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ABSTRACT

The classical random walk of which the one-step displacement variable $\mathbf{u}$ has a first finite negative moment is considered. The R.W. possesses an unique stationary distribution; $\mathbf{x}$ is a random variable with this distribution. It is assumed that the righthand and/or the lefthand tail of the distribution of $\mathbf{u}$ are heavy-tailed. For the type of heavy-tailed distribution considered in this study a contraction factor $\Delta(a)$ exists with $\Delta(a)\downarrow 0$ for $a\uparrow 1$, and $a\uparrow 1$ is equivalent with $\mathrm{E}\{\mathbf{u}\}\uparrow 0$. It is shown that $\Delta(a)\mathbf{x}$ converges in distribution for $a\uparrow 1$. It is the analysis of the tail of this limiting distribution of $\Delta(a)\mathbf{x}$ which is the main purpose of the present study in particular when $\mathbf{u}$ is a mix of stochastic variables $\mathbf{u}_i, i = 1, \dots, N$, each $\mathbf{u}_i$ having its own heavy tail characteristics for its right- and lefthand tails. For an important case it is shown that for the tail of the distribution of $\Delta(a)\mathbf{x}$ an asymptotic expression in the variables $\Delta(a)$ and t for $\Delta(a)\downarrow 0$ and $t\to\infty$ can be derived. For this asymptotic relation the dominating term is completely determined by the heavier tail of the $2N$ tails of the $\mathbf{u}_i$; the other terms of the asymptotic relation show the influence of less heavier tails and, depending on t, the terms may have a contribution which is not always negligible.

The study starts with the derivation of a functional equation for the L.S.-transform of the distribution of $\mathbf{x}$ and that of the excess distribution of the stationary idle time distribution. For several important cases this functional equation could be solved and thus has led to the above mentioned asymptotic result. The derivation of it required quite some preparation, because it needed an effective description of the heavy-tailed jump vector $\mathbf{u}$. It was obtained by prescribing the heavy-tailed distributions of $[\mathbf{u}]^+ = \max(0,\mathbf{u})$ and $[\mathbf{u}]^- = \min(0,\mathbf{u})$.

The random walk may serve as a model for the actual waiting process of a GI/G/1 queueing model; in that case the distribution of $\mathbf{u}$ is that of the difference of the service time and the interarrival time. The analysis of the present study then describes the heavy-traffic theory for the case with heavy-tailed service-and/or interarrival time distribution.

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1. Introduction

Let $\mathbf{u}_n, n = 0,1,2,\dots$, be a sequence of i.i.d. stochastic variables with

$$\mathbf{u}_n \in (-\infty,\infty) \text{ and } -\infty < \mathrm{E}\{\mathbf{u}_\mathrm{n}\} < 0. \tag{1.1}$$

The random walk $\{\mathbf{x}_n, n = 0,1,\dots\}$, generated by the $\mathbf{u}_n$-sequence, is recursively defined by:

$$\begin{aligned} &\mathbf{x}_{n+1} = [\mathbf{x}_n + \mathbf{u}_n]^+ \text{ for n} = 0,1,2,\dots, \\ &\mathbf{x}_0 := x_0 \ \geq 0; \end{aligned} \tag{1.2}$$

here x_0 is the initial state of the random walk.

Next to the $\mathbf{x}_n$-sequence we introduce the $\mathbf{y}_n$-sequence defined by

$$\mathbf{y}_n := -[\mathbf{x}_n + \mathbf{u}_n]^-, n = 0,1,2,\dots\ . \tag{1.3}$$

Here we use the notation: for real z

$$[z]^+ := \max(0, z); \quad [z]^- := \min(0, z). \tag{1.4}$$

The random walk $\{\mathbf{x}_n, n = 0, 1, 2, \dots\}$ is a wellknown stochastic process. Since the first moment of $\mathbf{u}_n$ is negative the $\mathbf{x}_n$-sequence possesses a unique stationary distribution and for $n \to \infty$ the distribution of $\mathbf{x}_n$ converges weakly to this distribution, i.e. $\mathbf{x}_n$ converges in distribution for $n \to \infty$. This implies that also $\mathbf{y}_n$ converges in distribution.

The present paper concerns the analysis of these limiting distributions whenever the jump vector $\mathbf{u}$ has regularly varying tails at $+\infty$ and/or $-\infty$, with an index between -1 and -2. Let $\mathbf{x}$ and $\mathbf{y}$ be stochastic variables with distributions the limiting distribution of $\mathbf{x}_n$ and $\mathbf{y}_n$, respectively. It will be shown that a function Δ of $\mathrm{E}\{\mathbf{u}_n\}$ exists which goes to zero for $\mathrm{E}\{\mathbf{u}_n\}\uparrow 0$ and such that $\Delta\mathbf{x}$, and similarly $\Delta\mathbf{y}$, converges for $\Delta \downarrow 0$ in distribution. These limiting distributions are studied in the present paper.

The random walk $\{\mathbf{x}_n, n = 0, 1, \dots\}$ introduced above is frequently used in the modelling of stochastic processes occurring in Operations Research. In particular in Queueing Theory it is used in the analysis of the actual waiting time process. Then $\mathbf{u}_n$ is the difference of two nonnegative stochastic variables $\boldsymbol{\tau}_n$ and $\boldsymbol{\sigma}_{n+1}$, the service time of the nth arriving customer and the interarrival time between the nth and $(n+1)$th arrival. In the context of Queueing Theory the study of the limiting distributions of $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ for $\Delta \downarrow 0$ is known as heavy-traffic analysis. In [13] it has been shown that these heavy-traffic results lead to excellent approximations even for traffic loads which cannot be considered to be heavy. It seems that the regularly varying right tail of the distribution of the jump vector causes this result for the case studied in [13], which is, however, not the most general case.

The present study contains new results, next to results which are taken from previous studies of the present author and his colleague Prof. Boxma, see the list of references.

We proceed with an overview of the following sections.

In Section 2 a functional equation is derived for $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{z}}\}$ where $\mathbf{z}$ is a stochastic variable with distribution the excess distribution of that of $\mathbf{y}$. This functional equation is the starting point for all the further analysis.

In Section 3 we outline the structure of the heavy-tailed distributions with support $(0, \infty)$, as they will be used in the present context. In Section 4 the heavy-tailed jump vector $\mathbf{u}$ is introduced. This is obtained by specifying the heavy-tailed distributions of $\mathbf{v} = [\mathbf{u}]^+ \equiv \max(0, \mathbf{u})$ and $\mathbf{w} = -\min(0, \mathbf{u}) = -[\mathbf{u}]^-$, and by noting that the L.S.-transform of the distribution of $\mathbf{u}$ is a linear combination of the L.S.-transforms of the excess distributions of those of $\mathbf{v}$ and $\mathbf{w}$.

Section 5 concerns the representation of the distribution of the jump vector $\mathbf{u}$ as a mix of N stochastic variables $\mathbf{u}_i$. The introduction of such a mix serves several purposes, theoretical as well as practical ones, see e.g. Sections 7 and 15. The distributions of the components $\mathbf{u}_i$ of $\mathbf{u}$ are assumed to depend on i and are generally heavy-tailed with different indices ν_{ij}, cf. (5.11).

In Section 6 we describe the solution of the functional equation and present expressions for $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}}\}$ and $\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{z}}\}$ in terms of the L.S.-transform of the distribution of $\mathbf{u}$.

In Section 7 the contraction factor $\Delta(a)$ is introduced as a root of the contraction equation with $\Delta(a)\downarrow 0$ for $a \uparrow 1$. Here a is a characteristic of the mix $\mathbf{u}_i, i = 1, \dots, N$. Actually

$$a = \sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{v}_i\} / \sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{w}_i\}, \tag{1.5}$$

with

$$p_i > 0, \quad \sum_{i=1}^{N} p_i = 1, \quad \mathbf{v}_i = \mathbf{u}_i^+, \quad \mathbf{w}_i = -\mathbf{u}_i^-.$$

It is further shown in Section 7 that the functional

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-r\Delta \mathbf{u}}\}}{r\Delta \mathrm{E}\{\mathbf{u}\}}, \mathrm{Re}\ r = 0, \Delta = \Delta(a), \tag{1.6}$$

converges for $a \uparrow 1$, see (7.17). Note that this limit contains only the first moments $\mathrm{E}\{\mathbf{v}_i\}, \mathrm{E}\{\mathbf{w}_i\}, p_i$ and the characteristics ν_{ij} and d_{ij} of the tails of $\mathbf{v}_i$ and $\mathbf{w}_i$. Using the result of Section 7 it is shown in Section 8 that the stochastic variables $\Delta(a)\mathbf{x}$ and $\Delta(a)\mathbf{z}$ both converge in distribution for $a \uparrow 1$.

In Section 9, 10 and 11 expressions for the L.S.-transforms of the limiting distributions of $\Delta(a)\mathbf{x}$ and $\Delta(a)\mathbf{z}$ for $a \uparrow 1$ are presented. These three sections concern the cases that the tail of $\mathbf{v}$ is heavier than that of $\mathbf{w}$, is lighter or the tails of $\mathbf{v}$ and $\mathbf{w}$ are balanced, i.e. they have the same ν-index. The results of these sections have been obtained in earlier studies; this also holds for the asymptotics for $t \to \infty$ of the tails of the limiting distributions of $\Delta(a)\mathbf{x}$ and $\Delta(a)\mathbf{z}$ which are mentioned in Section 11. Characteristic for these limiting results is that they are expressed in terms of the $p_i, \mathrm{E}\{\mathbf{v}_i\}$ and $\mathrm{E}\{\mathbf{w}_i\}$ and by characteristics of the heavier tail among those of $\mathbf{v}_i$ and $\mathbf{w}_i$. Of the lighter tails only the $\mathrm{E}\{\mathbf{v}_i\}$ or the $\mathrm{E}\{\mathbf{w}_i\}$ occur cf. Section 13. This is a phenomenon which is not unknown in teletraffic theory. Roughly it may be formulated as follows. When several traffic sources share a common facility then the most demanding source causes a more than "proportional" hindrance (congestion) to the less demanding ones.

Obviously the limiting distributions for $a \uparrow 1$ or $\Delta(a) \downarrow 0$ may be considered as the first term of an asymptotic series of the distribution of $\Delta(a)\mathbf{x}$ for $\Delta(a) \downarrow 0$. To obtain further terms of the asymptotic expansion we derive in Section 14 an asymptotic expansion with $\Delta(a) \downarrow 0$ for the expression in (1.6), see form. (14.5). This asymptotic expansion is used in Section 15 and 17 to obtain a second term of the asymptotic series in $\Delta(a)$ for $\Delta(a) \downarrow 0$ (expressions for higher order terms can be also obtained). Section 15 concerns the case that only the $\mathbf{v}_i$, $i = 1, \ldots, N-1$ are heavy-tailed distributions, i.e. $d_{2i} \equiv 0$. For both these cases the functional equation for the L.S. transform of the distribution of $\Delta(a)\mathbf{x}$ can be solved. From the knowledge of the L.S.-transform of $\Delta(a)\mathbf{x}$ we derive the tail asymptotics of the distribution of $\Delta(a)\mathbf{x}$. The result is an asymptotic relation in two variables i.e. $\Delta(a) \downarrow 0$ and $t \to \infty$. Formula (15.11) states here the result. The effect of the lighter tails among the $\mathbf{v}_i$ can be judged from this formula.

Analogous results have been obtained for the case that all $\mathbf{w}_i, i = 1, \ldots, N-1$, are heavy-tailed and the $\mathbf{v}_i$ are not heavy-tailed, i.e. all $d_{2i} = 0$.

It remains to discuss Section 16. This section relates the present analysis with the basic model of Queueing Theory. Here the jump vector $\mathbf{u}$ is the difference of two nonnegative stochastic variables $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. Section 16 describes the relations between the L.S.-transforms of $\mathbf{v}$ and $\mathbf{w}$ on the one hand and those of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ on the other hand. They are needed when applying the results of the preceding sections in Queueing Theory.

2. Derivation of the Boundary Value problem

In this section we derive a functional relation for the functions

$$\begin{aligned} X(r,\rho) &:= \sum_{n=0}^{\infty} r^n \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{x}_n}\} \text{ with } |r| < 1, \mathrm{Re}\ \rho \geq 0, \\ Y(r,\rho) &:= \sum_{n=0}^{\infty} r^n \mathrm{E}\{\mathrm{e}^{\rho \mathbf{y}_n}\} \text{ with } |r| < 1, \mathrm{Re}\ \rho \leq 0; \end{aligned} \tag{2.1}$$

note that $\mathbf{x}_n$ and $\mathbf{y}_n$ are both nonnegative.

We start from the following identity which is easily seen to be true. For real x we have, cf. (1.4),

$$1 + \mathrm{e}^{-\rho \mathbf{x}} = \mathrm{e}^{-\rho [\mathbf{x}]^+} + \mathrm{e}^{-\rho [\mathbf{x}]^-}. \tag{2.2}$$

Hence from (2.2) and (1.2), (1.3),

$$\begin{aligned} 1+\mathrm{e}^{-\rho[\mathbf{x}_n+\mathbf{u}_n]} &= \mathrm{e}^{-\rho[\mathbf{x}_n+\mathbf{u}_n]^+}+\mathrm{e}^{-\rho[\mathbf{x}_n+\mathbf{u}_n]^-} \\ &= \mathrm{e}^{-\rho\mathbf{x}_{n+1}}+\mathrm{e}^{\rho\mathbf{y}_n}. \end{aligned} \tag{2.3}$$

By taking expectations in (2.3) we have for $n=0,1,2,\dots$, and Re $\rho=0$,

$$1+\mathrm{E}\{\mathrm{e}^{-\rho(\mathbf{x}_n+\mathbf{u}_n)}\}=\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}_{n+1}}\}+\mathrm{E}\{\mathrm{e}^{\rho\mathbf{y}_n}\}. \tag{2.4}$$

Let $\mathbf{u}$ be a stochastic variable with the same distribution as $\mathbf{u}_n$. The $\mathbf{u}_n, n=0,1,2,\dots$, are independent, and since $\mathbf{x}_n$ depends only on $\mathbf{u}_0,\dots\mathbf{u}_{n-1}$, cf. (1.2), it follows that $\mathbf{x}_n$ and $\mathbf{u}_n$ are independent. Hence (2.4) may be rewritten as: for Re $\rho=0$ and $n=0,1,2,\dots$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}_{n+1}}\}=\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}_n}\}+1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{y}_n}\}. \tag{2.5}$$

Multiplying (2.5) by r^n and then summing over $n=0,1,2,\dots$, with $|r|<1$, leads to

$$[1-r\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}]X(r,\rho)=\mathrm{e}^{-\rho x_0}+\frac{r}{1-r}-rY(r,\rho). \tag{2.6}$$

From the results above it follows that for fixed $|r|<1$:

$$\begin{aligned} &\text{i}\quad X(r,\rho)\ \text{ is regular for Re } \rho>0,\ \text{ continuous and bounded for Re } \rho\le 0;\\ &\text{ii}\quad Y(r,\rho)\ \text{ is regular for Re } \rho\le 0,\ \text{ continuous and bounded for Re } \rho\le 0;\\ &\text{iii}\quad \text{for Re } \rho=0,\\ &\quad [1-r\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}](1-r)X(r,\rho)=(1-r)\mathrm{e}^{-\rho x_0}+r[1-(1-r)Y(r,\rho)]. \end{aligned} \tag{2.7}$$

It is evident that (2.7) formulates a Riemann Boundary Value (RBV) problem for the functions $X(r,\rho)$ and $Y(r,\rho)$ with $|r|<1$ and r fixed. The line of discontinuity of this RBV problem is here the imaginary axis and (2.7)iii is the boundary condition (for details, see [1]). Since this condition holds on the imaginary axis the Boundary Value problem is often also called a Wiener-Hopf problem, cf. [2].

From Fluctuation Theory, cf. [3, p. 152], it is known that the condition $-\infty<\mathrm{E}\{\mathbf{u}\}<0$, cf. (1.1), implies that $\mathbf{x}_n$, and also $\mathbf{y}_n$, both converge for $n\to\infty$ in distribution. Let $\mathbf{x}$ and $\mathbf{y}$ be stochastic variables with distribution the limiting distribution of $\mathbf{x}_n$ and of $\mathbf{y}_n$, respectively, for $n\to\infty$.

We may now write: for Re $\rho\ge 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}}\}=\lim_{n\to\infty}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}_n}\}=\lim_{r\uparrow 1}(1-r)X(r,\rho), \tag{2.8}$$

and for Re $\rho\le 0$,

$$\mathrm{E}\{\mathrm{e}^{\rho\mathbf{y}}\}=\lim_{n\to\infty}\mathrm{E}\{\mathrm{e}^{\rho\mathbf{y}_n}\}=\lim_{r\uparrow 1}(1-r)Yr,\rho). \tag{2.9}$$

In (2.8) and (2.9) the first equality signs follow from Feller's convergence theorem for L.S.-transforms of one sided distributions, the second equality signs result by using a well-known Abel theorem for generating functions, cf. [6].

