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A Control Problem for Affine Dynamical Systems on a Full-Dimensional Simplex

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Given an affine system on a simplex, the problem of reaching a particular facet of the simplex, using affine state feedback is studied. Necessary and sufficient conditions for the existence of a solution are derived in terms of linear inequalities on the input vectors at the vertices of the simplex. If these conditions are met, a constructive procedure yields an affine feedback control law, that solves this reachability problem.

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1 Introduction

The purpose of this paper is to present a result on reachability of a facet for an affine dynamical system on a full-dimensional simplex. This problem is motivated by the control of piecewise-linear hybrid systems.

Control of engineering systems is often carried out by computers. This implementation induces an interaction between the discrete dynamics of a computer program on the one hand, and the continuous dynamics of an engineering system on the other. Research in hybrid systems aims at the modeling of this interaction, and the development of systems and control theory for this class of systems. An overview of some of the current research in this area can be found in the conference proceedings [10, 15].

In [12, 13, 14], E.D. Sontag has introduced the class of piecewise-linear hybrid systems. A system of this class consists of an automaton and, for each discrete state, of an affine system on a polyhedral set. A simple case of a polyhedral set is a simplex, the $n$-dimensional generalization of the triangle in $\mathbb{R}^2$. The class of piecewise-linear hybrid systems has been analyzed by several authors, see e.g. [2, 3]. Also the reachability of general hybrid systems has received considerable interest, see e.g. [1, 8, 9]. A particular approach to the reachability problem was developed by the second co-author in [16]. In case of piecewise-linear hybrid systems, this method requires the solution of a reachability problem of an affine system, by steering the state to a particular facet of a polyhedral set. The latter problem is treated in
this paper, under the additional assumption that the polyhedral set under consideration is a simplex.

Given an affine system on a full-dimensional simplex, the main problem can be formulated as follows: determine necessary and sufficient conditions for the existence of an affine control law such that, independent of the initial state, all state-trajectories of the closed-loop system reach a particular facet of the simplex in finite time. In the solution of this problem, convexity arguments play an important role. The necessary condition may be derived for affine systems on arbitrary polytopes, using continuous (i.e. not necessarily affine) feedback, and consists of a set of linear inequalities for the input vectors at the vertices of the polytope. The sufficient condition for an affine system on a simplex using affine state feedback is based on an analysis of the dynamics of the corresponding closed-loop system. The necessary and sufficient conditions are identical in case of a simplex. For general polytopes the situation is more complicated; this problem will be treated in a future paper.

Once the linear inequalities on the input vectors at the vertices of the simplex are obtained, existing algorithms may be used to check the existence of a solution. For this purpose, computer programs have been developed, for example in the research groups of Verimag (N. Halbwachs, B. Jeannet) and of IRISA (D.K. Wilde), see [7, 18]. The final step is the computation of the affine control law. For this problem, a simple procedure is provided.

In the literature there are several publications on the invariance of linear systems on polyhedral sets, see [5, 17] and on invariance of piecewise-linear hybrid systems, see [4]. The problem treated in this paper differs from that of those references in that the trajectories of the system concerned need to reach a particular facet in finite time, hence the system is not invariant, and in that the conditions are more explicit than those in the literature.

The paper is organized as follows. The next section contains the problem formulation and terminology on polyhedral sets and simplices. Necessary conditions for existence of a continuous feedback law realizing the control objective are stated in Section 3. A sufficient condition for existence of an affine feedback meeting the control objective is stated in Section 4. In Section 5 computational issues for the control laws are discussed. Concluding remarks are stated in Section 6. Appendix A provides several technical results on simplices.

2 Problem formulation

Let $N \in \mathbb{N}$, and consider the $N$-dimensional space $\mathbb{R}^N$. Let $v_1, \ldots, v_{N+1}$ be $N + 1$ affinely independent points in $\mathbb{R}^N$, which means that there exists no hyperplane of $\mathbb{R}^N$, containing all these $N + 1$ points. The full-dimensional simplex $S_N$ is defined as the convex hull of $v_1, \ldots, v_{N+1}$. The points $v_1, \ldots, v_{N+1}$ are called the vertices of $S_N$.

For every $i \in \{1, \ldots, N+1\}$, the convex hull of the points $\{v_1, \ldots, v_{N+1}\}\{v_i\}$ is a facet of $S_N$, that will be denoted by $F_i$. Let $n_i$ be the normal vector of the corresponding hyperplane. By convention, $n_i$ points outward of the simplex $S_N$, and $\|n_i\| = 1$. There exists an $\alpha_i \in \mathbb{R}$ such that the hyperplane containing the points $\{v_1, \ldots, v_{N+1}\}\{v_i\}$ (and thus the facet $F_i$) is described by

$$\{x \in \mathbb{R}^N \mid n_i^T x = \alpha_i\}.$$  

This indicates that the simplex $S_N$ is located inside the half-space $\{x \in \mathbb{R}^N \mid n_i^T x \leq \alpha_i\}$. Since the same observation holds for every $i \in \{1, \ldots, N+1\}$, we obtain the following implicit description of the simplex $S_N$:

$$S_N = \{x \in \mathbb{R}^N \mid \forall i = 1, \ldots, N+1: n_i^T x \leq \alpha_i\}.$$  

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Some technical results needed in this paper regarding the sets of vertices and normal vectors of a full-dimensional simplex are collected in Appendix A.

On the full-dimensional simplex $S_N$, we consider an affine control system

$$\dot{x} = Ax + Bu + a, \quad x(0) = x_0,$$

with $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$, and $a \in \mathbb{R}^N$. So, on every time instant $t \in T$, the state $x \in \mathbb{R}^N$ is assumed to be contained in the simplex $S_N$. Also the input $u$ is assumed to take values in a polyhedral set $U \subset \mathbb{R}^m$ only. Note that the affine differential equation (1) only remains valid, as long as the state $x$ is contained in the simplex $S_N$. In a hybrid system, as soon as the state-trajectory reaches one of the facets of $S_N$, a discrete event occurs, together with a resetting of the state $x$. In general the state $x$ will leave the simplex $S_N$, and proceed to a different simplex, where it is ruled by a different type of affine dynamics. In this paper, we consider the control problem of steering the state of system (1) in finite time to one particular facet of the simplex $S_N$.

**Problem 2.1** For any initial state $x_0 \in S_N$, find a time-instant $T_0 \geq 0$ and an input function $u : [0, T_0] \rightarrow U$, such that

(i) $\forall t \in [0, T_0] : x(t) \in S_N$,

(ii) $x(T_0) \in F_1$, where $F_1$ is the facet of the simplex $S_N$, not containing the vertex $v_1$,

(iii) $n_1^T \dot{x}(T_0) > 0$, i.e. the velocity vector $\dot{x}(T_0)$ at the point $x(T_0) \in F_1$ has a positive component in the direction of $n_1$. This implies that in the point $x(T_0)$, the velocity vector $\dot{x}(T_0)$ points out of the simplex $S_N$.

