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ABSTRACT

We analyze the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. We show a sharp dichotomy in qualitative behavior, depending on the relative values of the weight parameters. For certain weight combinations, an individual flow with long-tailed traffic characteristics is effectively served at a *constant* rate. The effective service rate may be interpreted as the maximum average rate for the flow to be stable, which is only influenced by the traffic characteristics of the other flows through their average rates. In particular, the flow is essentially immune from excessive activity of flows with 'heavier'-tailed traffic characteristics. In many situations, the effective service rate is simply the link rate reduced by the aggregate average rate of the other flows. This confirms that GPS-based scheduling algorithms provide a potential mechanism for extracting significant multiplexing gains, while isolating individual flows. For other weight combinations however, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. The stark contrast in qualitative behavior illustrates the crucial importance of the weight parameters.

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1 Introduction

Measurements indicate that traffic in high-speed networks exhibits burstiness on a wide range of time scales, manifesting itself in long-range dependence and self-similarity, see for instance Leland *et al.* [34], Paxson & Floyd [41]. The occurrence of these phenomena is commonly attributed to extreme variability and long-tailed characteristics in the underlying activity patterns (connection times, file sizes, scene lengths), see for instance Beran *et al.* [5], Crovella & Bestavros [23], Willinger *et al.* [45]. This has triggered a lively interest in queueing models with long-tailed traffic characteristics.

Although the presence of long-tailed traffic characteristics is widely acknowledged, the practical implications for network performance and traffic engineering remain to be fully resolved. For moderate buffer sizes, the impact of long-tailed traffic characteristics is not as pronounced as found in theoretical studies for infinite buffers, see Grossglauser & Bolot [27], Heyman & Lakshman [28], Mandjes & Kim [35], Ryu & Elwalid [42]. For larger buffer sizes, flow control mechanisms play a critical role in preventing long-tailed activity patterns from overwhelming the buffer contents, see Arvidsson & Karlsson [4].

Scheduling and priority mechanisms also play a major role in controlling the effect of long-tailed traffic characteristics on network performance. The present paper specifically examines the effectiveness of Generalized Processor Sharing (GPS) in isolating long-tailed traffic flows. As a design paradigm, GPS is at the heart of commonly-used scheduling algorithms for high-speed switches, such as Weighted Fair Queueing, see for instance Parekh & Gallager [39, 40].

A basic approach in the analysis of long-tailed traffic phenomena is the use of fluid models with long-tailed arrival processes (e.g. On/Off sources with long-tailed On-periods). Fluid models are closely related to the ordinary single-server queue, thus bringing within reach the classical results on regularly-varying (Cohen [20]) or subexponential (Pakes [38], Veraverbeke [44]) behavior of the service and waiting-time distribution in the GI/G/1 queue. Those results are immediately applicable in the case of a single long-tailed arrival stream, see Boxma [12] and Choudhury & Whitt [18]. They are also of use when a single long-tailed stream is multiplexed with exponential streams, see Boxma [13] and Jelenković & Lazar [30]. We refer to Boxma & Dumas [16] for a comprehensive survey on fluid queues with long-tailed arrival processes. See also Jelenković [29] for an extensive list of references on subexponential queueing models.

The impact of priority and scheduling mechanisms on long-tailed traffic phenomena has received relatively little attention. Some recent studies have investigated the effect of the scheduling discipline on the waiting-time distribution in the classical M/G/1 queue, see for instance Anantharam [3]. For FCFS, it is well-known [20] that the waiting-time tail is regularly varying of index $1-\nu$ iff the service time tail is regularly varying of index $-\nu$. For

LCFS preemptive resume as well as for Processor Sharing, the waiting-time tail turns out to be regularly varying of the *same* index as the service time tail, see Boxma & Cohen [14] and Zwart & Boxma [51], although with different pre-factors. In the case of Processor Sharing with several customer classes, Zwart [48] showed that the sojourn time distribution of a class-i customer is regularly varying of index $-\nu_i$ iff the service time distribution of that class is regularly varying of index $-\nu_i$, *regardless* of the service time distributions of the other classes. In contrast, for two customer classes with ordinary non-preemptive priority, the tail behavior of the waiting- and sojourn time distributions is determined by the *heaviest* of the (regularly-varying) service time distributions, see Abate & Whitt [1] and Boxma *et al.* [15].

In the present paper, we consider the Generalized Processor Sharing (GPS) discipline. GPS-based scheduling algorithms, such as Weighted Fair Queueing, play a major role in achieving differentiated quality-of-service in integrated-services networks. The queueing analysis of GPS is extremely difficult. Interesting partial results for exponential traffic models were obtained in Bertsimas *et al.* [6], Dupuis & Ramanan [24], Massoulié [36], Zhang [46], and Zhang *et al.* [47].

Here, we focus on non-exponential traffic models, extending the results of [8]–[11]. We show that, for certain weight combinations, an individual flow with long-tailed traffic characteristics is effectively served at a *constant* rate. The effective service rate may be interpreted as the maximum average rate for the flow to be stable, which is only influenced by the traffic characteristics of the other flows through their average rates. In particular, the flow is essentially immune from excessive activity of flows with 'heavier'-tailed traffic characteristics. In many situations, the effective service rate is simply the link rate reduced by the aggregate average rate of the other flows. This is strongly reminiscent of the reduced-load equivalence established by Agrawal *et al.* [2]. For other weight combinations however, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness.

The remainder of the paper is organized as follows. In Section 2, we present a detailed model description. In Section 3, we consider a scenario where the traffic intensity of each flow is smaller than its weight in the GPS scheme. We start by deriving lower and upper bounds for the workload distribution of an individual flow. We show that the bounds, although quite crude by themselves, agree in terms of tail behavior, resulting in the exact workload asymptotics.

Next, we consider the general situation where the traffic intensity of a flow may be larger than its GPS weight. We start by discussing some stability issues, and then introduce a stability-related notion which plays a crucial role in the analysis. We distinguish between two cases, depending on whether a flow may be affected by other flows or not. These cases are examined in Sections 4 and 5, respectively. In both cases, we establish bounds for the workload distribution of an individual flow, which are now more complicated and rely on more refined GPS properties. As before though, the bounds coincide as far as tail behavior is concerned, thus yielding exact asymptotic results.

In Section 6, we consider a closely related model of two coupled processors. Using transform techniques, we obtain the exact workload asymptotics, which qualitatively agree with those of the GPS model. In Section 7, we make some concluding remarks.

2 Model description

Consider N traffic flows sharing a link of unit rate. Traffic from the flows is served in accordance with the Generalized Processor Sharing (GPS) discipline, which operates as follows. Flow i is assigned a weight ϕ_i, with $\sum_{i=1}^{N} \phi_i = 1$. If all the flows are backlogged at time t, then flow i is served at rate ϕ_i. If some of the flows are not backlogged, however, then the excess capacity is redistributed among the backlogged flows in proportion to their respective weights. We refer to Dupuis & Ramanan [24] for a formal description of the evolution of the backlog process.

Denote by $A_i(s,t)$ the amount of traffic generated by flow i during the time interval $(s,t]$. We assume that the process $A_i(s,t)$ is stationary. Denote by ρ_i the traffic intensity of flow i as will be defined below in detail for the two traffic scenarios that we consider. Denote by $V_i(t)$ the backlog (workload) of flow i at time t. Let $\mathbf{V}_i$ be a stochastic variable with as distribution the limiting distribution of $V_i(t)$ for $t \to \infty$ (assuming it exists). In the cases that we consider, the limiting distribution when it exists does not depend on the initial state of the system. As we are primarily interested in studying $\mathbf{V}_i$, we may thus assume without loss of generality that the system is initially empty, i.e., $V_j(0) = 0$ for all $j = 1, \ldots, N$.

Define $B_i(s,t)$ as the amount of service received by flow i during the time interval $(s,t]$. Then the following identity relation holds for all $0 \le s \le t$:

$$V_i(t) = V_i(s) + A_i(s,t) - B_i(s,t). \tag{1}$$

Denote by $A(s,t) := \sum_{i=1}^{N} A_i(s,t)$ the total amount of traffic generated during $(s,t]$. Define $\rho := \sum_{i=1}^{N} \rho_i$ as the total traffic intensity. Define $V(t) := \sum_{i=1}^{N} V_i(t)$ as the total workload at time t.

For any $c \ge 0$, denote by $V_i^c(t) := \sup_{0 \le s \le t} \{A_i(s,t) - c(t-s)\}$ the workload at time t in a queue of capacity c fed by flow i only (assuming $V_i^c(0) = 0$). For $c > \rho_i$, let $\mathbf{V}_i^c$ be a stochastic variable with as distribution the limiting distribution of $V_i^c(t)$ for $t \to \infty$. Define $B_i^c(s,t)$

as the amount of service received by flow i during $(s,t]$ in a queue of capacity c. Similarly to the identity relation above, for all $0 \leq s \leq t$:

$$V_i^c(t) = V_i^c(s) + A_i(s,t) - B_i^c(s,t). \tag{2}$$

Denote by $\mathbf{P}_i^c$ the busy period associated with the workload process $\mathbf{V}_i^c$. We occasionally use the short-hand notation $\mathbf{P}_i$ when the capacity c is clear from the context.

Before describing the traffic model, we first introduce some additional notation.
For any two real functions $g(\cdot)$ and $h(\cdot)$, we use the notational convention $g(x) \sim h(x)$ to denote $\lim_{x\to\infty} g(x)/h(x) = 1$, or equivalently, $g(x) = h(x)(1+o(1))$ as $x \to \infty$.
For any stochastic variable $\mathbf{X}$ with distribution function $F(\cdot)$, $\mathbb{E}\mathbf{X} < \infty$, denote by $F^r(\cdot)$ the distribution function of the residual lifetime of $\mathbf{X}$, i.e., $F^r(x) = \frac{1}{\mathbb{E}\mathbf{X}} \int_0^x (1-F(y))\mathrm{d}y$, and by $\mathbf{X}^r$ a stochastic variable with distribution $F^r(\cdot)$.
The classes of *long-tailed*, *subexponential*, *regularly varying*, *intermediately regularly varying*, and *dominatedly varying* distributions are denoted with the symbols $\mathcal{L}$, $\mathcal{S}$, $\mathcal{R}$, $\mathcal{IR}$, and $\mathcal{DR}$, respectively. The definitions of these classes may be found in Appendix A.

We now describe the two traffic scenarios that we consider.

2.1 Instantaneous input

Here, a flow generates instantaneous traffic bursts according to a renewal process. The interarrival times between bursts of flow i have distribution function $U_i(\cdot)$ with mean $1/\lambda_i$. The burst sizes of flow i have distribution $S_i(\cdot)$ with mean $\sigma_i < \infty$. Thus, the traffic intensity of flow i is $\rho_i = \lambda_i\sigma_i$.

We now state some results which will play a crucial role in the analysis.

Theorem 2.1 *(Pakes [38]) If $S_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c$, then*

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \sim \frac{\rho_i}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i^r > x\}.$$

Theorem 2.2 *(Zwart [49]) If $U_i(\cdot)$ is an exponential distribution, i.e., the arrival process is Poisson, $S_i(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then*

$$\mathbb{P}\{\mathbf{P}_i > x\} \sim \frac{c}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i > x(c-\rho_i)\}.$$

In fact, the preceding theorem can be extended to non-Poisson arrival processes, see Zwart [49]. In the analysis we will need a slight modification:

Theorem 2.3 *If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then*

$$\mathbb{P}\{\mathbf{P}_i^r > x\} \sim \frac{c}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i^r > x(c-\rho_i)\}.$$

Remark 2.1 Although Theorem 2.3 is only a minor extension of Theorem 2.2, the proof is new and might be of independent interest, see Appendix B. It directly uses Theorem 2.1 to derive the asymptotic behavior of the residual busy period. Note that if $S_i(\cdot) \in \mathcal{IR}$, then Theorem 2.2 implies Theorem 2.3. However, if we only assume $S_i^r(\cdot) \in \mathcal{IR}$, then we cannot directly use Theorem 2.2, since $S_i^r(\cdot) \in \mathcal{IR}$ does not necessarily imply $S_i(\cdot) \in \mathcal{IR}$.

2.2 Fluid input

Here, a flow generates traffic according to an On-Off process, alternating between On- and Off-periods. The Off-periods of flow i have distribution function $U_i(\cdot)$ with mean $1/\lambda_i$. The On-periods of flow i have distribution $S_i(\cdot)$ with mean $\sigma_i < \infty$. While On, flow i produces traffic at a constant rate r_i, so the mean burst size is $\sigma_i r_i$. The fraction of time that flow i is Off is

$$p_i = \frac{1/\lambda_i}{1/\lambda_i + \sigma_i} = \frac{1}{1+\lambda_i\sigma_i}.$$

The traffic intensity of flow i is

$$\rho_i = (1-p_i)r_i = \frac{\lambda_i\sigma_i r_i}{1+\lambda_i\sigma_i}.$$

We now state the analogues of Theorems 2.1, 2.2, and 2.3 in the case of On-Off processes.

Theorem 2.4 *(Jelenković & Lazar [30]) If $S_i^r(\cdot) \in \mathcal{S}$, and $\rho_i < c < r_i$, then*

$$\mathbb{P}\{\mathbf{V}_i^c > x\} \sim p_i\frac{\rho_i}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i^r > x/(r_i-c)\}.$$

Theorem 2.5 *(Boxma & Dumas [17], Zwart [49]) If $U_i(\cdot)$ is an exponential distribution, i.e., the Off-periods are exponentially distributed, $S_i(\cdot) \in \mathcal{IR}$, and $\rho_i < c < r_i$, then*

$$\mathbb{P}\{\mathbf{P}_i > x\} \sim p_i\frac{c}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i > x(c-\rho_i)/(r_i-\rho_i)\}.$$

In addition, the following minor extension of the preceding theorem holds:

Theorem 2.6 *If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c < r_i$, then*

$$\mathbb{P}\{\mathbf{P}_i^r > x\} \sim p_i\frac{c}{c-\rho_i}\mathbb{P}\{\mathbf{S}_i^r > x(c-\rho_i)/(r_i-\rho_i)\}.$$

Remark 2.2 Theorems 2.5 and 2.6 follow directly from Theorems 2.2 and 2.3 because of a useful equivalence relation observed by Boxma & Dumas [17] and Zwart [48]. The busy period in a fluid queue is equal in distribution to the busy period in a corresponding $G/G/1$ queue scaled by a factor $r_i/(r_i - c_i)$. The interarrival times in the $G/G/1$ queue are exactly the Off-periods in the fluid queue, and the service times correspond to the *net input* during the On-periods. Thus, with some minor abuse of notation, $\mathbb{P}\{\mathbf{P}_i > x\} = \mathbb{P}\{\mathbf{P}_i^{G/G/1} > x(r_i - c)/r_i\}$ for all values of x, with $U_i^{G/G/1}(\cdot) = U_i(\cdot)$ and $\mathbf{S}_i^{G/G/1} := (r_i - c)\mathbf{S}_i$.
From Theorem 2.2, noting that $c - \rho_i^{M/G/1} = (c - \rho_i)/p_i$ and $p_i r_i = r_i - \rho_i$,

$$\begin{aligned}\mathbb{P}\{\mathbf{P}_i^{M/G/1} > x(r_i - c)/r_i\} &\sim \frac{c}{c - \rho_i^{M/G/1}}\mathbb{P}\{\mathbf{S}_i^{M/G/1} > x(c - \rho_i^{M/G/1})(r_i - c)/r_i\} \\ &= p_i\frac{c}{c - \rho_i}\mathbb{P}\{\mathbf{S}_i > x(c - \rho_i)/(r_i - \rho_i)\},\end{aligned}$$

yielding Theorem 2.5.
In Boxma & Dumas [17], Theorem 2.6 was essentially obtained in this manner from a weaker version of Theorem 2.2 in De Meyer & Teugels [37] for the case $S_i(\cdot) \in \mathcal{R}$. Similarly, Theorem 2.6 for the residual busy period can be directly obtained from Theorem 2.3.
Alternatively, Theorem 2.6 can be proved by mimicking the proof of Theorem 2.3. The only difference is that in Equations (65) and (69), one uses Theorem 2.4 instead of Theorem 2.1, and replaces c in (67) with c/r.

3 Reduced-load equivalence

3.1 Bounds

We first derive bounds for the workload distribution which we will use in the next subsection to analyze the tail behavior. We focus on a particular yet arbitrary flow i. The bounds do not involve any specific assumptions regarding the traffic model. In particular, the bounds apply for the two traffic scenarios described in Subsections 2.1 and 2.2.
The bounds rely on the following two simple properties of the GPS discipline: (i) it is work-conserving, i.e., it serves at the full link rate whenever any of the flows is backlogged; (ii) it guarantees minimum rates $\phi_1, \dots, \phi_N$, i.e., it serves flow i at least at rate ϕ_i whenever flow i is backlogged.
From property (i),

$$V(t) = \sup_{0 \le s \le t} \{A(s,t) - (t - s)\} \qquad \text{for all } t \ge 0. \tag{3}$$

From property (ii),

$$V_i(t) \le V_i^{\phi_i}(t) \qquad \text{for all } t \ge 0. \tag{4}$$

In the remainder of the section, we make the following crucial assumption.

Assumption 3.1 *The traffic intensities and weights satisfy $\rho_i < \phi_i$ for all $i = 1, \dots, N$.*

Note that the above assumption ensures stability of the flows. For a formal stability proof, we refer to Dupuis & Ramanan [24].

We first present a lower bound for the workload distribution of flow i. For compactness, define $A_{-i}(s,t) := A(s,t) - A_i(s,t) = \sum_{j \neq i} A_j(s,t)$ as the aggregate amount of traffic generated by all flows other than i during the time interval $(s,t]$. Also, denote by $\rho_{-i} := \rho - \rho_i = \sum_{j \neq i} \rho_j$ the aggregate traffic intensity of these flows. For any $c \geq 0$, define $Z^c_{-i}(t) := \sup_{0 \leq s \leq t} \{c(t-s) - A_{-i}(s,t)\}$. For $c < \rho_{-i}$, let $\mathbf{Z}^c_{-i}$ be a stochastic variable with as distribution the limiting distribution of $Z^c_{-i}(t)$ for $t \to \infty$.

Lemma 3.1 *(Lower bound) For any $\delta > 0$,*

$$\mathbb{P}\{\mathbf{V}_i > x\} \geq \mathbb{P}\{\mathbf{V}_i^{1-\rho_{-i}+\delta} - \mathbf{Z}_{-i}^{\rho_{-i}-\delta} - \sum_{j \neq i} \mathbf{V}_j^{\phi_j} > x\}. \tag{5}$$

Proof

Using properties (3), (4) we obtain, for any θ,

$$\begin{aligned}
V_i(t) &= V(t) - \sum_{j \neq i} V_j(t) \\
&\geq V(t) - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&= \sup_{0 \leq s \leq t} \{A(s,t) - (t-s)\} - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&= \sup_{0 \leq s \leq t} \{A_i(s,t) - (1-\theta)(t-s) + A_{-i}(s,t) - \theta(t-s)\} - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&\geq \sup_{0 \leq s \leq t} \{A_i(s,t) - (1-\theta)(t-s)\} + \inf_{0 \leq s \leq t} \{A_{-i}(s,t) - \theta(t-s)\} - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&= \sup_{0 \leq s \leq t} \{A_i(s,t) - (1-\theta)(t-s)\} - \sup_{0 \leq s \leq t} \{\theta(t-s) - A_{-i}(s,t)\} - \sum_{j \neq i} V_j^{\phi_j}(t) \\
&= V_i^{1-\theta}(t) - Z^{\theta}_{-i}(t) - \sum_{j \neq i} V_j^{\phi_j}(t) \qquad \text{for all } t \geq 0.
\end{aligned}$$

In particular, taking $\theta = \rho_{-i} - \delta$, we have

$$V_i(t) \geq V_i^{1-\rho_{-i}+\delta}(t) - Z_{-i}^{\rho_{-i}-\delta}(t) - \sum_{j \neq i} V_j^{\phi_j}(t) \qquad \text{for all } t \geq 0.$$

Thus, in the stationary regime (5) holds.