With

$$\begin{aligned} X(\rho) &:= \lim_{r\uparrow 1}(1-r)X(r,\rho), \quad \text{Re } \rho\ge 0,\\ Y(\rho) &:= \lim_{r\uparrow 1}(1-r)Y(r,\rho), \quad \text{Re } \rho\le 0, \end{aligned} \tag{2.10}$$

we obtain from (2.7)iii: for Re $\rho = 0$,

$$[1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}]X(\rho) = 1 - Y(\rho),$$

or

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}}\} = \frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{y}}\}}{\rho}. \tag{2.11}$$

Because E$\{\mathbf{u}\}$ is finite, cf. (1.1), and Fluctuation Theory implies that $\mathbf{x}$ is finite with probability one, and hence this also holds for $\mathbf{y}$, it is seen by letting $\rho \to 0$ in (2.11) that, cf. (1.1),

$$-\infty < -\mathrm{E}\{\mathbf{y}\} = \mathrm{E}\{\mathbf{u}\} < 0. \tag{2.12}$$

Hence (2.11) may be rewritten as: for Re $\rho = 0$,

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}}X(\rho) = Z(-\rho), \tag{2.13}$$

with

$$Z(\rho) := \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{y}}\}}{\rho\mathrm{E}\{\mathbf{y}\}}, \mathrm{Re}\ \rho \geq 0. \tag{2.14}$$

Because $\mathbf{y}$ is a nonnegative stochastic variable, it is seen that $Z(\rho)$ is the L.S.-transform of a distribution with support contained in $(0,\infty)$, actually it is the excess distribution of $\mathbf{y}$.

From the definition of $X(\rho)$ and of $Z(\rho)$ and from (2.13) it is seen that the determination of $X(\rho)$ and $Z(\rho)$ leads to the following RBV-problem with Re $\rho = 0$ as the line of discontinuity.

i $X(\rho)$ is regular for Re $\rho > 0$, continuous and bounded for Re $\rho \geq 0, X(0) = 1$;

ii $Z(-\rho)$ is regular for Re $\rho < 0$, continuous and bounded for Re $\rho \leq 0, Z(0) = 1$;

iii for Re $\rho = 0$, (2.15)

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}}X(\rho) = Z(-\rho).$$

This Boundary Value problem has also been discussed in a slightly different setting in [7].

Remark 2.1 We define the nonnegative stochastic variable $\mathbf{z}$ as a variable with distribution the excess distribution of $\mathbf{y}$. Hence

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{z}}\} = \frac{1-\mathrm{E}\mathrm{e}^{-\rho\mathbf{y}}\}}{\rho\mathrm{E}\{\mathbf{y}\}}, \mathrm{Re}\ \rho \geq 0. \qquad \square \tag{2.16}$$

3. Heavy-tailed distribution on $[0,\infty)$

In this section we introduce a class of heavy-tailed distributions on $[0,\infty)$; this class is a subclass of distributions of regular variation at infinity, cf. [5].

Let $B(x)$ be a probability distribution with support $[0,\infty)$. Its L.S.-transform will be indicated by $\beta(\rho)$, Re $\rho \geq 0$, and its first moment β is assumed to be nonzero and finite; γ shall stand for the time unit.

The class of heavy-tailed distributions which we shall consider in the present study is characterized by their L.S.-transforms which have the following structure. For the L.S.-transform $\beta(\rho)$ holds: for Re $\rho \geq 0$,

$$1 - \frac{1-\beta(\rho)}{\rho\beta} = g(\gamma\rho) + (\gamma\rho)^{\nu-1}cL(\gamma\rho), \tag{3.1}$$

with

i. $c > 0$ is a constant,

ii. for the index $-\nu$ holds $1 < \nu \leq 2$,

iii. $g(0) = 0$ and there exist a $\delta > 0$ such that $g(\gamma\rho)$ is regular for Re $\rho > -\delta$,

iv. $L(\gamma\rho)$ is a regular function of ρ for Re $\rho > 0$, and continuous for Re $\rho \geq 0$, except possibly at $\rho = 0$, further $L(\gamma\rho) \to b > 0$ for $|\rho| \to 0$, Re $\rho \geq 0$, with $b = \infty$ if $\nu = 2$, and
$$\lim_{x\downarrow 0} \frac{L(\gamma\rho x)}{L(x)} = 1 \text{ for Re } \rho \geq 0; \rho \neq 0. \tag{3.2}$$

REMARK 3.1 Since ρ can be complex we have to define $\rho^{\nu-1}$ in (3.1). It is so defined that it is positive for positive ρ. □

Examples of heavy-tailed distributions with structure characterized by (3.1) and (3.2) have been discussed in Section 3 of [4]. We discuss here an example.

Let $B(\cdot)$ be a distribution for which holds:

$$1 - B(t) = q(\frac{\beta}{t})^{\nu} + G_2(t) \text{ for } t > T > 0, \tag{3.3}$$

and with $q > 0, 1 < \nu < 2$ and $G_2(t)$ such that: for a $\delta > 0$,

$$\int_T^{\infty} \mathrm{e}^{-\rho t} G_2(t)\mathrm{d}t$$

exists for Re $\rho > -\delta$.

Calculation of the L.S.-transform of $B(t)$ yields: for Re $\rho \geq 0$ and $T = \beta$,

$$1 - \frac{1-\beta(\rho)}{\rho\beta} = \chi_2(\beta\rho) + \frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}(\beta\rho)^{\nu-1}, \tag{3.4}$$

where for Re $\rho > -\delta$,

$$\chi_2(\beta\rho) := \int_0^{\beta}(1-\mathrm{e}^{-\rho t})(1-B(t))\frac{\mathrm{d}t}{\beta} + \int_{\beta}^{\infty} c(\frac{\beta}{t})^{\nu}\frac{\mathrm{d}t}{\beta} + \int_0^{\infty}(1-\mathrm{e}^{-\rho t})G_2(t)\frac{\mathrm{d}t}{\beta} + \frac{c}{\nu-1}\mathrm{e}^{-\beta\rho} + \frac{c\rho}{1-\nu}\int_0^{\beta}\mathrm{e}^{-\rho t}(\frac{t}{\beta})^{1-\nu}\mathrm{d}t. \tag{3.5}$$

Note that we have chosen $T = \beta$ in (3.4), this is not essential but simplifies somewhat the calculations; further $\Gamma(\cdot)$ denotes here the gamma function. From (3.4) and (3.5) it is readily seen that the L.S.-transform of $B(t)$ has the structure as characterized by (3.1) and (3.2).

From (3.4) it is seen that for $\rho > 0$ the following asymptotic results hold:

$$1 - \frac{1-\beta(\rho)}{\rho\beta} \sim \frac{c\pi}{\Gamma(\nu)\sin(\nu-1)\pi}(\beta\rho)^{\nu-1} \text{ for } \rho\downarrow 0. \tag{3.6}$$

From this result the asymptotic relation for $1 - B(t)$ with $t \to \infty$ can be readily obtained by applying a theorem of [5] and from (3.1) and (3.2) with $1 < \nu < 2$ it is seen that the heavy-tailed distributions which are characterized by (3.1) and (3.2) are distributions with a regularly varying tail at infinity and index $-\nu$ with $1 < \nu < 2$.

4. The heavy-tailed jump vector

The jump vector $\mathbf{u}$ occurring in the Boundary Value Problem (2.15) has in general a distribution with support $(-\infty, \infty)$, i.e. it has a lefthand and a righthand tail of which both or one or none may be heavy tails. In this section we given an approach to construct jump vectors with heavy tails.

It is assumed that

$$-\infty < \mathrm{E}\{\mathbf{u}\} < 0 \text{ and } \mathrm{E}\{|\mathbf{u}|\} < \infty. \tag{4.1}$$

Put

$$\mathbf{v} := [\mathbf{u}]^{+} \text{ and } \mathbf{w} := -[\mathbf{u}]^{-}, \tag{4.2}$$

and

$$a := \frac{\mathrm{E}\{\mathbf{v}\}}{\mathrm{E}\{\mathbf{w}\}} < 1.$$

Note that

$$\mathrm{E}\{\mathbf{u}\} = \mathrm{E}\{\mathbf{v}\} - \mathrm{E}\{\mathbf{w}\}. \tag{4.3}$$

We prove the following lemma concerning the L.S.-transform of the distributions of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Lemma 4.1 *For* Re $\rho = 0$,

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} = \frac{a}{a-1}\frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} - \frac{1}{a-1}\frac{1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{w}}\}}{-\rho\mathrm{E}\{\mathbf{w}\}}, \tag{4.4}$$

or equivalently

$$1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} = \frac{a}{a-1}\{1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}}\} - \frac{1}{a-1}[1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{w}}\}}{-\rho\mathrm{E}\{\mathbf{w}\}}]. \tag{4.5}$$

Proof. Obviously (4.5) and (4.4) are equivalent. From the identity (2.2) we have by using (4.2),

$$1 + \mathrm{e}^{-\rho\mathbf{u}} = \mathrm{e}^{-\rho\mathbf{v}} + \mathrm{e}^{\rho\mathbf{w}},$$

with probability one. So by taking expectations: for Re $\rho = 0$,

$$1 + \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\} = \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\} + \mathrm{E}\{\mathrm{e}^{\rho\mathbf{w}}\};$$

and so

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho} = \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho} - \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho\mathbf{w}}\}}{-\rho},$$

hence by using (4.2) and (4.3) the statement (4.4) follows. □

The relation (4.4) expresses the L.S.-transform of the distribution of $\mathbf{u}$ in terms of the L.S.-transforms of the excess distributions of the nonnegative stochastic variables $\mathbf{v}$ and $\mathbf{w}$. Evidently the tail of the distribution of $\mathbf{v}$ determines the tail behaviour of the left tail of the distribution of $\mathbf{u}$ so by specifying the tails of the distributions of $\mathbf{v}$ and $\mathbf{w}$ the right and left tail of the distribution of $\mathbf{u}$ are specified.

Let $\mathbf{v}_{\mathrm{e}}$ and $\mathbf{w}_{\mathrm{e}}$ be stochastic variables with distributions the excess distribution of $\mathbf{v}$ and of $\mathbf{w}$, respectively. It then follows from (4.4): for Re $\rho = 0$,

$$\begin{aligned}\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} &= \frac{1}{1-a}\frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{w}}\}}{-\rho\mathrm{E}\{\mathbf{w}\}} - \frac{a}{1-a}\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} \\ &= \frac{1}{1-a}\mathrm{E}\{\mathrm{e}^{-\rho(-\mathbf{w}_{\mathrm{e}})}\} - \frac{a}{1-a}\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}_{\mathrm{e}}}\} \\ &= \frac{1}{1-a}\int_{-\infty}^{+\infty}\mathrm{e}^{-\rho x}\mathrm{d}[\frac{1}{1-a}\,\mathrm{Pr}\,\{-\mathbf{w}_{\mathrm{e}} < x\} - \frac{a}{1-a}\,\mathrm{Pr}\,\{\mathbf{v}_{\mathrm{e}} < x\}] \\ &= \int_{-\infty}^{+\infty}\mathrm{e}^{-\rho x}\mathrm{d}U_{\mathrm{e}}(x),\end{aligned} \tag{4.6}$$

with for $-\infty < x < \infty$,

$$U_{\mathrm{e}}(x) := \frac{1}{1-a}\,\mathrm{Pr}\,\{-\mathbf{w}_{\mathrm{e}} < x\} - \frac{a}{1-a}\,\mathrm{Pr}\,\{\mathbf{v}_{\mathrm{e}} < x\}, \tag{4.7}$$

if we take $\mathbf{u}_{\mathrm{e}}(-\infty) := 0$. Because $\mathbf{v}_e$ and $\mathbf{w}_e$ are both positive with probability one we have from (4.7)

$$\begin{aligned}U_{\mathrm{e}}(x) &= \frac{1}{1-a}\,\mathrm{Pr}\,\{-\mathbf{w}_{\mathrm{e}} < x\} \qquad \text{for } x > 0, \\ &= \frac{1}{1-a} - \frac{a}{1-a}\,\mathrm{Pr}\,\{\mathbf{v}_{\mathrm{e}} < x\} \quad \text{for } x > 0.\end{aligned} \tag{4.8}$$

The heavy-tailed distributions of jump vectors which will be considered in the present study are characterized by taking: for Re $\rho \geq 0$,

$$\begin{aligned}1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} &= g_2(\gamma\rho) + c_2(\gamma\rho)^{\nu_2-1}L_2(\gamma\rho), \\ 1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho\mathrm{E}\{\mathbf{w}\}} &= g_1(\gamma\rho) + c_1(\gamma\rho)^{\nu_1-1}L_1(\gamma\rho),\end{aligned} \tag{4.9}$$

where $g_j(\cdot)$, c_j, ν_j and $L_j(\cdot), j = 1, 2$, have the same properties as the corresponding constants and functions in (3.2).

Hence from (4.5) and 4.6) it follows that the L.S.-transform of a heavy-tailed distributed jump vector $\mathbf{u}$ is represented as follows: for Re $\rho = 0$,

$$\begin{aligned}1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} &= \frac{a}{a-1}g_2(\gamma\rho) - \frac{1}{a-1}g_1(\gamma\bar{\rho}) \\ &+ \frac{ac_2}{a-1}(\gamma\rho)^{\nu_2-1}L_2(\gamma\rho) - \frac{c_1}{a-1}(\gamma\bar{\rho})^{\nu_1-1}L_1(\gamma\bar{\rho}).\end{aligned} \tag{4.10}$$

5. Mixing of heavy-tailed jump vectors

In this section we consider N heavy-tailed jump vectors $\mathbf{u}_i, i = 1 \ldots N$. With these vectors we construct the jump vector $\mathbf{u}$ defined by:

$$\mathbf{u} = \mathbf{u}_i \text{ with probability } p_i,\ i = 1, \ldots, N, \tag{5.1}$$

with

$$0 < p_i < 1\,,\ \sum_{i=1}^{N} p_i = 1 \text{ and } \mathrm{E}\{|\mathbf{u}_i|\} < \infty.$$

With

$$\mathbf{v}_i = [\mathbf{u}_i]^+ \text{ and } \mathbf{w}_i = -[\mathbf{u}_i]^-, \tag{5.2}$$

we have for Re $\rho = 0$,

$$1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}_i}\}}{\rho \mathrm{E}\{\mathbf{u}_i\}} = \frac{\mathrm{E}\{\mathbf{v}_i\}}{\mathrm{E}\{\mathbf{u}_i\}}[1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{v}_i}\}}{\rho \mathrm{E}\{\mathbf{v}_i\}}] - \frac{\mathrm{E}\{\mathbf{w}_i\}}{\mathrm{E}\{\mathbf{u}_i\}}[1 - \frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{w}_i}\}}{-\rho \mathrm{E}\{\mathbf{w}_i\}}] \tag{5.3}$$

and

$$\begin{aligned} 1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{v}_i}\}}{\rho \mathrm{E}\{\mathbf{v}_i\}} &= g_{2i}(\gamma\rho) + c_{2i}(\gamma\rho)^{\nu_{2i}-1} L_{2i}(\gamma\rho), \\ 1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{w}_i}\}}{\rho \mathrm{E}\{\mathbf{w}_i\}} &= g_{1i}(\gamma\rho) + c_{1i}(\gamma\rho)^{\nu_{1i}-1} L_{1i}(\gamma\rho), \end{aligned} \tag{5.4}$$

where the $g_{..}(\cdot), c_{..}, \nu_{..}$ and the $L_{..}(\gamma\rho)$, have the corresponding properties as the similar functions and constants in (3.2).