Furthermore, if possible this input function $u$ should be realized by the application of an affine feedback law

$$u(t) = Fx(t) + g,$$

with $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$, that is independent of the initial state $x_0$.

Note that in Problem 2.1 the choice of the exit facet $F_1$ is completely arbitrary. Without loss of generality the facet $F_1$ may be replaced by any other facet of $S_N$.

After application of the feedback law (2) to system (1) we obtain the closed-loop system

$$\dot{x} = (A + BF)x + (a + Bg), \quad x(0) = x_0,$$

hence the system dynamics remain affine after application of this feedback. This type of autonomous affine systems exhibits interesting convexity properties, for which it does not make any difference whether the state $x$ is restricted to the simplex $S_N$, or the whole space $\mathbb{R}^N$ is considered.

**Lemma 2.2** Consider the autonomous affine system in $\mathbb{R}^N$, given by $\dot{x} = Ax + a$, and let $p_1, p_2$ be two points in $\mathbb{R}^N$, and let $n \in \mathbb{R}^N$ be a nonzero vector in $\mathbb{R}^N$. If $n^T \dot{x} \mid_{p_1} = n^T(Ap_1 + a) < 0$ and $n^T \dot{x} \mid_{p_2} = n^T(Ap_2 + a) < 0$, then for any $\lambda \in (0, 1)$ also $n^T \dot{x} \mid_{\lambda p_1 + (1 - \lambda)p_2} = n^T(A(\lambda p_1 + (1 - \lambda)p_2) + a) < 0$. 

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**Proof:** $n^T \dot{x} \mid_{\lambda p_1 + (1-\lambda)p_2} = n^T (A(\lambda p_1 + (1-\lambda)p_2) + a) = \lambda n^T (Ap_1 + a) + (1-\lambda)n^T (Ap_2 + a) < 0$. 

Of course the result of Lemma 2.2 still holds if all ‘$<$’ signs are replaced by ‘$\leq$’, ‘$>$’, or ‘$\geq$’ signs. The lemma indicates that if in two points $p_1, p_2 \in \mathbb{R}^N$ the vector field of the velocity of an autonomous affine system points into the same direction w.r.t. a normal vector $n$, then in all points on the straight line, joining $p_1$ and $p_2$, the velocity vector field points into the same direction. In this paper, this simple observation will play a crucial role in the solution of Control Problem 2.1.

### 3 Necessary conditions for feedback control to a facet

In this section we will show how the convexity result of Lemma 2.2 may be used to obtain necessary conditions for the solution of Problem 2.1, by restricting our attention to the determination of a suitable control input at the vertices $v_1, \ldots, v_{N+1}$ of the simplex $S_N$.

First, we consider one of the facets of the simplex $S_N$, with $v_1$ as one of its vertices, say the facet $F_i$ ($i = 2, \ldots, N + 1$), defined as the convex hull of $\{v_1, \ldots, v_{N+1}\} \setminus \{v_i\}$, with normal vector $n_i$. The state variable $x$ is not allowed to leave the simplex $S_N$ through this facet. This can only be the case if in every point of this facet the vector field of the velocity of $x$ does not point out of the simplex $S_N$. So, in every point of $p \in F_i$ there should exist a $u \in U$, such that $n_i^T (Ap + Bu + a) \leq 0$. However, due to convexity it suffices to check this condition at the vertices of $F_i$ only.

**Lemma 3.1** Let $i \in \{2, \ldots, N + 1\}$. For any point $p \in F_i$ there exists an input $u \in U$ such that

$$n_i^T (Ap + Bu + a) \leq 0,$$

if and only if there exist $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{N+1} \in U$, such that for all $j \in \{1, \ldots, N + 1\} \setminus \{i\}$:

$$n_i^T (Av_j + Bu_j + a) \leq 0.$$  

**Proof:** Necessity of condition (5) is obvious because all vertices $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{N+1}$ are points in $F_i$. To prove sufficiency, take $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{N+1} \in [0, 1]$ such that $\sum_{j \neq i} \lambda_j = 1$ and $\sum_{j \neq i} \lambda_j v_j = p$. Choose $u = \sum_{j \neq i} \lambda_j u_j$; then $u \in U$ because $U$ is a polyhedral set. Furthermore, $n_i^T (Ap + Bu + a) = n_i^T \sum_{j \neq i} \lambda_j (Av_j + Bu_j + a) = \sum_{i \neq j} \lambda_j n_i^T (Av_j + Bu_j + a) \leq 0$. 

Note that Lemma 3.1 still holds if in (4) and (5) the ‘$\leq$’ signs are replaced by ‘$<$’ signs.

Next we consider the exit facet $F_1$. For this facet, the direction of the vector field is not completely determined by the description of Problem 2.1. It might be possible to solve this problem by steering the state into the simplex $S_N$ at one part of the facet $F_1$, and out of the simplex $S_N$ at another part. In this paper we want to avoid this situation by a slight modification of the problem description.

**Problem 3.2** Consider the same problem as described in Problem 2.1, and add the additional constraint that

(iv) $T_0 \geq 0$ is the smallest time-instant in the interval $[0, T_0]$ for which the state $x$ reaches the exit facet $F_1$, i.e. $T_0 = \min\{t \mid t \geq 0 \text{ and } x(t) \in F_1\}$. 


In combination with condition (iii), condition (iv) implies that in any point of the facet $F_1$ the input $u \in U$ should be chosen in such a way that the vector field of the velocity of $x$ has a positive component in the $n_1$-direction, so is pointing out of the simplex $S_N$. Using the same arguments as in Lemma 3.1, it suffices to verify this condition at the vertices of $F_1$ only.

**Lemma 3.3** For any point $p \in F_1$ there exists an input $u \in U$ such that

$$n_1^T(Ap + Bu + a) > 0,$$

if and only if there exist $u_2, \ldots, u_{N+1} \in U$, such that for all $j \in \{2, \ldots, N + 1\}$:

$$n_1^T(Av_j + Bu_j + a) > 0. \quad (6)$$

In Lemmas 3.1 and 3.3 conditions were derived on the existence of a set of inputs, that guarantee that the vector field of the velocity of $x$ can be put in the right direction at the boundaries of the simplex $S_N$. Note that for every single facet $F_i$ ($i = 1, \ldots, N+1$) this gives a different set of conditions on the inputs $u_j$ ($j \neq i$) at the vertices $v_1, \ldots, v_i-1, v_i+1, \ldots, v_{N+1}$ of $F_i$. So, for solving Problem 3.2 it seems sufficient that for every facet $F_i$ a different $N$-tuple of inputs $u_j \in U$ ($j \neq i$) is found, satisfying the inequalities (5) and (6), respectively. However, as soon as we apply state feedback, the input at every vertex of $S_N$ is fixed, and inequalities (5) and (6) have to be solved simultaneously. This leads to the following set of necessary conditions for the existence of a closed-loop solution to Problem 3.2.