□

We now provide an upper bound for the workload distribution of flow i. For any $c \geq 0$, define $V^c_{-i}(t) := \sup_{0 \leq s \leq t} \{A_{-i}(s,t) - c(t-s)\}$ as the workload at time t in a queue of capacity c fed by all flows other than i. For $c > \rho_{-i}$, let $\mathbf{V}^c_{-i}$ be a stochastic variable with as distribution the limiting distribution of $V^c_{-i}(t)$ for $t \to \infty$.

Lemma 3.2 *(Upper bound) For any $0 < \delta < 1 - \rho$,*

$$\mathbb{P}\{\mathbf{V}_i > x\} \leq \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_i^{1-\rho_{-i}-\delta} + \mathbf{V}_{-i}^{\rho_{-i}+\delta} > x\}. \tag{6}$$

Proof

Using property (3), we have, for any θ,

$$\begin{aligned} V_i(t) &\leq V(t) \\ &= \sup_{0 \leq s \leq t} \{A(s,t) - (t-s)\} \\ &= \sup_{0 \leq s \leq t} \{A_i(s,t) - (1-\theta)(t-s) + A_{-i}(s,t) - \theta(t-s)\} \\ &\leq \sup_{0 \leq s \leq t} \{A_i(s,t) - (1-\theta)(t-s)\} + \sup_{0 \leq s \leq t} \{A_{-i}(s,t) - \theta(t-s)\} \\ &= V_i^{1-\theta}(t) + V_{-i}^{\theta}(t) \qquad \text{for all } t \geq 0. \end{aligned}$$

Invoking property (4), and taking $\theta = \rho_{-i} + \delta$, we obtain

$$V_i(t) \leq \min\{V_i^{\phi_i}(t), V_i^{1-\rho_{-i}-\delta}(t) + V_{-i}^{\rho_{-i}+\delta}(t)\} \qquad \text{for all } t \geq 0.$$

Thus, in the stationary regime (6) holds.

□

3.2 Asymptotic behavior

We now use the bounds from the previous subsection to determine the tail distribution of the workload. Denote by $c_i := 1 - \rho_{-i} = 1 - \sum_{j \neq i} \rho_j$ the link rate reduced by the aggregate average rate of all flows other than i. We consider a specific flow i which satisfies the following three properties for $c = c_i$.

Property 3.1 $\mathbb{P}\{\mathbf{V}^c_i > x\} \in \mathcal{L}$, *i.e.,*

$$\lim_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}^c_i > x - y\}}{\mathbb{P}\{\mathbf{V}^c_i > x\}} = 1, \text{ for all real } y.$$

Property 3.2 *For any $\theta > 0$,*

$$\liminf_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}^{c+\theta}_i > x\}}{\mathbb{P}\{\mathbf{V}^c_i > x\}} = F^c_i(\theta),$$

with $\lim_{\theta \downarrow 0} F^c_i(\theta) = 1.$

Property 3.3 *For any* $0 < \theta < c - \rho_i$,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^{c-\theta} > x\}}{\mathbb{P}\{\mathbf{V}_i^c > x\}} = G_i^c(\theta) < \infty,$$

with $\lim_{\theta\downarrow 0} G_i^c(\theta) = 1$.

According to Theorem 2.1, in case of instantaneous input, flow i satisfies Properties 3.1, 3.2, and 3.3 for any $c > \rho_i$ if $S_i^r(\cdot) \in \mathcal{S}$.
According to Theorem 2.4, in case of fluid input, flow i satisfies Property 3.1 for any $r_i > c > \rho_i$ if $S_i^r(\cdot) \in \mathcal{S}$, and Properties 3.2 and 3.3 if $S_i^r(\cdot) \in \mathcal{IR}$.

We now give the main result of this section.

Theorem 3.1 *Consider a flow i which satisfies Properties 3.1, 3.2, and 3.3 for* $c = c_i$. *If Assumption 3.1 holds, then*

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{V}_i^{c_i} > x\}.$$

Proof
(Lower bound) From Lemma 3.1, using independence, for any $\delta > 0$ and y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\geq \mathbb{P}\{\mathbf{V}_i^{c_i+\delta} > x + y, \mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leq y\} \\ &= \mathbb{P}\{\mathbf{V}_i^{c_i+\delta} > x + y\}\mathbb{P}\{\mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leq y\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \geq \frac{\mathbb{P}\{\mathbf{V}_i^{c_i+\delta} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x + y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{c_i} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \mathbb{P}\{\mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leq y\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}$ satisfies Properties 3.1 and 3.2,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \geq F_i^{c_i}(\delta)\mathbb{P}\{\mathbf{Z}_{-i}^{\rho_{-i}-\delta} + \sum_{j\neq i} \mathbf{V}_j^{\phi_j} \leq y\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \geq 1.$$

(Upper bound) From Lemma 3.2, using independence, for any $0 < \delta < 1 - \rho$ and y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\leq \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_i^{c_i-\delta} > x - y \text{ or } \mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\} \\ &\leq \mathbb{P}\{\mathbf{V}_i^{c_i-\delta} > x - y\} + \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\} \\ &= \mathbb{P}\{\mathbf{V}_i^{c_i-\delta} > x - y\} + \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x\}\mathbb{P}\{\mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \leq \frac{\mathbb{P}\{\mathbf{V}_i^{c_i-\delta} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x - y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{c_i} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} + \frac{\mathbb{P}\{\mathbf{V}_i^{\phi_i} > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \mathbb{P}\{\mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}$ satisfies Properties 3.1 and 3.3,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \leq G_i^{c_i}(\delta) + G_i^{c_i}(c_i - \phi_i)\mathbb{P}\{\mathbf{V}_{-i}^{\rho_{-i}+\delta} > y\}.$$

Passing $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{c_i} > x\}} \leq 1.$$

□

Theorem 3.1 states that the workload of an individual flow i (with long-tailed traffic characteristics) is asymptotically equivalent to that in an isolated system. In the isolated system, flow i is served at a *constant* rate, which is equal to the link rate reduced by the aggregate average rate of all other flows. The result suggests that the most likely way for flow i to build a large queue is that the flow itself generates a large burst, or experiences a long On-period, while all other flows show roughly average behavior, each flow j consuming a fraction ρ_j of the link rate. During that period, flow i then receives service approximately at rate $c_i = 1 - \sum_{j\neq i} \rho_j$.

Thus, asymptotically, the workload of flow i is only affected by the traffic characteristics of the other flows through their aggregate average rate. In particular, flow i is essentially immune from excessive activity of other flows, even when those have 'heavier'-tailed traffic characteristics.

The result is reminiscent of the 'reduced-load equivalence' established by Agrawal *et al.* [2] and a result derived in Jelenković & Lazar [30] for multiplexing exponential with subexponential flows. In these scenarios, the total workload is asymptotically equivalent to that in a reduced system. The reduced system consists of a single dominant flow i served at the link rate reduced by the aggregate average rate of all other flows. However, these results do require bounding conditions on the variability of the other flows. Here, such conditions are not needed because of the properties of the GPS discipline. In fact, we have only used the following two properties of the GPS discipline in establishing Theorem 3.1: (i) it is work-conserving; (ii) it guarantees minimum rates $\phi_1, \ldots, \phi_N$. Thus, the result does not rely on the specific way in which excess capacity is redistributed in GPS, but holds for any rate sharing algorithm with the above two properties. Also, the workload is not significantly influenced by the exact values of the GPS weights (as long as they are larger than the average flow rates as stipulated in Assumption 3.1).

Now suppose each of the flows were served in isolation. Then the required capacity to achieve similar tail behavior is $\sum_{i=1}^{N} c_i = \sum_{i=1}^{N} (1 - \sum_{j \neq i} \rho_j) = \sum_{i=1}^{N} (1 - \rho + \rho_i) = 1 + (N-1)(1-\rho)$. The latter quantity may typically be expected to be substantially larger than 1. This confirms that GPS-based scheduling algorithms provide an effective mechanism for extracting significant multiplexing gains, while isolating indvidual flows.

To conclude the section, we briefly discuss the significance of Assumption 3.1. The assumption that $\rho_i < \phi_i$ for all $i = 1, \dots, N$ implies two crucial properties: (i) flow i is always guaranteed to receive service at a stable rate, even when other flows generate large bursts or experience long On-periods; (ii) when flow i generates a large burst, or experiences a long On-period, and thus builds up a large queue, all other flows j continue to be served at a stable rate, demanding a fraction ρ_j of the link rate.

If Assumption 3.1 is relaxed, then two complicating situations may arise: (i) when other flows generate large bursts or experience long On-periods, flow i may not receive service at a stable rate, and thus build up a large queue; (ii) when flow i generates a large burst, or experiences a long On-period, not all other flows j may continue to be served at a stable rate, so some may consume *less* than a fraction ρ_j of the link rate.

In scenario (i), the tail behavior of flow i may potentially be affected by other flows with 'heavier'-tailed traffic characteristics, which drastically complicates the analysis. In case (ii), precisely what rate the other flows will get, depends on the exact values of the GPS weights (or the detailed mechanics of the rate sharing algorithm in general). We will examine these scenarios in detail in the next sections when we relax Assumption 3.1.

4 Generalized reduced-load equivalence

4.1 Stability issues

We now relax the assumption that the traffic intensities and weights satisfy $\rho_i < \phi_i$ for all $i = 1, \dots, N$, so that stability of the flows is not automatically ensured. In case $\sum_{i=1}^{N} \rho_i < 1$, all flows will remain stable, because the GPS discipline is work-conserving. However, the scenario $\sum_{i=1}^{N} \rho_i > 1$ may occur as well now. In that case, at least one of the flows will be unstable, while others may still be stable.

We now identify which flows are stable and which ones are unstable. To avoid technical subtleties, flow i is considered 'stable' if the mean service rate is ρ_i, see also Remark 4.1 below. For ease of presentation, we assume the flows are indexed such that

$$\frac{\rho_1}{\phi_1} \leq \dots \leq \frac{\rho_N}{\phi_N}.$$

Lemma 4.1 *With the above ordering, the set of stable flows is* $S^* = \{1, \ldots, K^*\}$, *with*

$$K^* = \max_{k=1,\ldots,N} \left\{ k : \frac{\rho_k}{\phi_k} \leq \frac{1 - \sum\limits_{j=1}^{k-1} \rho_j}{\sum\limits_{j=k}^{N} \phi_j} \right\}.$$

Proof

See Appendix C.

□

It may be verified that $K^* = N$ (i.e. all the flows receive a stable service rate) iff $\sum\limits_{i=1}^{N} \rho_i \leq 1$. By definition, each of the stable flows $i \in S^*$ receives a mean service rate ρ_i. Each of the unstable flows $i \notin S^*$ receives a mean service rate $\phi_i R < \rho_i$, with

$$R = \frac{1}{\sum\limits_{j \notin S^*} \phi_j} \left(1 - \sum_{j \in S^*} \rho_j \right).$$

To understand the above formula, notice that the stable flows consume an average aggregate rate $\sum\limits_{j \in S^*} \rho_j$, leaving an average rate $1 - \sum\limits_{j \in S^*} \rho_j$ for the unstable flows, which is shared in proportion to the weights ϕ_i.

We now introduce a stability-related notion which will play a fundamental role in the analysis. Define γ_{iE} as the mean rate at which flow i would receive service if the flows $j \in E$ were to continuously claim their full share of the link rate (while the remaining flows $j \notin E$ still acted 'normally'). (With minor abuse of notation we write γ_{ij} for $\gamma_{i\{j\}}$ and abbreviate γ_{ii} to γ_i.) Now observe that the flows $j \in E$ would in fact show such greedy behavior if they were unstable (which they need not be in reality). So we may determine γ_{iE} by forcing the flows $j \in E$ into the set of unstable flows, and then apply Lemma 4.1. The set of flows which would receive a stable service rate if the flows $j \in E$ were to continuously claim their full share of the link rate, is then $S_E = \{1, \ldots, K_E^*\} \setminus E$, with

$$K_E^* = \max_{k=1,\ldots,N} \left\{ k : \frac{\rho_k}{\phi_k} \leq \frac{1 - \sum\limits_{j=1}^{k-1} \rho_j \mathrm{I}_{\{j \notin E\}}}{\sum\limits_{j=k}^{N} \phi_j \mathrm{I}_{\{j \notin E\}} + \sum\limits_{j \in E} \phi_j} \right\}.$$

Thus, $\gamma_{iE} = \rho_i$ for all $i \in S_E$, and $\gamma_{iE} = \phi_i R_E < \rho_i$ for all $i \notin S_E$, with

$$R_E = \frac{1}{\sum\limits_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} \rho_j \right).$$

To explain the above formula, observe that the flows $j \in S_E$ by definition receive an average aggregate rate $\sum_{j \in S_E} \rho_j$, leaving an average rate $1 - \sum_{j \in S_E} \rho_j$ for the flows $j \notin S_E$, which is shared in proportion to the weights ϕ_i.

Remark 4.1 For later purposes, we find it convenient to label flow i as 'stable' if the mean service rate is ρ_i. In fact, the latter condition is necessary for stability in the usual sense, but not entirely sufficient. A sufficient condition is $\rho_i < \gamma_i$. Indeed, if the queue of flow i never emptied, then it would receive a mean service rate γ_i, so that γ_i is the critical mean rate for stability.

4.2 Bounds

We first derive bounds for the workload distribution which we will use in the next subsection to analyze the tail behavior. We focus on a particular yet arbitrary flow i for which we assume $\rho_i < \gamma_i$ to ensure stability.
We first introduce some additional notation. For any subset $E \subseteq \{1, \ldots, N\}$, define

$$\gamma_{iE}(\delta) = (1-\delta)\gamma_{iE} = (1-\delta)\rho_i \qquad \text{for all } i \in S_E,$$

and

$$\gamma_{iE}(\delta) = \phi_i R_E(\delta) \qquad \text{for all } i \notin S_E,$$

with

$$R_E(\delta) = \frac{1}{\sum\limits_{j \notin S_E} \phi_j} \left(1 - \sum_{j \in S_E} \gamma_{jE}(\delta)\right) = \frac{1}{\sum\limits_{j \notin S_E} \phi_j} \left(1 - (1-\delta) \sum_{j \in S_E} \rho_j\right).$$

Note that $\sum_{i=1}^{N} \gamma_{iE}(\delta) = 1$ for all values of δ (unless $E = \emptyset$).

We now state some preliminary results which will play a crucial role in deriving the bounds.

Lemma 4.2 *Let $E, S, T \subseteq \{1, \ldots, N\}$ be sets with $S_E \subseteq S$, $S \cap T = \emptyset$.*
Then

$$\sum_{j \in S} B_j(r,t) \geq \sum_{j \in S} \inf_{r \leq s \leq t} \left\{ A_j(r,s) + \frac{\gamma_{jE}(\delta)}{\sum\limits_{k \notin T} \gamma_{kE}(\delta)} \left[t - s - \sum_{k \in T} B_k(s,t) \right] \right\},$$

for all $\delta \geq \delta_0$ for some $\delta_0 < 0$.

Proof
The proof follows immediately from combining Lemma's D.1 and D.2 in Appendix D.

□

Lemma 4.3 *Let $E, S \subseteq \{1, \ldots, N\}$ be sets with $E \neq \emptyset$, $S_E \subseteq S$. Then*

$$\sum_{j \in S} B_j(r,t) \geq \sum_{j \in S} \inf_{r \leq s \leq t} \{A_j(r,s) + \gamma_{jE}(\delta)(t-s)\},$$

for all $\delta \geq \delta_0$ for some $\delta_0 < 0$.

Proof

The statement follows immediately from Lemma 4.2 when taking $T = \emptyset$ so that $\sum_{k \in T} B_k(s,t) = 0$ and $\sum_{k \notin T} \gamma_{kE}(\delta) = \sum_{k=1}^{N} \gamma_{kE}(\delta) = 1$.

□

Lemma 4.4 *Let $E, S \subseteq \{1, \ldots, N\}$ be sets with $E \neq \emptyset$, $S_E \subseteq S$. Then*

$$\sum_{j \in S} V_j(t) \leq \sum_{j \in S} V_j^{\gamma_{jE}(\delta)}(t),$$

for all $\delta \geq \delta_0$ for some $\delta_0 < 0$.

Proof

Using the identity relation (1), Lemma 4.3, and the assumption that $V_j(0) = 0$ for all $j = 1, \ldots, N$,

$$\begin{aligned}
\sum_{j \in S} V_j(t) &= \sum_{j \in S} [A_j(0,t) - B_j(0,t)] \\
&\leq \sum_{j \in S} [A_j(0,t) - \inf_{0 \leq s \leq t} \{A_j(0,s) + \gamma_{jE}(\delta)(t-s)\}] \\
&= \sum_{j \in S} \sup_{0 \leq s \leq t} \{A_j(s,t) - \gamma_{jE}(\delta)(t-s)\} \\
&= \sum_{j \in S} V_j^{\gamma_{jE}(\delta)}(t).
\end{aligned}$$

□

We first present a lower bound for the workload distribution of flow i. For any $c \geq 0$, define $Z_j^c(r) := \sup_{s \geq r} \{c(s-r) - A_j(r,s)\}$. For $c < \rho_j$, let $\mathbf{Z}_j^c$ be a stochastic variable with as distribution the distribution of $Z_j^c(r)$ (which in fact does not depend on r because the process $A_j(s,t)$ is stationary).

Lemma 4.5 *(Lower bound) For $\delta > 0$ sufficiently small,*

$$\mathbb{P}\{\mathbf{V}_i > x\} \geq \mathbb{P}\{\mathbf{V}_i^{\gamma_i(\delta)} - \sum_{j \neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} > x\}. \tag{7}$$

Proof

From (1),

$$V_i(t) \geq A_i(r,t) - B_i(r,t) \tag{8}$$

for all $0 \leq r \leq t$.

Note that $\sum_{j=1}^{N} B_j(r,t) \leq t - r$, so that

$$B_i(r,t) \leq t - r - \sum_{j \neq i} B_j(r,t). \tag{9}$$

By definition, $i \notin S_i$. Hence, from Lemma 4.3, for any $\delta \geq 0$,

$$\sum_{j \neq i} B_j(r,t) \geq \sum_{j \neq i} \inf_{r \leq s \leq t} \{A_j(r,s) + \gamma_{ji}(\delta)(t-s)\}. \tag{10}$$

Combining (8), (9), (10), for any $\delta \geq 0$ and $0 \leq r \leq t$,

$$\begin{aligned} V_i(t) &\geq A_i(r,t) - (t-r) + \sum_{j \neq i} \inf_{r \leq s \leq t} \{A_j(r,s) + \gamma_{ji}(\delta)(t-s)\} \\ &= A_i(r,t) - \gamma_i(\delta)(t-r) - \sum_{j \neq i} \gamma_{ji}(\delta)(t-r) + \sum_{j \neq i} \inf_{r \leq s \leq t} \{A_j(r,s) + \gamma_{ji}(\delta)(t-s)\} \\ &\geq A_i(r,t) - \gamma_i(\delta)(t-r) + \sum_{j \neq i} \inf_{r \leq s \leq t} \{A_j(r,s) - \gamma_{ji}(\delta)(s-r)\} \\ &= A_i(r,t) - \gamma_i(\delta)(t-r) - \sum_{j \neq i} \sup_{r \leq s \leq t} \{\gamma_{ji}(\delta)(s-r) - A_j(r,s)\} \\ &\geq A_i(r,t) - \gamma_i(\delta)(t-r) - \sum_{j \neq i} Z_j^{\gamma_{ji}(\delta)}(r). \end{aligned} \tag{11}$$

Define $r^* := \arg \sup_{0 \leq r \leq t} \{A_i(r,t) - \gamma_i(\delta)(t-r)\}$, so that $V_i^{\gamma_i(\delta)}(t) = A_i(r^*,t) - \gamma_i(\delta)(t-r^*)$. Taking $r = r^*$ in (11) then yields

$$V_i(t) \geq V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} Z_j^{\gamma_{ji}(\delta)}(r^*).$$

By definition, $\gamma_{ji}(\delta) = \rho_j(1-\delta)$ for all $j \in S_i$. Also, $\gamma_{ji}(\delta) > \gamma_{ji}$ with $\gamma_{ji}(\delta) \downarrow \gamma_{ji}$ for $\delta \downarrow 0$ for all $j \notin S_i$. In particular, $\gamma_i(\delta) > \rho_i$, because $\gamma_i > \rho_i$. Since $\gamma_{ji} < \rho_j$ for $j \notin S_i$, $j \neq i$, we also have that for δ sufficiently small, $\gamma_{ji}(\delta) < \rho_j(1-\delta)$ for $j \notin S_i$, $j \neq i$. Hence, for δ sufficiently small, $\gamma_{ji}(\delta) \leq \rho_j(1-\delta)$ for all $j \neq i$, so that

$$V_i(t) \geq V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} Z_j^{\rho_j(1-\delta)}(r^*),$$

as $Z_j^c(r)$ is increasing in c.