From (5.1) we obtain: for Re $\rho = 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\} = \sum_{i=1}^{N} p_i \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}_i}\}, \tag{5.5}$$

from which it readily follows that: for Re $\rho = 0$,

$$\begin{aligned} &\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\}}{\rho \mathrm{E}\{\mathbf{u}\}} = \sum_{i=1}^{N} p_i \frac{E\{\mathbf{u}_i\}}{\mathrm{E}\{\mathbf{u}\}} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}_i}\}}{\rho \mathrm{E}\{\mathbf{u}_i\}} \\ &= \sum_{i=1}^{N} [\frac{p_i \mathrm{E}\{\mathbf{v}_i\}}{\mathrm{E}\{\mathbf{u}\}} \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{v}_i}\}}{\rho \mathrm{E}\{\mathbf{v}_i\}} - \frac{p_i \mathrm{E}\{\mathbf{w}_i\}}{\mathrm{E}\{\mathbf{u}\}} \frac{1-\mathrm{E}\{\mathrm{e}^{\rho \mathbf{w}_i}\}}{-\rho \mathrm{E}\{\mathbf{w}_i\}}]. \end{aligned} \tag{5.6}$$

Put

$$a := \frac{\sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{v}_i\}}{\sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{w}_i\}}, \tag{5.7}$$

and for $i = 1, \dots, N$,

$$q_{2i}(a) := \frac{p_i \mathrm{E}\{\mathbf{v}_i\}}{\sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{v}_i\}} \text{ and } q_{1i}(a) := \frac{p_i \mathrm{E}\{\mathbf{w}_i\}}{\sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{w}_i\}}. \tag{5.8}$$

Because

$$\mathrm{E}\{\mathbf{u}\} = \sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{v}_i\} - \sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{w}_i\}, \tag{5.9}$$

it is seen from (5.5) that: for Re $\rho = 0$,

$$\begin{aligned} 1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\}}{\rho \mathrm{E}\{\mathbf{u}\}} &= \frac{a}{a-1} \sum_{i=1}^{N} q_{2i}(a)[1 - \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho \mathbf{v}_i}\}}{\rho \mathrm{E}\{\mathbf{v}_i\}}] \\ &- \frac{1}{a-1} \sum_{i=1}^{N} q_{1i}(a)[1 - \frac{\mathrm{E}\{\mathrm{e}^{\rho \mathbf{w}_i}\}}{-\rho \mathrm{E}\{\mathbf{w}_i\}}]. \end{aligned} \tag{5.10}$$

Inserting the expressions (5.4) into (5.10) leads to: for Re $\rho = 0$,

$$\begin{aligned} 1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\}}{\rho \mathrm{E}\{\mathbf{u}\}} &= \frac{1}{a-1}\sum_{i=1}^{N}\{a q_{2i}(a) g_{2i}(\gamma\rho) - q_{1i}(a) g_{1i}(\gamma\bar{\rho})\} \\ &+ \frac{1}{a-1}\sum_{i=1}^{N}\{a q_{2i}(a) c_{2i}(\gamma\rho)^{\nu_{2i}-1} L_{2i}(\gamma\rho) - q_{1i}(a) c_{1i}(\gamma\bar{\rho})^{\nu_{1i}-1} L_{1i}(\gamma\bar{\rho})\}. \end{aligned} \tag{5.11}$$

The relation (5.11) is the one which will be used in modelling a teletraffic problem, see Section 17. In the analysis above it has been assumed that $\mathrm{E}\{\mathbf{v}_i\}$ and $\mathrm{E}\{\mathbf{w}_i\}, i = 1, 2, \ldots, N$ are all finite. It is here not assumed that $\mathrm{E}\{\mathbf{v}_i\} < \mathrm{E}\{\mathbf{w}_i\}$, but when using (5.10) and (5.11) it will usually be assumed that

$$a < 1, \tag{5.12}$$

or equivalently

$$\sum_{i=1}^{N} p_i [\mathrm{E}\{\mathbf{v}_i\} - \mathrm{E}\{\mathbf{w}_i\}] < 1. \tag{5.13}$$

6. On the stationary distribution

In this section we solve the Boundary Value Problem (2.15) for the case that the jump vector $\mathbf{u}$ has the following representation, cf. Section 5.

For Re $\rho = 0$,

$$\begin{aligned} 1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{\rho \mathbf{u}}\}}{\rho \mathrm{E}\{\mathbf{u}\}} &= \frac{1}{a-1}\sum_{i=1}^{N}\{a q_{2i}(a)(\gamma\rho) g_{2i}(\gamma\rho) - q_{1i}(a) g_{1i}(\gamma\bar{\rho})\} \\ &+ \frac{1}{a-1}\sum_{i=1}^{N}[a q_{2i}(a) c_{2i}(\gamma\rho)^{\nu_{2i}-1} - q_{1i}(a) c_{1i}(\gamma\bar{\rho})^{\nu_{2i}-1}], \end{aligned} \tag{6.1}$$

with

$$0 < a = \sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{v}_i\} / \sum_{i=1}^{N} p_i \mathrm{E}\{\mathbf{w}_i\} < 1. \tag{6.2}$$

The condition (6.2) implies that the $\mathbf{x}_n$-sequence, cf. Section 1, possesses a stationary distribution.

The functional relation of the present Boundary Value Problem reads, cf. (2.15)iii, for Re $\rho = 0$,

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}}\}}{\rho \mathrm{E}\{\mathbf{u}\}} X(\rho) = Z(-\rho). \tag{6.3}$$

This functional relation has been studied in [7]. From (6.1) it is seen that $\mathrm{E}\{\mathbf{u}\}$ exists and is finite for $1 < \mu < \min(\nu_{ij}, j = 1, 2; i = 1, \ldots, N)$. Using this and (6.1) it follows from the results obtained in [7] that the integral

$$H(\rho) := \frac{1}{2\pi \mathrm{i}} \int_{\xi = -\mathrm{i}\infty}^{\mathrm{i}\infty} \{\log \frac{1 - \mathrm{E}\{\mathrm{e}^{-\xi \mathbf{u}}\}}{\xi \mathrm{E}\{\mathbf{u}\}}\} \frac{\rho}{\xi(\xi - \rho)} \mathrm{d}\xi, \tag{6.4}$$

with $|\rho| < \infty$ is well defined as a Cauchy principal value integral at infinity, and also as a singular Cauchy integral at $\xi = \rho$ if Re $\rho = 0$, cf. [1]. Further the logarithm of the integral satisfies the Hölder condition on the imaginary axis. So that the Plemelj formulas show that: for Re $\rho = 0$,

$$\begin{aligned} H^-(\rho) &:= \lim_{\substack{t\to\rho \\ \text{Re } t>0}} H(t) = -\frac{1}{2}\log\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} + H(\rho), \\ H^+(\rho) &:= \lim_{\substack{t\to\rho \\ \text{Re } t<0}} H(t) = \frac{1}{2}\log\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} + H(\rho), \end{aligned} \tag{6.5}$$

and

$$H^+(\rho) - H^-(\rho) = \log\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}}.$$

By using these relations it is readily verified that

$$\begin{aligned} X(\rho) &= \mathrm{e}^{H(\rho)} && \text{for Re } \rho > 0, \\ &= \mathrm{e}^{H^-(\rho)} && \text{for Re } \rho = 0, \\ Z(-\rho) &= \mathrm{e}^{H(\rho)} && \text{for Re } \rho < 0, \\ &= \mathrm{e}^{H^+(\rho)} && \text{for Re } \rho = 0, \end{aligned} \tag{6.6}$$

satisfies all the conditions of the Boundary Value Problem (2.15). Because (6.2) implies that $\mathrm{E}\{\mathbf{u}\} < 0$ the limiting distribution of $\mathbf{x}_n$ for $n \to \infty$ is uniquely determined, hence

$$\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}}\} \text{ and } [1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{y}}\}]/[\rho\mathrm{E}\{\mathbf{y}\}],\ \text{Re } \rho \le 0,$$

are uniquely determined and should satisfy the conditions (2.15). Hence we have

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{x}}\} &= X(\rho),\ \text{Re } \rho \ge 0, \\ \frac{1-\mathrm{E}\{\mathrm{e}^{\rho\mathbf{y}}\}}{-\rho\mathrm{E}\{\mathbf{y}\}} &= Z(-\rho),\ \rho \le 0, \end{aligned} \tag{6.7}$$

with $X(\rho)$ and $Z(-\rho)$ given by (6.6).

With a view to future applications of the results above we rewrite the formulas as follows.

Replace in the formulas above ρ by $r\Delta$ with $\Delta > 0$. We then obtain:

$$\begin{aligned} H(r\Delta) &= \frac{1}{2\pi\mathrm{i}}\int_{\eta=-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\frac{1-\mathrm{E}\{\mathrm{e}^{-\eta\Delta\mathbf{u}}\}}{\eta\Delta\mathrm{E}\{\mathbf{u}\}}\frac{r}{(\eta-r)\eta}\mathrm{d}\eta, \\ H^\pm(r\Delta) &= \pm\log\frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}} + H(r\Delta); \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\} &= \mathrm{e}^{H(r\Delta)} && \text{for Re } r > 0, \\ &= \mathrm{e}^{H^-(r\Delta)} && \text{for Re } r = 0, \\ \frac{1-\mathrm{E}\{\mathrm{e}^{r\Delta\mathbf{y}}\}}{-r\Delta\mathrm{E}\{\mathbf{y}\}} &= \mathrm{e}^{H(r\Delta)} && \text{for Re } r < 0, \\ &= \mathrm{e}^{H^+(r\Delta)} && \text{for Re } r = 0, \end{aligned} \tag{6.9}$$

with for Re $r = 0$,

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}}\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\} = \frac{1-\mathrm{E}\{\mathrm{e}^{r\Delta\mathbf{y}}\}}{-r\Delta\mathrm{E}\{\mathbf{y}\}}. \tag{6.10}$$

7. Limiting distributions

In this section we continue the analysis of the jump vector $\mathbf{u}$, which has been introduced in Section 5 as a mixture of $\mathbf{N}$ heavy-tailed jump vectors $\mathbf{u}_i$, $i = 1, \dots, \mathbf{N}$, cf. (5.10) and (5.11). However, we shall here restrict the generality of the distributions of the $\mathbf{u}_i$ slightly.

The first restriction concerns the distributions of $\mathbf{u}_N$. From now on we take: for Re $\rho = 0$,

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho \mathbf{u}_N}\}}{\rho \mathrm{E}\{\mathbf{u}_N\}} = \frac{\mathrm{E}\{\mathbf{v}_N\}}{\mathrm{E}\{\mathbf{u}_N\}} \frac{1}{1 + \rho \mathrm{E}\{\mathbf{v}_N\}} - \frac{\mathrm{E}\{\mathbf{w}_N\}}{\mathrm{E}\{\mathbf{u}_N\}} \frac{1}{1 - \rho \mathrm{E}\{\mathbf{w}_N\}} . \tag{7.1}$$

Obviously, (7.1) implies that the distribution of $\mathbf{u}_N$ has no heavy-tails and that $\mathbf{v}_N$ and $\mathbf{w}_N$ are both negative exponentially distributed. Hence in the representation (5.11) we have to take for Re $\rho = 0$,

$$\begin{aligned} g_{2N}(\gamma\rho) &= \frac{\rho \mathrm{E}\{\mathbf{v}_N\}}{1 + \rho \mathrm{E}\{\mathbf{v}_N\}} \quad , \quad g_{1N}(\gamma\rho) = \frac{\rho \mathrm{E}\{\mathbf{w}_N\}}{1 + \rho \mathrm{E}\{\mathbf{w}_N\}}, \\ c_{2N} &= 0 \quad , \quad c_{1N} = 0. \end{aligned} \tag{7.2}$$

Again it will be assumed that (5.12), i.e.

$$a < 1 \tag{7.3}$$

holds.

The second restriction is:

$$\nu := \min(\nu_{ji},\ j = 1, 2;\ i = 1, \dots, \mathbf{N} - 1) < 2. \tag{7.4}$$

Obviously, (7.4) implies that at least one of the v_{ji} is less than 2.

The third restriction concerns the functions $L_{ji}(\gamma\rho)$. It will be assumed, cf. (3.2)iv, that: for $|\rho| \to 0$, Re $\rho \geq 0$, $j = 1, 2$; $i = 1, \dots, N - 1$,

$$L_{ji}(\gamma\rho) \to b_{ji} \quad \text{with} \quad 0 < \mathrm{b_{ji}} < \infty. \tag{7.5}$$

This restriction is introduced to guarantee that the quotient of two $L_{ji}(\cdot)$ functions has a limit and that this limit is finite.

For some further comments concerning the assumptions see Remark 7.1 at the end of this section.

With

$$L(x) := \sum_{i=1}^{N-1} c_{2i}(x) L_{2i}(x) + \sum_{i=1}^{N-1} (-1)^{\nu_{1i}} c_{1i} L_{1i}(x),\ x \geq 0, \tag{7.6}$$

we define the contraction coefficient $\Delta(a)$ to be that root of the contraction equation

$$x^{v-1} |L(x)| = 1 - a, \quad x > 0, \tag{7.7}$$

which goes to zero for $a \uparrow 1$.

Take for $a < 1$,

$$\rho = r\Delta(a), \tag{7.8}$$

then we obtain from (5.11) and (7.2) for Re $r = 0$ and $\Delta \equiv \Delta(a)$:

$$
\begin{aligned}
1 \quad - \frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}} &= \frac{1}{a-1}\sum_{i=1}^{N}[aq_{2i}(a)g_{2i}(\gamma r\Delta) - q_{1i}(a)g_{1i}(\gamma\bar{r}\Delta)] \\
&- \sum_{i=1}^{N-1} aq_{2i}(a)c_{2i}(\gamma r)^{\nu_{2i}-1}\frac{L_{2i}(\gamma r\Delta)}{L_{2i}(\Delta)}\ \frac{L_{2i}(\Delta)}{|L(\Delta)|}\ \Delta^{\nu_{2i}-\nu} \\
&+ \sum_{i=1}^{N-1} aq_{1i}(a)c_{1i}(\gamma\bar{r})^{\nu_{1i}-1}\frac{L_{1i}(\gamma\bar{r}\Delta)}{L_{1i}(\Delta)}\ \frac{L_{1i}(\Delta)}{|L(\Delta)|}\ \Delta^{\nu_{1i}-\nu}.
\end{aligned}
\tag{7.9}
$$

We consider the relation (7.9) for $a\uparrow 1$, so that $\Delta = \Delta(a)\downarrow 0$. To realize this limit we rewrite (5.7) as:

$$
a = \frac{p_N\mathrm{E}\{\mathbf{v}_N\} + \sum_{i=1}^{N-1} p_i\mathrm{E}\{\mathbf{v}_i\}}{p_N\mathrm{E}\{\mathbf{w}_N\} + \sum_{i=1}^{N-1} p_i\mathrm{E}\{\mathbf{w}_i\}} < 1.
\tag{7.10}
$$

The limit $a\uparrow 1$ is now realized by keeping all the expectations in (7.10) constant except $\mathrm{E}\{\mathbf{v}_N\}$ which is considered as a variable for which holds

$$
\mathrm{E}\{\mathbf{v}_N\} \uparrow \mathrm{E}\{\mathbf{w}_N\} + \sum_{i=1}^{N-1}\frac{p_i}{p_N}[\mathrm{E}\{\mathbf{w}_i\} - \mathrm{E}\{\mathbf{v}_i\}].
\tag{7.11}
$$

To distinguish the notation for the case $a<1$ and $a=1$ we write:

$$
\begin{aligned}
&\hat{v}_i := \mathrm{E}\{\mathbf{v}_i\}, \qquad \hat{w}_i := \mathrm{E}\{\mathbf{w}_i\}, \quad i = 1,2,\dots,N-1, \\
&\hat{w}_N := \mathrm{E}\{\mathbf{w}_N\}, \quad \hat{v}_N := \hat{w}_N + \sum_{i=1}^{N-1}\frac{p_i}{p_N}[\hat{w}_i - \hat{v}_i].
\end{aligned}
\tag{7.12}
$$

We further define, cf. (5.8), for $i = 1,2,\dots,N$,

$$
q_{2i} := \lim_{a\uparrow 1}\ q_{2i}(a) = \frac{p_i\hat{v}_i}{\sum_{i=1}^{N} p_i\hat{v}_i}\ ,
\tag{7.13}
$$

$$
q_{1i} := \lim_{a\uparrow 1}\ q_{1i}(a) = \frac{p_i\hat{w}_i}{\sum_{i=1}^{N} p_i\hat{w}_i}\ ,
$$

and, (cf. 7.5) for $j=1,2;\ i=1,\dots,N-1$,

$$
d_{ji} := \lim_{a\uparrow 1} c_{ji}\ \frac{L_{ji}(\Delta)}{|L(\Delta)|}\ .
\tag{7.14}
$$

To consider (7.9) for $a\uparrow 1$, i.e. $\Delta(a)\downarrow 0$, we have to investigate for $a\uparrow 1$,

$$g_{ji}(\gamma r\Delta)/(1-a), \quad j=1,2;\ i=1,\dots,N. \tag{7.15}$$

From the conditions (3.2) together with (7.4), (7.5) and (7.7) it is readily seen that the expression (7.15) tends to zero for $a\uparrow 1$.