**Proposition 3.4** Consider the affine dynamical system $\dot{x}(t) = Ax(t) + Bu(t) + a$, with $x \in S_N$ and $u \in U$. Assume that there exists a continuous function $f : S_N \rightarrow U$, such that the state feedback control law $u(t) = f(x(t))$ solves Problem 3.2, i.e. irrespective of the initial state $x_0 \in S_N$, the closed-loop system

$$\dot{x} = Ax + Bf(x) + a, \quad x(0) = x_0,$$

has a solution $x$, satisfying the conditions (i)–(iv) of Problems 2.1 and 3.2. Then there exist $u_1, \ldots, u_{N+1} \in U$ such that

1. $n_1^T(Av_j + Bu_j + a) > 0$ for $j = 2, \ldots, N + 1$,

2. $n_1^T(Av_i + Bu_i + a) \leq 0$ for $i = 2, \ldots, N + 1$, and there exists an $i \in \{2, \ldots, N + 1\}$ such that $n_1^T(Av_1 + Bu_1 + a) < 0$,

3. $n_i^T(Av_j + Bu_j + a) \leq 0$ for all $i, j = 2, \ldots, N + 1$ with $i \neq j$.

**Proof:** Suppose that the continuous function $f : S_N \rightarrow U$ generates a feedback law $u(t) = f(x(t))$, that solves Control Problem 3.2. We show that the inputs $u_j = f(v_j)$ ($j = 1, \ldots, N + 1$) satisfy (1), (2), and (3).

1. For every $j \in \{2, \ldots, N + 1\}$, $v_j \in F_1$. So, as soon as the vertex $v_j$ is reached, the state trajectory should leave the simplex $S_N$ with a positive velocity in the $n_1$-direction. This implies that $n_1^T(Av_j + Bf(v_j) + a) > 0$, so by the definition of $u_j$: $n_1^T(Av_j + Bu_j + a) > 0$ for $j = 2, \ldots, N + 1$.

2. Consider the closed-loop system $\dot{x}(t) = Ax(t) + Bf(x(t)) + a$, with initial value $x(0) = v_1$. The corresponding solution $x$ can only leave the simplex $S_N$ through the facet $F_1$. So in $v_1$, the velocity vector does not point out of the simplex $S_N$. Therefore $n_1^T(Av_1 + Bu_1 + a) = n_1^T(Av_1 + Bf(v_1) + a) \leq 0$ for $i = 2, \ldots, N + 1$. Furthermore, if $n_1^T(Av_1 + Bu_1 + a) = 0$
for \( i = 2, \ldots, N + 1 \), then \( Ax + Bu + a = 0 \) because the vectors \( n_2, \ldots, n_{N+1} \) constitute a basis of \( \mathbb{R}^N \) (see Lemma A.2). This implies that \( v_1 \) is a fixed point of the closed-loop system \( \dot{x} = Af(x) + a \), and the corresponding solution \( x(t) \equiv v_1 \) will never leave the simplex \( S_N \) through the facet \( F_1 \). This contradicts the assumption that \( u(t) = f(x(t)) \) solves Control Problem 3.2.

(3): Since for \( N = 1 \) Condition (3) is void, we assume, without loss of generality, that \( N > 1 \). We prove (3) by contradiction. Suppose that there exist \( i, j \in \{2, \ldots, N + 1\} \), \( i \neq j \), such that \( n_i^T(Av_j + Bu_j + a) > 0 \). Define the function

\[
h : S_N \rightarrow \mathbb{R} : h(x) = n_i^T(Ax + Bf(x) + a).
\]

Then \( h \) is continuous, and by assumption \( h(v_j) > 0 \). So, there exists \( \delta > 0 \), such that for all \( x \in S_N \), with \( \|x - v_j\| < \delta \): \( h(x) > 0 \).

Let \( 0 < \varepsilon < \min \left( \frac{\delta}{\sum_{k=1}^{N+1} \|v_k\|}, 1 \right) \), and define

\[
p := (1 - \varepsilon)v_j + \frac{\varepsilon}{N - 1} \sum_{k=1,k \neq i,j}^{N+1} v_k.
\]

Then \( p \) is a convex combination of \( v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{N+1} \), with only strictly positive coefficients. So \( p \in F_i \), but \( p \) does not belong to any of the other facets of \( S_N \). Furthermore,

\[
\|p - v_j\| = \left\| (1 - \varepsilon)v_j + \left( \frac{\varepsilon}{N - 1} \sum_{k=1,k \neq i,j}^{N+1} v_k \right) - v_j \right\|
\]

\[
= \left\| -\varepsilon v_j + \frac{\varepsilon}{N - 1} \sum_{k=1,k \neq i,j}^{N+1} v_k \right\| \leq \varepsilon \cdot \sum_{k=1,k \neq i}^{N+1} \|v_k\| < \delta,
\]

so \( h(p) > 0 \). Therefore, the trajectory of the closed-loop system \( \dot{x} = Ax + Bf(x) + a \), with initial value \( x(0) = p \in F_i \), will immediately leave the simplex \( S_N \) through the facet \( F_i \) because \( n_i^T \dot{x}(0) = h(p) > 0 \). This contradicts the fact that the feedback law \( u(t) = f(x(t)) \) is a solution to Problem 3.2, irrespective of the initial state \( x_0 \in S_N \). We conclude that \( n_i^T(Av_j + Bu_j + a) \leq 0 \) for all \( i, j \in \{2, \ldots, N + 1\} \) with \( i \neq j \).

The necessary conditions derived in Proposition 3.4 may be generalized from full-dimensional simplices \( S_N \) in \( \mathbb{R}^N \) to arbitrary full-dimensional polytopes \( P_N \) in \( \mathbb{R}^N \). The main observation is that conditions (1) to (3) characterize the direction of the vector field of \( \dot{x} \), the derivative of the state of the closed-loop system, at the vertices of \( S_N \). So, after a suitable reformulation the same conditions remain valid for general full-dimensional polytopes.