Note that r^*, $V_i^{\gamma_i(\delta)}(t)$ only depend on $A_i(s,t)$, not on $A_j(s,t)$, $j \neq i$, and are thus independent of $Z_j^{\rho_j(1-\delta)}(r^*)$. Hence, for $\delta > 0$ sufficiently small,

$$\begin{aligned} \mathbb{P}\{V_i(t) > x \mid r^*\} &\geq \mathbb{P}\{V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} Z_j^{\rho_j(1-\delta)}(r^*) > x \mid r^*\} \\ &= \mathbb{P}\{V_i^{\gamma_i(\delta)}(t) - \sum_{j \neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} > x \mid r^*\}. \end{aligned}$$

Thus, in the stationary regime (7) holds for $\delta > 0$ sufficiently small.

□

We now provide an upper bound for the workload distribution of flow i. For any subset $E \subseteq \{1, \ldots, N\}$, define $\psi_{iE} = \phi_i / \sum\limits_{j \notin S_E} \phi_j$.

Lemma 4.6 *(Upper bound) For $\delta > 0$ sufficiently small,*

$$\mathbb{P}\{\mathbf{V}_i > x\} \leq \mathbb{P}\{\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \psi_{iE} \sum_{j \in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \text{ for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\}. \quad (12)$$

Proof

Define $r^* := \sup\{r \leq t | V_i(r) = 0\}$. Then $V_i(r^*) = 0$, so from (1),

$$V_i(t) \leq A_i(r^*, t) - B_i(r^*, t). \quad (13)$$

Also, $V_i(r) > 0$ for all $r \in (r^*, t]$, i.e., flow i is continuously backlogged during the interval $(r^*, t]$. Hence, by definition of the GPS discipline,

$$B_i(r^*, t) \geq \frac{\phi_i}{\phi_j} B_j(r^*, t)$$

for all $j = 1, \ldots, N$, and

$$\sum_{j=1}^{N} B_j(r^*, t) = t - r^*.$$

Thus, for any subset $S \subseteq \{1, \ldots, N\}$,

$$B_i(r^*, t) \geq \frac{\phi_i}{\sum\limits_{j \notin S} \phi_j} \sum_{j \notin S} B_j(r^*, t), \quad (14)$$

and

$$\sum_{j \notin S} B_j(r^*, t) = t - r^* - \sum_{j \in S} B_j(r^*, t). \quad (15)$$

Substituting (15) into (14), using (1),

$$\begin{aligned} B_i(r^*,t) &\geq \frac{\phi_i}{\sum\limits_{j\notin S}\phi_j}\left[t-r^*-\sum_{j\in S}B_j(r^*,t)\right] \\ &= \frac{\phi_i}{\sum\limits_{j\notin S}\phi_j}\left[t-r^*-\sum_{j\in S}[V_j(r^*)+A_j(r^*,t)-V_j(t)]\right] \\ &\geq \frac{\phi_i}{\sum\limits_{j\notin S}\phi_j}\left[t-r^*-\sum_{j\in S}[V_j(r^*)+A_j(r^*,t)]\right]. \end{aligned}$$

In particular, for any subset $E \subseteq \{1,\ldots,N\}$, $\delta > 0$, using Lemma 4.4,

$$B_i(r^*,t) \geq \psi_{iE}\left[t-r^*-\sum_{j\in S_E}[V_j^{\rho_j(1+\delta)}(r^*)+A_j(r^*,t)]\right]. \tag{16}$$

Substituting (16) into (13),

$$\begin{aligned} V_i(t) &\leq A_i(r^*,t)-\psi_{iE}\left[t-r^*-\sum_{j\in S_E}[V_j^{\rho_j(1+\delta)}(r^*)+A_j(r^*,t)]\right] \\ &= A_i(r^*,t)-\psi_{iE}\left(1-\sum_{j\in S_E}\rho_j(1+\delta)\right)(t-r^*) \\ &+ \psi_{iE}\left[\sum_{j\in S_E}[V_j^{\rho_j(1+\delta)}(r^*)+A_j(r^*,t)]-\sum_{j\in S_E}\rho_j(1+\delta)(t-r^*)\right] \\ &= A_i(r^*,t)-\gamma_{iE}(-\delta)(t-r^*)+\psi_{iE}\sum_{j\in S_E}[V_j^{\rho_j(1+\delta)}(r^*)+A_j(r^*,t)-\rho_j(1+\delta)(t-r^*)] \\ &\leq V_i^{\gamma_{iE}(-\delta)}(t)+\psi_{iE}\sum_{j\in S_E}V_j^{\rho_j(1+\delta)}(t). \end{aligned}$$

From the definition it is easily seen that for $\delta > 0$, $\gamma_{iE}(-\delta) < \gamma_{iE}$ with $\gamma_{iE}(\delta) \uparrow \gamma_{iE}$ for $\delta \downarrow 0$. Since $\gamma_{iE} > \rho_i$, we have that $\gamma_{iE}(-\delta) > \rho_i$ for δ sufficiently small, and hence $\mathbf{V}_i^{\gamma_{iE}(-\delta)}$ is well-defined.

Thus, in the stationary regime (12) holds for $\delta > 0$ sufficiently small.

□

4.3 Asymptotic behavior

We now use the bounds from the previous subsection to determine the tail distribution of the workload. As before, we consider a specific flow i which satisfies Properties 3.1, 3.2 and 3.3, but now for $c = \gamma_i$.

We make the following assumption.

Assumption 4.1 *At least one of the following two conditions holds:*
(i) $\rho_i < \phi_i$*;*
(ii) for all sets $E \not\ni i$ *with* $\gamma_{iE\cup\{i\}} \leq \rho_i$*, for any* $\delta > 0$*,* $\prod_{j\in E} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\} = o(\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\})$ *as* $x \to \infty$*. In addition, flow* i *satisfies the following property for* $c = \gamma_i$*.*

Property 4.1 $\mathbb{P}\{\mathbf{V}_i^c > x\} \in \mathcal{DR}$*, i.e.,*

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i^c > \eta x\}}{\mathbb{P}\{\mathbf{V}_i^c > x\}} = H_i^c(\eta) < \infty, \text{ for some real } \eta \in (0,1)$$

(which implies the property holds for all $\eta > 0$*).*

As will be formally shown below, the above assumption ensures that flow i is not significantly affected by flows with 'heavier'-tailed traffic characteristics. Specifically, the assumption implies that temporary instability caused by activity of other flows does not substantially influence the workload of flow i compared to the contribution of flow i itself. Condition (i) in fact guarantees unconditional stability of flow i, regardless of the activity of the other flows. Note that the inequality $\gamma_{iE\cup\{i\}} \leq \rho_i$ implies that flow i would be pushed into instability if the flows $j \in E$ continuously claimed their full share of the link rate. Thus, condition (ii) guarantees that only sets of flows with 'combined lighter tails', could potentially drive flow i into instability. Or equivalently, sets of flows with 'combined heavier tails' *cannot* drive flow i into instability.

According to Theorems 2.1 and 2.4, if $S_j^r(\cdot) \in \mathcal{IR}$, then for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}_j^c > x\} \sim K_j^c \mathbb{P}\{\mathbf{S}_j^r > x\}$ for some constant $0 < K_j^c < \infty$. If $S_j(\cdot)$ is light-tailed, i.e., $\mathbb{P}\{\mathbf{S}_j > x\} = o(\mathrm{e}^{-\kappa_1 x})$ for some $\kappa_1 > 0$, then for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}_j^c > x\} = o(\mathrm{e}^{-\kappa_2 x})$ for some $\kappa_2 > 0$. Thus, a sufficient requirement for condition (ii) of Assumption 4.1 to hold is $S_i^r(\cdot) \in \mathcal{IR}$, and for all sets $E \subseteq \{1, \ldots, N\}$ with $\gamma_{iE\cup\{i\}} \leq \rho_i$, either $S_j(\cdot)$ is light-tailed for some $j \in E$, or $S_j^r(\cdot) \in \mathcal{IR}$ for all $j \in E$ and $\prod_{j\in E} \mathbb{P}\{\mathbf{S}_j^r > x\} = o(\mathbb{P}\{\mathbf{S}_i^r > x\})$ as $x \to \infty$.

Now consider the special case where the flows $j \in R$ have regularly varying tails of index $-\nu_j$, while the others $j \notin R$ have exponential tails. In that case, for flows $i \in R$, the sufficient condition indicated above may be expressed as follows: for all sets $E \subseteq R$ with $\gamma_{iE\cup\{i\}} \leq \rho_i$, $\sum_{j\in E} (\nu_j - 1) > \nu_i - 1$.

We now give the main result of this section.

Theorem 4.1 *Consider a flow* i *which satisfies Properties 3.1, 3.2, 3.3 for* $c = \gamma_i$*. If Assumption 4.1 holds, then*

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}.$$

Before giving the formal proof of Theorem 4.1, we first provide an intuitive explanation. Theorem 4.1 states that the workload of an individual flow i (with long-tailed traffic characteristics) is asymptotically equivalent to that in an isolated system. In the isolated system, flow i is served at a *constant* rate γ_i, which is equal to the average rate that flow i would receive if it continuously claimed its full share of the link rate. The result suggests that the most likely way for flow i to build a large queue is that the flow itself generates a large burst, or experiences a long On-period, while all other flows show roughly average behavior. During that period, flow i then receives service approximately at rate γ_i.
Thus, asymptotically, the workload of flow i is only affected by the traffic characteristics of the other flows through their average rates. In particular, flow i is largely insensitive to extreme activity of other flows, even when those have 'heavier'-tailed traffic characteristics.

We now briefly discuss the significance of Assumption 4.1. As mentioned earlier, the assumption ensures that flow i is not significantly affected by flows with 'heavier'-tailed traffic characteristics. If Assumption 4.1 does *not* hold, then there exists some set E with heavier combined tails than flow i and $\gamma_{iE\cup\{i\}} \leq \rho_i$. We conjecture that the tail distribution of $\mathbf{V}_i$ in that case is determined by the set E^* with the 'heaviest' tails, i.e., $\prod_{j\in E} \mathbb{P}\{\mathbf{S}_j^r > x\} = o(\prod_{j\in E^*} \mathbb{P}\{\mathbf{S}_j^r > x\})$ for all $E \neq E^*$ with $\gamma_{iE\cup\{i\}} \leq \rho_i$. The tail distribution of $\mathbf{V}_i$ is then *heavier* than when flow i were served in isolation at a stable rate. The most likely way for flow i to build a large queue is that the flows $j \in E^*$ generate large bursts, or experience long On-periods, while the other flows, including flow i itself, show roughly average behavior. Flow i then receives service approximately at rate $\gamma_{iE^*} \leq \rho_i$, so that the queue will roughly grow at rate $\rho_i - \gamma_{iE^*}$ for a substantial period of time. In the next section we investigate this scenario in detail for the case where the 'dominant' set E^* consists of just a single flow k^*.
For Theorem 4.1 to hold in case of fluid input, we need besides stability, i.e., $\rho_i < \gamma_i$, also $r_i > \gamma_i$ as implicitly required in Properties 3.1, 3.2, and 3.3. If Assumption 4.1 does *not* hold, then we expect the tail behavior of $\mathbf{V}_i$ in case $r_i < \gamma_i$ is still determined by the set E^* as described above. We will prove this in the next section for the case where the 'dominant' set E^* consists of just a single flow. If Assumption 4.1 *does hold*, however, then we conjecture that, possibly under some additional conditions, the tail behavior is determined by the set E^* with the heaviest tails for which either (i) $\gamma_{iE} < \rho_i$, if $i \notin E$ or (ii) $\gamma_{iE} < r_i$, if $i \in E$. The tail distribution of $\mathbf{V}_i$ is then *lighter* than when flow i were served in isolation. The most likely way for flow i to build a large queue is still that the flows $j \in E^*$ generate large bursts or experience long On-periods, while the other flows show roughly average behavior.

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1

(Lower bound) From Lemma 4.5, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\geq \mathbb{P}\{\mathbf{V}_i^{\gamma_i(\delta)} > x + y, \sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\} \\ &= \mathbb{P}\{\mathbf{V}_i^{\gamma_i(\delta)} > x + y\}\mathbb{P}\{\sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \geq \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i(\delta)} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x + y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x + y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \mathbb{P}\{\sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}$ satisfies Properties 3.1 and 3.2,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \geq F_i^{\gamma_i}(\gamma_i(\delta) - \gamma_i)\mathbb{P}\{\sum_{j\neq i} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \geq 1.$$

(Upper bound) We first consider the case that condition (i) of Assumption 4.1 applies. From property (4) and Lemma 4.6, taking $E = \{i\}$, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned}\mathbb{P}\{\mathbf{V}_i > x\} &\leq \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_i^{\gamma_i(-\delta)} + \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > x\} \\ &\leq \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x, \mathbf{V}_i^{\gamma_i(-\delta)} > x - y \text{ or } \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\} \\ &\leq \mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\} + \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x, \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\} \\ &= \mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\} + \mathbb{P}\{\mathbf{V}_i^{\phi_i} > x\}\mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\}.\end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leq \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} + \frac{\mathbb{P}\{\mathbf{V}_i^{\phi_i} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\}.$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}$ satisfies Properties 3.1 and 3.3,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leq G_i^{\gamma_i}(\gamma_i - \gamma_i(-\delta)) + G_i^{\gamma_i}(\gamma_i - \phi_i)\mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\}.$$

Passing $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leq 1.$$

We now consider the case that condition (ii) of Assumption 4.1 applies.

Let us index the sets $E \ni i$ for which $\gamma_{iE} > \rho_i$ as $E_1, \ldots, E_M$. Note that $M \geq 1$ as $\gamma_i > \rho_i$.

From Lemma 4.6, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned}
\mathbb{P}\{\mathbf{V}_i > x\} &\leq \mathbb{P}\{\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \sum_{j\in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \text{ for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\} \\
&= \mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} + \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > x, \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} + \sum_{j\in S_{E_m}} \mathbf{V}_j^{\rho_j(1+\delta)} > x \; \forall m = 1, \ldots, M\} \\
&\leq \mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y \text{ or } \sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y, \\
&\quad \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N \text{ or } \exists j_m \in S_{E_m} : \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > x/N \; \forall m = 1, \ldots, M\} \\
&\leq \mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\} + \mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y, \exists m : \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\} \\
&+ \mathbb{P}\{\exists j_m \in S_{E_m} : \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > x/N \; \forall m = 1, \ldots, M\} \\
&\leq \mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\} + \mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\} \sum_{m=1}^{M} \mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\} \\
&+ \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} \prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x/N\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} &\leq \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i(-\delta)} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x - y\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \\
&+ \mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\} \sum_{m=1}^{M} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \\
&+ \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x/N\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}}.
\end{aligned}$$

Using the fact that $\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}$ satisfies Properties 3.1, 3.3, and 4.1,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} &\leq G_i^{\gamma_i}(\gamma_i - \gamma_i(-\delta)) \\
&+ H_i^{\gamma_i}(1/N)\mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\} \sum_{m=1}^{M} G_i^{\gamma_i}(\gamma_i - \gamma_{iE_m}(-\delta)) \\
&+ H_i^{\gamma_i}(1/N) \sum_{j_1\in S_{E_1},\ldots,j_M\in S_{E_M}} \limsup_{x\to\infty} \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}}.
\end{aligned}$$

Now consider a set $\{j_1, \ldots, j_M\}$ with $j_1 \in S_{E_1}, \ldots, j_M \in S_{E_M}$. By definition, $j_1 \notin E_1, \ldots, j_M \notin E_M$, so that $\{i, j_1, \ldots, j_M\} \neq E_1, \ldots, E_M, \{i\}$. Consequently, $\gamma_{i\{i,j_1,\ldots,j_M\}} \leq \rho_i$. Condition (ii) of Assumption 4.1 then implies that

$$\limsup_{x\to\infty} \frac{\prod_{j\in\{j_1,\ldots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} = 0.$$

Hence,

$$\begin{aligned}\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} &\leq G_i^{\gamma_i}(\gamma_i - \gamma_i(-\delta)) \\ &+ H_i^{\gamma_i}(1/N)\mathbb{P}\{\sum_{j\in S_i} \mathbf{V}_j^{\rho_j(1+\delta)} > y\} \sum_{m=1}^{M} G_i^{\gamma_i}(\gamma_i - \gamma_{iE_m}(-\delta)).\end{aligned}$$

Passing $y \to \infty$, and then $\delta \downarrow 0$, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{V}_i^{\gamma_i} > x\}} \leq 1.$$

□

5 Induced burstiness

As in the previous sections, we focus on a particular flow i for which we assume $\rho_i < \gamma_i$ to ensure stability. However, now we consider the case that Assumption 4.1 does *not* hold. Instead, we assume there exists a 'dominant' set E^* consisting of just a single flow k^*. Thus, $\gamma_{ik^*} < \rho_i$, which in fact implies that $\gamma_{jk^*} < \rho_j$ for all $j > i$. In addition, we assume that $\gamma_{jk^*} > \rho_j$ for all $j < i$, so that $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$.

5.1 Bounds

We start with deriving bounds for the workload distribution which we will use in the next subsection to analyze the tail behavior. We first introduce some additional notation. Denote $\psi_i := \phi_i / \sum_{j=i}^{N} \phi_j$, $\chi_i := \phi_i/(\phi_{k^*} + \sum_{j=i}^{N} \phi_j)$, $\xi_i := 1 - \sum_{j=1, j\neq k^*}^{i-1} \rho_j - \rho_i/\psi_i$. It is easily verified from the stability condition $\rho_i < \gamma_i$ that $\rho_i < \psi_i \left(1 - \sum_{j=1}^{i-1} \rho_j\right)$, so that $\xi_i > \rho_{k^*}$. Define

$$Q_{k^*}^{\delta}(t) := \sup_{0\leq s\leq t} \{\psi_i[B_{k^*}^{\gamma_{k^*}(\delta)}(s,t) - \gamma_{k^*}(\delta)(t-s)] + (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))(t-s)\}, \quad (17)$$

with $B_k^{\gamma_{k^*}(\delta)}(s,t)$ as in (2).

As $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$,

$$\begin{aligned}
\gamma_{ik^*}(\delta) &= \frac{\phi_i}{\phi_{k^*} + \sum\limits_{j=i}^{N} \phi_j} \left(1 - (1-\delta) \sum_{j=1, j \neq k^*}^{i-1} \rho_j \right) \\
&= \psi_i \left(1 - \frac{\phi_{k^*}}{\phi_{k^*} + \sum\limits_{j=i}^{N} \phi_j} \right) \left(1 - (1-\delta) \sum_{j=1, j \neq k^*}^{i-1} \rho_j \right) \\
&= \psi_i \left(1 - \gamma_{k^*}(\delta) - (1-\delta) \sum_{j=1, j \neq k^*}^{i-1} \rho_j \right).
\end{aligned}$$

Thus, (17) may be rewritten as

$$Q^{\delta}_{k^*}(t) = \psi_i \sup_{0 \leq s \leq t} \{ B^{\gamma_{k^*}(\delta)}_{k^*}(s,t) - c(\delta)(t-s) \}, \tag{18}$$

with

$$c(\delta) := \gamma_{k^*}(\delta) + (\gamma_{ik^*}(\delta) - \rho_i(1-\delta))/\psi_i = (1-\delta)\xi_i + \delta.$$

For δ not too small, let $\mathbf{Q}^{\delta}_{k^*}$ be a stochastic variable with as distribution the limiting distribution of $Q^{\delta}_{k^*}(t)$ for $t \to \infty$.