Define for $j=1,2$; $i=1,\dots,N-1$,

$$\begin{aligned} \varepsilon_{ji} &= 1 \quad \text{if} \quad \nu_{ji}=\nu, \\ &= 0 \quad \text{if} \quad \nu_{ji}>\nu. \end{aligned} \tag{7.16}$$

From the results above it is seen that for $a\uparrow 1$, i.e. $\Delta\downarrow 0$, the lefthand side of (7.9) has a limit for which holds: for $\mathrm{Re}\, r=0$,

$$\lim_{a\uparrow 1} \frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}} = 1+\sum_{i=1}^{N-1}[\varepsilon_{2i}q_{2i}d_{2i}(\gamma r)^{\nu_{2i}-1}-\varepsilon_{1i}q_{1i}d_{1i}(\gamma\bar{r})^{\nu_{1i}-1}], \tag{7.17}$$

with

$$\sum_{i=1}^{N} q_{ji}=1 \quad \text{for} \quad j=1,2;\ i=1,\dots,N. \tag{7.18}$$

The relation (7.17) is actually the most important result of the present section, it will play a central role in the next section. However, also the relation (7.9) with $\Delta(a)>0$ is of great interest, cf. Section 15.

REMARK 7.1. We make here some remarks concerning the three assumptions used in the derivation of (7.17).

The assumption (7.1) has been introduced to realize the limit $a\uparrow 1$. For if $\mathbf{v}_N$ would be heavy-tailed then it could be that g_{2N}, c_{2N} and $L_{2N}(.)$ depend on $\mathrm{E}\{\mathbf{v}_N\}$; so if $\mathrm{E}\{\mathbf{v}_N\}$ is a variable then the limit for $a\uparrow 1$ could be difficult to establish. Actually (7.17) holds without the assumption (7.1). Take in (7.1) $\mathbf{w}_N=0$ with probability one. Let $p_N\downarrow 0$ and let one of the other p_i increase such that $a\uparrow 1$, then it is readily seen that (7.17) also holds without assumption (7.1); so it also applies for $N=1$.

The condition (7.4) excludes the case that all $\nu_{ji}=2$. This can actually better be discussed separately, cf. [9].

The condition (7.5) is a rather strong condition at least from a theoretical viewpoint, see [4, 9] and [14]. However, for our purpose it is useful, see Section 17. □

8. Heavy traffic theorems

In this section we consider the solution of the Boundary Value Problem (2.15), see (6.7), for the case of a jump vector for which the limiting relation (7.17) holds. First we consider the question whether in (6.8) with $\Delta=\Delta(a)$, $a<1$ as defined in the preceding section, the limit for $\Delta\equiv\Delta(a)\downarrow 0$, i.e. $a\uparrow 1$, and the integration can be interchanged.

Put for $N=2,3,\dots$ and $\mathrm{Re}\, r=0$,

$$\begin{aligned} \varphi_N(r) :=& \sum_{i=1}^{N-1}[\varepsilon_{2i}q_{2i}d_{2i}(\gamma r)^{\nu_{2i}-1}-\varepsilon_{1i}q_{1i}d_{1i}(\gamma\bar{r})^{\nu_{1i}-1}] \\ =& (\gamma r)^{\nu-1}\sum_{i=1}^{N-1}\varepsilon_{2i}q_{2i}d_{2i}-(\gamma\bar{r})^{\nu-1}\sum_{i=1}^{N-1}\varepsilon_{1i}q_{1i}d_{1i}, \end{aligned} \tag{8.1}$$

see (7.4) and (7.17). For ρ a complex number we define

$$\hat{H}(\rho) := \frac{1}{2\pi\mathrm{i}} \int_{\eta=-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\{1+\varphi_N(\eta)\} \frac{\rho}{(\eta-\rho)} \frac{\mathrm{d}\eta}{\eta} . \tag{8.2}$$

With $\varphi_N(r)$, $\mathrm{Re}\, r = 0$, as given by (8.1) it is not difficult to verify that the integral exists as a principal value integral at infinity, that the integrand satisfies a Hölder condition on $\mathrm{Re}\,\eta = 0$ and that the integral is well defined as a singular Cauchy integral whenever $\mathrm{Re}\,\rho = 0$, cf. [1]. By using the Plemelj formulas we obtain, for $\mathrm{Re}\,\rho = 0$,

$$\begin{aligned} H^-(\rho) &:= \lim_{\substack{t\to\rho \\ \mathrm{Re}\ t>0}} H(t) = -\frac{1}{2}\log\{1+\varphi_N(\rho)\} + \hat{H}(\rho), \\ H^+(\rho) &:= \lim_{\substack{t\to\rho \\ \mathrm{Re}\ t<0}} H(t) = \frac{1}{2}\log\{1+\varphi_N(\rho)\} + \hat{H}(\rho). \end{aligned} \tag{8.3}$$

Write the integral in (6.8) with $\Delta = \Delta(a)$ as

$$\begin{aligned} H(r\Delta) \;&=\; \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}R}^{+\mathrm{i}R} [\log\{\frac{1-\mathrm{E}\{\mathrm{e}^{-\eta\Delta\mathbf{u}}\}}{\eta\Delta\mathrm{E}\{\mathbf{u}\}}\}] \;\frac{r}{\eta-r}\;\frac{\mathrm{d}\eta}{\eta} \\ &+ \frac{1}{2\pi\mathrm{i}} \int_{\eta\in F(R)} [\log\{\frac{1-\mathrm{E}\{\mathrm{e}^{-\eta\Delta\mathbf{u}}\}}{\eta\Delta\mathrm{E}\{\mathbf{u}\}}\}] \;\frac{r}{\eta-r}\;\frac{\mathrm{d}\eta}{\eta}, \end{aligned} \tag{8.4}$$

with

$$F(R) := \{\eta : \mathrm{Re}\,\eta = 0,\ |\eta| \ge R\} \quad \text{and} \quad R > 0.$$

By using (7.9) it is seen that the absolute value of the second integral in (8.4) can be made arbitrarily small uniformly in R for R sufficiently large. Hence since the integral in (8.4) exists, it follows that for $a \uparrow 1$, i.e. $\Delta = \Delta(a) \downarrow 0$,

$$H(r\Delta) \to \hat{H}(r). \tag{8.5}$$

The result leads to the following heavy-traffic Theorem 8.1.

THEOREM 8.1 *For a jump vector* $\mathbf{u}$ *which is a mixing of* $\mathbf{N}$ *heavy-tailed stochastic variables* $\mathbf{u}_i$, *as defined in Section 6, the stochastic variables* $\Delta(a)\mathbf{x}$ *and* $\Delta(a)\mathbf{z}$ *both converge in distribution for* $a \uparrow 1$. *With* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{z}}$ *stochastic variables with distributions the limiting distributions of* $\Delta(a)\mathbf{x}$ *and* $\Delta(a)\mathbf{z}$ *we have:*

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}}\} \;&=\; \lim_{a\uparrow 1} \mathrm{E}\{\mathrm{e}^{-\rho\Delta(a)\mathbf{x}}\} \;=\; \mathrm{e}^{\hat{H}(\rho)} \quad \text{for} \quad \mathrm{Re}\,\rho > 0, \\ &\qquad\qquad\qquad\qquad\quad\;\;\, = \mathrm{e}^{\hat{H}^-(\rho)} \quad \text{for} \quad \mathrm{Re}\,\rho = 0, \\ \mathrm{E}\{\mathrm{e}^{\rho\hat{\mathbf{z}}}\} \;&=\; \lim_{a\uparrow 1} \mathrm{E}\{e^{\rho\Delta(a)\mathbf{z}}\} \;=\; \mathrm{e}^{\hat{H}(\rho)} \quad \text{for} \quad \mathrm{Re}\,\rho < 0, \\ &\qquad\qquad\qquad\qquad\quad\;\;\, = \mathrm{e}^{\hat{H}^+(\rho)} \quad \text{for} \quad \mathrm{Re}\,\rho = 0. \end{aligned} \tag{8.6}$$

PROOF. The proof follows immediately from (2.16), (6.9), (8.5) and Feller's continuity theorem, cf. [6], for the L.S.-transforms of distributions with support contained in $[0,\infty)$, note that $\hat{H}^{\pm}(0) = 0$ and that $H(\rho)$ is continuous for $\operatorname{Re}\rho \geq 0$ as well as for $\operatorname{Re}\rho \leq 0$. □

COROLLARY 8.1 *For* $\operatorname{Re} r = 0$ *we have*

$$[1+\varphi_N(r)]\mathrm{E}\{\mathrm{e}^{-r\hat{\mathbf{x}}}\} = \mathrm{E}\{\mathrm{e}^{r\hat{\mathbf{z}}}\}. \tag{8.7}$$

PROOF. The relation (8.7) follows immediately from the above theorem and (2.8), (2.13), (2.16), with $\rho = r\Delta(a)$, $\operatorname{Re} r = 0$. □

The function $\varphi_N(\rho)$, $\operatorname{Re}\rho = 0$, has been defined in (8.1), and as a function of ρ it is a rather complicated function; note that $\rho^{\nu-1}$ is defined by its principal value, i.e. it is positive for $\rho > 0$.

In our further analysis we shall distinguish three cases. Put

$$B_j := \sum_{i=1}^{N-1} \varepsilon_{ji} q_{ji} d_{ji}, \quad j = 1, 2, \tag{8.8}$$

then we consider the following cases separately

$$\begin{aligned} &\text{i.} \quad B_2 > 0, \quad B_1 = 0,\\ &\text{ii.} \quad B_2 = 0, \quad B_1 > 0,\\ &\text{iii.} \quad B_2 > 0, \quad B_1 > 0. \end{aligned} \tag{8.9}$$

Note that $B_2 > 0$, $B_1 = 0$ implies that all $\nu_{1i} > \nu$, and analogously for case (8.9)ii, whereas (8.9)iii implies that at least one ν_{2i} is equal to at least one ν_{1i}, $i = 1, \ldots, N-1$, cf. (7.4) and (7.16).

The three cases distinguished will be referred to as case I, II and III. Actually the dichotomy of these three cases resembles that of the queueing models M/G/1, GI/M/1 and GI/G/1.

9. RESULTS FOR CASE I

We refer to (8.9) for the specification of case I. It follows that for this case we have: for $\operatorname{Re} r = 0$, cf. (8.1),

$$\varphi_N(r) = (\gamma r)^{\nu-1} B_2, \tag{9.1}$$

and so we have from (8.2): for $\operatorname{Re}\rho \geq 0$,

$$\hat{H}(\rho) = \frac{1}{2\pi\mathrm{i}} \int_{\eta=-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 + (\gamma\eta)^{\nu-1} B_2\}] \frac{\rho}{\eta - \rho} \frac{\mathrm{d}\eta}{\eta}\ . \tag{9.2}$$

The function $1 + (\gamma\eta)^{\nu-1} B_2$, $1 < \nu < 2$, is regular for $\operatorname{Re}\eta > 0$, and for $\operatorname{Re}\eta \geq 0$, continuous and nonzero. Consequently, the integrand in (9.2) is regular for $\operatorname{Re}\eta > 0$, continuous and nonzero for $\operatorname{Re}\eta \geq 0$, except for a single pole at $\eta = \rho$. Note that $1 < \nu < 2$ implies that $\eta = 0$ is not a pole of the integrand. Note further that we do not need here to define the principal value of the integrand in (9.2) because it does not influence the value of $\exp \hat{H}(\rho)$. We evaluate the integral in (9.2) by contour integration in the right semi-plane; the contour being formed by the line piece $(-\mathrm{i}R,\ \mathrm{i}R)$ and the semi-circle $\eta = \mathrm{Re}^{\,\mathrm{i}\omega}$, $|\omega| \le \frac{1}{2}\pi$, $R > 0$.

By applying Cauchy's theorem and by noting that for R sufficiently large the contribution of the integral along the semi-circle tends to zero for $R \to \infty$ we obtain from (8.6): for Re $\rho \ge 0$,

$$\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}/\gamma}\} = e^{-\log(1+B_2\rho^{\nu-1})} = \frac{1}{1+B_2\rho^{\nu-1}}. \tag{9.3}$$

Actually (9.3) follows from the above only for Re $\rho > 0$. However, since $\hat{H}(\rho)$ is continuous for Re $\rho \ge 0$ and the last member is also continuous for Re $\rho \ge 0$, the relation (9.3) also holds for Re $\rho = 0$.

We next determine $\mathrm{E}\{e^{\rho\mathbf{z}}\}$ for Re $\rho \le 0$. Because the righthand side of (9.3) is equal to $[1+\varphi_N(\rho)]^{-1}$ for Re $\rho = 0$ it results from (8.7) that: for Re $\rho = 0$,

$$\mathrm{E}\{e^{\rho\hat{\mathbf{z}}/\gamma}\} = 1. \tag{9.4}$$

From (9.4) and the inversion theorem for characteristic functions, cf. [1], it follows that $\hat{\mathbf{z}} = 0$ with probability one. Hence the following theorem has been proved.

THEOREM 9.1 *For case I, i.e.* $\nu < \nu_{1i}$, $i = 1, \dots, N-1$, *holds:*

$$\mathrm{E}\{e^{-\rho\hat{\mathbf{x}}/\gamma}\} = \frac{1}{1+B_2\rho^{\nu-1}}, \quad \mathrm{Re}\,\rho \ge 0, \tag{9.5}$$

$$\Pr\{\hat{\mathbf{z}} = 0\} = 1.$$

REMARK 9.1. For $q_{21} = 1$ Theorem 9.1 has been derived in previous studies of the author, e.g. [9]. The interesting case is here actually that with $N = 2$, and $d_{12} = 0$, i.e. the jump vector $\mathbf{u}$ is a mix of $\mathbf{u}_1$ and $\mathbf{u}_2$, with the right tail of $\mathbf{u}_1$ the heavier one and $\mathbf{u}_2$ has only light tails.

For the heavy-traffic case we then have

$$1 = a = \frac{p_1\hat{v}_1 + p_2\hat{v}_2}{p_1\hat{w}_1 + p_2\hat{w}_2} = \frac{p_1\hat{v}_1}{p_1\hat{v}_1 + p_2\hat{v}_2} + \frac{p_2\hat{v}_2}{p_1\hat{v}_1 + p_2\hat{v}_2} = q_{21} + q_{22}.$$

Hence q_{21} is the heavy-tailed part of a and q_{22} the light tailed part or *the best effort traffic* of a; here we misuse slightly the term best effort traffic.

Further note that (9.5) may be rewritten as: for Re $\rho \ge 0$,

$$\mathrm{E}\{\exp[-\rho\hat{\mathbf{x}}/(q_{21}d_{21})^{1/(\nu-1)}\gamma]\} = \frac{1}{1+\rho^{\nu-1}}.$$

From this expression it is seen that the distribution of $\hat{\mathbf{x}}$ is very sensitive to the value of q_{21}, or equivalently q_{22}, because $1/(\nu-1) > 1$ and $0 < q_{21} < 1$. This may be also seen from the asymptotic expression for the tail of $\hat{\mathbf{x}}$, see (12.8). □

REMARK 9.2. The generalization of the case of Remark 9.1 is simple, i.e., if $N > 2$ and as before $d_{1i} = 0$, $\nu_{2i} > \nu_{21}$, $i = 2, \dots, N-1$, then the only change to be made in the formulas of the preceding remark is to replace the first formula there by

$$1 = a = \sum_{i=1}^{N} q_{2i}.$$

The formula for the L.S.-transform of $\hat{\mathbf{x}}$ does not change.

It is hence seen that whenever $\mathbf{u}$ is a mixture of stochastic variables of which only the righthand tails are heavy-tailed then the heavy-traffic limiting distribution depends on all the characteristics of the most-heavy-tailed component and only on the traffic loads q_{2i} of those components with the less heavier tails.