**Proposition 3.5** Let \( P_N \) be a full-dimensional polytope in \( \mathbb{R}^N \) with vertices \( v_1, \ldots, v_M \), \( (M \geq N + 1) \). Let \( F_1, \ldots, F_L \) denote the facets of \( P_N \), with normal vectors \( n_1, \ldots, n_L \), respectively, pointing out of the polytope \( P_N \). For \( i \in \{1, \ldots, L\} \), let \( V_i \subset \{1, \ldots, M\} \) be the index set such that \( \{v_j \mid j \in V_i\} \) is the set of vertices of the facet \( F_i \). Conversely, for every \( j \in \{1, \ldots, M\} \), the set \( W_j \subset \{1, \ldots, L\} \) contains the indices of all facets of which \( v_j \) is a vertex. Assume that \( F_1 \) is the exit facet of \( P_N \). If Control Problem 3.2, with \( S_N \) replaced by \( P_N \), is solvable by a continuous state feedback \( f \), then there exist inputs \( u_1, \ldots, u_M \in U \) such that

\((1*) \forall j \in V_1: n_i^T(Av_j + Bu_j + a) > 0,\)
\((2^*)\) \forall i \in \{2, \ldots, L\} \forall j \in V_i: n_i^T(Av_j + Bu_j + a) \leq 0,

\((3^*)\) \forall j \in \{1, \ldots, M\}\setminus V_1 \exists i \in W_j: n_i^T(Av_j + Bu_j + a) < 0.

In comparison with the original formulation of Proposition 3.4, conditions (1) and \((1^*)\) are identical, condition (3) and the first part of condition (2) are replaced by \((2^*)\), and the second part of (2) is now condition \((3^*)\). The proofs of \((1^*)\) and \((2^*)\) may be carried out analogously to the proof of Proposition 3.4, because these proofs do not rely on the fact that \(S_N\) is a simplex. Only the proof of \((3^*)\) requires a slight modification, based on the following result.

**Lemma 3.6** Let \(v \in \mathbb{R}^N\) be a vertex of a full-dimensional polytope in \(\mathbb{R}^N\). Let \(F_1, \ldots, F_K\) denote all facets of \(P_N\), containing \(v\). Then the normal vectors \(n_1, \ldots, n_K\) of \(F_1, \ldots, F_K\) generate \(\mathbb{R}^N\).

**Proof:** Let \(F_{K+1}, \ldots, F_L\) denote the other facets of \(P_N\), with normal vectors \(n_{K+1}, \ldots, n_L\), and assume without loss of generality that all normal vectors are pointing out of the polytope \(P_N\). Then there exist \(\alpha_1, \ldots, \alpha_L \in \mathbb{R}\) such that

\[
P_N = \left\{ x \in \mathbb{R}^N \mid \forall i = 1, \ldots, L: n_i^T x \leq \alpha_i \right\}
\]

is the implicit description of \(P_N\). At the vertex \(v\) we know that \(n_i^T v = \alpha_i\) for \(i = 1, \ldots, K\), and \(n_i^T v < \alpha_i\) for \(i = K + 1, \ldots, L\).

Suppose that \(n_1, \ldots, n_K\) do not generate \(\mathbb{R}^N\). Then there exists a nonzero vector \(n \in \mathbb{R}^N\) such that \(n_i^T n = 0\) for \(i = 1, \ldots, K\). Hence, there exists a \(\delta > 0\) such that for all \(\varepsilon \in (-\delta, \delta)\) the point \(v + \varepsilon n \in P_N\). This indicates that \(v\) is a convex combination of two other points in \(P_N\), and thus contradicts the fact that \(v\) is a vertex of \(P_N\) (see e.g. [11, p. 162]).

Using Lemma 3.6, the proof of Condition \((3^*)\) becomes obvious. If there exists a \(j \in \{1, \ldots, M\}\setminus V_1\) such that \(n_i^T(Av_j + Bu_j + a) = 0\) for all \(i \in W_j\), then \(Av_j + Bu_j + a = 0\) because the set \(\{n_i \mid i \in W_j\}\) generates \(\mathbb{R}^N\). This implies that the vertex \(v_j \notin F_1\) is a fixed point of the closed-loop system, which contradicts the assumption that the feedback \(u(t) = f(x(t))\) solves Control Problem 3.2.

### 4 Control to a facet by affine state feedback

In the previous section we derived some necessary conditions for the solution of Control Problem 3.2 by continuous static state feedback. Nevertheless, the conditions that were obtained in Propositions 3.4 and 3.5 seem to have a strong open-loop control character: they are formulated as a set of inequalities on the inputs to the system at the vertices of the simplex \(S_N\) or the polytope \(P_N\). In this section we restrict our attention to simplices again, and show that if an \(N+1\)-tuple of inputs exists, satisfying inequalities (1), (2), and (3) of Proposition 3.4, then it may be realized by the application of an affine static state feedback law. Moreover, such an affine state feedback law is also a solution to Control Problem 3.2. The proof of this result consists of two parts. First it is shown that every trajectory of the closed-loop system cannot leave the simplex \(S_N\) through one of the facets \(F_2, \ldots, F_{N+1}\). Subsequently it is proven that every trajectory will reach the exit facet \(F_1\) in finite time. These observations indicate that for the solution of Problem 3.2 it is not necessary to consider the whole class of continuous static state feedback laws. Instead it is sufficient to confine ourselves to affine static state feedback.
**Remark 4.1** The results of this section are stated and proved for full-dimensional simplices only. Generalization of these results to general full-dimensional polytopes seems difficult, because the design of the static state feedback and also several of the proofs in this section rely on the fact that the state set $S_N$ is assumed to be a full-dimensional simplex.

Assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that conditions (1), (2), and (3) of Proposition 3.4 are satisfied. Since input $u_i$ should be applied at the moment that the state vector reaches vertex $v_i$, we first search for an affine feedback control law $u = Fx + g$, with $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$ such that

$$u_i = Fv_i + g, \quad (i = 1, \ldots, N+1).$$

Transposition of (7) yields the equalities $v_i^TF^T + g^T = u_i^T$, $(i = 1, \ldots, N+1)$, and by collecting all equalities in a matrix, the following linear equation has to be solved for $F$ and $g$:

$$
\begin{pmatrix}
  v_1^T & 1 \\
  \vdots & \vdots \\
  v_{N+1}^T & 1
\end{pmatrix}
\begin{pmatrix}
  F^T \\
  g^T
\end{pmatrix}
= 
\begin{pmatrix}
  u_1^T \\
  \vdots \\
  u_{N+1}^T
\end{pmatrix}.
$$

(8)