We first present a lower bound for the workload distribution of flow i.

Lemma 5.1 *(Lower bound) For $\delta > 0$ sufficiently small,*

$$\mathbb{P}\{\mathbf{V}_i > x\} \geq \mathbb{P}\{\mathbf{Q}^{\delta}_{k^*} - \sum_{j \neq k^*} \mathbf{Z}^{\rho_j(1-\delta)}_j > x\}. \tag{19}$$

Proof
See Appendix E.

□

We now provide an upper bound for the workload distribution of flow i. Define $s^* := \sup\{s \leq t | V^{\gamma_{k^*}(-\delta)}_{k^*}(s) = 0\}$ (or, equivalently, $s^* := \arg \sup_{0 \leq s \leq t} \{A_{k^*}(s,t) - \gamma_{k^*}(-\delta)(t-s)\}$). Denote

$$W^{\delta}_j(t) := V^{\rho_j(1+\delta)}_j(t) + \frac{\phi_{k^*}}{\sum\limits_{j=i}^{N} \phi_j} V^{\rho_j(1+\delta)}_j(s^*).$$

For $\delta > 0$, let $\mathbf{W}^{\delta}_j$ be a stochastic variable with the limiting distribution of $W^{\delta}_j(t)$ for $t \to \infty$.

Lemma 5.2 *(Upper bound) For $\delta > 0$ sufficiently small,*

$$\mathbb{P}\{\mathbf{V}_i > x\} \leq \mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \chi_i \sum_{j=1, j \neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x\}. \tag{20}$$

Proof

See Appendix F.

□

5.2 Asymptotic behavior

We now use the bounds from the previous subsection to determine the tail distribution of the workload. We first prove an auxiliary lemma. For conciseness, denote $\mathbf{P}_{k^*}^r := (\mathbf{P}_{k^*}^{\gamma_{k^*}})^r$, $\mathbf{Q}_{k^*} := \mathbf{Q}_{k^*}^0$.

Lemma 5.3 *If $S_{k^*}^r(\cdot) \in \mathcal{IR}$, $U_{k^*}(\cdot)$ is an exponential distribution, and $r_{k^*} > \gamma_{k^*}$ in case of fluid input, then*

$$\mathbb{P}\{\mathbf{Q}_{k^*} > x\} \sim \frac{\gamma_{k^*} - \rho_{k^*}}{\gamma_{k^*}} \frac{\rho_{k^*}}{\xi_i - \rho_{k^*}} \mathbb{P}\{\mathbf{P}_{k^*}^r > x/(\rho_i - \gamma_{ik^*})\}, \tag{21}$$

with $\mathbb{P}\{\mathbf{P}_{k^*}^r > x/(\rho_i - \gamma_{ik^*})\}$ as in Theorems 2.3 and 2.6, respectively.

Proof

Notice from (18) that, up to a multiplicative factor ψ_i, $Q_{k^*}^{\delta}(t)$ represents the workload at time t in a queue of capacity $c(\delta)$ fed by the departure process of a queue of capacity $\gamma_{k^*}(\delta)$ fed by flow k^*. The departure process of the latter queue is an On-Off process with as On- and Off-periods the busy and idle periods associated with the workload process $V_{k^*}^{\gamma_{k^*}(\delta)}(t)$. During the On-periods, traffic is generated at constant rate $\gamma_{k^*}(\delta)$ (for δ sufficiently small in case of fluid input so that $\gamma_{k^*}(\delta) < r_{k^*}$). The fraction of Off-time is $1 - \rho_{k^*}/\gamma_{k^*}(\delta)$. The On- and Off-periods are independent because $U_{k^*}(\cdot)$ is an exponential distribution. As in the proof of Theorem 2.6, it may be shown that $S_{k^*}^r(\cdot) \in \mathcal{IR}$ implies $P_{k^*}^r(\cdot) \in \mathcal{IR}$. Hence, from Theorem 2.4,

$$\mathbb{P}\{\mathbf{Q}_{k^*}^{\delta} > x\} \sim \frac{\gamma_{k^*}(\delta) - \rho_{k^*}}{\gamma_{k^*}(\delta)} \frac{\rho_{k^*}}{(1-\delta)\xi_i + \delta - \rho_{k^*}} \mathbb{P}\{\mathbf{P}_{k^*}^r > x/(\rho_i(1-\delta) - \gamma_{ik^*}(\delta))\}, \tag{22}$$

and in particular (21) follows.

□

In accordance with the discussion in the previous section, we make the following assumption.

Assumption 5.1 *In addition to* $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$*, each of the following two conditions holds:*

(i) $\mathbb{P}\{\mathbf{S}_i^r > x\} = o(\mathbb{P}\{\mathbf{S}_{k^*}^r > x\})$ *as* $x \to \infty$*;*

(ii) For all sets $E \not\ni i$, $E \neq \{k^*\}$*, with* $\gamma_{iE\cup\{i\}} \leq \rho_i$*, for any* $\delta > 0$, $\prod_{j\in E} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\} = o(\mathbb{P}\{\mathbf{Q}_{k^*} > x\})$ *as* $x \to \infty$*.*

As indicated earlier, if $S_j^r(\cdot) \in \mathcal{IR}$, then according to Theorems 2.1 and 2.4, for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}_j^c > x\} \sim K_j^c \mathbb{P}\{\mathbf{S}_j^r > x\}$ for some constant $0 < K_j^c < \infty$. If $S_j(\cdot)$ is light-tailed, i.e., $\mathbb{P}\{\mathbf{S}_j > x\} = o(\mathrm{e}^{-\kappa_1 x})$ for some $\kappa_1 > 0$, then for $c > \rho_j$, $\mathbb{P}\{\mathbf{V}_j^c > x\} = o(\mathrm{e}^{-\kappa_2 x})$ for some $\kappa_2 > 0$. Also, according to Theorems 2.3 and 2.6, if $S_{k^*}^r(\cdot) \in \mathcal{IR}$, then $\mathbb{P}\{\mathbf{Q}_{k^*} > x\} \sim K\mathbb{P}\{\mathbf{S}_{k^*}^r > x\}$ for some constant $K > 0$. Thus, a sufficient requirement for condition (ii) of Assumption 5.1 to hold is $S_{k^*}^r(\cdot) \in \mathcal{IR}$, and for all sets $E \subseteq \{1, \ldots, N\}$, $E \neq \{k^*\}$, with $\gamma_{iE\cup\{i\}} \leq \rho_i$, either $S_j(\cdot)$ is light-tailed for some $j \in E$, or $S_j^r(\cdot) \in \mathcal{IR}$ for all $j \in E$ and $\prod_{j\in E} \mathbb{P}\{\mathbf{S}_j^r > x\} = o(\mathbb{P}\{\mathbf{S}_{k^*}^r > x\})$ as $x \to \infty$.
Now consider the special case where the flows $j \in R$, in particular flow k^*, have regularly varying tails with index $-\nu_j$, while the others $j \notin R$ have exponential tails. In that case, the sufficient condition indicated above may be expressed as follows: for all sets $E \subseteq R$ with $\gamma_{iE\cup\{i\}} \leq \rho_i$, $\sum_{j\in E} (\nu_j - 1) > \nu_{k^*} - 1$. Condition (i) then reduces to $i \notin R$ or $\nu_i > \nu_{k^*}$.

We now give the main result of this section.

Theorem 5.1 *If* $S_{k^*}^r(\cdot) \in \mathcal{IR}$, $U_{k^*}(\cdot)$ *is an exponential distribution, and Assumption 5.1 holds, then*

$$\mathbb{P}\{\mathbf{V}_i > x\} \sim \mathbb{P}\{\mathbf{Q}_{k^*} > x\}.$$

Before giving the formal proof of Theorem 5.1, we first provide an intuitive interpretation. As alluded to earlier, the result suggests that the most likely way for flow i to build a large queue is that flow k^* generates a large burst or experiences a long On-period, while the other flows, including flow i itself, show roughly average behavior. Specifically, when flow k^* generates a large amount of traffic, so it becomes backlogged for a long period of time, it receives service approximately at rate γ_{k^*}. Thus it experiences a busy period as if it were served at constant rate γ_{k^*}.
During that congestion period, the flows $j \neq k^*$ receive service approximately at rate γ_{jk^*}, while they generate traffic at average rate ρ_j. Thus, the queue of flow $j \notin S_{k^*} = \{i, \ldots, N\}$, in particular flow i, grows roughly at rate $\rho_j - \gamma_{jk^*} > 0$.
By the time the long congestion period ends, the flows $j \geq i$ have built large queues, and then start to receive service approximately at rate $\frac{\phi_j}{\sum_{j=i}^{N} \phi_j}(1 - \sum_{j=1, j\neq k^*}^{i-1} \rho_j)$. The queue of flow i

then starts to drain roughly at rate $\psi_i(1 - \sum_{j=1, j\neq k^*}^{i-1} \rho_j) - \rho_i = \psi_i \xi_i$, and is the first to empty among the flows $j \geq i$.

In conclusion, the queue of flow i grows at rate $\rho_i - \gamma_{ik^*}$ when flow k^* is backlogged. When flow k^* is not backlogged, the queue of flow i drains at rate $\psi_i \xi_i$.

Thus, the queue of flow i behaves as that of a queue of capacity $\psi_i \xi_i$ fed by an On-Off process with as On- and Off-periods the busy and idle periods of flow k^* when served at constant rate γ_{k^*}. During the On-periods, traffic is produced at rate $\rho_i - \gamma_{ik^*} + \psi_i \xi_i = \psi_i \gamma_{k^*}$. This is reflected in Theorem 5.1 if we use Lemma 5.3 to interpret the right-hand side.

In preparation for the proof of Theorem 5.1, we first state an auxiliary lemma.

Lemma 5.4 *If $S^r_{k^*}(\cdot) \in \mathcal{IR}$, and $\mathbb{P}\{\mathbf{S}^r_i > x\} = o(\mathbb{P}\{\mathbf{S}^r_{k^*} > x\})$ as $x \to \infty$, then for any $c > \rho_i$, $\mathbb{P}\{\mathbf{V}^c_i > x\} = o(\mathbb{P}\{\mathbf{Q}_{k^*} > x\})$ as $x \to \infty$.*

Proof

For any $\epsilon > 0$, construct the stochastic variable $\mathbf{S}^\epsilon_i$ with distribution

$$\mathbb{P}\{\mathbf{S}^\epsilon_i > x\} = \min\{1, \mathbb{P}\{\mathbf{S}_i > x\} + \epsilon \mathbb{P}\{\mathbf{S}_{k^*} > x\}\}.$$

Denote by $V^{c,\epsilon}_i(t)$ the workload at time t in a queue of capacity c fed by flow i where the stochastic variable $\mathbf{S}_i$ in the arrival process is replaced by $\mathbf{S}^\epsilon_i$. For $\epsilon > 0$ sufficiently small, let $\mathbf{V}^{c,\epsilon}_i$ be a stochastic variable with as distribution the limiting distribution of $V^{c,\epsilon}_i(t)$ for $t \to \infty$. (Note that $\mathbb{E}\mathbf{S}^\epsilon_i \leq \mathbb{E}\mathbf{S}_i + \epsilon \mathbb{E}\mathbf{S}_{k^*}$, so that the queue is stable for $\epsilon > 0$ sufficiently small.)

Clearly, $\mathbf{S}^\epsilon_i$ is stochastically larger than $\mathbf{S}_i$, so that for $\epsilon > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}^c_i > x\} \leq \mathbb{P}\{\mathbf{V}^{c,\epsilon}_i > x\}. \tag{23}$$

Also,

$$\mathbb{P}\{(\mathbf{S}^\epsilon_i)^r > x\} \sim \epsilon \frac{\mathbb{E}\mathbf{S}_{k^*}}{\mathbb{E}\mathbf{S}^\epsilon_i} \mathbb{P}\{\mathbf{S}^r_{k^*} > x\},$$

which implies that $\mathbb{P}\{(\mathbf{S}^\epsilon_i)^r > x\} \in \mathcal{IR}$. Hence, by Theorems 2.1, 2.3, 2.4, 2.6, and Lemma 5.3,

$$\limsup_{x \to \infty} \frac{\mathbb{P}\{\mathbf{V}^{c,\epsilon}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \leq \epsilon K, \tag{24}$$

for some finite constant K independent of ϵ.

The lemma follows by combining (23) and (24) and letting $\epsilon \downarrow 0$.

□

We now give the proof of Theorem 5.1.

Proof

Using (21) and the fact that $P^r_{k^*}(\cdot) \in \mathcal{IR}$ (which implies $P^r_{k^*}(\cdot) \in \mathcal{L}$), for any y,

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x-y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = 1, \tag{25}$$

and

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = F < \infty. \tag{26}$$

Also using (22),

$$\lim_{x\to\infty} \frac{\mathbb{P}\{\mathbf{Q}^\delta_{k^*} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = G(\delta), \tag{27}$$

with $\lim_{\delta\to 0} G(\delta) = 1$.

(Lower bound) From Lemma 5.1, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned} \mathbb{P}\{\mathbf{V}_i > x\} &\geq \mathbb{P}\{\mathbf{Q}^\delta_{k^*} > x+y, \sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\} \\ &= \mathbb{P}\{\mathbf{Q}^\delta_{k^*} > x+y\}\mathbb{P}\{\sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\}. \end{aligned}$$

Thus

$$\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \geq \frac{\mathbb{P}\{\mathbf{Q}^\delta_{k^*} > x+y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x+y\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x+y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \mathbb{P}\{\sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\}.$$

Using (25), (27),

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \geq G(\delta)\mathbb{P}\{\sum_{j\neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} \leq y\}.$$

Letting $y \to \infty$, and then $\delta \downarrow 0$, we have

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \geq 1.$$

(Upper bound) Let us index the sets $E \ni i$ for which $\gamma_{iE} > \rho_i$ as $E_1, \ldots, E_M$. Note that $M \geq 1$ as $\gamma_i > \rho_i$. It is easily verified from the fact that $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$ that $k^* \notin E_m$, $k^* \in S_{E_m}$ for all $m = 1, \ldots, M$.

From Lemmas 4.6, 5.2, using independence, for $\delta > 0$ sufficiently small and any y,

$$\begin{aligned} \mathbb{P}\{\mathbf{V}_i > x\} \leq \mathbb{P}\{&\mathbf{Q}^{-\delta}_{k^*} + \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}^\delta_j > x, \\ &\mathbf{V}_i^{\gamma_{iE}(-\delta)} + \sum_{j\in S_E} \mathbf{V}_j^{\rho_j(1+\delta)} > x \text{ for all sets } E \ni i \text{ with } \gamma_{iE} > \rho_i\} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x, \\
&\qquad \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} + \sum_{j\in S_{E_m}} \mathbf{V}_j^{\rho_j(1+\delta)} > x \ \forall m = 1, \dots, M\} \\
&= \mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} + \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > x, \\
&\qquad \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} + \mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} + \sum_{j\in S_{E_m}, j\neq k^*} \mathbf{V}_j^{\rho_j(1+\delta)} > x \ \forall m = 1, \dots, M\} \\
&\leq \mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} > x - y \text{ or } \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y, \\
&\qquad \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N \text{ or } \mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N \text{ or} \\
&\qquad \exists j_m \in S_{E_m}, j_m \neq k^* : \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > x/N \ \forall m = 1, \dots, M\} \\
&\leq \mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} > x - y\} + \mathbb{P}\{\exists m : \mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\} \\
&+ \mathbb{P}\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N, \mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\} \\
&+ \mathbb{P}\{\exists j_m \in S_{E_m}, j_m \neq k^* : \mathbf{V}_{j_m}^{\rho_{j_m}(1+\delta)} > x/N \ \forall m = 1, \dots, M\} \\
&\leq \mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} > x - y\} + \sum_{m=1}^{M} \mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\} \\
&+ \mathbb{P}\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N\}\mathbb{P}\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\} \\
&+ \sum_{j_1\in S_{E_1}\setminus\{k^*\}, \dots, j_M\in S_{E_M}\setminus\{k^*\}} \ \prod_{j\in\{j_1,\dots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x/N\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} &\leq \frac{\mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta} > x - y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x - y\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x - y\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} + \sum_{m=1}^{M} \frac{\mathbb{P}\{\mathbf{V}_i^{\gamma_{iE_m}(-\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \\
&+ \frac{\mathbb{P}\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \mathbb{P}\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1} \mathbf{W}_j^{\delta} > y\} \\
&+ \sum_{j_1\in S_{E_1}\setminus\{k^*\}, \dots, j_M\in S_{E_M}\setminus\{k^*\}} \frac{\prod_{j\in\{j_1,\dots,j_M\}} \mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} \frac{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}}.
\end{aligned}
$$

According to Theorems 2.3, 2.6, and Lemma 5.3,

$$
\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{V}_{k^*}^{\rho_{k^*}(1+\delta)} > x/N\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x/N\}} = H(\delta) < \infty.
$$

Using (25), (26), and (27), and Lemma 5.4,

$$\begin{aligned}\limsup_{x\to\infty}\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} &\leq G(-\delta) + FH(\delta)\mathbb{P}\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1}\mathbf{W}_j^{\delta} > y\} \\ &+ F\sum_{j_1\in S_{E_1}\backslash\{k^*\},\ldots,j_M\in S_{E_M}\backslash\{k^*\}}\limsup_{x\to\infty}\frac{\prod_{j\in\{j_1,\ldots,j_M\}}\mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}}.\end{aligned}$$

Now consider a set $\{j_1,\ldots,j_M\}$ with $j_1\in S_{E_1}\backslash\{k^*\},\ldots,j_M\in S_{E_M}\backslash\{k^*\}$. By definition $j_1\notin E_1,\ldots,j_M\notin E_M$, so that $\{i,j_1,\ldots,j_M\}\neq E_1,\ldots,E_M,\{k^*\}$. Consequently, $\gamma_{i\{i,j_1,\ldots,j_M\}}\leq\rho_i$. Condition (ii) of Assumption 5.1 then implies that

$$\limsup_{x\to\infty}\frac{\prod_{j\in\{j_1,\ldots,j_M\}}\mathbb{P}\{\mathbf{V}_j^{\rho_j(1+\delta)} > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} = 0.$$

Hence,

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \leq G(-\delta) + FH(\delta)\limsup_{x\to\infty}\mathbb{P}\{\mathbf{V}_i^{\rho_i(1+\delta)} + \sum_{j=1, j\neq k^*}^{i-1}\mathbf{W}_j^{\delta} > y\}.$$

Passing $y\to\infty$, and then $\delta\downarrow 0$, we obtain

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{\mathbf{V}_i > x\}}{\mathbb{P}\{\mathbf{Q}_{k^*} > x\}} \leq 1.$$

□

6 Two coupled processors with regularly varying service times

In the previous sections we have investigated the tail behavior of the workload distribution of an individual flow under GPS, exploiting sample path lower and upper bounds. In the present section we obtain qualitatively similar results for a closely related model of two coupled processors using analytic methods. These methods do not seem to apply for $N > 2$ flows. However, for $N = 2$, they do provide detailed results which seem hard to obtain using sample path techniques. Furthermore, the methods to be presented here seem to be of independent interest.