These observations concern a limiting result for $a \uparrow 1$. For the case $1 > 1 - a >> 0$ a more refined analysis is required for the tail analysis of the distribution of $\mathbf{u}$, see herefor Section 15. □

10. Results for case II

For case II we have from (8.1) and (8.9): for Re $\rho = 0$,

$$\begin{aligned} &\log[1 + \varphi_N(\rho)] = \log\{1 - (\gamma\bar{r})^{\nu-1} B_1\} \\ &= \log \frac{1 - (\gamma\bar{r})^{\nu-1} B_1}{1 - \omega_1 \gamma \bar{r}} + \log\{1 - \omega_1 \gamma \bar{r}\} \end{aligned} \tag{10.1}$$

with

$$\omega_1 := B_1^{1/(\nu-1)} > 0. \tag{10.2}$$

From (8.2) we have for Re $\rho > 0$,

$$\begin{aligned} \hat{H}(\rho) &= \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 - (\gamma\bar{\eta})^{\nu-1} B_1\}] \frac{\rho}{\eta - \rho} \frac{\mathrm{d}\eta}{\eta} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 - (\gamma\eta)^{\nu-1} B_1\}] \frac{\rho}{\eta + \rho} \frac{\mathrm{d}\eta}{\eta} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{\frac{1 - B_1(\gamma\eta)^{\nu-1}}{1 - \omega_1\gamma\eta}\} + \log(1 - \omega_1\gamma\eta)] \frac{\rho}{\eta + \rho} \frac{\mathrm{d}\eta}{\eta}. \end{aligned} \tag{10.3}$$

The function $[1 - B_1(\gamma\eta)^{\nu-1}]/(1 - \omega_1\gamma\eta)$ is regular for Re $\eta > 0$, continuous and nonzero for Re $\eta \geq 0$. We apply Cauchy's theorem with a contour consisting of $(-\mathrm{i}R, \mathrm{i}R)$ and Re $^{\mathrm{i}\varphi}$, $|\varphi| \leq \frac{1}{2}\pi$, with $R > 0$ to the last integral in (10.3) and with integrand formed by the first term of the integrand of that integral. This integral has no pole in Re $\eta \geq 0$ and the contribution of this integral along the semi-circle of this contour tends to zero for $R \to \infty$. Hence this integral is zero and we have for Re $\rho > 0$:

$$\hat{H}(\rho) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 - \omega_1\gamma\eta\}] \frac{\rho}{\eta + \rho} \frac{\mathrm{d}\eta}{\eta}. \tag{10.4}$$

The integrand in (10.4) has only one pole in Re $\eta \leq 0$, viz. $\eta = -\rho$ (note $\eta = 0$ is not a pole). Applying Cauchy's theorem for a contour consisting of $(-\mathrm{i}R, \mathrm{i}R)$ and Re $^{\mathrm{i}\psi}$, $\frac{1}{2}\pi \leq \psi < 1\frac{1}{2}\pi$, with $R > 0$ leads easily to the result that for Re $\rho \geq 0$, $R \to \infty$,

$$\hat{H}(\rho) = -\log\{1 + \omega_1\gamma\rho\}. \tag{10.5}$$

From this and (8.6) it follows that

$$\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}/\gamma}\} = \frac{1}{1+\omega_1\rho} \text{ for } \mathrm{Re}\ \rho \geq 0. \tag{10.6}$$

That (10.6) also holds for Re $\rho = 0$ follows as in the preceding section.

Next we derive an expression for the L.S.-transform of the distribution of $\hat{\mathbf{z}}$. From (8.7) and (10.6) we have: for Re $\rho = 0$,

$$\frac{1-B_1\bar{\rho}^{\nu-1}}{1+\omega_1\rho} = \mathrm{E}\{\mathrm{e}^{\rho\hat{\mathbf{z}}/\gamma}\},$$

or

$$\frac{1-B_1\rho^{\nu-1}}{1-\omega_1\rho} = \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{z}}/\gamma}\},\ \mathrm{Re}\ \ \rho = 0. \tag{10.7}$$

From (10.2) it is seen that the lefthand side of (10.7) is regular for Re $\rho > 0$, continuous for Re $\rho \geq 0$. Because $\hat{\mathbf{z}}$ is a nonnegative stochastic variable, the righthand side of (10.7) is regular for Re $\rho > 0$, continuous for Re $\rho \geq 0$. Consequently, analytic continuation shows that (10.7) holds for Re $\rho \geq 0$. The results derived above lead to the following theorem.

Theorem 10.1 *For the case II i.e.* $\nu < \nu_{2\mathrm{i}}, i = 1, \dots, N-1$, *holds: for* Re $\rho \geq 0$,

$$\begin{aligned}\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}/\gamma}\} &= \frac{1}{1+\omega_1\rho},\\ \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{z}}/\gamma}\} &= \frac{1-B_1\rho^{\nu-1}}{1-\omega_1\rho} = \frac{1-(\omega_1\rho)^{\nu-1}}{1-\omega_1\rho},\end{aligned}$$

and $\hat{\mathbf{x}}/\gamma$ *is negative exponentially distributed with first moment* $\omega_1 = B_1^{1/(\nu-1)}$.

Corollary 10.1 *For* $0 < \delta < 1$ *the function*

$$\frac{1-\rho^{\delta}}{1-\rho},\ \mathrm{Re}\ \ \rho \geq 0,$$

is the L.S.-transform of a probability distribution with support $[0,\infty)$*; also*

$$\frac{1+\rho^{1/n}+\cdots+\rho^{(m-1)/n}}{1+\rho^{1/n}+\cdots+\rho^{(n-1)/n}}, \mathrm{Re}\ \ \rho \geq 0,\ \ 1 \leq m < n,$$

m *and* n *integers with g.c.d.* $(m,n) = 1$*, is the L.S.-transform of such a distribution.*

Proof. The first statement of the corollary follows immediately from the theorem above.

To prove the second statement take $\delta = m/n$ with m and n as specified above. Further take $x = \rho^{1/n} > 0$ so defined that it is positive for $\rho > 0$. Then

$$\frac{1-\rho^{m/n}}{1-\rho} = \frac{1-x^m}{1-x^n} = \frac{1+x+\cdots+x^{m-1}}{1+x+\cdots+x^{n-1}},$$

and replacing x by $\rho^{1/n}$ leads to the statement.

□

Remark 10.1 The L.S.-transform mentioned in Corollary 10.1 has been discussed in [10]. There it is shown that it is the L.S.-transform of an infinitely divisible distribution, a result which also follows immediately from (2.16) and (6.6). □

11. Results for case III

The case III occurs whenever $B_1 = B_2$, cf. (8.9), i.e. when,

$$\nu = \min\{\nu_{2\mathrm{i}}, i = 1, \ldots, N-1\} = \min\{\nu_{1\mathrm{i}}, i = 1, \ldots, N-1\}. \tag{11.1}$$

For the present case we have, cf. (8.1), for Re $\rho = 0$,

$$\varphi_N(\rho) = (\gamma\rho)^{\nu-1} B_2 - (\gamma\bar{\rho})^{\nu-1} B_1. \tag{11.2}$$

For the case I and II discussed in the previous sections the integral in (8.2) could be evaluated, for the present case this does not seem possible. For the present case we have in [9] the complex integrals transformed into real integrals. In the integral (8.2) we have replaced η by is (i the imaginary unit) and we then obtain for Re $r > 0, 1 < \nu < 2$,

$$\begin{aligned} \hat{H}(r) &= F_1(r) + F_2(r), \\ \hat{H}(-r) &= F_1(r) - F_2(r), \end{aligned} \tag{11.3}$$

with

$$\begin{aligned} F_1(r) &:= -\frac{1}{\pi}\int_0^\infty \{\arctan \frac{A(rs)^{\nu-1}}{1 + B(rs)^{\nu-1}}\}\frac{1}{1+s^2}\frac{\mathrm{d}s}{s}, \\ F_2(r) &:= -\frac{1}{2\pi}\int_0^\infty [\log\{1 + 2B(rs)^{\nu-1} + C(rs)^{2(\nu-1)}\}]\frac{1}{1+s^2}\mathrm{d}s, \end{aligned} \tag{11.4}$$

where

$$\begin{aligned} A &:= (B_2 + B_1)\sin\frac{1}{2}(\nu-1)\pi > 0, \\ B &:= (B_2 - B_1)\cos\frac{1}{2}(\nu-1)\pi, \\ C &:= (B_2 - B_1)^2 + 4B_1B_2\sin^2\frac{1}{2}(\nu-1)\pi > 0. \end{aligned} \tag{11.5}$$

Further it has been shown in [9] that for Re $\rho \geq 0$,

$$\begin{aligned} \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}}\} &= \mathrm{e}^{F_1(\rho)+F_2(\rho)}, \\ \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{z}}}\} &= \mathrm{e}^{F_1(\rho)-F_2(\rho)}. \end{aligned} \tag{11.6}$$

12. Asymptotic results for heavy-traffic

In [9] several calculations have been made in order to compare the results obtained for the L.S.-transforms of the distributions of $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ for the three cases I, II, III. However, the comparison is hampered due to the fact that the integral for case III cannot be evaluated explicitly. We, therefor, will compare here the asymptotic expressions for the tails of the various distributions.

We first remark that a noticeable difference between the results for the distribution of $\hat{\mathbf{x}}$ is of a similar type as the difference between the results for the stationary waiting time distributions for the classical queuing models M/G/1, GI/M/1 and GI/G/1. Here also explicit results are known for the M/G/1 and the GI/M/1; for the GI/G/1 only integral expressions are available. For case II and for GI/M/1 the distribution of $\hat{\mathbf{x}}$ is negative exponential.

Next, we consider the asymptotic comparison.

For case I we have from Theorem 9.1:

$$\mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}/\gamma}\} = \frac{1}{1 + B_2\rho^{\nu-1}} \text{ for Re } \rho \geq 0. \tag{12.1}$$

From (12.1) it follows: for Re $\rho > 0$,

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\rho t}[1 - \Pr\{\hat{\mathbf{x}} < \gamma t\}]\gamma \mathrm{d}t &= \frac{1}{\rho}[1 - \mathrm{E}\{\mathrm{e}^{-\rho\hat{\mathbf{x}}}\}] \\ &= \frac{B_2\rho^{\nu-2}}{1 + B_2\rho^{\nu-1}}. \end{aligned} \tag{12.2}$$

The righthand side of (12.2) can be continued analytically into the ρ-plane slitted along the negative real axis, i.e. along $-\infty < \rho < 0$.

For $|B_2\rho^{\nu-1}| < 1$ and $|\arg\rho| < \pi$ we have

$$\frac{B_2\rho^{\nu-2}}{1 + B_2\rho^{\nu-1}} = \sum_{n=0}^{\infty}(-1)^n B_2^{n+1}\rho^{(n+1)(\nu-1)-1}. \tag{12.3}$$

The L.S.-transform (12.3) is of a type for which from the behaviour for $|\rho|\downarrow 0$ on the slitted ρ-plane an asymptotic expression for the tail of the distribution of $\hat{\mathbf{x}}$ can be obtained, see therefor [10, vol. II, p. 159], Theorem 2; cf. also [5]. Actually, it follows by applying this theorem to (12.2), (cf. also [9]) that for $t \to \infty, 1 < \nu < 2$, and for every $M = 1, 2, \dots$,

$$\Pr\{\hat{\mathbf{x}} \geq \gamma t\} = \sum_{n=0}^{M}(-1)^n \frac{B_2^{n+1}}{\Gamma(1 - (n+1)(\nu-1))} t^{-(n+1)(\nu-1)} + o\left(\left(\frac{1}{t}\right)^{(M+1)(\nu-1)}\right). \tag{12.4}$$

Here $\Gamma(\cdot)$ is the gamma-function and

$$\Gamma(1 - (n+1)(\nu-1)) = \frac{\pi}{\Gamma((n+1)(\nu-1))\sin(n+1)(\nu-1)\pi}.$$

For case II we have from Theorem 9.1 for Re $\rho \geq 0$,

$$\int_0^\infty \mathrm{e}^{-\rho t}[1 - \Pr\{\mathbf{z} < \gamma t\}]\gamma \mathrm{d}t = \frac{B_1\rho^{\nu-2} - \omega_1}{1 - \omega_1\rho}. \tag{12.5}$$

Hence for $|\omega_1\rho| < 1$, $|\arg\rho| < \pi$ we have

$$\int_0^\infty \mathrm{e}^{-\rho t}\Pr\{\mathbf{z} > \gamma t\}\gamma \mathrm{d}t = B_1\sum_{n=0}^{\infty}\omega_1^n\rho^{n+\nu-2} - \frac{\omega_1}{1 - \omega_1\rho}. \tag{12.6}$$

Applying again Theorem 2, vol. II, p. 159 of [10] yields that for $t \to \infty, 1 < \nu < 2$ and $M = 0, 1, \dots$,

$$\Pr\{\mathbf{z} > \gamma t\} = \sum_{n=0}^{M}\frac{\omega_1^{n+\nu-1}}{\Gamma(2 - n - \nu)}\left(\frac{1}{t}\right)^{n+\nu-1} + o\left(\left(\frac{1}{t}\right)^{M+\nu-1}\right). \tag{12.7}$$

In [9] for the case III asymptotic relations for the tails of the distribution of $\hat{\mathbf{x}}$ and of $\hat{\mathbf{z}}$ have been obtained; however, no asymptotic series but only the first term of it have been derived. We list them here together with the first term of the asymptotic series for the cases I and II.

$$\begin{aligned} &\text{i} && \Pr\{\hat{\mathbf{x}} \geq \gamma t\} \sim \frac{B_2}{\Gamma(2-\nu)}\left(\frac{1}{t}\right)^{\nu-1} && \text{for case I,} \\ &\text{ii} && \Pr\{\hat{\mathbf{z}} \geq \gamma t\} \sim \frac{B_1}{\Gamma(2-\nu)}\left(\frac{1}{t}\right)^{\nu-1} && \text{for case II,} \\ &\text{iii} && \Pr\{\hat{\mathbf{x}} \geq \gamma t\} \sim \frac{\max(B_1, B_2)}{\Gamma(2-\nu)}\left(\frac{1}{t}\right)^{\nu-1} && \text{for case III,} \\ & && \Pr\{\hat{\mathbf{z}} \geq \gamma t\} \sim \frac{\min(B_1, B_2)}{\Gamma(2-\nu)}\left(\frac{1}{t}\right)^{\nu-1} && \text{for case III,} \end{aligned} \tag{12.8}$$

here γ is the time unit.

Note that

$$\begin{aligned} &\Pr\{\hat{\mathbf{x}} \geq t\gamma\} = \mathrm{e}^{-B_1 t} &&, t \geq 0, \quad \text{for case II,} \\ &\Pr\{\hat{\mathbf{z}} > t\} = 0 &&, t \geq 0, \quad \text{for case I.} \end{aligned} \tag{12.9}$$

The relations (12.8) make a good comparison possible. The difference between the cases I and II is fairly obvious because they concern the cases that the right tail of the jump vector is heavier than its left tail and conversely, see also the above mentioned comparison with the M/G/1 and GI/M/1 queuing model. However, when both tails of the jump vector $\mathbf{u}$ are equally heavy then the difference between I and II, so far it concerns the tail of $\hat{\mathbf{x}}$ is determined by the maximum of B_1 and B_2. It should be noted, cf. (8.8), that the B_j are in some sense averages in which the traffic ratios as well as the intensities of the tails c_{ji} and their ratios

$$L_{ij}(\Delta)/|L(\Delta)|$$

are incorporated.

Further the case III with $B_2 > B_1$ is rather obvious but somewhat amazing if $B_2 < B_1$, a phenomenon which has also been noticed in [11].

13. Some further remarks on the heavy-traffic distribution

In the preceding Sections $8, \dots, 12$, heavy-traffic theorems have been derived. It is of great interrest to see how the traffic characteristics of the distribution of $\mathbf{u}_i$, the i-th component of the jump vector $\mathbf{u}$, influences the heavy-traffic limiting distribution.

It is seen that of the characteristics of the distribution of $\mathbf{u}_i$ only $\mathrm{E}\{\mathbf{v}_i\}$ and $\mathrm{E}\{\mathbf{w}_i\}$ always occur in the limiting distribution; the characteristics of the tails of the distribution of $\mathbf{u}_i$ do not occur if $\nu_{2i} > \nu$ and $\nu_{1i} > \nu$. Only the tail characteristics of the most heavy tail determine the tail of the limiting distribution. These results for the limiting distributions, i.e. $\Delta(a)\downarrow 0$, have been proved in the preceding sections.

In the following sections we do not consider the limiting case $\Delta(a)\downarrow 0$, but we shall analyze the relevant distributions for $0 < \Delta(a) \ll 1$, i.e. we start with an asymptotic analysis for $\Delta(a)\downarrow 0$.

14. Approximations

Can the heavy-traffic limit distribution be used as an approximation for the distribution of $\mathbf{x}$ whenever $a < 1$?

For the case of the M/G/1 queueing model with the service time distribution heavy-tailed it has been shown in [13] that the question just posed has a surprisingly affirmative answer. The equivalent approach of [13] may be certainly used here. The quality of such an approximation will be discussed later. Here we shall present a derivation of an approximation which seems to generalize slightly the one in [13] and which takes also into account the variability of the tails of the $\mathbf{u}_i$-variables which compose the jump vector $\mathbf{u}$.

We start here from the relation (7.9) with $\Delta \equiv \Delta(a)$ as defined via (7.7). It is assumed that a is close to one so that $\Delta > 0$ is close to zero. However,

$$[\Delta(a)]^{\nu_{ji}-\nu} \quad \text{with} \quad \nu_{ji} > \nu, \quad j = 1, 2; \ i = 1, \dots, N-1,$$

may be not so close to zero since $0 < \nu_{ji} - \nu < 1$.