Note however that this equation has a unique solution. Indeed, because of Lemma A.1, the vectors $v_2 - v_1, v_3 - v_1, \ldots, v_{N+1} - v_1$ are linearly independent and thus

$$
\det
\begin{pmatrix}
  v_1^T & 1 \\
  \vdots & \vdots \\
  v_{N+1}^T & 1
\end{pmatrix}
= \det
\begin{pmatrix}
  0 & v_2^T - v_1^T & 1 \\
  \vdots & \vdots & \vdots \\
  v_{N+1}^T - v_1^T & 0 & 0
\end{pmatrix}
= (-1)^N \det
\begin{pmatrix}
  v_2^T - v_1^T \\
  \vdots \\
  v_{N+1}^T - v_1^T
\end{pmatrix}
\neq 0,
$$

so this square $(N + 1) \times (N + 1)$ matrix is invertible. Furthermore, since the input set $U$ is a polyhedral set, the input determined by the feedback control law $u = Fx + g$ belongs to $U$ for all $x \in S_N$:

**Lemma 4.2** Let $u_1, \ldots, u_{N+1} \in U$, and let $F \in \mathbb{R}^{m \times N}, g \in \mathbb{R}^m$, such that for all $i = 1, \ldots, N+1$: $u_i = Fv_i + g$. Then

$$
\forall x \in S_N : u = Fx + g \in U.
$$

(9)

**Proof:** Let $x \in S_N$. Then there exist $\lambda_1, \ldots, \lambda_{N+1} \in [0,1]$, with $\sum_{i=1}^{N+1} \lambda_i = 1$ such that $x = \sum_{i=1}^{N+1} \lambda_i v_i$. Then

$$
u = Fx + g = F \sum_{i=1}^{N+1} \lambda_i v_i + \sum_{i=1}^{N+1} \lambda_i g = \sum_{i=1}^{N+1} \lambda_i (Fv_i + g) = \sum_{i=1}^{N+1} \lambda_i u_i \in U.\quad \blacksquare
$$

**Proposition 4.3** Consider the dynamical system $\dot{x}(t) = Ax(t) + Bu(t) + a$, with $x \in S_N$ and $u \in U$, and assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that

(2') $n_i^T(Av_i + Bu_i + a) \leq 0$ for $i = 2, \ldots, N + 1$,

(3) $n_i^T(Av_j + Bu_j + a) \leq 0$ for all $i, j = 2, \ldots, N + 1$ with $i \neq j$. 
Let $F, g$ be the corresponding solution of (8), and apply the feedback law $u(t) = Fx(t) + g$. Then $u(t) \in U$, and for any initial state $x_0 \in S_N$, the state $x$ of the closed-loop system
\[ \dot{x} = (A + BF)x + (a + Bg), \quad x(0) = x_0, \tag{10} \]
can only leave the simplex $S_N$ through the facet $F_1$.

Note that conditions (2') and (3) are not only sufficient for the construction of an affine feedback law $u = Fx + g$, but also necessary; condition (3) literally occurs in Proposition 3.4 and (2') is just slightly weaker than condition (2) of Proposition 3.4. Condition (1) of the feedback law is that a solution of the closed-loop dynamics cannot leave the simplex $S_N$ through one of the facets $F_2, \ldots, F_{N+1}$. Whether or not the state trajectory leaves the simplex through the facet $F_1$ will be discussed in Theorem 4.6.

In the proof of Proposition 4.3 we need the following version of Gronwall’s Lemma.

**Lemma 4.4** ([6, p. 19]) Let $\varphi \in C^+[0, b]$, where $C^+[0, b]$ denotes the set of continuous nonnegative functions on the interval $[0, b]$. If a function $f \in C^+[0, b]$ satisfies
\[ f(t) \leq \varphi(t) + K \int_0^t f(s) \, ds, \quad (t \in [0, b]), \tag{11} \]
for some fixed $K > 0$, then
\[ f(t) \leq \varphi(t) + K \int_0^t e^{K(t-s)} \varphi(s) \, ds, \quad (t \in [0, b]). \tag{12} \]
In particular, if there exist $K > 0$ and $\beta > 0$ such that $f \in C^+[0, b]$ satisfies (11) with $\varphi(t) = \beta t$, then
\[ f(t) \leq \frac{\beta}{K} (e^{Kt} - 1), \quad (t \in [0, b]). \tag{13} \]

**Proof of Proposition 4.3:** Let $w \in \mathbb{R}^N$ be the vector $w = \sum_{j=2}^{N+1} (v_j - v_1)$, and define $n := \frac{1}{\|n\|} w$. Then $\|n\| = 1$, and according to Lemma A.4, $n_T^i n < 0$ for $i = 2, \ldots, N + 1$.

Let $\varepsilon > 0$, and consider the following perturbed closed-loop dynamics
\[ \dot{x}_\varepsilon = (A + BF)x_\varepsilon + (a + Bg) + \varepsilon n. \tag{14} \]
We first show that a trajectory $x_\varepsilon(t)$, with $x_\varepsilon(0) \in S_N$ and satisfying (14), cannot leave the simplex $S_N$ through one of the facets $F_2, \ldots, F_{N+1}$. Indeed, let $i \in \{2, \ldots, N + 1\}$, and $p \in F_i$. Since $p$ is a convex combination of the vertices $v_1, \ldots, v_i, v_{i+1}, \ldots, v_{N+1}$, there exist $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{N+1} \in [0, 1]$ such that $\sum_{j \neq i} \lambda_j = 1$ and $p = \sum_{j \neq i} \lambda_j v_j$. Then
\[
\left. n_T^i \dot{x}_\varepsilon \right|_p = n_T^i ((A + BF)p + (a + Bg) + \varepsilon n) \\
= n_T^i \left( (A + BF) \sum_{j \neq i} \lambda_j v_j + \sum_{j \neq i} \lambda_j (Bg + a) + \varepsilon n \right) \\
= n_T^i \sum_{j \neq i} \lambda_j ((A + BF)v_j + Bg + a) + \varepsilon n_T^i n \\
= \sum_{j \neq i} \lambda_j n_T^i (Av_j + Bu_j + a) + \varepsilon n_T^i n < 0,
\]
because $\sum_{j \neq i} \lambda_j n_i^T (Av_j + Bu_j + a) \leq 0$ by conditions (2') and (3) and $n_i^T n < 0$. So, unless $p \in F_1$, the velocity vector field of (14) is pointing strictly into the simplex $S_N$. This implies that any solution $x_t$ of the perturbed closed-loop system (14) can only leave the simplex $S_N$ through the facet $F_1$. By contradiction we will prove that the same is true for the unperturbed closed-loop system.