We consider the following model of two coupled M/G/1 queues, Q_1 and Q_2. Q_i, $i = 1, 2$, receives a Poisson arrival stream of customers of type i with arrival rate λ_i and with required amounts of service that are i.i.d. random variables with distribution $B_i(\cdot)$, with mean β_i and Laplace-Stieltjes transform (LST) $\beta_i\{s\}$. $\mathbf{B}_i$ denotes a random variable with distribution $B_i(\cdot)$, $i = 1, 2$. Denote the traffic intensity at Q_i by $\rho_i := \lambda_i\beta_i$. The arrival processes at the two queues, and the families of required service amounts in both streams, are independent

of each other. Whenever there is work of both types, each server serves its own queue at speed 1. However, the speed of server i increases to $r_i^* \geq 1$ if the other server is idle, $i = 1, 2$. In a sense, the servers are coupled, and a server with no work at its own queue is able to assist the other server. Notice that $r_1^* = r_2^* = 2$ corresponds to GPS with $\phi_1 = \phi_2 = \frac{1}{2}$.
This coupled-processors model has been analyzed by Fayolle & Iasnogorodski [25] and by Konheim, Meilijson & Melkman [33] in the case of negative exponentially distributed service requests, and by Cohen & Boxma [22] for generally distributed service requests. In the latter case, the joint distribution of the workloads in both queues was obtained by formulating and solving a Wiener-Hopf boundary value problem.
We shall exploit that joint workload distribution for an investigation of the influence of heavy-tailed service request distributions on the tail behavior of the workload distributions. We study this tail behavior under the assumption of regularly varying service request distributions (see Appendix A for the definition). For the special case $r_2^* = 1$ (i.e., Q_2 is not affected by Q_1) this analysis has already been performed in [11]. In [22] a distinction is made between the special case $1/r_1^* + 1/r_2^* = 1$ (which corresponds directly to GPS) and the case $1/r_1^* + 1/r_2^* \neq 1$. We concentrate on the latter more general case, which is of more interest for our purposes; see also Remark 6.4.
We shall distinguish between two cases: $\rho_1 < 1$ and $\rho_1 > 1$. In the former case, server 1 is able to handle its offered traffic, even if server 2 were never idle. In constrast, in the latter case, server 1 needs the assistance of server 2. We shall see that the tail behavior of the workload at Q_1 is not affected by a heavier-tailed service time distribution at Q_2 if $\rho_1 < 1$, but that it *is* affected if $\rho_1 > 1$. The workload asymptotics for the cases $\rho_1 < 1$ and $\rho_1 > 1$ will be analyzed in Subsections 6.2 and 6.3, respectively.

6.1 Preliminary results

This subsection contains some preparations for the further analysis. We restrict ourselves to the steady-state situation. The ergodicity conditions are discussed in Section III.3.7 of [22]; here it suffices to observe that at least one of the conditions $\rho_1 < 1$, $\rho_2 < 1$ should be satisfied, but not necessarily both.
In the sequel, $\mathbf{V}_i$ denotes the steady-state workload at Q_i. For Re $s_1 \geq 0$, Re $s_2 \geq 0$, let

$$\begin{aligned}
\psi(s_1, s_2) &= \mathbb{E}[\mathrm{e}^{-s_1 \mathbf{V}_1 - s_2 \mathbf{V}_2}], \\
\psi_1(s_2) &= \mathbb{E}[\mathrm{e}^{-s_2 \mathbf{V}_2} \mathrm{I}_{\{\mathbf{V}_1 = 0\}}], \\
\psi_2(s_1) &= \mathbb{E}[\mathrm{e}^{-s_1 \mathbf{V}_1} \mathrm{I}_{\{\mathbf{V}_2 = 0\}}], \\
\psi_0 &= \mathbb{P}\{\mathbf{V}_1 = 0, \mathbf{V}_2 = 0\}.
\end{aligned}$$

We concentrate on $\psi(s, 0)$, the LST of the workload distribution at Q_1. From (2.16) of Chapter III.3 of [22] (in the sequel we omit mentioning Chapter III.3 when referring to formulas from [22]) it follows that, for Re $s \geq 0$,

$$\psi(s,0) \;=\; \mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] \;=\; \frac{(1-\rho_1)s}{s-\lambda_1(1-\beta_1\{s\})}[\frac{\psi_1(0)}{1-\rho_1}+\frac{r_1^*-1}{1-\rho_1}(\psi_0-\psi_2(s))]. \tag{28}$$

We now discuss $\psi_2(s)$. According to (6.21) and (6.22) of [22],

$$\frac{1}{r_2^*}[\psi_2(\delta_1(w))-\psi_0] = \frac{1}{r_1^* r_2^*}\frac{\psi_0}{1-1/r_1^*-1/r_2^*}[1-\mathrm{e}^{-R_1(w)+R_2(w)}], \quad \text{Re } w \geq 0. \tag{29}$$

In the remainder of this section we specify $\delta_1(w)$ and $R_i(w)$, $i=1,2$. For later use we also introduce their companion functions $P_i(w)$, $i=1,2$:

$$P_i(w) \;:=\; \sum_{n=1}^{\infty}\frac{b_i^n}{n}\mathbb{E}[\mathrm{e}^{-w\sigma_n^{(i)}}\mathrm{I}_{\{\sigma_n^{(i)}<0\}}], \quad \text{Re } w \geq 0, \tag{30}$$

$$R_i(w) \;:=\; \sum_{n=1}^{\infty}\frac{b_i^n}{n}\mathbb{E}[\mathrm{e}^{-w\sigma_n^{(i)}}\mathrm{I}_{\{\sigma_n^{(i)}>0\}}], \quad \text{Re } w \geq 0, \tag{31}$$

and

$$b_1 \;:=\; \rho_1(1-\frac{1}{r_2^*})+\frac{\rho_2}{r_2^*}, \tag{32}$$

$$b_2 \;:=\; \rho_2(1-\frac{1}{r_1^*})+\frac{\rho_1}{r_1^*}, \tag{33}$$

and for $i=1,2$,

$$\sigma_n^{(i)} := \mathbf{X}_{i1}+\ldots+\mathbf{X}_{in},$$

with $\mathbf{X}_{11},\ldots,\mathbf{X}_{1n}$ i.i.d. and $\mathbf{X}_{21},\ldots,\mathbf{X}_{2n}$ i.i.d., and

$$\begin{aligned}\mathbf{X}_{11} \;&=\; \hat{\mathbf{P}}_1 \quad w.p.\ \pi_1 := \frac{\rho_1}{b_1}(1-\frac{1}{r_2^*}),\\ &\quad -\hat{\mathbf{P}}_2 \quad w.p.\ 1-\pi_1 = \frac{\rho_2}{b_1 r_2^*},\end{aligned} \tag{34}$$

$$\begin{aligned}\mathbf{X}_{21} \;&=\; \hat{\mathbf{P}}_1 \quad w.p.\ \pi_2 := \frac{\rho_1}{b_2 r_1^*},\\ &\quad -\hat{\mathbf{P}}_2 \quad w.p.\ 1-\pi_2 = \frac{\rho_2}{b_2}(1-\frac{1}{r_1^*});\end{aligned} \tag{35}$$

here $\hat{\mathbf{P}}_i$ denotes a busy period in a single-server queue that has exactly the same traffic characteristics as Q_i and has service speed 1, and that *starts with an exceptional first service* $\mathbf{B}_i^r$ (a residual service time as defined in Section 2).

The function $\delta_1(w)$ plays a key role in the analysis of this coupled-processors model.

$$f_1(s,w) \;:=\; \lambda_1(1-\beta_1\{s\})-s+w \tag{36}$$

has for Re $w \geq 0$, $w \neq 0$, exactly one zero $s=\delta_1(w)$ in Re $s \geq 0$, and this zero has multiplicity one.

$f_1(s,0)$ has for $\rho_1 < 1$ exactly one zero $s = \delta_1(0) = 0$ in Re $s \geq 0$, with multiplicity one;
$f_1(s,0)$ has for $\rho_1 = 1$ exactly one zero $s = \delta_1(0) = 0$ in Re $s \geq 0$, with multiplicity two;
$f_1(s,0)$ has for $\rho_1 > 1$ two zeroes $s = \delta_1(0) > 0$ and $s = \epsilon_1(0) = 0$ in Re $s \geq 0$, each with multiplicity one.
Similarly $\delta_2(w)$ is defined for Re $w \geq 0$ as zero of the function

$$f_2(s,w) \;:=\; \lambda_2(1-\beta_2\{s\}) - s - w.$$

The different behavior of $\delta_1(w)$ for w near 0 for $\rho_1 < 1$ and $\rho_1 > 1$ will be reflected in very different behavior of the workload at Q_1 for these two cases, to be discussed in the next two subsections.
In the analysis we will use the following crucial lemma (Lemma 2.2 in [16], which is an extension of Theorem 8.1.6 in [7]), to link the regularly varying tail behavior of a distribution function $\mathbb{P}\{\mathbf{Y} > t\}$ for $t \to \infty$ to the behavior of its LST $f(s)$.

Lemma 6.1 *Let* $\mathbf{Y}$ *be a non-negative random variable with LST* $f(s)$, $l(t)$ *a slowly varying function,* $\nu \in (n, n+1)$ *(*$n \in \mathbb{N}$*) and* $C \geq 0$. *Then the following statements are equivalent:*
(i) $\mathbb{P}\{\mathbf{Y} > t\} = [C + \mathrm{o}(1)]t^{-\nu}l(t), \quad t \to \infty;$
(ii) $\mathbb{E}[\mathbf{Y}^n] < \infty$ *and* $f(s) - \sum_{j=0}^{n} \frac{\mathbb{E}[\mathbf{Y}^j](-s)^j}{j!} = (-1)^n, \, (1-\nu)[C + \mathrm{o}(1)]s^{\nu}l(1/s), \quad s \downarrow 0.$

6.2 Workload asymptotics for the case $\rho_1 < 1$

In this subsection and the next one it is assumed that the service time distribution at Q_i is regularly varying at infinity of index $-\nu_i$ (see Appendix A for the definition of regularly, and slowly, varying functions):

$$\mathbb{P}\{\mathbf{B}_i > t\} \sim \frac{C_i}{-, \, (1-\nu_i)} t^{-\nu_i} l_i(t), \quad t \to \infty, \quad i = 1,2; \tag{37}$$

here $l_i(\cdot)$, $i = 1, 2$, are slowly varying functions.
Let us assume that $1 < \nu_i < 2$; larger values of ν_i can be handled with minor adaptations. According to Lemma 6.1, (37) with $1 < \nu_i < 2$ is equivalent with

$$\frac{1-\beta_i\{s\}}{\beta_i s} = 1 - \frac{C_i}{\beta_i} s^{\nu_1 - 1} l_i(\frac{1}{s}), \quad s \downarrow 0. \tag{38}$$

We intend to show that, if $\rho_1 < 1$, then the tail of the workload distribution at Q_1 is regularly varying at infinity of index $1 - \nu_1$. That would also be the case if Q_2 were not present, cf. [21]; i.e., in the case $\rho_1 < 1$, the tail behavior of $\mathbf{V}_1$ is not really influenced by Q_2.

Remark 6.1 *We also assume in this subsection that* $\rho_2 < 1$. *In Remark 6.3 it is indicated that the results to a large extent remain true if* $\rho_2 \geq 1$.

Our approach is as follows. Formula (28) expresses $\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}]$ into $\psi_2(s)$. Formula (29) expresses $\psi_2(s)$, or rather $\psi_2(\delta_1(w))$, into $R_1(w)$ and $R_2(w)$. We use these formulas to derive the behavior of $\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}]$ for $s \downarrow 0$. Lemma 6.1 then yields $\mathbb{P}\{\mathbf{V}_1 > t\}$ for $t \to \infty$. Therefore we now concentrate on the behavior of $R_i(w)$ and, first, $\delta_1(w)$ for $w \downarrow 0$.

Let $\mathbf{P}_1$ denote a random variable with distribution the steady-state distribution of a busy period at Q_1 in isolation, i.e., an M/G/1 queue with arrival rate λ_1 and service time distribution $B_1(\cdot)$. Comparing (36) with the Takács equation for the busy period LST $\mathbb{E}[\mathrm{e}^{-w\mathbf{P}_1}]$, cf. p. 250 of Cohen [21], it is seen that

$$\delta_1(w) = w + \lambda_1(1 - \mathbb{E}[\mathrm{e}^{-w\mathbf{P}_1}]).$$

De Meyer and Teugels [37] have proven that $\mathbb{P}\{\mathbf{P}_1 > t\}$ is regularly varying at infinity of index $-\nu_1$ iff $\mathbb{P}\{\mathbf{B}_1 > t\}$ is regularly varying at infinity of index $-\nu_1$, and if either holds then

$$\mathbb{P}\{\mathbf{P}_1 > t\} \sim \frac{1}{1-\rho_1}\mathbb{P}\{\frac{\mathbf{B}_1}{1-\rho_1} > t\}, \quad t \to \infty. \tag{39}$$

Lemma 6.1 then gives the behavior of $\mathbb{E}[\mathrm{e}^{-wP_1}] - 1$ for $w \downarrow 0$. We conclude that, if (37) holds for $i = 1$, then

$$\delta_1(w) - \frac{w}{1-\rho_1} \sim -\lambda_1 C_1 \frac{w^{\nu_1}}{(1-\rho_1)^{\nu_1+1}} l_1(\frac{1}{w}), \quad w \downarrow 0. \tag{40}$$

In addition, using (36), we have for $\delta_1^{-1}(s) = s - \lambda_1(1 - \beta_1\{s\})$:

$$\delta_1^{-1}(s) - (1-\rho_1)s \sim \lambda_1 C_1 s^{\nu_1} l_1(\frac{1}{s}), \quad s \downarrow 0.$$

In the study of $R_i(w)$, a key role is played by the LST of $\hat{\mathbf{P}}_1$, a busy period at Q_1 in isolation that starts with a *residual* service time. From (6.4) of [22],

$$\mathbb{E}[\mathrm{e}^{-w\hat{\mathbf{P}}_1}] = \frac{1 - \beta_1\{\delta_1(w)\}}{\beta_1\delta_1(w)}, \quad \mathrm{Re}\ w \geq 0.$$

It is now readily verified that

$$1 - \mathbb{E}[\mathrm{e}^{-w\hat{\mathbf{P}}_1}] \sim \frac{C_1}{\beta_1}(\frac{w}{1-\rho_1})^{\nu_1 - 1} l_i(\frac{1}{\delta_1(w)}), \quad w \downarrow 0,$$

and hence, using Lemma 6.1, $\mathbb{P}\{\hat{\mathbf{P}}_1 > t\}$ is seen to be regularly varying at infinity of index $1 - \nu_1$:

$$\mathbb{P}\{\hat{\mathbf{P}}_1 > t\} \sim \frac{C_1}{\beta_1,\ (2-\nu_1)}((1-\rho_1)t)^{1-\nu_1} l_1(t), \quad t \to \infty. \tag{41}$$

The difference with (39) is caused by the *residual* service time with which the busy period starts; it is regularly varying of one index higher than an ordinary service time.

We are now ready to study the tail behavior of $R_i(w)$. Observe that $R_i(w)$ is the LST of

$$r_i(t) := \sum_{n=1}^{\infty} \frac{b_i^n}{n} \mathbb{P}\{0 < \mathbf{X}_{i1} + \ldots + \mathbf{X}_{in} < t\}, \quad t > 0.$$

Consider

$$R_i(0) - r_i(t) = \sum_{n=1}^{\infty} \frac{b_i^n}{n} \mathbb{P}\{\mathbf{X}_{i1} + \ldots + \mathbf{X}_{in} > t\}, \quad t > 0. \tag{42}$$

Using properties of long-tailed random variables, it is shown in [11] that

$$\mathbb{P}\{\mathbf{X}_{21} + \ldots + \mathbf{X}_{2n} > t\} \sim n\pi_2 \mathbb{P}\{\hat{\mathbf{P}}_1 > t\}. \tag{43}$$

We conclude from (41), (42) and (43) that

$$R_2(0) - r_2(t) \sim \frac{b_2 \pi_2}{1 - b_2} \mathbb{P}\{\hat{\mathbf{P}}_1 > t\} \sim \frac{1}{(1 - b_2) r_1^*} \frac{\lambda_1 C_1}{, (2 - \nu_1)} ((1 - \rho_1) t)^{1 - \nu_1} l_1(t), \quad t \to \infty. \tag{44}$$

Again applying Lemma 6.1,

$$R_2(w) - R_2(0) \sim -\frac{\lambda_1 C_1}{(1 - b_2) r_1^*} (\frac{w}{1 - \rho_1})^{\nu_1 - 1} l_1(\frac{1}{w}), \quad w \downarrow 0. \tag{45}$$

Similarly, it is seen that

$$\mathbb{P}\{\mathbf{X}_{11} + \ldots + \mathbf{X}_{1n} > t\} \sim n\pi_1 \mathbb{P}\{\hat{\mathbf{P}}_1 > t\},$$

leading to

$$R_1(w) - R_1(0) \sim -\frac{\lambda_1 C_1 (1 - \frac{1}{r_2^*})}{1 - b_1} (\frac{w}{1 - \rho_1})^{\nu_1 - 1} l_1(\frac{1}{w}), \quad w \downarrow 0. \tag{46}$$

It follows from (29), (45) and (46) after a lengthy calculation that

$$\psi_2(\delta_1(w)) - \psi_2(0) \sim -\frac{\lambda_1 C_1 (1 - \rho_2)}{(1 - b_2) r_1^*} (\frac{w}{1 - \rho_1})^{\nu_1 - 1} l_1(\frac{1}{w}), \quad w \downarrow 0. \tag{47}$$

Finally, see (40),

$$\psi_2(s) - \psi_2(0) \sim -\frac{\lambda_1 C_1 (1 - \rho_2)}{(1 - b_2) r_1^*} s^{\nu_1 - 1} l_1(\frac{1}{s}), \quad s \downarrow 0. \tag{48}$$

Using (28), (38) and (48), and the fact that (cf. (2.23) of [22], or take $s = 0$ in (28)),

$$\frac{\psi_1(0)}{1 - \rho_1} + \frac{r_1^* - 1}{1 - \rho_1} [\psi_0 - \psi_2(0)] = 1, \tag{49}$$

it follows that

$$\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] - 1 \sim -[\frac{1}{1 - \rho_1} - \frac{(r_1^* - 1)(1 - \rho_2)}{1 - \rho_1} \frac{1}{(1 - b_2) r_1^*}] \lambda_1 C_1 s^{\nu_1 - 1} l_1(\frac{1}{s}), \quad s \downarrow 0.$$

Using (33), we can rewrite this into

$$\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] - 1 \sim -\frac{\lambda_1 C_1}{K - \rho_1} s^{\nu_1 - 1} l_1(\frac{1}{s}), \quad s \downarrow 0, \tag{50}$$

with $K := \rho_2 + (1 - \rho_2) r_1^*$. Applying Lemma 6.1 once more, we have proven the main result of this subsection:

Theorem 6.1 *If* $\mathbb{P}\{\mathbf{B}_1 > t\} \in \mathcal{R}_{-\nu_1}$, $1 < \nu_1 < 2$, *as given in (37), and if* $\rho_1 < 1$, $\rho_2 < 1$, *then* $\mathbb{P}\{\mathbf{V}_1 > t\} \in \mathcal{R}_{1-\nu_1}$, *as given below:*

$$\mathbb{P}\{\mathbf{V}_1 > t\} \sim \frac{1}{K-\rho_1}, \frac{\lambda_1 C_1}{(2-\nu_1)} t^{1-\nu_1} l_1(t) \sim \frac{\rho_1}{K-\rho_1} \mathbb{P}\{\mathbf{B}_1^r > t\}, \quad t \to \infty.$$

The above theorem implies that (cf. [20]) $\mathbb{P}\{\mathbf{V}_1 > t\}$ behaves exactly as if Q_1 is an M/G/1 queue in isolation, with *constant* server speed K. K has the interpretation of a reduced-load equivalence. Indeed, if $\mathbf{V}_1$ is large, then Q_2 operates at low speed 1, and in that case Q_2 is empty with probability $1-\rho_2$; so Q_1 operates at speed r_1^* a fraction $1-\rho_2$ of the time. Hence K can be interpreted as the average available service speed for Q_1 when its workload is large. The distribution of $\mathbf{B}_2$, and its tail behavior, play no role in this result, and neither does r_2^*. Qualitatively, these results correspond to those of Theorem 3.1 for the GPS model.

In the present subsection we have assumed that $\rho_1 < 1$, i.e., server 1 would have been able to handle all the work in its queue without any assistance of server 2 (i.e., without periods of high speed r_1^*). It makes sense that in this case the tail behavior of $\mathbf{V}_1$ is not really influenced by Q_2, except for the factor K. In the next subsection we shall see that this is different when $\rho_1 > 1$. The boundary case $\rho_1 = 1$ is more delicate; it is not discussed in this paper.

