}_2^r > x\}$.

(Lower bound) Using Lemmas 3.1, 4.1 and the fact that $P_2^r(\cdot) \in \mathcal{IR}$,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_1 > x\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} &\ge \mathbb{P}\left\{\mathbf{Z}_1^{\rho_1-\epsilon} \le y\right\} \liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}^{-\epsilon} > x+y\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} \\
&= \mathbb{P}\left\{\mathbf{Z}_1^{\rho_1-\epsilon} \le y\right\} \frac{\phi_2-\rho_2}{\phi_2} \frac{\rho_2}{1-\rho_1-\rho_2-\epsilon} \\
&\qquad \liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{P}_2^r > (x+y)/(\rho_1-\epsilon-\phi_1)\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} \\
&= \frac{\phi_2-\rho_2}{\phi_2} \frac{\rho_2}{1-\rho_1-\rho_2-\epsilon} K\left(\frac{\rho_1-\phi_1}{\rho_1-\epsilon-\phi_1}\right) \mathbb{P}\left\{\mathbf{Z}_1^{\rho_1+\epsilon} \le y\right\},
\end{aligned}$$

with $\lim_{\alpha\downarrow 1} K(\alpha) = 1$.

Thus, letting $y \to \infty$, then $\epsilon \downarrow 0$,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_1 > x\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} \ge \frac{\phi_2-\rho_2}{\phi_2} \frac{\rho_2}{1-\rho_1-\rho_2}.$$

(Upper bound) Using Theorems 2.3, 2.6 and Lemma 4.2,

$$\mathbb{P}\left\{\mathbf{V}_1^{\rho_1+\epsilon} > \delta x\right\} = o(\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}) \quad \text{as} \quad x \to \infty. \tag{22}$$

Using (22), Lemmas 3.2, 4.1, and the fact that $P_2^r(\cdot) \in \mathcal{IR}$,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_1 > x\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} &\le \limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}^\epsilon > (1-\delta)x\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} + \limsup_{x\to\infty} \frac{\mathbb{P}\left\{\mathbf{V}_1^{\rho_1+\epsilon} > \delta x\right\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} \\
&= \frac{\phi_2-\rho_2}{\phi_2} \frac{\rho_2}{1-\rho_1-\rho_2+\epsilon} \limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{P}_2^r > (1-\delta)x/(\rho_1+\epsilon-\phi_1)\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1-\phi_1)\}} \\
&= \frac{\phi_2-\rho_2}{\phi_2} \frac{\rho_2}{1-\rho_1-\rho_2+\epsilon} K\left(\frac{(1-\delta)(\rho_1-\phi_1)}{\rho_1+\epsilon-\phi_1}\right),
\end{aligned}$$

with $\lim_{\alpha \uparrow 1} K(\alpha) = 1$.
Thus, letting $\delta, \epsilon \downarrow 0$,

$$\limsup_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}_1 > x\}}{\mathbb{P}\{\mathbf{P}_2^r > x/(\rho_1 - \phi_1)\}} \leq \frac{\phi_2 - \rho_2}{\phi_2} \frac{\rho_2}{1 - \rho_1 - \rho_2}.$$

□

5 Conclusion

We analyzed the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, play a major role in achieving differentiated quality-of-service in integrated-services networks. We proved that, in certain scenarios, a flow may be severely influenced by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. This phenomenon contrasts with previous results which show that, under certain conditions, an individual flow with long-tailed traffic characteristics is effectively served at a constant rate. In particular, the flow is then largely insensitive to extreme activity of flows with 'heavier'-tailed traffic characteristics. The sharp dichotomy in qualitative behavior highlights the critical role of the weight parameters in isolating individual flows.

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A Definitions

Definition A.1 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *long-tailed* ($F(\cdot)\in\mathcal{L}$) if

$$\lim_{x\to\infty}\frac{1-F(x-y)}{1-F(x)}=1,\quad \text{for all real } y.$$

Definition A.2 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *subexponential* ($F(\cdot)\in\mathcal{S}$) if

$$\lim_{x\to\infty}\frac{1-F^{2*}(x)}{1-F(x)}=2,$$

where $F^{2*}(\cdot)$ is the 2-fold convolution of $F(\cdot)$ with itself, i.e., $F^{2*}(x)=\int_0^\infty F(x-y)F(\mathrm{d}y)$.

The class of subexponential distributions was introduced by Chistyakov [11]. The definition is motivated by the simplification of the asymptotic analysis of the convolution tails. A well-known subclass of $\mathcal{S}$ is the class $\mathcal{R}$ of *regularly-varying* distributions (which contains the Pareto distribution):

Definition A.3 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *regularly varying of index $-\nu$* ($F(\cdot)\in\mathcal{R}_{-\nu}$) if

$$F(x)=1-\frac{l(x)}{x^\nu},\quad \nu\geq 0,$$

where $l:\mathbb{R}_+\to\mathbb{R}_+$ is a function of slow variation, i.e., $\lim_{x\to\infty} l(\eta x)/l(x)=1$, $\eta>1$.

The class of regularly-varying functions was introduced by Karamata [19]; a key reference is Bingham *et al.* [4]. It is easily seen that $\mathcal{R}\subset\mathcal{S}\subset\mathcal{L}$. Examples of subexponential distributions which do not belong to $\mathcal{R}$ include the Weibull, lognormal, and Benktander distributions (see Klüppelberg [20]). A useful extension of $\mathcal{R}$ is the class $\mathcal{IR}$ of *intermediately regularly-varying* distributions:

Definition A.4 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *intermediately regularly varying* ($F(\cdot)\in\mathcal{IR}$) if

$$\lim_{\eta\uparrow 1}\limsup_{x\to\infty}\frac{1-F(\eta x)}{1-F(x)}=1.$$

A further extension is the class $\mathcal{DR}$ of *dominatedly varying* distributions (see Cline [12]; $\mathcal{R}\subset\mathcal{IR}\subset(\mathcal{DR}\cap\mathcal{L})\subset\mathcal{S}$):

Definition A.5 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *dominatedly varying* ($F(\cdot)\in\mathcal{DR}$) if

$$\limsup_{x\to\infty}\frac{1-F(\eta x)}{1-F(x)}<\infty,\quad \text{for some real } \eta\in(0,1).$$