Consider the affine dynamical system

$$\dot{x} = (A + BF)x + (a + Bg),$$

with $x \in \mathbb{R}^N$. So for a moment we assume that differential equation (15) does not only hold on the simplex $S_N$, but on the whole space $\mathbb{R}^N$. Suppose that there exists $x_0 \in S_N$ such that the solution $x(t)$ of (15) with initial value $x(0) = x_0$ leaves $S_N$ through the facet $F_i$ with $i \in \{2, \ldots, N+1\}$ before it has reached the facet $F_1$. Then there exists $t_0 > 0$ such that

(i) $\beta := n_i^T x(t_0) > \alpha_i$,

(ii) $\forall t \in [0, t_0]: n_i^T x(t) < \alpha_1$.

Define $\gamma := \max \{n_i^T x(t) \mid t \in [0, t_0]\}$ and $K := \|A + BF\| = \max \{\|(A + BF)x\| \mid \|x\| = 1\}$. Let $0 < \varepsilon < \frac{K}{(e^{\alpha_1} - 1)} \cdot \min \left(\frac{1}{2}(\beta - \alpha_i), \frac{1}{2}(\alpha_1 - \gamma)\right)$. We will compare the solutions $x_\varepsilon(t)$ of the perturbed system (14) and $x(t)$ of the unperturbed system (15), both with initial value $x_0 \in S_N$, on the interval $[0, t_0]$. For all $t \in [0, t_0]$ we have

$$x(t) - x_\varepsilon(t) = \int_0^t \dot{x}(s) - \dot{x}_\varepsilon(s) \, ds$$

$$= \int_0^t (A + BF)x(s) + (a + Bg) - ((A + BF)x_\varepsilon(s) + (a + Bg) + \varepsilon n) \, ds$$

$$= \int_0^t (A + BF)(x(s) - x_\varepsilon(s)) \, ds - \int_0^t \varepsilon n \, ds,$$

hence

$$\|x(t) - x_\varepsilon(t)\| \leq \int_0^t \|A + BF\| \|x(s) - x_\varepsilon(s)\| \, ds + \int_0^t \varepsilon \cdot \|n\| \, ds$$

$$\leq \varepsilon \cdot t + K \int_0^t \|x(s) - x_\varepsilon(s)\| \, ds.$$

Next we apply Lemma 4.4 with $f(t) = \|x(t) - x_\varepsilon(t)\|$ and $\varphi(t) = \varepsilon t$, and find that for all $t \in [0, t_0]$:  

$$\|x(t) - x_\varepsilon(t)\| \leq \frac{\varepsilon}{K}(e^{Kt} - 1).$$

So, in particular

$$\|x(t) - x_\varepsilon(t)\| \leq \frac{\varepsilon}{K}(e^{Kt_0} - 1) < \min \left(\frac{1}{2}(\beta - \alpha_i), \frac{1}{2}(\alpha_1 - \gamma)\right), \quad (t \in [0, t_0]).$$

This implies that for every $t \in [0, t_0]$ the solution $x_\varepsilon$ satisfies

$$n_i^T x_\varepsilon(t) = n_i^T (x_\varepsilon(t) - x(t)) + n_i^T x(t) \leq n_1 \cdot \|x_\varepsilon(t) - x(t)\| + \gamma$$

$$< 1 \cdot \frac{1}{2}(\alpha_1 - \gamma) + \gamma < \alpha_1,$$
i.e., on the interval \([0,t_0]\), the solution \(x_\varepsilon\) does not reach the facet \(F_1\). Since \(x_\varepsilon\) can only leave the simplex \(S_N\) through this facet, this indicates that \(x_\varepsilon(t) \in S_N\) for all \(t \in [0,t_0]\). In combination with the observation that
\[
\begin{align*}
n_i^T x_\varepsilon(t_0) &= n_i^T (x_\varepsilon(t_0) - x(t_0)) + n_i^T x(t_0) = n_i^T (x_\varepsilon(t_0) - x(t_0)) + \beta \\
&\geq \beta - ||n_i|| \cdot ||x_\varepsilon(t_0) - x(t_0)|| > \beta - 1 \cdot \frac{1}{2} (\beta - \alpha_i) > \alpha_i
\end{align*}
\]
we obtain a contradiction. We conclude that every solution of the closed-loop system (15), starting in a point \(x_0 \in S_N\) can only leave the simplex \(S_N\) through the exit facet \(F_1\).

Proposition 4.3 does not yet describe a complete solution to Problem 3.2, using affine state feedback. Although this result guarantees that the state of the closed-loop system cannot leave the simplex \(S_N\) through one of the facets \(F_2, \ldots, F_{N+1}\), we still have to check that the state reaches the exit facet \(F_1\) in finite time. For this purpose we need conditions (1) and (2) of Proposition 3.4:

**Lemma 4.5** Consider the system \(\dot{x} = Ax + Bu + a\) with \(x \in S_N\) and \(u \in U\), and assume that there exist inputs \(u_1, \ldots, u_{N+1} \in U\) such that conditions (1) and (2) of Proposition 3.4 are satisfied, i.e.

1. \(n_i^T (Av_j + Bu_j + a) > 0\) for \(j = 2, \ldots, N + 1,\)
2. \(n_i^T (Av_i + Bu_i + a) \leq 0\) for \(i = 2, \ldots, N + 1,\) and there exists an \(i \in \{2, \ldots, N + 1\}\) such that \(n_i^T (Av_i + Bu_i + a) < 0.\)

Let \(F, g\) be the corresponding solution of (8), and apply the feedback law \(u = Fx + g\). Then \(u \in U\) and in every \(x_0 \in S_N\), the closed-loop dynamics satisfy
\[
n_i^T \dot{x} |_{x_0} = n_i^T ((A + BF)x_0 + (a + Bg)) > 0.
\]

So in every point of the simplex \(S_N\), the state of the closed-loop system is moving with a strictly positive speed in the direction of the exit facet \(F_1\).