6.3 Workload asymptotics for the case $\rho_1 > 1$

Starting-point for studying the tail behavior of the workload $\mathbf{V}_1$ if $\rho_1 > 1$ is again Relation (28) for its LST, but we can no longer use (29) for the term $\psi_2(s)$ which is contained in it. The reason for this is the following. We want to let $s \to 0$, but $\delta_1(w) \to \delta_1(0) \neq 0$ for $w \to 0$ if $\rho_1 > 1$. Let us therefore take a closer look at the zeroes of $f_1(s,w)$, cf. (36). In [22] it is observed that $\frac{\mathrm{d}}{\mathrm{d}s} f_1(s,w)$ has, for real $s \geq 0$, no zero if $\rho_1 < 1$, one zero $s_0 = 0$ if $\rho_1 = 1$, and one zero $s_0 > 0$ if $\rho_1 > 1$. If $\rho_1 \geq 1$, then the point $w_0 := s_0 - \lambda_1(1-\beta_1\{s_0\})$ is a second-order branch-point of the analytic continuation of $\delta_1(w)$, Re $w \geq 0$, into Re $w < 0$. For $\rho_2 < 1$, $\rho_1 \geq 1$, and $w \in [w_0, 0]$, the *two* zeroes of $f_1(s,w)$ in $[0, \delta_1(0)]$ will be denoted by $\epsilon_1(w)$ and $\delta_1(w)$, and such that

$\epsilon_1(w)$ maps $[w_0, 0]$ one-to-one onto $[0, s_0]$,

$\delta_1(w)$ maps $[w_0, 0]$ one-to-one onto $[s_0, \delta_1(0)]$.

If $\rho_1 \geq 1$, then (6.24) of [22] yields:

$$[(1-\frac{1}{r_1^*})\frac{w}{\delta_2(w)} - \frac{1}{r_1^*}\frac{w}{\epsilon_1(w)}][\frac{1}{r_2^*}(\psi_2(\epsilon_1(w)) - \psi_0) - \frac{\psi_0}{r_1^* r_2^*}\frac{1}{1-1/r_1^*-1/r_2^*}] =$$
$$- \quad [(1-\frac{1}{r_2^*})\frac{w}{\epsilon_1(w)} - \frac{1}{r_2^*}\frac{w}{\delta_2(w)}][\frac{1}{r_1^*}(\psi_1(\delta_2(w)) - \psi_0) - \frac{\psi_0}{r_1^* r_2^*}\frac{1}{1-1/r_1^*-1/r_2^*}]. \quad (51)$$

To determine the behavior of $\psi_2(\epsilon_1(w))$ for $w \uparrow 0$ (which eventually will give us the behavior of $\mathbb{E}[e^{-s\mathbf{V}_1}]$ for $s \downarrow 0$, hence that of $\mathbb{P}\{\mathbf{V}_1 > t\}$ for $t \to \infty$), we need to determine the behavior, for $w \uparrow 0$, of $\epsilon_1(w)$, $\delta_2(w)$ and $\psi_1(\delta_2(w))$ – the terms that appear in (51). Take $w < 0$, $w \uparrow 0$. Then (cf. (40)):

$$\epsilon_1(w) = \frac{-w}{\rho_1 - 1} + \lambda_1 C_1 \frac{(-w)^{\nu_1}}{(\rho_1 - 1)^{\nu_1+1}} l_1(\frac{-1}{w}), \quad w \uparrow 0. \tag{52}$$

In view of the symmetry between the two regularly-varying-tail assumptions in (37) and between the definitions of $\delta_1(w)$ and $\delta_2(w)$, it is readily seen from (40) that

$$\delta_2(w) = \frac{-w}{1 - \rho_2} - \lambda_2 C_2 \frac{(-w)^{\nu_2}}{(1 - \rho_2)^{\nu_2+1}} l_2(\frac{-1}{w}), \quad w \uparrow 0. \tag{53}$$

We again assume that $\rho_2 < 1$; this time (with $\rho_1 > 1$), that is necessary for ergodicity to hold. For $\rho_2 < 1$, $\psi_1(\delta_2(w))$ is specified by Formula (6.23) of [22] (notice the symmetry with (29)):

$$\frac{1}{r_1^*}[\psi_1(\delta_2(w)) - \psi_0] = \frac{1}{r_1^* r_2^*} \frac{\psi_0}{1 - 1/r_1^* - 1/r_2^*}[1 - e^{P_1(w) - P_2(w)}], \quad \mathrm{Re}\, w \le 0. \tag{54}$$

We now turn to the study of $P_i(w) - P_i(0)$, $i = 1, 2$. Compared to the study of $R_i(w) - R_i(0)$, there is a small difference. If $\rho_1 > 1$ then the busy period $\hat{\mathbf{P}}_1$ is defective: $\mathbb{P}\{\hat{\mathbf{P}}_1 < \infty\} = \frac{1}{\rho_1}$. Similar to the derivation (for $r_2^* = 1$) in Section IV of [11], we get, with $p_i(t) := \sum_{n=1}^{\infty} \frac{b_i^n}{n} \mathbb{P}\{-t < \mathbf{X}_{i1} + \ldots + \mathbf{X}_{in} < 0\}$: For $t \to \infty$,

$$P_2(0) - p_2(t) \sim \frac{b_2(1 - \pi_2)}{1 - b_2(\frac{\pi_2}{\rho_1} + 1 - \pi_2)} \mathbb{P}\{\hat{\mathbf{P}}_2 > t\} \;=\; \frac{\rho_2}{1 - \rho_2} \mathbb{P}\{\hat{\mathbf{P}}_2 > t\}. \tag{55}$$

Using the counterpart of (41) for $\hat{\mathbf{P}}_2$, and Lemma 6.1, it finally follows that

$$P_2(w) - P_2(0) \sim -\frac{\lambda_2 C_2}{1 - \rho_2}(\frac{-w}{1 - \rho_2})^{\nu_2 - 1} l_2(\frac{-1}{w}), \quad w \uparrow 0. \tag{56}$$

Similarly,

$$P_1(0) - p_1(t) \sim \frac{b_1(1 - \pi_1)}{1 - b_1(\frac{\pi_1}{\rho_1} + 1 - \pi_1)} \mathbb{P}\{\hat{\mathbf{P}}_2 > t\} \;=\; \frac{\rho_2}{1 - \rho_2} \mathbb{P}\{\hat{\mathbf{P}}_2 > t\},$$

so that (56) also holds for $P_1(w) - P_1(0)$. Combining this with (54) gives:

$$\psi_1(\delta_2(w)) - \psi_1(0) = o(w^{\nu_2 - 1} l_2(\frac{-1}{w})), \quad w \uparrow 0. \tag{57}$$

Remark 6.2 *It follows from (57) that* $\mathbb{P}\{\mathbf{V}_1 = 0, \mathbf{V}_2 > t\} = o(t^{1-\nu_2} l_2(t))$, $t \to \infty$. *This may be surprising in view of the fact that if* Q_2 *were an M/G/1 queue in isolation, then* $\mathbb{P}\{\mathbf{V}_2 > t\} \sim C t^{1-\nu_2} l_2(t)$, *cf. [20]. The explanation is the following. The workload at* Q_1 *has a positive drift* $\rho_1 - 1$ *when* $\mathbf{V}_2 > 0$. *Therefore* $\mathbb{P}\{\mathbf{V}_1 = 0 | \mathbf{V}_2 > t\} = o(1)$ *for* $t \to \infty$*: When the workload at* Q_2 *is very large, it is highly unlikely that* Q_1 *is empty.*

The above result for the behavior of $\psi_1(\delta_2(w))$ for $w \uparrow 0$ allows us to determine the behavior of $\psi_2(\epsilon_1(w))$ for $w \uparrow 0$. Using Relation (51) between $\psi_2(\epsilon_1(w))$ and $\psi_1(\delta_2(w))$, along with the asymptotic results (52) and (53) for $\epsilon_1(w)$ and $\delta_2(w)$, it follows after some calculations:

$$\begin{aligned} \psi_2(\epsilon_1(w)) - \psi_2(0) \quad \sim \quad & -\frac{\rho_1 - 1}{(1-b_2)r_1^*}\lambda_2 C_2 (\frac{-w}{1-\rho_2})^{\nu_2 - 1} l_2(\frac{-1}{w}) \mathrm{I}_{\{\nu_1 > \nu_2\}} \\ & - \quad \frac{(1-\rho_2)}{(1-b_2)r_1^*}\lambda_1 C_1 (\frac{w}{1-\rho_1})^{\nu_1 - 1} l_1(\frac{-1}{w}) \mathrm{I}_{\{\nu_1 < \nu_2\}}, \quad w \uparrow 0. \end{aligned} \tag{58}$$

Using (52) once more,

$$\begin{aligned} \psi_2(s) - \psi_2(0) \quad \sim \quad & -\frac{\rho_1 - 1}{(1-b_2)r_1^*}\lambda_2 C_2 (s\frac{\rho_1 - 1}{1-\rho_2})^{\nu_2 - 1} l_2(\frac{1}{s}) \mathrm{I}_{\{\nu_1 > \nu_2\}} \\ & - \quad \frac{(1-\rho_2)}{(1-b_2)r_1^*}\lambda_1 C_1 s^{\nu_1 - 1} l_1(\frac{1}{s}) \mathrm{I}_{\{\nu_1 < \nu_2\}}, \quad s \downarrow 0. \end{aligned} \tag{59}$$

Finally we are ready to determine the tail behavior of the workload $\mathbf{V}_1$ at Q_1. The LST of $\mathbf{V}_1$ is given by (28). The first factor in its righthand side is the LST of the workload distribution at Q_1 in isolation, with a server that always has speed 1 (the Pollaczek-Khintchine workload LST in the M/G/1 queue); this factor would give a $t^{1-\nu_1}$ tail behavior, cf. [20]. Using (49) and (59), the second factor in the righthand side of (28) is seen to yield either a $t^{1-\nu_1}$ or a $t^{1-\nu_2}$ tail behavior. To see which of the terms dominates, we have to distinguish between three cases: $\nu_1 < \nu_2$, $\nu_1 > \nu_2$ and $\nu_1 = \nu_2$.

Case 1: $\nu_1 < \nu_2$. In this case the heavier tail of $\mathbf{B}_1$ dominates, and Formula (50) still holds when $\rho_1 > 1$:

$$\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] - 1 \sim -\frac{\lambda_1 C_1}{K - \rho_1} s^{\nu_1 - 1} l_1(\frac{1}{s}), \quad s \downarrow 0, \tag{60}$$

with $K = \rho_2 + (1-\rho_2)r_1^*$.

Case 2: $\nu_1 > \nu_2$. In this case the heavier tail of $\mathbf{B}_2$ dominates, resulting in:

$$\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] - 1 \sim -\frac{1 - \frac{1}{r_1^*}}{1 - b_2}\lambda_2 C_2 (s\frac{\rho_1 - 1}{1-\rho_2})^{\nu_2 - 1} l_2(\frac{1}{s}), \quad s \downarrow 0. \tag{61}$$

Case 3: $\nu_1 = \nu_2$. In this case, addition of the righthand sides of (60) and (61) gives the right asymptotic behavior of $\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] - 1$.

Applying Lemma 6.1 again, we have proven the main theorem of this subsection:

Theorem 6.2 *If* $\mathbb{P}\{\mathbf{B}_i > t\} \in \mathcal{R}_{-\nu_i}$, $1 < \nu_i < 2$, $i = 1,2$, *as given in (37), and if* $\rho_1 > 1$, *then* $\mathbb{P}\{\mathbf{V}_1 > t\} \in \mathcal{R}_{1-\min\{\nu_1,\nu_2\}}$*:*
If $\nu_1 < \nu_2$, *then*

$$\mathbb{P}\{\mathbf{V}_1 > t\} \sim \frac{1}{K - \rho_1}, \frac{\lambda_1 C_1}{(2-\nu_1)} t^{1-\nu_1} l_1(t) \sim \frac{\rho_1}{K - \rho_1}\mathbb{P}\{\mathbf{B}_1^r > t\}, \quad t \to \infty;$$

If $\nu_1 > \nu_2$*, then*

$$\mathbb{P}\{\mathbf{V}_1 > t\} \sim \frac{1-\frac{1}{r_1^*}}{1-b_2}\frac{\lambda_2 C_2}{,\,(2-\nu_2)}(t\frac{\rho_1 - 1}{1-\rho_2})^{1-\nu_2} l_2(t), \quad t \to \infty; \tag{62}$$

If $\nu_1 = \nu_2$*, then*

$$\mathbb{P}\{\mathbf{V}_1 > t\} \sim \frac{1}{K-\rho_1}\frac{\lambda_1 C_1}{,\,(2-\nu_1)} t^{1-\nu_1} l_1(t) \;+\; \frac{1-\frac{1}{r_1^*}}{1-b_2}\frac{\lambda_2 C_2}{,\,(2-\nu_2)}(t\frac{\rho_1 - 1}{1-\rho_2})^{1-\nu_2} l_2(t), \quad t \to \infty.$$

The above result implies the following. If the tail of $\mathbf{B}_1$ is heavier than that of $\mathbf{B}_2$, then $\mathbb{P}\{\mathbf{V}_1 > t\}$ behaves exactly as if Q_1 is an M/G/1 queue in isolation, with *constant* server speed K (which is the average available speed for Q_1 if it is non-empty). But if the tail of $\mathbf{B}_2$ is heavier than that of $\mathbf{B}_1$ and $\rho_1 > 1$ (server 1 needs the help of server 2), then the tail behavior of $\mathbf{B}_2$ determines that of $\mathbb{P}\{\mathbf{V}_1 > t\}$. Qualitatively, these results reflect those of Theorems 4.1 and 5.1 for the GPS model.

In particular, Formula (62) has a similar interpretation as the result of Theorem 5.1. Notice that the workload at Q_1 has a positive drift $\rho_1 - 1$ during the busy periods $\mathbf{P}_2$ of Q_2, and a negative drift $\rho_1 - r_1^*$ during the ($\exp(\lambda_2)$ distributed) idle periods of Q_2. Thus the workload at Q_1 behaves as that in a queue fed by a single On-Off flow, with Off-periods $\exp(\lambda_2)$ distributed, and On-periods distributed like the busy periods of Q_2 with speed 1 (as observed above, Q_2 operates for a long time at low speed if $\mathbf{V}_1 > t >> 0$). During Off-periods, the buffer content $\mathbf{V}$ of the fluid queue decreases at rate $r_1^* - \rho_1$. During On-periods, the buffer content $\mathbf{V}$ increases at rate $\rho_1 - 1$. Using Theorems 2.2 and 2.4, we see that this behavior is reflected in Formula (62).

Remark 6.3 *Let us go back to the case* $\rho_1 < 1$ *of the previous subsection, and let us now take* $\rho_2 > 1$*. In the analysis of Subsection 6.2, one change occurs.* $\hat{\mathbf{P}}_2$ *now is defective;* $\mathbb{P}\{\hat{\mathbf{P}}_2 < \infty\} = \frac{1}{\rho_2}$*. The first part of (44) now changes, in accordance with (55), into:*

$$R_2(0) - r_2(t) \sim \frac{b_2\pi_2}{1 - b_2(\pi_2 + \frac{1-\pi_2}{\rho_2})}\mathbb{P}\{\hat{\mathbf{P}}_1 > t\}, \quad t \to \infty.$$

Similarly,

$$R_1(0) - r_1(t) \sim \frac{b_1\pi_1}{1 - b_1(\pi_1 + \frac{1-\pi_1}{\rho_2})}\mathbb{P}\{\hat{\mathbf{P}}_1 > t\}, \quad t \to \infty.$$

The two constants in the righthand sides both equal $\frac{\rho_1}{1-\rho_1}$*. Finally it follows, cf. (57), that*

$$\psi_2(\delta_1(w)) - \psi_2(0) = \mathrm{o}(w^{\nu_1 - 1} l_1(\frac{1}{w})), \quad w \downarrow 0. \tag{63}$$

Using (28), (38) and (63) (instead of (47)), it follows that

$$\mathbb{E}[\mathrm{e}^{-s\mathbf{V}_1}] - 1 \sim -\frac{\lambda_1 C_1}{1-\rho_1} s^{\nu_1 - 1} l_1(\frac{1}{s}), \quad s \downarrow 0,$$

and hence, cf. Theorem 6.1,

$$\mathbb{P}\{\mathbf{V}_1 > t\} \sim \frac{\rho_1}{1-\rho_1} \mathbb{P}\{\mathbf{B}_1^r > t\}, \quad t \to \infty.$$

Remark 6.4 *Interestingly, both Theorem 6.1 and Theorem 6.2 agree with the results in [11] where* $r_2^* = 1$ *was chosen. The explanation is that, if* $\mathbf{V}_1 > t >> 0$*, then* Q_2 *operates at low speed all the time. Also notice that, although we have excluded the case* $1/r_1^* + 1/r_2^* = 1$*, this case does not seem to represent a limiting case in Theorem 6.1 and Theorem 6.2 for* $1/r_1^* + 1/r_2^* \to 1$*. This suggests that those theorems remain valid if* $1/r_1^* + 1/r_2^* = 1$*.*

7 Conclusion

We analyzed the queueing behavior of long-tailed traffic flows under the Generalized Processor Sharing (GPS) discipline. We showed a sharp dichotomy in qualitative behavior, depending on the relative values of the weight parameters. For certain weight combinations, an individual flow with long-tailed traffic characteristics is effectively served at a *constant* rate. The effective service rate may be interpreted as the maximum average rate for the flow to be stable, which is only influenced by the traffic characteristics of the other flows through their average rates. This indicates that GPS-based scheduling algorithms offer a potential mechanism for obtaining substantial multiplexing gains, while protecting indvidual flows. For other weight combinations however, a flow may be strongly affected by the activity of 'heavier'-tailed flows, and may inherit their traffic characteristics, causing induced burstiness. The stark contrast in qualitative behavior highlights the great significance of the weight parameters.

In the present paper we focused on the workload of an individual flow at a single node. Some of the results may be extended to 'bottle-neck nodes' in feed-forward networks [43]. It would also be interesting to examine delays or loss probabilities in case of finite buffers. With class aggregation, the flows that we considered may actually be *macro*-flows, each consisting of several *micro*-flows, which at a lower level may be served on a FCFS basis, or also according to GPS. It would be interesting to investigate the behavior of the micro-flows in such hierarchical situations.

In Section 5 we found that a light-tailed flow whose weight is 'too small' could be strongly affected by a heavy-tailed flow. The case of a light-tailed flow whose weight is 'large enough' to be protected is a topic of current research.

A final issue concerns the behavior of an On-Off flow whose peak rate r_i is smaller than the effective service rate γ_i. In that case, other flows too need to show anomalous activity for the workload of flow i to grow, which means that the tail behavior may become 'less heavy-tailed' or even light-tailed. This phenomenon may be viewed as somewhat dual to the induced burstiness described above.

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A Definitions

Definition A.1 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *long-tailed* ($F(\cdot) \in \mathcal{L}$) if

$$\lim_{x\to\infty} \frac{1-F(x-y)}{1-F(x)} = 1, \quad \text{for all real } y.$$

Definition A.2 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *subexponential* ($F(\cdot) \in \mathcal{S}$) if

$$\lim_{x\to\infty} \frac{1-F^{2*}(x)}{1-F(x)} = 2,$$

where $F^{2*}(\cdot)$ is the 2-fold convolution of $F(\cdot)$ with itself, i.e., $F^{2*}(x) = \int_0^x F(x-y)F(\mathrm{d}y)$.

A relevant subclass of $\mathcal{S}$ is the class $\mathcal{R}$ of *regularly-varying* distributions (which contains the Pareto distribution):

Definition A.3 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *regularly varying of index* $-\nu$ ($F(\cdot) \in \mathcal{R}_{-\nu}$) if

$$F(x) = 1 - \frac{l(x)}{x^{\nu}}, \quad \nu \geq 0,$$

where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of slow variation, i.e., $\lim_{x\to\infty} l(\eta x)/l(x) = 1$, $\eta > 1$.