From (7.9) we obtain for $a < 1$, $\Delta = \Delta(a)$ and $\mathrm{Re}\, r = 0$,

$$\begin{aligned}\frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}} &= 1+\frac{1}{1-a}\sum_{i=1}^{N}[aq_{2i}(a)g_{2i}(\gamma r\Delta)-q_{1i}(a)g_{1i}(\gamma\bar{r}\Delta)]\\ &+\sum_{i=1}^{N-1}aq_{2i}(a)c_{2i}(\gamma r)^{\nu_{2i}-1}\frac{L_{2i}(\gamma r\Delta)}{L_{2i}(\Delta)}\ \frac{L_{2i}(\Delta)}{|L(\Delta)|}\ \Delta^{\nu_{2i}-\nu}\\ &-\sum_{i=1}^{N-1}q_{1i}(a)c_{1i}(\gamma\bar{r})^{\nu_{1i}-1}\frac{L_{1i}(\gamma\bar{r}\Delta)}{L_{1i}(\Delta)}\ \frac{L_{1i}(\Delta)}{|L(\Delta)|}\ \Delta^{\nu_{1i}-\nu}.\end{aligned} \tag{14.1}$$

From this relation we derive an asymptotic expression in $\Delta=\Delta(a)\downarrow 0$ for $a\uparrow 1$.

From (3.2)iii it is seen that for $\Delta\downarrow 0$, $j=1,2$; $i=1,\dots,N$,

$$g_{ji}(\gamma r\Delta)=\gamma r O(\Delta)\quad\text{for}\quad \mathrm{Re}\, r>-\delta_{ji}. \tag{14.2}$$

From (7.5) and (7.7) we have

$$1-a=\Delta^{\nu-1}L(\Delta); \tag{14.3}$$

and so it follows for $\Delta\downarrow 0$ and Re $r>\delta_{ji}$ that:

$$\frac{1}{1-a}\ g_{ji}(\gamma r\Delta)=\gamma r O(\Delta^{2-\nu}). \tag{14.4}$$

Because $1>\nu_{ji}-\nu>0$ for $\nu_{ji}\neq\nu$ we may write by using (7.5) and (7.15) that: for Re $r=0$ and $\Delta=\Delta(a)\downarrow 0$,

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}}=1+\sum_{i=1}^{N-1}[aq_{2i}d_{2i}(\gamma r)^{\nu_{2i}-1}\Delta^{\nu_{2i}-\nu}-q_{1i}d_{1i}(\gamma\bar{r})^{\nu_{1i}-1}\Delta^{\nu_{1i}-\nu}]+\gamma r O(\Delta^{2-\nu}). \tag{14.5}$$

Note that $2-\nu>\nu_{ji}-\nu$ since $0<\nu_{ji}<2$. Further we may replace a in the first sum of (14.5) by 1 because of (14.3), but for the present we prefer to keep a here.

The relation (14.5) is the starting point for our approximations. Actually, this relation has to be inserted into (6.8), and if the integral can be evaluated we can obtain $\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\}$, Re $r\geq 0$, from (6.9). It is this approach which we shall follow in the next section for the case that

$$d_{1i}=0,\quad i=1,\dots,N. \tag{14.6}$$

This assumption implies that the lefthand tails of all the distributions of the stochastic variable $\mathbf{u}_i, i=1,\dots,N-1$, are not heavy-tailed. It is shown that this assumption leads to an attractive result which is conjectured to be most useful for practical purposes cf. the results obtained in [13].

15. Heavy traffic approximation for heavy-tailed upward jump

The assumption (14.6) implies that of the jump vector $\mathbf{u}$ only the right tail is heavy. For this case we shall derive an asymptotic expression in $\Delta(a)$ for $a\uparrow 1$ as well as in t for $t\to\infty$ for the tail of the stationary distribution of the random walk. We start here from the asymptotics for $a\uparrow 1$, cf. (14.5) and (14.6), i.e. for $\Delta=\Delta(a)\downarrow 0$, and Re $\rho=0$,

$$\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\Delta\mathbf{u}}\}}{\rho\Delta\mathrm{E}\{\mathbf{u}\}} \;=\; 1+\sum_{i=1}^{N-1} aq_{2i}d_{2i}(\gamma\rho)^{\nu_{2i}-1}\Delta^{\nu_{2i}-\nu}+\gamma\rho O(\Delta^{2-\nu}), \tag{15.1}$$

with

$$\nu=\min(\nu_{2i},\ i=1,\dots,N-1).$$

For the sake of simplicity it is assumed that

$$\nu=\nu_{21}<\nu_{22}<\ldots<\nu_{2N-1}. \tag{15.2}$$

Obviously the sum in (15.1) can be continued analytically for Re $\rho\geq 0$, it can also be continued analytically on the whole ρ-plane slitted along the negative real axis $\rho<0$.

Put

$$\gamma\rho=r\mathrm{e}^{i\varphi},\quad r\geq 0,\quad |\varphi|\leq\frac{1}{2}\pi, \tag{15.3}$$

then, since $\nu>1$ and all $\nu_{2i}<2$,

$$\begin{aligned}\omega_N(\rho;\Delta):=\ &\sum_{i=1}^{N-1} aq_{2i}d_{2i}\Delta^{\nu_{2i}-\nu}(\gamma\rho)^{\nu_{2i}-1}=\\ &\sum_{i=1}^{N-1} aq_{2i}d_{2i}\Delta^{\nu_{2i}-\nu}r^{\nu_{2i}-1}[\cos(\nu_{2i}-1)\varphi+\mathrm{i}\sin(\nu_{2i}-1)\varphi].\end{aligned} \tag{15.4}$$

Hence for Re $\rho\geq 0$, and $\Delta(a)>0$,

$$1+\omega_n(\rho;\Delta)\neq 0, \tag{15.5}$$

and this function is regular for Re $\rho>0$, and continuous for Re $\rho\geq 0$.

Next we consider the function $H(r\Delta)$, cf. (6.8), with $\Delta=\Delta(a)$, $a<1$, and with the integrand given by (15.1). So for $\Delta=\Delta(a)>0$, cf. (6.8), and $r>0$,

$$\begin{aligned}H(r\Delta)\ &=\frac{1}{2\pi i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\{1+\omega_N(\eta;\Delta)\}+\gamma\eta O(\Delta^{2-\nu})]\frac{r}{\eta-r}\,\frac{\mathrm{d}\eta}{\eta}\\ &=\frac{1}{2\pi i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\{1+\omega_N(\eta;\Delta)\}]\frac{r}{\eta-r}\,\frac{\mathrm{d}\eta}{\eta}\\ &+\frac{1}{2\pi i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\{1+\frac{\gamma\eta O(\Delta^{2-\nu})}{1+\omega_N(\eta;\Delta)}\}]\frac{r}{\eta-r}\,\frac{\mathrm{d}\eta}{\eta}.\end{aligned} \tag{15.6}$$

From the properties of $\omega_N(\gamma;\Delta)$, mentioned above, see (15.5) it is seen that the second integral in (15.6) can be evaluated by contour integration in the right half plane, cf. the evaluation of (9.2). It is seen that for Re $r \geq 0$,

$$\frac{1}{2\pi i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}[\log\{1+\omega_N(\eta;\Delta)\}]\frac{r}{\eta-r}\,\frac{\mathrm{d}\eta}{\eta} = -\log\{1+\omega_N(r;\Delta)\}. \tag{15.7}$$

Next, we consider the third integral in (15.6). The expression (15.6) is obtained from (6.8) with $[1-\mathrm{E}\{\mathrm{e}^{-\eta\Delta\mathbf{u}}\}]\eta\Delta\mathrm{E}\{\mathbf{u}\}$ given by (15.1). It is actually a principal value integral at infinity, i.e. for every $\varepsilon > 0$ an $R(\varepsilon) > 0$ exists such that, cf. (8.4), for Re $r > 0$, with $F_\varepsilon := \{\eta : |\eta| > R(\varepsilon),\ \mathrm{Re}\,\eta = 0\}$,

$$\Big|\frac{1}{2\pi i}\int_{\eta\in F_\varepsilon}[\log\{1+\frac{\gamma\eta O(\Delta^{2-\nu})}{1+\omega_N(\eta;\Delta)}\}]\frac{r}{\eta-r}\,\frac{\mathrm{d}\eta}{\eta}\Big| < \varepsilon. \tag{15.8}$$

Obviously we have for Re $r > 0$, $\Delta = \Delta(a)\downarrow 0$,

$$\frac{1}{2\pi i}\int_{-\mathrm{i}R(\varepsilon)}^{\mathrm{i}R(\varepsilon)}[\log\{1+\frac{\gamma\eta O(\Delta^{2-\nu})}{1+\omega_N(\eta;\Delta)}\}]\,\frac{r}{\eta-r}\,\frac{\mathrm{d}\eta}{\eta} = O(\Delta^{2-\nu}). \tag{15.9}$$

Hence from (15.6), . . . , (15.9) we obtain: for Re $r > 0$ and $\Delta = \Delta(a)\downarrow 0$,

$$H(r,\Delta) = -\log\{1+\omega_N(r;\Delta)\} + O(\Delta^{2-\nu}), \tag{15.10}$$

and so from (6.9) and (15.10): for Re $r > 0$, $\Delta\downarrow 0$,

$$\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\} = \frac{1}{1+\omega_N(r;\Delta)}\{1+O(\Delta^{2-\nu})\}, \tag{15.11}$$

or by using (15.5),

$$\Big|\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\} - \frac{1}{1+\omega_N(r;\Delta)}\Big| = \frac{O(\Delta^{2-\nu})}{|1+\omega_N(r;\Delta)|} = O(\Delta^{2-\nu}). \tag{15.12}$$

Remark 15.1. The lefthand side of (15.9) is as a function of Δ of order $O(\Delta^{2-\nu})$, as a function of r it is regular for Re $r > 0$ and Δ sufficiently close to zero because then the argument of the logarithm is never zero, cf. (15.5). The relation (15.12) shows that there will be no need to stress the dependence of the order term in (15.12) on r. □

From (15.12) it is seen by letting $\Delta\downarrow 0$, i.e. $a\uparrow 1$, that the first statement of Theorem 9.1 is again obtained.

The relation (15.12) is used to derive an asymptotic expression for the event $\mathbf{x}/\gamma > t$ for $t \to \infty$. Herefor we start from

$$\Big|\frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\}}{r} - \frac{1}{r}\,\frac{\omega_N(r;\Delta)}{1+\omega_N(r;\Delta)}\Big| = O(\Delta^{2-\nu}), \tag{15.13}$$

for Re $r \geq 0$, $\Delta\downarrow 0$. Note that the continuity of both terms in the lefthand side of (15.12) for Re $r \geq 0$, implies that (15.12) also holds for Re $r \geq 0$.

The derivation of the asymptotic expression proceeds along the same lines as in Section 12; i.e. we have to derive an asymptotic expression for the second term in (15.13) for $|r|\downarrow 0$, $|\arg r| < \pi$ on the r-plane slitted along $r < 0$ and then to use Theorem 2 of [10], vol. II, p. 159, which can be applied here.

From (15.4) we have: for Re $r \geq 0$, $|r|\downarrow 0$,

$$\frac{1}{r}\omega(r;\Delta) = O((\gamma r)^{\nu-2}), \tag{15.14}$$

$$\frac{1}{r}\,\frac{\omega(r;\Delta)}{1+\omega(r;\Delta)} = \frac{1}{r}\omega(r;\Delta) - \frac{1}{r}\omega^2(r;\Delta) + O((\gamma r)^{3\nu-4}),$$

with

$$\frac{1}{r\gamma}\omega(r;\Delta) = aq_{21}d_{21}(\gamma r)^{\nu-2} + \sum_{i=2}^{N} aq_{2i}d_{2i}(\gamma r)^{\nu_{2i}-2}\Delta^{\nu_{2i}-\nu}. \tag{15.15}$$

From (15.13) and (15.15) we have for Re $r > 0$, $|r|\downarrow 0$,

$$\begin{aligned}\frac{1-\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}}\}}{r} &= \int_0^\infty \mathrm{e}^{-rt}\,\Pr\{\mathbf{x}/\gamma \geq t\}\mathrm{d}t = \frac{1}{r}\omega(r/\gamma;\Delta)\\ &= \frac{1}{r}\omega(r/\gamma;\Delta) + O(r^{\nu-2}) + O(\Delta^{2-\nu})\\ &= aq_{21}d_{21}r^{\nu-2} + \sum_{i=2}^{N} aq_{2i}d_{2i}r^{\nu_{2i}-2}\Delta^{\nu_{2i}-\nu} + O(r^{\nu-2}) + O(\Delta^{2-\nu}).\end{aligned} \tag{15.16}$$

We may apply Theorem 2 of [10, p. 159] to derive from (15.16) an asymptotic expression, since it is readily seen that the conditions to hold in order to apply this theorem actually do hold, cf. [9]. To apply this theorem note that the term r^{λ} in the L-transform for $|r|\downarrow 0$ yields a term $t^{-\lambda-1}/\Gamma(-\lambda)$ in the asymptotic representation for $t\to\infty$ of the original function; whenever λ is a nonnegative integer then $\Gamma^{-1}(-\lambda) := 0$.

We obtain from (15.16); for $t\to\infty$, $\Delta\downarrow 0$, or $a\uparrow 1$,

$$\begin{aligned}\Pr\{\Delta\mathbf{x}/\gamma \geq t\} = {}& aq_{21}d_{21}\frac{t^{-(\nu-1)}}{\Gamma(2-\nu)} + \sum_{i=2}^{N-1} aq_{2i}d_{2i}\frac{t^{-(\nu_{2i}-1)}}{\Gamma(2-\nu_{2i})}\Delta^{\nu_{2i}-\nu} + O(t^{2-\nu})\\ &+O(\Delta^{2-\nu}) = aq_{21}d_{21}\frac{t^{-(\nu-1)}}{\Gamma(2-\nu)}\{1 + \sum_{i=1}^{N-1}\frac{q_{2i}d_{2i}}{q_{21}d_{21}}\,\frac{\Gamma(2-\nu)}{\Gamma(2-\nu_{2i})}\\ &t^{-(\nu_{2i}-\nu)}\Delta^{\nu_{2i}-\nu} + O(t^{1-\nu}))\} + O(\Delta^{2-\nu}).\end{aligned} \tag{15.17}$$

It should be noted that all terms in the righthand side of (15.17) are positive, and hence it is seen that for $a\uparrow 1$ the heavy-traffic Theorem 9.1 underestimates the probability in (15.17). However, (15.17) over estimates this probability. This may be seen by using the second relation of (15.14) in (15.13) and then proceeding as above to obtain the asymptotic relation for $t\to\infty$. We shall not give here the expression for this more detailed asymptotic relation but do note the '−' sign in the second relation of (15.14).

In the expression for the heavy-traffic distribution of $\hat{\mathbf{x}}/\gamma$ only the index ν of the most heavy tail of the distribution of the $\mathbf{u}_i$, $i = 1, \dots, N-1$, occurs. In the asymptotic expression (15.17) with $a \uparrow 1$ or $\Delta(a) \downarrow 0$ all the indices of the righthand tails of the $\mathbf{u}_i$ occur, and so (15.17) may be used to assess the influence of the less heavy tails. For instance if ν_{22} is only slightly larger than ν then $\Delta^{\nu_{22}-\nu}$ for $0 < \Delta < 1$ may still be not so small, i.e. close to one, and the same applies for $r^{-(\nu_{22}-\nu)}$ even for t not so large, i.e. the second term of the expression in curled brackets in the righthand side of (15.17) may not be small as compared to one, in particular if also $q_{22}d_{22} > q_{21}d_{21}$.

The relation (15.17) is actually an important result of the present study. In [13] an analogous relation has been derived, and it turned out that it was most useful as an approximation even for a not so close to one. It is conjectured that this also applies here. However, its verification requires the numerical inversion of the expression for $H(r)$, cf. (6.4) and (6.6), see therefor [15].

16. The traffic model

The random walk $\mathbf{x}_n$, $n = 0, 1, \dots$, introduced in Section 1 is used to model the actual waiting time of the nth arriving customer in a GI/G/1 queueing model with $\boldsymbol{\tau}_n$ the service time of this customer and $\boldsymbol{\sigma}_{n+1}$ the interarrival time between the nth and $(n+1)$th arriving customer. In the recurrence relation (1.2) we then have $\mathbf{u}_n = \boldsymbol{\tau}_n - \boldsymbol{\sigma}_{n+1}$.