**Proof:** Let \(x_0 \in S_N\). Then \(x_0\) is a convex combination of the vertices of \(S_N\), and there exist \(\lambda_1, \ldots, \lambda_{N+1} \in [0,1]\) such that \(\sum_{j=1}^{N+1} \lambda_j = 1\) and \(\sum_{j=1}^{N+1} \lambda_j v_j = x_0\). So
\[
n_i^T ((A + BF)x_0 + (a + Bg)) = n_i^T \left((A + BF) \sum_{j=1}^{N+1} \lambda_j v_j + \sum_{j=1}^{N+1} \lambda_j (a + Bg)\right)
= n_i^T \left(\sum_{j=1}^{N+1} \lambda_j ((A + BF)v_j + Bg + a)\right)
= n_i^T \left(\sum_{j=1}^{N+1} \lambda_j (Av_j + Bu_j + a)\right)
= \lambda_1 n_i^T (Av_1 + Bu_1 + a) + \sum_{j=2}^{N+1} \lambda_j n_i^T (Av_j + Bu_j + a).
\]

According to condition (1), \(n_i^T (Av_j + Bu_j + a) > 0\) for \(j = 2, \ldots, N + 1,\) so it suffices to show that \(n_i^T (Av_1 + Bu_1 + a) > 0.\) For this purpose we combine Lemma A.3 with condition
(2). Since $n_1$ is a negative linear combination of $n_2, \ldots, n_{N+1}$, there exist $\mu_2, \ldots, \mu_{N+1} < 0$ such that

$$n_1 = \sum_{i=2}^{N+1} \mu_i n_i.$$ 

So, according to condition (2):

$$n_1^T (Av_1 + Bu_1 + a) = \sum_{i=2}^{N+1} \mu_i n_i^T (Av_1 + Bu_1 + a) > 0.$$ 

**Theorem 4.6** Consider the system $\dot{x}(t) = Ax(t) + Bu(t) + a$ with $x \in S_N$ and $u \in U$, and assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that conditions (1), (2), and (3) of Proposition 3.4 are satisfied. Let $F, g$ be the corresponding solution of (8). Then, irrespective of the initial state $x_0 \in S_N$, the feedback control law $u(t) = Fx(t) + g$ is a solution to Control Problem 3.2.

**Proof:** First note that according to Lemma 4.2 the feedback $u = Fx + g$ is admissible: for all $x \in S_N$: $u = Fx + g \in U$. Furthermore, Proposition 4.3 states that the control law $u = Fx + g$ guarantees that the state $x$ can only leave the simplex $S_N$ through the facet $F_1$. On this exit facet $F_1$, the velocity vector field $\dot{x} = (A + BF)x + (a + Bg)$ points out of the simplex $S_N$, because of Lemma 4.5. So it suffices to show that the exit facet $F_1$ is reached within finite time.

For this purpose, we consider the affine function

$$h : S_N \rightarrow \mathbb{R} : h(x) = n_1^T((A + BF)x + (a + Bg)).$$

Since the simplex $S_N$ is compact, $h$ attains a maximum $c_2$ and a minimum $c_1$, and, according to Lemma 4.5, $c_1 > 0$. Let $x(t) \in S_N$ be a solution of the closed-loop system $\dot{x} = (A + BF)x + (a + Bg)$, and define

$$y(t) := n_1^T x(t).$$

Then $\dot{y}(t) = n_1^T \dot{x}(t) = h(x(t)) \geq c_1$. So $y(t) \geq y(0) + c_1 t$, with $y(0) = n_1^T x(0) \leq \alpha_1$, because the state is initially located in the simplex $S_N$. However, since $y$ is strictly increasing at a rate of at least $c_1$, there exists $0 \leq T_0 < \infty$, such that $y(T_0) = \alpha_1$. At that time, $n_1^T x(T_0) = \alpha_1$, so $x(T_0) \in F_1$, and indeed the exit facet $F_1$ is reached within finite time.

Combining Proposition 3.4 and Theorem 4.6, we obtain

**Corollary 4.7** Consider the affine dynamical system $\dot{x}(t) = Ax(t) + Bu(t) + a$, with $x \in S_N$ and $u \in U$. Then the following statements are equivalent:

(i) There exists a continuous function $f : S_N \rightarrow U$ such that the feedback control $u(t) = f(x(t))$ solves Problem 3.2 for every initial state $x(0) = x_0 \in S_N$.

(ii) There exist $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$ such that the affine feedback control $u(t) = Fx(t) + g$ solves Problem 3.2 for every initial state $x(0) = x_0 \in S_N$.

(iii) There exist $u_1, \ldots, u_{N+1} \in U$ such that

1. $n_1^T(Av_j + Bu_j + a) > 0$ for $j = 2, \ldots, N + 1$,
n^T_i (Av_1 + Bu_1 + a) \leq 0 \text{ for } i = 2, \ldots, N + 1, \text{ and there exists an } i \in \{2, \ldots, N + 1\}
\text{such that } n^T_i (Av_1 + Bu_1 + a) < 0,
\item \( n^T_i (Av_j + Bu_j + a) \leq 0 \) for all \( i, j = 2, \ldots, N + 1 \) with \( i \neq j \).
\end{enumerate}

Corollary 4.7 indicates that for the solution of Problem 3.2 by static state feedback it is not necessary to consider the whole class of continuous state feedbacks. If a solution to the problem exists, it may always be realized by an affine static state feedback.

Note that the considerations in the proof of Theorem 4.6 also give rise to an upper bound for the time \( T_0 \) at which the exit facet is reached. The function \( y(t) = n^T_1 x(t) \) satisfies \( n^T_1 v_1 \leq y(0) \leq \alpha_1 \) and \( y(T_0) = \alpha_1 \), and its minimal rate of increase \( c_1 \) is given by
\[
c_1 = \min \{ n^T_1 ((A + BF)x + (a + Bg)) \mid x \in S_N \} = \min \{ n^T_1 ((A + BF)v_i + (a + Bg)) \mid i = 1, \ldots, N + 1 \}.
\]

This implies that
\[
T_0 \leq \frac{\alpha_1 - n^T_1 v_1}{c_1},
\]
where \( c_1 \) is easily computed with formula (18). Note however that the upper bound on \( T_0 \) is conservative, because \( T_0 \) depends both on the initial state \( x_0 \) and on the time-varying value \( n^T_1 ((A + BF)x + (a + Bg)) \) of the growth of \( n^T_1 x \) along the solution trajectory \( x(t) \in S_N \). Furthermore, if the solvability conditions (1), (2), and (3) of Proposition 3.4 admit some freedom in the choice of the inputs \( u_1, \ldots, u_{N+1} \), this may be used to decrease the upper bound for \( T_0 \) in (19), by increasing \( c_1 \). Since \( u_i = Fv_i + g \), formula (18) may be written as
\[
c_1 = \min \{ n^T_1 (Av_i + Bu_i + a) \mid i = 1, \ldots, N + 1 \}.
\]

This indicates that the upper bound in (19) may be optimized by solving a constrained max-min problem for the inputs \( u_1, \ldots, u_{N+1} \) at the vertices of the simplex \( S_N \).