A key reference is Bingham *et al.* [7]. It is easily seen that $\mathcal{R} \subset \mathcal{S} \subset \mathcal{L}$. Examples of subexponential distributions which do not belong to $\mathcal{R}$ include the Weibull, lognormal, and Benktander distributions (see Klüppelberg [32]). A minor extension of $\mathcal{R}$ is the class $\mathcal{IR}$ of *intermediately regularly-varying* distributions:

Definition A.4 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *intermediately regularly varying* ($F(\cdot) \in \mathcal{IR}$) if

$$\lim_{\eta\uparrow 1} \limsup_{x\to\infty} \frac{1-F(\eta x)}{1-F(x)} = 1.$$

A further extension is the class $\mathcal{DR}$ of *dominatedly varying* distributions (see Cline [19]; $\mathcal{R} \subset \mathcal{IR} \subset (\mathcal{DR} \cap \mathcal{L}) \subset \mathcal{S}$):

Definition A.5 A distribution function $F(\cdot)$ on $[0,\infty)$ is called *dominatedly varying* ($F(\cdot) \in \mathcal{DR}$) if

$$\limsup_{x\to\infty} \frac{1-F(\eta x)}{1-F(x)} < \infty, \quad \text{for some real } \eta \in (0,1).$$

B Proof of Theorem 2.3

Theorem 2.3 If $U_i(\cdot)$ is an exponential distribution, $S_i^r(\cdot) \in \mathcal{IR}$, and $\rho_i < c$, then

$$\mathbb{P}\{\mathbf{P}_i^r > x\} \sim \frac{c}{c-\rho_i} \mathbb{P}\{\mathbf{S}_i^r > x(c-\rho_i)\}.$$

Proof

For compactness, we suppress the subscript i, e.g., $V^c(t) \equiv V_i^c(t)$, $\rho \equiv \rho_i$, etc.

For $0 < \delta < c - \rho$, define

$$L^\delta(t) = \sup_{0 \leq s \leq t} \{B^c(s,t) - (c-\delta)(t-s)\},$$

with $B^c(s,t)$ as in (2).

Observe that $L^\delta(t)$ and $V^c(t)$ represent the workload processes in a priority queue with service rate c and arrival processes $\delta(t-s)$ and $A(s,t)$, respectively, with $L^\delta(t)$ having lower priority. Since the total workload does not depend on the priority mechanism, the sum of the workloads equals

$$L^\delta(t) + V^c(t) = V^{c-\delta}(t) = \sup_{0 \leq s \leq t} \{A(s,t) - (c-\delta)(t-s)\}. \tag{64}$$

(Upper bound) By the previous equality and Theorem 2.1, in steady state,

$$\begin{aligned} \mathbb{P}\{\mathbf{L}^\delta > \delta x\} &\leq \mathbb{P}\{\mathbf{V}^{c-\delta} > \delta x\} \\ &\sim \frac{\rho}{c-\delta-\rho} \mathbb{P}\{\mathbf{S}^r > \delta x\}. \end{aligned} \tag{65}$$

Let $\mathbf{P}^{b,r}$ be the past lifetime of the busy period associated with $V^c(t)$ in steady state. By symmetry, $\mathbf{P}^{b,r}$ is equal in distribution to $\mathbf{P}^r$. Hence,

$$\mathbb{P}\{\mathbf{L}^\delta > x\} \geq \mathbb{P}\{\mathbf{V}^c > 0, \mathbf{P}^{b,r} > x/\delta\} = \mathbb{P}\{\mathbf{V}^c > 0\}\mathbb{P}\{\mathbf{P}^r > x/\delta\}, \tag{66}$$

where we use the fact that in steady state $\mathbf{P}^{b,r}$ is independent of the event $\{\mathbf{V}^c > 0\}$.

Since the busy period $\mathbf{P}$ is larger than the time $\mathbf{S}/c$ it takes to serve a single service request, it easily follows that

$$\mathbb{P}\{\mathbf{P}^r > x\} \geq \frac{\mathbb{E}\mathbf{S}}{c\mathbb{E}\mathbf{P}} \mathbb{P}\{\mathbf{S}^r > cx\}, \tag{67}$$

which, in conjunction with (65), (66) and $S^r(\cdot) \in \mathcal{IR}$, implies that $P^r(\cdot) \in \mathcal{DR}$, and therefore $P^r(\cdot) \in \mathcal{S}$.

Now observe that $L^\delta(t)$ may also be interpreted as the workload at time t in a queue with constant service rate $c - \delta$ fed by an On-Off process with On- and Off-periods equal to the busy and idle periods associated with the workload process $V^c(t)$, respectively. During the On-periods, traffic is produced at constant rate c. The fraction Off-time is $1 - \rho/c$. The

On- and Off-periods are independent because $U(\cdot)$ is an exponential distribution. Hence, by Theorem 2.4,

$$\mathbb{P}\{\mathbf{L}^\delta > \delta x\} \sim \frac{c-\rho}{c}\frac{\rho}{c-\delta-\rho}\mathbb{P}\{\mathbf{P}^r > x\}. \tag{68}$$

Now, (65) and (68) yield

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{P}^r > x\}}{\mathbb{P}\{\mathbf{S}^r > \delta x\}} \le \frac{c}{c-\rho},$$

and the upper bound follows by letting $\delta \uparrow c - \rho$.

(Lower bound) From (64), in steady state, for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\{\mathbf{L}^\delta > \delta x\} &= \mathbb{P}\{\mathbf{V}^{c-\delta} - \mathbf{V}^c > \delta x\} \\ &\ge \mathbb{P}\{\mathbf{V}^{c-\delta} > (1+\epsilon)\delta x, \mathbf{V}^c \le \epsilon\delta x\} \\ &\ge \mathbb{P}\{\mathbf{V}^{c-\delta} > (1+\epsilon)\delta x\} - \mathbb{P}\{\mathbf{V}^c > \epsilon\delta x\}. \end{aligned} \tag{69}$$

Hence, by (68), (69), and Theorem 2.1,

$$\liminf_{x\to\infty} \frac{\mathbb{P}\{\mathbf{P}^r > x\}}{\mathbb{P}\{\mathbf{S}^r > (1+\epsilon)\delta x\}} \ge \frac{c}{c-\rho} - \frac{c(c-\delta-\rho)}{(c-\rho)^2} \limsup_{x\to\infty} \frac{\mathbb{P}\{\mathbf{S}^r > \epsilon\delta x/(r-c)\}}{\mathbb{P}\{\mathbf{S}^r > (1+\epsilon)\delta x\}},$$

which, by letting first $\delta \uparrow c - \rho$ and then $\epsilon \downarrow 0$ completes the proof of the lower bound.

□

C Stability issues

We now identify which flows are stable and which ones are unstable. Flow i is considered 'stable' if the mean service rate is ρ_i. For ease of presentation, we assume the flows are indexed such that

$$\frac{\rho_1}{\phi_1} \le \ldots \le \frac{\rho_N}{\phi_N}.$$

Define S^* as the set of stable flows. Denote by γ_i the mean service rate for flow i (assuming it exists).

We have $\gamma_i \le \rho_i$ for all $i = 1, \ldots, N$, with equality for all $i \in S^*$. Also, if $j \notin S^*$, then $\frac{\gamma_i}{\phi_i} \le \frac{\gamma_j}{\phi_j}$ for all $i = 1, \ldots, N$.

In particular, we have $\frac{\gamma_i}{\phi_i} = \frac{\gamma_j}{\phi_j}$ for any pair of flows $i, j \notin S^*$, so $\gamma_i = \phi_i R$ for all $i \notin S^*$ for some $R \ge 1$. To determine R, observe that $\sum\limits_{i=1}^{N} \gamma_i = 1$ if $S^* \ne \{1, \ldots, N\}$, which gives

$$R = \frac{1}{\sum\limits_{j\notin S^*} \phi_j} \left(1 - \sum_{j\in S^*} \rho_j\right).$$

We first prove a lemma that characterizes the structure of the set S^*.

Lemma C.1 *With the above ordering of the flows, the set S^* is of the form $\{1, \ldots, K\}$ for some K.*

Proof

Suppose not, i.e., there are flows i and j, with $i < j$, $i \notin S^*$, and $j \in S^*$. Then we have $\gamma_i < \rho_i$, $\gamma_j = \rho_j$, and $\frac{\gamma_i}{\phi_i} \geq \frac{\gamma_j}{\phi_j}$. Thus, $\frac{\rho_i}{\phi_i} > \frac{\rho_j}{\phi_j}$, which would contradict the ordering of the flows.

□

We now prove an auxiliary lemma.

Lemma C.2 *With the above ordering of the flows, if*

$$\rho_k > \frac{\phi_k}{\sum_{j=k}^{N} \phi_j} \left(1 - \sum_{j=1}^{k-1} \rho_j\right), \tag{70}$$

then

$$\rho_{k+1} > \frac{\phi_{k+1}}{\sum_{j=k+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{k} \rho_j\right). \tag{71}$$

Proof

First observe the equivalence relation

$$\rho_k > \frac{\phi_k}{\sum_{j=k}^{N} \phi_j} \left(1 - \sum_{j=1}^{k-1} \rho_j\right) \Longleftrightarrow \rho_k > \frac{\phi_k}{\sum_{j=k+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{k} \rho_j\right). \tag{72}$$

The proof then immediately follows from the fact that $\frac{\rho_k}{\phi_k} \leq \frac{\rho_{k+1}}{\phi_{k+1}}$.

□

The next lemma now identifies the set of stable flows.

Lemma 4.1

With the above ordering of the flows, the set of stable flows is $S^* = \{1, \ldots, K^*\}$, with

$$K^* = \max_{k=1,\ldots,N} \left\{ k : \frac{\rho_k}{\phi_k} \leq \frac{1 - \sum_{j=1}^{k-1} \rho_j}{\sum_{j=k}^{N} \phi_j} \right\}.$$

Proof

By Lemma C.1, the set S^* is of the form $\{1, \ldots, L\}$ for some L, so it suffices to show that $L = K^*$. First observe that

$$\rho_{L+1} > \gamma_{L+1} = \frac{\phi_{L+1}}{\sum\limits_{j=L+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{L} \rho_j\right).$$

By Lemma C.2 and the definition of K^*, this implies $L \geq K^*$.

We also have $\gamma_L = \rho_L$ and $\frac{\gamma_L}{\phi_L} \leq \frac{\gamma_{L+1}}{\phi_{L+1}}$. Thus,

$$\rho_L \leq \frac{\phi_L}{\phi_{L+1}} \gamma_{L+1} = \frac{\phi_L}{\sum\limits_{j=L+1}^{N} \phi_j} \left(1 - \sum_{j=1}^{L} \rho_j\right),$$

which is equivalent to

$$\rho_L \leq \frac{\phi_L}{\sum\limits_{j=L}^{N} \phi_j} \left(1 - \sum_{j=1}^{L-1} \rho_j\right).$$

By Lemma C.2 and the definition of K^*, this implies $L \leq K^*$.

□

D Basic GPS inequalities

Lemma D.1 *Let $S, T \subseteq \{1, \ldots, N\}$ be sets with $S \cap T = \emptyset$, and let α_j, $j \in S$, be numbers such that*

$$\alpha_i \frac{\sum\limits_{j \notin S \cup T} \phi_j}{\phi_i} \leq 1 - \sum_{j \in S} \alpha_j \tag{73}$$

for all $i \in S$.

Then

$$\sum_{j \in S} B_j(r,t) \geq \sum_{j \in S} \inf_{r \leq s \leq t} \{A_j(r,s) + \alpha_j [t - s - \sum_{k \in T} B_k(s,t)]\}$$

for all $0 \leq r \leq t$.

Proof

For given values of r, t, define

$$v^* := \max_{r \leq v \leq t} \{v : \sum_{j \in S} B_j(r,v) \geq \sum_{j \in S} \inf_{r \leq s \leq v} \{A_j(r,s) + \alpha_j [v - s - \sum_{k \in T} B_k(s,v)]\}\}.$$

We need to show that $v^* = t$. Suppose not, i.e., $v^* < t$. Then there must be some flow i^* for which

$$B_{i^*}(r,v) < \inf_{r \le s \le v} \{A_{i^*}(r,s) + \alpha_{i^*}[v - s - \sum_{k \in T} B_k(s,v)]\}$$

for all $v \in (v^*, w^*)$ for some $w^* > v^*$.
Define

$$u^* := \max_{r \le u \le w^*} \{u : B_{i^*}(r,u) \ge \inf_{r \le s \le u} \{A_{i^*}(r,s) + \alpha_{i^*}[u - s - \sum_{k \in T} B_k(s,u)]\}\}.$$

First observe that

$$B_{i^*}(u^*, w^*) \le \alpha_{i^*}[w^* - u^* - \sum_{k \in T} B_k(u^*, w^*)], \tag{74}$$

because otherwise

$$\begin{aligned}
B_{i^*}(r,w^*) &= B_{i^*}(r,u^*) + B_{i^*}(u^*,w^*) \\
&> \inf_{r \le s \le u^*} \{A_{i^*}(r,s) + \alpha_{i^*}[u^* - s - \sum_{k \in T} B_k(s,u^*)]\} + \alpha_{i^*}[w^* - u^* - \sum_{k \in T} B_k(u^*,w^*)] \\
&= \inf_{r \le s \le u^*} \{A_{i^*}(r,s) + \alpha_{i^*}[w^* - s - \sum_{k \in T} B_k(s,w^*)]\} \\
&\ge \inf_{r \le s \le w^*} \{A_{i^*}(r,s) + \alpha_{i^*}[w^* - s - \sum_{k \in T} B_k(s,w^*)]\},
\end{aligned}$$

contradicting the definition of w^*.
Further observe that

$$B_{i^*}(r,u) < \inf_{r \le s \le u} \{A_{i^*}(r,s) + \alpha_{i^*}[u - s - \sum_{k \in T} B_k(s,u)]\} \le A_{i^*}(r,u)$$

for all $u \in (u^*, w^*)$, so that flow i^* must be continuously backlogged during the interval (u^*, w^*).
Hence, by definition of the GPS discipline,

$$B_{i^*}(u^*, w^*) \ge \frac{\phi_{i^*}}{\phi_j} B_j(u^*, w^*) \tag{75}$$

for all $j = 1, \ldots, N$, and

$$\sum_{j=1}^{N} B_j(u^*, w^*) = w^* - u^*. \tag{76}$$

Using (73), (74), (75),

$$\sum_{j \notin S \cup T} B_j(u^*, w^*) \quad \le \quad \frac{\sum_{j \notin S \cup T} \phi_j}{\phi_{i^*}} B_i(u^*, w^*)$$

$$
\begin{aligned}
&\leq \quad \alpha_{i^*} \frac{\sum_{j \notin S \cup T} \phi_j}{\phi_{i^*}} [w^* - u^* - \sum_{k \in T} B_k(u^*, w^*)] \\
&\leq \quad (1 - \sum_{j \in S} \alpha_j)[w^* - u^* - \sum_{k \in T} B_k(u^*, w^*)]. \qquad (77)
\end{aligned}
$$

Combining (76), (77),

$$
\begin{aligned}
\sum_{j \in S} B_j(u^*, w^*) &= \sum_{j=1}^{N} B_j(u^*, w^*) - \sum_{k \in T} B_k(u^*, w^*) - \sum_{j \notin S \cup T} B_j(u^*, w^*) \\
&\geq w^* - u^* - \sum_{k \in T} B_k(u^*, w^*) - (1 - \sum_{j \in S} \alpha_j)[w^* - u^* - \sum_{k \in T} B_k(u^*, w^*)] \\
&= \sum_{j \in S} \alpha_j [w^* - u^* - \sum_{k \in T} B_k(u^*, w^*)]. \qquad (78)
\end{aligned}
$$

Note that $u^* \leq v^*$, so by the definition of v^*,

$$
\sum_{j \in S} B_j(r, u^*) \geq \sum_{j \in S} \inf_{r \leq s \leq u^*} \{A_j(r, s) + \alpha_j [u^* - s - \sum_{k \in T} B_k(s, u^*)]\}. \qquad (79)
$$

From (78), (79),

$$
\begin{aligned}
\sum_{j \in S} B_j(r, w^*) &= \sum_{j \in S} B_j(r, u^*) + \sum_{j \in S} B_j(u^*, w^*) \\
&\geq \sum_{j \in S} \inf_{r \leq s \leq u^*} \{A_j(r, s) + \alpha_j [u^* - s - \sum_{k \in T} B_k(s, u^*)]\} \\
&+ \sum_{j \in S} \alpha_j [w^* - u^* - \sum_{k \in T} B_k(u^*, w^*)] \\
&= \sum_{j \in S} \inf_{r \leq s \leq u^*} \{A_j(r, s) + \alpha_j [w^* - s - \sum_{k \in T} B_k(s, w^*)]\} \\
&\geq \sum_{j \in S} \inf_{r \leq s \leq w^*} \{A_j(r, s) + \alpha_j [w^* - s - \sum_{k \in T} B_k(s, w^*)]\},
\end{aligned}
$$

contradicting the definition of v^*, so we must have $v^* = t$ as required.

□

We now prove that $\alpha_j = \gamma_{jE}(\delta) / \sum_{k \notin T} \gamma_{kE}(\delta)$, $j \in S$, with $S_E \subseteq S$, $S \cap T = \emptyset$ satisfy (73) for all $\delta \geq \delta_0$ for some $\delta_0 < 0$.

Lemma D.2 *Let $E, S, T \subseteq \{1, \ldots, N\}$ be sets with $S_E \subseteq S$, $S \cap T = \emptyset$. Then*

$$
\frac{\gamma_{iE}(\delta)}{\sum_{k \notin T} \gamma_{kE}(\delta)} \frac{\sum_{j \notin S \cup T} \phi_j}{\phi_i} \leq 1 - \sum_{j \in S} \frac{\gamma_{jE}(\delta)}{\sum_{k \notin T} \gamma_{kE}(\delta)} \qquad (80)
$$

for all $i \in S$ and $\delta \geq \delta_0$ for some $\delta_0 < 0$.

Proof

Note that

$$1 - \sum_{j \in S} \frac{\gamma_{jE}(\delta)}{\sum\limits_{k \notin T} \gamma_{kE}(\delta)} = \frac{\sum\limits_{j \notin S \cup T} \gamma_{jE}(\delta)}{\sum\limits_{k \notin T} \gamma_{kE}(\delta)}.$$

Using the definition of $\gamma_{jE}(\delta)$ and the fact that $S_E \subseteq S$,

$$\sum_{j \notin S \cup T} \gamma_{jE}(\delta) = \frac{\sum\limits_{j \notin S \cup T} \phi_j}{\sum\limits_{j \notin S_E} \phi_j} \left(1 - (1-\delta) \sum_{j \in E} \rho_j\right).$$

Thus, to prove (80), we need to show that

$$\gamma_{iE}(\delta) \leq \frac{\phi_i}{\sum\limits_{j \notin S_E} \phi_j} (1 - (1-\delta) \sum_{j \in S_E} \rho_j).$$

for all $i \in S$.

By definition, the above inequality holds with equality for all $i \in S \setminus S_E$.

From the definition of S_E and the equivalence relation (72),

$$\frac{\rho_i}{\phi_i} \leq \max_{j \in S_E} \frac{\rho_j}{\phi_j} \leq \frac{1 - \sum\limits_{j \in S_E} \rho_j}{\sum\limits_{j \notin S_E} \phi_j}$$

for all $i \in S_E$.

Hence, for all $i \in S_E$ and $\delta \geq \delta_0$,

$$\gamma_{iE}(\delta) = (1-\delta)\rho_i \leq \frac{\phi_i}{\sum\limits_{j \notin S_E} \phi_j} \left(1 - (1-\delta) \sum_{j \in S_E} \rho_j\right),$$

for some $\delta_0 < 0$.