The $\boldsymbol{\tau}_n$, $n = 0, 1, \dots$, and the $\boldsymbol{\sigma}_n$, $n = 0, 1, \dots$, are both sequences of i.i.d. stochastic variables, moreover, these sequences are assumed to be independent sequences. Let $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ be independent variables with distributions $B(\cdot)$ and $A(\cdot)$, respectively, where $B(\cdot)$ is the distribution of $\boldsymbol{\tau}_n$ and $A(\cdot)$ that of $\boldsymbol{\sigma}_n$. Then $\mathbf{u}$ in (2.5) may be written as

$$\mathbf{u} = \boldsymbol{\tau} - \boldsymbol{\sigma}. \tag{16.1}$$

We also have, cf. (4.2),

$$\mathbf{u} = \mathbf{v} - \mathbf{w}. \tag{16.2}$$

In order to be able to apply the results of the analysis of the preceding section we first need the relations which express the distribution of $\mathbf{v}$ in that of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, and similarly for $\mathbf{w}$. Concerning these relations we formulate the following

Lemma 16.1 *For* $0 < \mathrm{E}\{\boldsymbol{\tau}\} < \mathrm{E}\{\boldsymbol{\sigma}\}$ *holds:*

$$\Pr\{\mathbf{v} \geq v\} = \Pr\{\boldsymbol{\tau} \geq v\} \int_0^\infty \frac{1 - B(v+\sigma)}{1 - B(v)}\, dA(\sigma), \quad v \geq 0, \tag{16.3}$$

$$\Pr\{\mathbf{w} \geq w\} = \Pr\{\boldsymbol{\tau} \geq w\} \int_0^\infty \frac{1 - A(w+\tau)}{1 - A(w)}\, dB(\tau), \quad w \geq 0,$$

$$\mathrm{E}\{\mathbf{v}\} = \mathrm{E}\{\boldsymbol{\tau}\} \Pr\{\tilde{\boldsymbol{\tau}} \geq \boldsymbol{\sigma}\}, \tag{16.4}$$

$$\mathrm{E}\{\mathbf{w}\} = \mathrm{E}\{\boldsymbol{\sigma}\} \Pr\{\tilde{\boldsymbol{\sigma}} \geq \boldsymbol{\tau}\};$$

here $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\sigma}}$ have the excess distribution of that of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, respectively; further,

$$\mathrm{E}\{\mathbf{v}\} - \mathrm{E}\{\boldsymbol{\tau}\} = \mathrm{E}\{\mathbf{w}\} - \mathrm{E}\{\boldsymbol{\sigma}\} < 0. \tag{16.5}$$

PROOF. The first relation of (16.3) follows immediately from

$$\mathbf{v} \geq v \Longleftrightarrow \max(0, \boldsymbol{\tau} - \boldsymbol{\sigma}) \geq v, \quad v > 0,$$

and analogously for the second relation.

The relations in (16.4) follow from those in (16.3) by using

$$\mathrm{E}\{\mathbf{v}\} = \int_0^\infty \Pr\{\mathbf{v} \geq v\}\mathrm{d}v,$$

and

$$\Pr\{\tilde{\boldsymbol{\tau}} < \tau\} := \frac{1}{\mathrm{E}\{\boldsymbol{\tau}\}} \int_0^\tau [1 - B(y)]\mathrm{d}y,$$

and the analogous expressions for $\mathbf{w}$ and $\boldsymbol{\sigma}$. Finally, (16.5) follows from (16.1), (16.2) and (16.4). □

Put, cf. (16.5),

$$a := \frac{\mathrm{E}\{\mathbf{v}\}}{\mathrm{E}\{\mathbf{u}\}} < 1 \quad \text{and} \quad \tilde{a} := \frac{\mathrm{E}\{\boldsymbol{\tau}\}}{\mathrm{E}\{\boldsymbol{\sigma}\}} < 1, \tag{16.6}$$

then from (16.4),

$$a = \tilde{a}\frac{\Pr\{\tilde{\boldsymbol{\tau}} \geq \boldsymbol{\sigma}\}}{\Pr\{\tilde{\boldsymbol{\sigma}} \geq \boldsymbol{\tau}\}} < 1, \tag{16.7}$$

and from (16.5),

$$\frac{1-\tilde{a}}{1-a} = \Pr\{\tilde{\boldsymbol{\sigma}} \geq \boldsymbol{\tau}\} \quad \text{and} \quad \frac{a}{\tilde{a}}\frac{1-\tilde{a}}{1-a} = \Pr\{\tilde{\boldsymbol{\tau}} \geq \sigma\}. \tag{16.8}$$

Note that a simple calculation shows that

$$\begin{aligned} \Pr\{\tilde{\boldsymbol{\sigma}} \geq \boldsymbol{\tau}\} &= 1 - \frac{1}{\mathrm{E}\{\boldsymbol{\sigma}\}} \int_0^\infty \{1 - A(\sigma)\}\{1 - B(\sigma)\}\mathrm{d}\sigma, \\ \Pr\{\tilde{\boldsymbol{\tau}} \geq \boldsymbol{\sigma}\} &= 1 - \frac{1}{\mathrm{E}\{\boldsymbol{\tau}\}} \int_0^\infty \{1 - A(\sigma)\}\{1 - B(\sigma)\}\mathrm{d}\sigma, \end{aligned} \tag{16.9}$$

and

$$a < \tilde{a}, \quad a \uparrow 1 \Longleftrightarrow \tilde{a} \uparrow 1, \quad a = 1 \Longleftrightarrow \tilde{a} = 1. \tag{16.10}$$

The formulas (16.6), . . . , (16.10) describe some relations between a and $\tilde{a}$.
Next we derive the relations between the distributions of $\mathbf{v}$ and $\boldsymbol{\tau}$, in particular those between

$$\Pr\{\mathbf{v} \geq v\} \quad \text{and} \quad \Pr\{\boldsymbol{\tau} \geq v\} \quad \text{for} \quad v \to \infty,$$

for the case that the distribution $B(\cdot)$ of $\boldsymbol{\tau}$ has a heavy tail, cf. (3.1); analogous relations for $\mathbf{w}$ and $\boldsymbol{\sigma}$ are then obtained.
From Lemma 4.1 we obtain by contour integration in the right semi-plane for $\operatorname{Re}\rho > 0$,

$$\frac{a}{a-1}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} = \frac{-1}{2\pi i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{1-\mathrm{E}\{\mathrm{e}^{-\eta\mathbf{u}}\}}{\eta\mathrm{E}\{\mathbf{u}\}}\,\frac{\mathrm{d}\eta}{\eta-\rho}. \tag{16.11}$$

The following identity is readily verified: for $\operatorname{Re}\rho = 0$,

$$\begin{aligned}\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} &= \frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}\mathrm{E}\{\mathrm{e}^{\rho\boldsymbol{\sigma}}\}}{\rho\mathrm{E}\{\mathbf{u}\}}\\ &= \frac{\mathrm{E}\{\boldsymbol{\tau}\}}{\mathrm{E}\{\mathbf{u}\}}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}}{\rho\mathrm{E}\{\boldsymbol{\tau}\}} - \frac{\mathrm{E}\{\boldsymbol{\sigma}\}}{\mathrm{E}\{\mathbf{u}\}}\,\frac{1-\mathrm{E}\{\mathrm{e}^{\rho\boldsymbol{\sigma}}\}}{-\rho\mathrm{E}\{\boldsymbol{\sigma}\}}\\ &+ \rho\,\frac{\mathrm{E}\{\boldsymbol{\tau}\}\mathrm{E}\{\boldsymbol{\sigma}\}}{\mathrm{E}\{\tau\}-\mathrm{E}\{\boldsymbol{\sigma}\}}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}}{\rho\mathrm{E}\{\boldsymbol{\tau}\}}\,\frac{1-\mathrm{E}\{\mathrm{e}^{\rho\boldsymbol{\sigma}}\}}{-\rho\mathrm{E}\{\boldsymbol{\sigma}\}}.\end{aligned} \tag{16.12}$$

From (16.11) and (16.12) we can eliminate the expression in the righthand side of (16.11) by using again contour integration in the right semi-plane. We obtain by using (16.8): for $\operatorname{Re}\rho > 0$,

$$\frac{a}{a-1}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} = \frac{\tilde{a}}{\tilde{a}-1}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}}{\rho\mathrm{E}\{\boldsymbol{\tau}\}} + \frac{\tilde{a}}{\tilde{a}-1}\,I(\rho), \tag{16.13}$$

$$I(\rho) := \frac{1}{2\pi i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{1-\mathrm{E}\{\mathrm{e}^{-\eta\boldsymbol{\tau}}\}}{\eta\mathrm{E}\{\boldsymbol{\tau}\}}\,\frac{1-\mathrm{E}\{\mathrm{e}^{\eta\boldsymbol{\sigma}}\}}{-\eta\mathrm{E}\{\boldsymbol{\sigma}\}}\,\frac{\eta}{\eta-\rho}\,\mathrm{d}\eta. \tag{16.14}$$

Analogously, we have: for $\operatorname{Re}\rho > 0$,

$$\frac{1}{a-1}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho\mathrm{E}\{\mathbf{w}\}} = \frac{1}{\tilde{a}-1}\,\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\sigma}}\}}{\rho\mathrm{E}\{\boldsymbol{\sigma}\}} - \frac{\tilde{a}}{\tilde{a}-1}\,I(-\rho). \tag{16.15}$$

The first two terms in (16.13) are continuous for $\operatorname{Re}\rho \geq 0$, so by letting $\rho\downarrow 0$, we obtain,

$$\begin{aligned}\frac{\tilde{a}}{a}\,\frac{1-a}{1-\tilde{a}}\{1+\lim_{\rho\downarrow 0} I(\rho)\} &= 1,\\ \frac{1-a}{1-\tilde{a}}\{1-\tilde{a}\lim_{\rho\downarrow 0} I(-\rho)\} &= 1.\end{aligned} \tag{16.16}$$

From (16.13), . . . , (16.16), we obtain for Re $\rho \geq 0$,

$$1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} = \frac{\tilde{a}}{a}\,\frac{1-a}{1-\tilde{a}}\,\{1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}}{\rho\mathrm{E}\{\boldsymbol{\tau}\}}\} + \frac{I(0+)-I(\rho)}{1+I(0+)}\,, \tag{16.17}$$

$$1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho\mathrm{E}\{\mathbf{w}\}} = \frac{1-a}{1-\tilde{a}}\,\{1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\sigma}}\}}{\rho\mathrm{E}\{\boldsymbol{\sigma}\}}\} + \tilde{a}\frac{I(0-)-I(-\rho)}{1+\tilde{a}I(0-)}\,.$$

Replace in the integrand of (16.14),

$$\frac{\eta}{\eta-\rho} \quad \text{by} \quad 1+\frac{\rho}{\eta-\rho}\,,$$

it is then seen that: for Re $\rho \geq 0$, and $|\rho|\downarrow 0$,

$$I(0+)-I(\rho)=O(\rho), \tag{16.18}$$

$$I(0-)-I(-\rho)=O(\rho),$$

The relations (16.17) express the L.S.-transforms of the distribution of $\mathbf{v}$ and of that of $\mathbf{w}$ in terms of the L.S.-transforms of the distribution of the service – and of the interarrival time $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, respectively.

17. Heavy-tailed service- and interarrival time distribution

In this section it is assumed that both the service time– and the interarrival time distribution are heavy-tailed, i.e. for Re $\rho \geq 0$,

$$1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\tau}}\}}{\rho\mathrm{E}\{\boldsymbol{\tau}\}} = \tilde{g}_2(\gamma\rho)+\tilde{c}_2(\gamma\rho)^{\tilde{\nu}_2-1}\tilde{L}_2(\gamma\rho), \tag{17.1}$$

$$1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\boldsymbol{\sigma}}\}}{\rho\mathrm{E}\{\boldsymbol{\sigma}\}} = \tilde{g}_1(\gamma\rho)+\tilde{c}_1(\gamma\rho)^{\tilde{\nu}_1-1}\tilde{L}_1(\gamma\rho),$$

with the symbols in the righthand side of (17.1) defined as the corresponding ones in (3.2); however, we exclude here the case $\nu = 2$, so that

$$1<\tilde{\nu}_2<2, \quad 1<\tilde{\nu}_1<2. \tag{17.2}$$

From (16.17) and (17.1) we obtain: for Re $\rho \geq 0$,

$$\begin{aligned} &\text{i.} \quad 1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{v}}\}}{\rho\mathrm{E}\{\mathbf{v}\}} = \frac{\tilde{a}}{a}\,\frac{1-a}{1-\tilde{a}}\,\tilde{g}_2(\gamma\rho)+\frac{\tilde{a}}{a}\,\frac{1-a}{1-\tilde{a}}\,\tilde{c}_2(\gamma\rho)^{\tilde{\nu}_2-1}\tilde{L}_2(\gamma\rho)+\frac{I(0+)-I(\rho)}{1+I(0+)}\,, \\ &\text{ii.} \quad 1-\frac{1-\mathrm{E}\{\mathrm{e}^{-\rho\mathbf{w}}\}}{\rho\mathrm{E}\{\mathbf{w}\}} = \frac{1-a}{1-\tilde{a}}\,\tilde{g}_1(\gamma\rho)+\frac{1-a}{1-\tilde{a}}\,\tilde{c}_1(\gamma\rho)^{\tilde{\nu}_1-1}\tilde{L}_1(\gamma\rho)+\tilde{a}\frac{I(0-)-I(-\rho)}{1+\tilde{a}I(0-)}\,. \end{aligned} \tag{17.3}$$

By writing in (16.14):

$$\frac{\eta}{\eta-\rho} = 1 + \frac{\rho}{\eta-\rho} \;,$$

it is seen that: for Re $\rho \geq 0$, $|\rho|\downarrow 0$,

$$\frac{I(0+) - I(\rho)}{1 + I(0+)} = O(\rho), \tag{17.4}$$

and hence this term is of the same order as the first term in the righthand side of (17.3) and of smaller order than the second term in the righthand side of (17.3) since $1 < \tilde{v}_2 < 2$, cf. (17.3). However, with the information so far obtained it is not yet possible to decide whether the lefthand side of (17.3) can be characterized as in (4.10), i.e. with $g_2(\cdot)$, c_2, $L_2(\cdot)$ satisfying the conditions (3.2). This is due to the term (17.4) in (17.3). For c_2 and $L_2(\cdot)$ there is no problem. The problem is that in general the term in (17.4) is not necessarily regular for Re $\rho > -\delta$ for a $\delta > 0$. Similar remarks apply for the second expression in (17.3).

Actually, for our asymptotic analysis this lack of regularity at $\rho = 0$ does not prevent an asymptotic analysis as shown in Section 7. There we needed of $g_{2i}(\cdot)$ only that it is of $O(\rho)$ at $\rho = 0$. The last term in (17.3)i, as well as that in (17.3)ii has this property, and similar as the $g_{2i}(\cdot)$, $g_{1i}(\cdot)$, they do not contribute to the ultimate limiting results, see herefor the next section.

18. Comparison of limits

In this section we shall compare the limiting behaviour of the jump vector $\mathbf{u}$ when starting from the heavy-tail representation of $\mathbf{v}$ and $\mathbf{w}$ to that obtained by starting from a heavy-tail description of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$.

We start with the heavy-tail description of the distribution of $\mathbf{v}$ and $\mathbf{w}$, as given by (4.6), so that we have, cf. (4.10): for $a < 1$ and Re $\rho = 0$,

$$\begin{aligned} 1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} &= \frac{a}{a-1}\, g_2(\gamma\rho) - \frac{1}{a-1}\, g_1(\gamma\bar{\rho}) \\ &+ \frac{ac_2}{a-1}\, (\gamma\rho)^{\nu_2-1} L_2(\gamma\rho) - \frac{c_1}{a-1}\, (\gamma\bar{\rho})^{\nu_1-1} L_1(\gamma\bar{\rho}). \end{aligned} \tag{18.1}$$

Put

$$L(x) := ac_2 L_2(x) + (-1)^{\nu} c_1 L_1(x), x \geq 0, \tag{18.2}$$

where

$$\nu := \min(\nu_1, \nu_2). \tag{18.3}$$

Let $\Delta(a)$ be that zero of

$$x^{v-1}|L(x)| = 1 - a, \quad x > 0, \tag{18.4}$$

which tends to zero for $a \uparrow 1$.