□

E Proof of Lemma 5.1

Lemma 5.1

(Lower bound) For $\delta > 0$ sufficiently small,

$$\mathbb{P}\{\mathbf{V}_i > x\} \geq \mathbb{P}\{\mathbf{Q}_{k^*}^{\delta} - \sum_{j \neq k^*} \mathbf{Z}_j^{\rho_j(1-\delta)} > x\}. \tag{81}$$

Proof

Define

$$s^* := \arg \sup_{0 \le s \le t} \{\psi_i[B_{k^*}^{\gamma_{k^*}(\delta)}(s,t) - \gamma_{k^*}(\delta)(t-s)] + (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))(t-s)\}, \tag{82}$$

so that

$$Q_{k^*}^{\delta}(t) = \psi_i[B_{k^*}^{\gamma_{k^*}(\delta)}(s^*,t) - \gamma_{k^*}(\delta)(t-s^*)] + (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))(t-s^*). \tag{83}$$

The fact that s^* is the maximizing argument in (82) has the following two implications. First of all,

$$B_{k^*}^{\gamma_{k^*}(\delta)}(s^*,u) \ge (\gamma_{k^*}(\delta) - (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))/\psi_i)(u-s^*) \tag{84}$$

for all $s^* \le u \le t$. Second, for $\delta > 0$ sufficiently small,

$$V_{k^*}^{\gamma_{k^*}(\delta)}(s^*) = 0, \tag{85}$$

because otherwise

$$B_{k^*}^{\gamma_{k^*}(\delta)}(s^* - \Delta, t) = B_{k^*}^{\gamma_{k^*}(\delta)}(s^*,t) + \Delta\gamma_{k^*}(\delta),$$

contradicting the optimality of s^* as $\gamma_{ik^*}(\delta) < \rho_i(1-\delta)$ for $\delta > 0$ sufficiently small.
Using (2), (85),

$$B_{k^*}^{\gamma_{k^*}(\delta)}(s^*,u) = A_{k^*}(s^*,u) - V_{k^*}^{\gamma_{k^*}(\delta)}(u) \tag{86}$$

for all $u \ge s^*$.
Combining (84), (86),

$$A_{k^*}(s^*,u) \ge (\gamma_{k^*}(\delta) - (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))/\psi_i)(u-s^*)$$

for all $s^* \le u \le t$, so that

$$\sup_{s^* \le u \le t} \{(\gamma_{k^*}(\delta) - (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))/\psi_i)(u-s^*) - A_{k^*}(s^*,u)\} = 0. \tag{87}$$

Taking $u = t$ in (86),

$$\begin{aligned} B_{k^*}^{\gamma_{k^*}(\delta)}(s^*,t) &= A_{k^*}(s^*,t) - V_{k^*}^{\gamma_{k^*}(\delta)}(t) \\ &= A_{k^*}(s^*,t) - \sup_{0 \le u \le t} \{A_{k^*}(u,t) - \gamma_{k^*}(\delta)(t-u)\} \\ &\le A_{k^*}(s^*,t) - \sup_{s^* \le u \le t} \{A_{k^*}(u,t) - \gamma_{k^*}(\delta)(t-u)\} \\ &= \gamma_{k^*}(\delta)(t-s^*) - \sup_{s^* \le u \le t} \{\gamma_{k^*}(\delta)(u-s^*) - A_{k^*}(s^*,u)\}. \end{aligned} \tag{88}$$

As in the proof of Lemma 4.5,

$$\begin{aligned} V_i(t) &\geq A_i(r,t) - (t-r) + \sum_{j \neq i} B_j(r,t) \\ &= A_i(r,t) - (t-r) + \sum_{j=1}^{i-1} B_j(r,t) + \sum_{j=i+1}^{N} B_j(r,t). \end{aligned} \tag{89}$$

According to Lemma 4.2, taking $E = T = \{1, \ldots, i-1\}$, $S = \{i+1, \ldots, N\}$, so that $\gamma_{jE}(\delta) = \phi_j R_E(\delta)$ for all $j \in E$,

$$\sum_{j=i+1}^{N} B_j(r,t) \geq \sum_{j=i+1}^{N} \inf_{r \leq s \leq t} \{A_j(r,s) + \frac{\phi_j}{\sum_{j=i}^{N} \phi_j}[t - s - \sum_{j=1}^{i-1} B_j(s,t)]\}. \tag{90}$$

Substituting (90) into (89),

$$\begin{aligned} V_i(t) &\geq A_i(r,t) - (t-r) + \sum_{j=1}^{i-1} B_j(r,t) \\ &+ \sum_{j=i+1}^{N} \inf_{r \leq s \leq t} \{A_j(r,s) + \frac{\phi_j}{\sum_{j=i}^{N} \phi_j}[t - s - \sum_{j=1}^{i-1} B_j(s,t)]\} \\ &\geq A_i(r,t) + \frac{\phi_i}{\sum_{j=i}^{N} \phi_j}[\sum_{j=1}^{i-1} B_j(r,t) - (t-r)] \\ &+ \sum_{j=i+1}^{N} \inf_{r \leq s \leq t} \{A_j(r,s) + \frac{\phi_j}{\sum_{j=i}^{N} \phi_j}[\sum_{j=1}^{i-1} B_j(r,s) - (s-r)]\} \\ &\geq A_i(r,t) + \frac{\phi_i}{\sum_{j=i}^{N} \phi_j}[\sum_{j=1}^{i-1} B_j(r,t) - (t-r)] \\ &+ \sum_{j=i+1}^{N} \inf_{r \leq s \leq t} \{A_j(r,s) - \frac{\phi_j}{\phi_i}\rho_i(1-\delta)(s-r)\} \\ &+ \frac{\sum_{j=i+1}^{N} \phi_j}{\sum_{j=i}^{N} \phi_j} \inf_{r \leq s \leq t} \{\sum_{j=1}^{i-1} B_j(r,s) - (1 - \rho_i(1-\delta)/\psi_i)(s-r)\}. \end{aligned}$$

Noting that $\rho_i/\phi_i \leq \rho_j/\phi_j$ for all $j = i+1, \ldots, N$,

$$V_i(t) \geq A_i(r,t) + \psi_i[\sum_{j=1}^{i-1} B_j(r,t) - (t-r)]$$

$$
\begin{aligned}
&+ \quad \sum_{j=i+1}^{N} \inf_{r \le s \le t} \{A_j(r,s) - \rho_j(1-\delta)(s-r)\} \\
&+ \quad (1-\psi_i) \inf_{r \le s \le t} \{\sum_{j=1}^{i-1} B_j(r,s) - (1 - \rho_i(1-\delta)/\psi_i)(s-r)\}.
\end{aligned}
$$

By assumption, $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$. Hence, from Lemma 4.3, for any y,

$$
\sum_{j=1}^{i-1} B_j(r,s) \ge \sum_{j=1}^{i-1} \inf_{r \le u \le s} \{A_j(r,u) + \gamma_{jk^*}(\delta)(s-u)\}.
$$

Thus

$$
\begin{aligned}
V_i(t) \quad \ge \quad & A_i(r,t) + \psi_i[\sum_{j=1}^{i-1} \inf_{r \le u \le t} \{A_j(r,u) + \gamma_{jk^*}(\delta)(t-u)\} - (t-r)] \\
+ \quad & \sum_{j=i+1}^{N} \inf_{r \le s \le t} \{A_j(r,s) - \rho_j(1-\delta)(s-r)\} \\
+ \quad & (1-\psi_i) \inf_{r \le s \le t} \{\sum_{j=1}^{i-1} \inf_{r \le u \le s} \{A_j(r,u) + \gamma_{jk^*}(\delta)(s-u)\} - (1 - \rho_i(1-\delta)/\psi_i)(s-r)\} \\
\ge \quad & A_i(r,t) - \gamma_{ik^*}(\delta)(t-r) \\
+ \quad & \psi_i[\inf_{r \le u \le t} \{A_{k^*}(r,u) - \gamma_{k^*}(\delta)(u-r)\} + \sum_{j=1, j \ne k^*} \inf_{r \le u \le t} \{A_j(r,u) - \rho_j(1-\delta)(u-r)\}] \\
+ \quad & \sum_{j=i+1}^{N} \inf_{r \le s \le t} \{A_j(r,s) - \rho_j(1-\delta)(s-r)\} \\
+ \quad & (1-\psi_i)[\inf_{r \le u \le s \le t} \{A_{k^*}(r,u) - \gamma_{k^*}(\delta)(u-r) + (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))/\psi_i)(s-r)\} \\
+ \quad & \sum_{j=1, j \ne k^*}^{i-1} \inf_{r \le u \le t} \{A_j(r,u) - \rho_j(1-\delta)(u-r)\}] \\
\ge \quad & \inf_{r \le s \le t} \{A_i(r,s) - \rho_i(1-\delta)(s-r)\} + (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))(t-r) \\
+ \quad & \psi_i \inf_{r \le u \le t} \{A_{k^*}(r,u) - \gamma_{k^*}(\delta)(u-r)\} + \sum_{j=i+1}^{N} \inf_{r \le s \le t} \{A_j(r,s) - \rho_j(1-\delta)(s-r)\} \\
+ \quad & (1-\psi_i) \inf_{r \le u \le t} \{A_{k^*}(r,u) - (\gamma_{k^*}(\delta) - (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))/\psi_i)(u-r)\} \\
+ \quad & \sum_{j=1, j \ne k^*}^{i-1} \inf_{r \le u \le t} \{A_j(r,u) - \rho_j(1-\delta)(u-r)\} \\
= \quad & (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))(t-r) - \psi_i \sup_{r \le u \le t} \{\gamma_{k^*}(\delta)(u-r) - A_{k^*}(r,u)\} \\
- \quad & (1-\psi_i) \sup_{r \le u \le t} \{(\gamma_{k^*}(\delta) - (\rho_i(1-\delta) - \gamma_{ik^*}(\delta))/\psi_i)(u-r) - A_{k^*}(r,u)\} \\
- \quad & \sum_{j \ne k^*} \sup_{r \le s \le t} \{\rho_j(1-\delta)(s-r) - A_j(r,s)\}.
\end{aligned}
$$

Noting that

$$\sup_{r\le s\le t}\{\rho_j(1-\delta)(s-r)-A_j(r,s)\}\le\sup_{s\ge r}\{\rho_j(1-\delta)(s-r)-A_j(r,s)\}=Z_j^{\rho_j(1-\delta)}(r),$$

taking $r=s^*$, and using (83), (87), (88),

$$\begin{aligned}V_i(t) &\ge (\rho_i(1-\delta)-\gamma_{ik^*}(\delta))(t-s^*)-\psi_i\sup_{s^*\le u\le t}\{\gamma_{k^*}(\delta)(u-s^*)-A_{k^*}(s^*,u)\}-\sum_{j\ne k^*}Z_j^{\rho_j(1-\delta)}(s^*)\\ &\ge Q_{k^*}^{\delta}(t)-\sum_{j\ne k^*}Z_j^{\rho_j(1-\delta)}(s^*).\end{aligned}$$

Note that s^*, $Q_{k^*}^{\delta}(t)$ only depend on $A_{k^*}(s,t)$, not on $A_j(s,t)$, $j\ne k^*$, and are thus independent of $Z_j^{\rho_j(1-\delta)}(s^*)$ for all $j\ne k^*$. Hence, for $\delta>0$ sufficiently small,

$$\begin{aligned}\mathbb{P}\{V_i(t)>x|s^*\} &\ge \mathbb{P}\{Q_{k^*}^{\delta}(t)-\sum_{j\ne i,k^*}Z_j^{\rho_j(1-\delta)}(s^*)>x|s^*\}\\ &= \mathbb{P}\{Q_{k^*}^{\delta}(t)-\sum_{j\ne i,k^*}\mathbf{Z}_j^{\rho_j(1-\delta)}>x\}.\end{aligned}$$

Thus, in the stationary regime (81) holds.

□

F Proof of Lemma 5.2

Lemma 5.2

(Upper bound) For any $\delta>0$,

$$\mathbb{P}\{\mathbf{V}_i>x\}\le\mathbb{P}\{\mathbf{Q}_{k^*}^{-\delta}+\mathbf{V}_i^{\rho_i(1+\delta)}+\chi_i\sum_{j=1,j\ne k^*}^{i-1}\mathbf{W}_j^{\delta}>x\}. \tag{91}$$

Proof

From the definition of s^* as given in Section 5, we have $V_{k^*}^{\gamma_{k^*}(-\delta)}(s^*)=0$, and $V_{k^*}^{\gamma_{k^*}(-\delta)}(s)>0$ for all $s\in(s^*,t]$, i.e., flow k^* is continuously backlogged during the interval $(s^*,t]$, so that

$$B_{k^*}^{\gamma_{k^*}(-\delta)}(s^*,u)=\gamma_{k^*}(-\delta)(u-s^*) \tag{92}$$

for all $s^*\le u\le t$, and

$$\begin{aligned}B_{k^*}^{\gamma_{k^*}(-\delta)}(u,s^*) &= V_{k^*}^{\gamma_{k^*}(-\delta)}(u)+A_{k^*}(u,s^*)-V_{k^*}^{\gamma_{k^*}(-\delta)}(s^*)\\ &= V_{k^*}^{\gamma_{k^*}(-\delta)}(u)+A_{k^*}(u,s^*)\end{aligned} \tag{93}$$

for all $0\le u\le s^*$.
Define $r^*:=\sup\{r\le t|V_i(r)=0\}$.

By assumption, $S_{k^*} = \{1, \ldots, i-1\} \setminus \{k^*\}$.
Thus, as in the proof of Lemma 4.6,

$$V_i(t) \leq A_i(r^*, t) - B_i(r^*, t), \tag{94}$$

with for any $\delta > 0$, using Lemma 4.4,

$$B_i(r^*, t) \geq \chi_i[t - r^* - \sum_{j=1, j \neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, t)]]. \tag{95}$$

Also,

$$B_i(r^*, t) = B_i(r^*, s) + B_i(s, t), \tag{96}$$

with

$$\begin{aligned} B_i(r^*, s) &\geq \psi_i[s - r^* - \sum_{j=1}^{i-1}[V_j(r^*) + A_j(r^*, s) - V_j(s)]] \\ &\geq \psi_i[s - r^* - \sum_{j=1}^{i-1}[V_j^{\gamma_{jk^*}(\delta)}(r^*) + A_j(r^*, s) - V_j(s)]], \end{aligned} \tag{97}$$

and

$$\begin{aligned} B_i(s, t) &\geq \chi_i[t - s - \sum_{j=1, j \neq k^*}^{i-1} [V_j(s) + A_j(s, t) - V_j(t)]] \\ &\geq \chi_i[t - s - \sum_{j=1, j \neq k^*}^{i-1} [V_j(s) + A_j(s, t)]] \end{aligned} \tag{98}$$

for all $r^* \leq s \leq t$.
Substituting (97), (98) into (96),

$$\begin{aligned} B_i(r^*, t) &\geq \psi_i[s - r^* - \sum_{j=1}^{i-1}[V_j^{\gamma_{jk^*}(\delta)}(r^*) + A_j(r^*, s) - V_j(s)]] \\ &+ \chi_i[t - s - \sum_{j=1, j \neq k^*}^{i-1} [V_j(s) + A_j(s, t)]] \\ &\geq (\psi_i - \chi_i)[s - r^* - \sum_{j=1, j \neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, s)]] \\ &- \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s)] \\ &+ \chi_i[s - r^* - \sum_{j=1, j \neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, s) - V_j(s)]] \\ &+ \chi_i[t - s - \sum_{j=1, j \neq k^*}^{i-1} [V_j(s) + A_j(s, t)]] \end{aligned}$$

$$
\begin{aligned}
&= \chi_i[t - r^* - \sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, t)]] \\
&- (\psi_i - \chi_i) \sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, s) - \rho_j(1+\delta)(s - r^*)] \\
&- \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s) - \gamma_{k^*}(-\delta)(s - r^*)]. \qquad (99)
\end{aligned}
$$

From (95), (99),

$$
\begin{aligned}
B_i(r^*, t) &\geq \chi_i[t - r^* - \sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, t)]] \\
&- (\psi_i - \chi_i) \sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, s) - \rho_j(1+\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&- \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s) - \gamma_{k^*}(-\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}}. \qquad (100)
\end{aligned}
$$

Substituting (100) into (94),

$$
\begin{aligned}
V_i(t) &\leq A_i(r^*, t) + \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s) - \gamma_{k^*}(-\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&+ (\psi_i - \chi_i)[\sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, s) - \rho_j(1+\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&+ \chi_i[\sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, t)] - (t - r^*)] \\
&= \sup_{r^* \leq s \leq t} \{A_i(s, t) - \rho_i(1+\delta)(t - s)\} + (\rho_i(1+\delta) - \gamma_{ik^*}(-\delta))(t - r^*) \\
&+ \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s) - \gamma_{k^*}(-\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&+ (\psi_i - \chi_i)[\sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, s) - \rho_j(1+\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&+ \chi_i \sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(r^*) + A_j(r^*, t) - \rho_j(1+\delta)(t - r^*)] \\
&\leq V_i^{\rho_i(1+\delta)}(t) + (\rho_i(1+\delta) - \gamma_{ik^*}(-\delta))(t - r^*) \\
&+ \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s) - \gamma_{k^*}(-\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&+ (\psi_i - \chi_i) \sum_{j=1, j\neq k^*}^{i-1} V_j^{\rho_j(1+\delta)}(s)\mathrm{I}_{\{s>r^*\}} + \chi_i \sum_{j=1, j\neq k^*}^{i-1} V_j^{\rho_j(1+\delta)}(t) \\
&= V_i^{\rho_i(1+\delta)}(t) + (\rho_i(1+\delta) - \gamma_{ik^*}(-\delta))(t - r^*) \\
&+ \psi_i[V_{k^*}^{\gamma_{k^*}(-\delta)}(r^*) + A_{k^*}(r^*, s) - \gamma_{k^*}(-\delta)(s - r^*)]\mathrm{I}_{\{s>r^*\}} \\
&+ \chi_i \sum_{j=1, j\neq k^*}^{i-1} [V_j^{\rho_j(1+\delta)}(t) + \frac{\phi_{k^*}}{\sum_{j=i}^{N} \phi_j} V_j^{\rho_j(1+\delta)}(s)\mathrm{I}_{\{s>r^*\}}].
\end{aligned}
$$

Taking $s = \max\{r^*, s^*\}$, using (92), (93),

$$\begin{aligned} V_i(t) &\leq \psi_i[B_{k^*}^{\gamma_{k^*}(-\delta)}(r^*, t) - \gamma_{k^*}(-\delta)(t - r^*)] + (\rho_i(1+\delta) - \gamma_{ik^*}(-\delta))(t - r^*) \\ &+ V_i^{\rho_i(1+\delta)}(t) + \chi_i \sum_{j=1, j \neq k^*}^{i-1} W_j^\delta(t) \\ &\leq Q_{k^*}^{-\delta}(t) + V_i^{\rho_i(1+\delta)}(t) + \chi_i \sum_{j=1, j \neq k^*}^{i-1} W_j^\delta(t). \end{aligned}$$

Note that s^*, $Q_k^{-\delta}(t)$ only depend on $A_{k^*}(s,t)$, not on $A_j(s,t)$, $j \neq k^*$, and are thus independent of $V_i^{\rho_i(1+\delta)}(t)$, $V_j^{\rho_j(1+\delta)}(s^*)$, and $V_j^{\rho_j(1+\delta)}(t)$ for all $j = 1, \ldots, i-1$, $j \neq k^*$. Hence, for any $\delta > 0$,

$$\begin{aligned} \mathbb{P}\{V_i(t) > x | s^*\} &\leq \mathbb{P}\{Q_{k^*}^{-\delta}(t) + V_i^{\rho_i(1+\delta)}(t) + \chi_i \sum_{j=1, j \neq k^*}^{i-1} W_j^\delta(t) > x | s^*\} \\ &= \mathbb{P}\{Q_{k^*}^{-\delta}(t) + V_i^{\rho_i(1+\delta)}(t) + \chi_i \sum_{j=1, j \neq k^*}^{i-1} \mathbf{W}_j^\delta > x\}. \end{aligned}$$

Thus, in the stationary regime (91) holds.

□