Put

$$\rho = r\Delta(a), \tag{18.5}$$

then with $\Delta \equiv \Delta(a)$: for Re $r = 0$,

$$\begin{aligned} 1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}} &= \frac{1}{a-1}\,[ag_2(\gamma r\Delta) - g_1(\gamma\bar{r}\Delta)] \\ &- ac_2(\gamma r)^{v_2-1}\Delta^{\nu_2-\nu}\frac{L_2(\gamma r\Delta)}{|L(\Delta)|} + c_1(\gamma\bar{r})^{\nu_1-1}\Delta^{\nu_1-\nu}\,\frac{L_1(\gamma\bar{r}\Delta)}{|L(\Delta)|}. \end{aligned} \tag{18.6}$$

From $g_2(0)$ and the regularity of $g_2(\rho)$ at $\rho = 0$ follows by using (18.5) that: for Re $r = 0$,

$$\frac{g_2(\gamma r\Delta)}{1-a} = O(\gamma r\frac{\Delta}{1-a}) = O(\gamma r\Delta^{2-\nu}) \quad \text{for} \quad \Delta\downarrow 0,$$

and similarly for $g_1(\gamma\bar{r}\Delta)$; note that $L(\Delta) \neq 0$ for $\Delta\downarrow 0$, cf. (3.2)iv. By letting $a\uparrow 1$ or equivalently, $\Delta\downarrow 0$, we obtain: for Re $r = 0$,

$$\lim_{a\uparrow 1}\{1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}}\} = 1 + \varepsilon_{2\nu}d_2(\gamma r)^{\nu-1} - \varepsilon_{1\nu}\mathrm{d}_1(\gamma\bar{r})^{\nu-1} \tag{18.7}$$

where,

$$\begin{aligned} \varepsilon_{2\nu} &= 1 \quad \text{if} \quad \nu = \nu_2, \quad & \varepsilon_{1\nu} &= 1 \quad \text{if} \quad \nu = \nu_1, \\ &= 0 \quad \text{if} \quad \nu \neq \nu_2, & &= 0 \quad \text{if} \quad \nu \neq \nu_1, \end{aligned} \tag{18.8}$$

and

$$d_2 := c_2 \lim_{\Delta\downarrow 0} \frac{L_2(\gamma r\Delta)}{|L(\Delta)|}, \quad d_1 := c_1 \lim_{\Delta\downarrow 0} \frac{L_1(\gamma\bar{r}\Delta)}{|L(\Delta)|}.$$

Next we start from the heavy-traffic representation (17.1) for $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ with

$$\tilde{\nu}_2 = \nu_2 \quad \text{and} \quad \tilde{\nu}_1 = \nu_1. \tag{18.9}$$

From (17.1) we have derived the expressions (17.3) for the heavy-traffic representations of the distributions of $\mathbf{v}$ and $\mathbf{w}$ in terms of those of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. We substitute these expressions (17.3) into (4.5). This leads to: for Re $\rho = 0$,

$$\begin{aligned} 1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-\rho\mathbf{u}}\}}{\rho\mathrm{E}\{\mathbf{u}\}} &= \frac{\tilde{a}}{\tilde{a}-1}\,\tilde{g}_2(\gamma\rho) - \frac{1}{\tilde{a}-1}\,\tilde{g}_1(\gamma\bar{\rho}) \\ &+ \frac{\tilde{a}}{\tilde{a}-1}\,\tilde{c}_2(\gamma\rho)^{\nu_2-1}\tilde{L}_2(\gamma\rho) - \frac{1}{\tilde{a}-1}\,\tilde{c}_1(\gamma\bar{\rho})^{\nu_1-1}\tilde{L}_1(\gamma\bar{\rho}) \\ &+ \frac{\tilde{a}}{\tilde{a}-1}\,\frac{I(0+) - I(\rho)}{1 + I(0+)} - \frac{\tilde{a}}{\tilde{a}-1}\,\frac{I(0-) - I(\bar{\rho})}{1 + I(0+)}. \end{aligned} \tag{18.10}$$

Put

$$\tilde{L}(x) = \tilde{a}\tilde{c}_2\tilde{L}_2(x) + (-1)^\nu \tilde{c}_1\tilde{L}_1(x), \quad x \geq 0. \tag{18.11}$$

Let $\Delta(\tilde{a})$ be that zero of

$$x^{\nu-1}|\tilde{L}(x)| = 1 - \tilde{a}, \quad x > 0, \tag{18.12}$$

which tends to zero for $\tilde{a} \uparrow 1$.

Because $\tilde{g}_2(\gamma\rho)$, $\tilde{g}_1(\gamma\rho)$ are both of $O(\rho)$ for $\mathrm{Re}\,\rho \geq 0$ and $|\rho| \to 0$, and also $I(0+) - I(\rho)$, and $I(0-) - I(\tilde{\rho})$ are both of $O(\rho)$ for $\mathrm{Re}\,\rho \geq 0$, and $|\rho| \to 0$, it follows with

$$\rho = r\tilde{\Delta}(\tilde{a}), \quad \mathrm{Re}\,r = 0,$$

that for $\mathrm{Re}\,r = 0$,

$$\lim_{\tilde{a}\uparrow 1} \{1 - \frac{1 - \mathrm{E}\{\mathrm{e}^{-r\tilde{\Delta}\mathbf{u}}\}}{r\tilde{\Delta}u}\} = 1 + \varepsilon_{2\nu}\tilde{d}_2(\gamma r)^{\nu_2-1} - \varepsilon_{1\nu}\tilde{d}_1(\gamma\bar{r})^{\nu_1-1}, \tag{18.13}$$

where

$$\tilde{d}_2 := \tilde{c}_2 \lim_{\tilde{\Delta}\downarrow 0} \frac{\tilde{L}_2(\gamma r\tilde{\Delta})}{|\tilde{L}(\tilde{\Delta})|}\ , \quad \tilde{d}_1 := \tilde{c}_1 \lim_{\tilde{\Delta}\downarrow 0} \frac{\tilde{L}_1(\gamma\bar{r}\tilde{\Delta})}{|\tilde{L}(\tilde{\Delta})|}\ .$$

We next compare the results (18.7) to (18.13). Take $c_j = \tilde{c}_j$, $L_j(\cdot) = \tilde{L}_j(\cdot)$, $j = 1,2$, and $g_j(\cdot) = \tilde{g}_j(\cdot)$, $j = 1,2$, but instead of (3.2)iii, it is assumed that $\tilde{g}_j(\rho)$ is regular for $\mathrm{Re}\,\rho > 0$, continuous for $\mathrm{Re}\,\rho \geq 0$ and $O(\rho)$ for $|\rho| \to 0$. It is then readily verified that the results (18.13) and (18.7) are equivalent, because of (16.8), . . . , (16.10) and the fact that

$$\Delta(a)/\tilde{\Delta}(\tilde{a}) \to 1 \quad \text{for} \quad \tilde{a} \uparrow 1.$$

In traffic modelling the distributions of the service time and of the interarrival time are usually the input data. From the results of the present section it is seen that an analysis based on $\mathbf{v} = \max(0, \boldsymbol{\tau} - \boldsymbol{\sigma})$ and $\mathbf{w} = -\min(0, \boldsymbol{\tau} - \boldsymbol{\sigma})$ leads to equivalent heavy-traffic results as it could be expected but also to a somewhat simpler analysis, in particular if one is interested in asymptotic heavy-traffic results for mixed traffic cf. Section 5.

19. Heavy traffic approximation for heavy-tailed downward jump

In this section it is assumed that the jump vector does not have a heavy upward tail. This means that we have to take

$$\mathrm{d}_{2\mathrm{i}} = 0,\ i = 1, \dots, N-1. \tag{19.1}$$

This implies that (14.5) can be rewritten as: for $\mathrm{Re}\,r = 0$ and $\Delta = \Delta(a)\downarrow 0$,

$$\frac{1 - \mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{u}}\}}{r\Delta\mathrm{E}\{\mathbf{u}\}} = 1 - \sum_{i=1}^{N-1} q_{1\mathrm{i}}\mathrm{d}_{1\mathrm{i}}(\gamma\bar{r})^{\nu_{1\mathrm{i}}-1}\Delta^{\nu_{1\mathrm{i}}-\nu} + \gamma r O(\Delta^{2-\nu}). \tag{19.2}$$

For the sake of simplicity we take $N = 3$ and $\nu_{11} < \nu_{12}$. From the result to be obtained it is easily seen how the result for general $N = 4, 5, \dots$, will read, cf. Remark 19.1.

Put, cf. (8.8),

$$B_{11} := q_{11}\mathrm{d}_{11}, \quad B_{12} := q_{12}\mathrm{d}_{12},$$
$$\nu = \min(\nu_{11}, \nu_{12}) = \nu_{11} < \nu_{12} \text{ and } \omega_{12} := B_{11}^{1/(\nu-1)}. \tag{19.3}$$

Next we consider the function $H(r\Delta)$, cf. (6.8), with $\Delta = \Delta(a), a < 1$ and with the integrand given by (19.2). Hence for $\mathrm{Re}\, r > 0$ and $\Delta(a)\downarrow 0$,

$$\begin{aligned} H(\gamma r\Delta) &= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 - B_{11}\bar{\eta}^{\nu-1} - B_{12}\bar{\eta}^{\nu_{12}-1}\Delta^{\nu_{12}-\nu} + \gamma\eta O(\Delta^{2-\nu})\}]\frac{r}{\eta - r}\frac{\mathrm{d}\eta}{\eta} \\ &= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{1 - B_{11}\eta^{\nu-1} - B_{12}\eta^{\nu_{12}-1}\Delta^{\nu_{12}-\nu} + \gamma\bar{\eta} O(\Delta^{2-\nu})\}]\frac{r}{\eta + r}\frac{\mathrm{d}\eta}{\eta}. \end{aligned} \tag{19.4}$$

Note that

$$1 - B_{11}\eta^{\nu-1} - B_{12}\eta^{\nu_{12}-1}\Delta^{\nu_{12}-\nu},$$

is regular for $\mathrm{Re}\,\eta > 0$. For $\Delta = \Delta(a) > 0$ but sufficiently small it is seen that

$$B_{11}\eta^{\nu-1} = 1 - B_{12}\eta^{\nu_{12}-1}\Delta^{\nu_{12}-\nu}, \ \mathrm{Re}\,\eta > 0, \tag{19.5}$$

has only one zero in $\mathrm{Re}\,\eta > 0$, note that $\nu > \nu_{12}$ and apply Rouché's theorem to a contour consisting of the interval $(-\mathrm{i}R, \mathrm{i}R)$ and the semi circle in the right half plane with center the origin and radius R with $R \to \infty$.

Denote this zero by η_0, note that $\eta_0 > 0$, and so

$$\begin{aligned} \eta_0 &= \frac{1}{\omega_{11}}[1 - B_{12}\eta_0^{\nu_{12}-1}\Delta^{\nu_{12}-\nu}]^{\frac{1}{\nu-1}} \\ &= \frac{1}{\omega_{11}}[1 - \frac{B_{12}}{\nu-1}\eta_0^{\nu_{12}-1}\Delta^{\nu_{12}-\nu}] + \frac{1}{\omega_{11}}o(\Delta^{\nu_{12}-\nu}) \\ &= \frac{1}{\omega_{11}}[1 - \frac{B_{12}}{\nu-1}\Delta^{\nu_{12}-\nu}(\frac{1}{\omega_{11}})^{\nu_{12}-1}(1 + O(\Delta^{\nu_{12}-\nu})] \\ &= \frac{1}{\omega_{11}}[1 - \frac{B_{12}}{\nu-1}(\frac{1}{B_{11}})^{\frac{\nu_{12}-1}{\nu-1}}\Delta^{\nu_{12}-\nu}] + O(\Delta^{\nu_{12}-\nu}). \end{aligned} \tag{19.6}$$

We next rewrite (19.1): for $\mathrm{Re}\, r > 0, \ \Delta = \Delta(a)\downarrow 0$,

$$\begin{aligned} H(\gamma r\Delta) &= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\{\frac{1 - B_{11}\eta^{\nu-1} - B_{12}\eta^{\nu_{12}-1}\Delta^{\nu_{12}-\nu}}{\eta - \eta_0}\} + \log(\eta - \eta_0)]\frac{r}{\eta + r}\frac{\mathrm{d}\eta}{\eta} \\ &= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log(\eta - \eta_0)\frac{r}{\eta + r}\frac{\mathrm{d}\eta}{\eta} = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log(\eta - \eta_0)\mathrm{d}\log\frac{\eta}{r + \eta} \\ &= [\log\frac{\eta}{\eta + r}\log(\eta - \eta_0)]_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} - \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [\log\frac{\eta}{\eta + r}]\frac{\mathrm{d}\eta}{\eta - \eta_0} \\ &= \log\frac{\eta_0}{\eta_0 + r} = \log\frac{1}{1 + r/\eta_0}. \end{aligned} \tag{19.7}$$

In (19.7) the first equality sign follows immediately from (6.8) and (19.2). To motivate the second equality sign in (19.7) note that the first logarithm of the integrand is regular for $\mathrm{Re}\,\eta > 0$, and so since $\mathrm{Re}\,r > 0$ its contribution to the integral is zero, as it is readily seen by contour integration in the right half plane. The third equality sign follows by differentiation of $\log\{\eta/(r+\eta)\}$. The fourth equality sign stems from partial integration. Here the limits of the term between square brackets for $\eta \to \pm\mathrm{i}\infty$ are zero. Since $\log\{\eta/(\eta+r)\}, \mathrm{Re}\,r > 0$, is regular for $\mathrm{Re}\,\eta > 0$, the last integral in (19.5) can be evaluated by contour integration in the right half plane and this leads to the fifth equality sign, and the last equality sign is trivial.

As in Section 10 we obtain from (6.9), for $\mathrm{Re}\,r \geq 0, \Delta \equiv \Delta(a)\downarrow 0$

$$\mathrm{E}\{\mathrm{e}^{-r\Delta\mathbf{x}/\gamma}\} = \frac{1}{1 + r/\eta_0}, \tag{19.8}$$

with, cf. (19.6), $1/\eta_0$ given by: for $\Delta \downarrow 0$,

$$\frac{1}{\eta_0} = \omega_{11}[1 + \frac{1}{\nu-1} B_{12} (\frac{1}{B_{11}})^{(\nu_{12}-1)/(\nu-1)} \Delta^{\nu_{12}-\nu}] + o(\Delta^{\nu_{12}-\nu}), \tag{19.9}$$

whenever

$$\nu = \nu_{11} < \nu_{12} < \cdots < \nu_{1N-1}.$$

The relation implies that $\Delta\mathbf{x}/\gamma$ is negative exponentially distributed with first moment $1/\eta_0$.

Remark 19.1 The derivation above can be repeated for general N, and it is seen that the result (19.8) is again obtained with $1/\eta$ given by

$$\frac{1}{\eta_0} = \omega_{11}[1 + \frac{1}{\nu-1} \sum_{i=2}^{N} B_{12} (\frac{1}{B_{11}})^{(\nu_{1i}-1)/(\nu-1)} \Delta^{\nu_{1i}-\nu}] + o(\Delta^{\nu_{12}-\nu}).$$

□

References

1. Cohen, J.W. and Boxma, O.J., Boundary Value Problems in Queueing System Analysis, North-Holland Publ. Co., Amsterdam, 1983.
2. Noble, B., Methods Based on the Wiener-Hopf Technique, Pergamon Press, London, 1958.
3. Cohen, J.W., The Single Server Queue, North-Holland Publ. Co., Amsterdam, 2nd edition 1982.
4. Boxma, O.J. and Cohen, J.W., Heavy-traffic analysis for the GI/G/1 queue with heavy-tailed distributions, *Queueing Systems* **33** (1999) 177–204.
5. Bingham, N.H., Goldie, C.M., Teugels, J.L., Regular Variation, Cambridge Univ. Press, Cambridge, 1987.
6. Feller, W., An Introduction to Probability Theory and its Applications, Wiley, New York, 1971, 2nd ed.
7. Cohen, J.W., Complex functions in queueing theory, *AEÜ, Int. Journ. Electr. Commun.* Vol 47 (1993) 300–310.
8. Lukacs, E., Characteristic Functions, Griffin, London, 1960.
9. Cohen, J.W., *Heavy-traffic limit theorems for the heavy-tailed GI/G/I queue*, Report PNA-R9719, CWI, December 1997, Amsterdam.
10. Doetsch, G., Handbuch der Laplace Transformation, Vol. I, II, III, Birkhäuser Verlag Basel, 1950–1956.
11. Krishnan, K.R., Neidhart, A.L., Erramilli, A., Scaling analysis in traffic management of self-similar processes, *Teletraffic Contributions for the Information Age*, eds. V. Ramaswami, P.E. Wirth, Vol. 26, p. 1087–1097.
12. Cohen, J.W., Some results on regular variation for distributions in queueing and fluctuation theory, *J.A.P.* **10** (1973) 343–353.
13. Boxma, O.J., Cohen, J.W., The M/G/1 queue with heavy-tailed service time distribution, *IEEE Journ. Selected Areas in Communications* **16** (1998) 749–763.
14. Cohen, J.W. *The ν-stable Lévy motion in heavy-traffic analysis of queueing models with heavy-tailed distributions*, Report PNA-R9808, CWI August 1998, Amsterdam.
15. Abate, J and Whitt, W., Numerical inversion of Laplace transforms of probability distributions, *ORSA J. Computing* **7** (1995) 36–43.
16. Bondeson, L., Generalized , -convolution, Springer Verlag, Berlin, 1992.