5 Computational issues

At first sight, the necessary and sufficient conditions (1)–(3) of Proposition 3.4 and Theorem 4.6 for the solvability of Control Problem 3.2 do not seem easy to check. This is mainly due to the fact that condition (2) consists of a set of non-strict inequalities, of which at least one inequality has to be strict. However, this condition may be rewritten as follows:

2bis) \( n^T_{i,j} (Av_1 + Bu_1 + a) \leq 0 \) for \( i = 2, \ldots, N + 1 \), and
\[
\sum_{i=2}^{N+1} n^T_i (Av_1 + Bu_1 + a) < 0.
\]

With this reformulation of (2), the set of all solvability conditions becomes a set of strict and non-strict linear inequalities. Since also the input set \( U \) is assumed to be polyhedral, this has the advantage that the existence of a solution \( u_1, \ldots, u_{N+1} \in U \) may be checked, using existing software for polyhedral sets, like e.g. [7, 18]. This verification is further facilitated by the fact that the inequalities of conditions (1)–(3) are decoupled. Indeed, by reordering these inequalities, we obtain the following two conditions, completely equivalent with (1)–(3):

(1) \( n^T_{i,j} (Av_1 + Bu_1 + a) \leq 0 \) for \( i = 2, \ldots, N + 1 \), and
\[
\sum_{i=2}^{N+1} n^T_i (Av_1 + Bu_1 + a) < 0.
\]
(II) For \( j = 2, \ldots, N + 1 \): 
\[
n_1^T (Av_j + Bu_j + a) > 0, \quad \text{and} \\
n_i^T (Av_j + Bu_j + a) \leq 0 \quad \text{for all } \ i \in \{2, \ldots, N + 1\} \setminus \{j\}.
\]
So the existence of an \( N + 1 \)-tuple of inputs \( u_1, \ldots, u_{N+1} \in U \), satisfying (I) and (II) may be checked for every of the inputs \( u_j \) separately, by only considering the inequalities in which \( u_j \) occurs. In this way, the problem of verification is split up into \( N + 1 \) considerably smaller subproblems.

If the inputs \( u \) are unconstrained, i.e. if \( U = \mathbb{R}^m \), and if the matrix \( B \) is right-invertible, then the existence of a solution to (I) and (II) is automatically guaranteed. Intuitively this is clear because in this situation we have full control, which makes it possible to prescribe the vector field of \( \dot{x} \) at the vertices of \( S_N \) completely. Alternatively, the same observation can be made directly from the inequalities in (I) and (II). For this purpose we first rewrite (I) as

\[
\begin{pmatrix}
n_2^T \\
\vdots \\
n_{N+1}^T
\end{pmatrix} Bu_1 \leq -
\begin{pmatrix}
n_2^T \\
\vdots \\
n_{N+1}^T
\end{pmatrix} (Av_1 + a), \quad \text{and}
\]

\[
(1 \cdots 1)
\begin{pmatrix}
n_2^T \\
\vdots \\
n_{N+1}^T
\end{pmatrix} Bu_1 < -(1 \cdots 1)
\begin{pmatrix}
n_2^T \\
\vdots \\
n_{N+1}^T
\end{pmatrix} (Av_1 + a).
\]

According to Lemma A.2, the matrix \((n_2 | \cdots | n_{N+1})^T\) is invertible, so if \( \text{Range}(B) = \mathbb{R}^N \), it is obvious that both inequalities may be satisfied by a suitable choice of \( u_1 \). The argument for the other inputs \( u_2, \ldots, u_{N+1} \) is completely similar. The only differences are some modifications of the inequality signs, and the use of a different matrix of normal vectors (for the input \( u_j \), \( j = 2, \ldots, N + 1 \), one has to use a matrix consisting of all normal vectors \( n_1, \ldots, n_{N+1} \), except \( n_j \); Lemma A.2 indicates that this matrix is invertible).

To illustrate the role of the inequalities in (I) and (II) we end this section with a small example.

**Example 5.1** Let \( N = 2 \), and let the simplex \( S_2 \) be the triangle in \( \mathbb{R}^2 \) with vertices \( v_1 = (-1,0)^T \), \( v_2 = (1,1)^T \), and \( v_3 = (1,-1)^T \) (see Figure 1). The normal vectors on the three facets \( F_1 \), \( F_2 \), and \( F_3 \) of \( S_2 \) are \( n_1 = (1,0)^T \), \( n_2 = \frac{1}{\sqrt{5}}(-1,-2)^T \), and \( n_3 = \frac{1}{\sqrt{5}}(-1,2)^T \), respectively. On the simplex \( S_2 \) we consider the system

\[
\dot{x} = \begin{pmatrix}
-1 & -1 \\
-2 & 1
\end{pmatrix} x + \begin{pmatrix}
2 \\
-2
\end{pmatrix} u + \begin{pmatrix}
3 \\
1
\end{pmatrix},
\]

with state \( x \in S_2 \) and scalar input \(-1 \leq u \leq 1\). We want to construct an affine feedback law \( u = Fx + g \) such that the state of the closed-loop system can only leave the simplex \( S_2 \) through the facet \( F_1 \), the vertical line segment between the vertices \( v_2 \) and \( v_3 \).

For the existence of a solution it is necessary and sufficient that there exist an input \( u_1 \) at vertex \( v_1 \) satisfying (I), and inputs \( u_2, u_3 \) at the vertices \( v_2, v_3 \) satisfying (II). So, for \( u_1 \) the following inequalities should hold:

1. \( n_2^T Bu_1 \leq -n_2^T (Av_1 + a) \), so \( u_1 \leq 5 \),
2. \( n_3^T Bu_1 \leq -n_3^T (Av_1 + a) \), so \( u_1 \geq \frac{4}{5} \),
(3) \( n_1^T B u_1 + n_2^T B u_2 < -n_1^T (A v_1 + a) - n_2^T (A v_1 + a) \), so \( u_1 \geq -2 \).

and, additionally, \(-1 \leq u_1 \leq 1\). Therefore all conditions are satisfied for \( u_1 \in [1/3, 1] \). For \( u_2 \) we have

(1) \( n_1^T B u_2 > -n_1^T (A v_2 + a) \), so \( u_2 > -\frac{1}{2} \),

(2) \( n_2^T B u_2 \leq -n_2^T (A v_2 + a) \), so \( u_2 \geq -\frac{1}{6} \),

with \(-1 \leq u_2 \leq 1\). Hence \( u_2 \in [-\frac{1}{6}, 1] \). Finally, \( u_3 \) has to satisfy

(1) \( n_1^T B u_3 > -n_1^T (A v_3 + a) \), so \( u_3 > -\frac{3}{4} \),

(2) \( n_2^T B u_3 \leq -n_2^T (A v_3 + a) \), so \( u_3 \leq -\frac{1}{2} \),

and \(-1 \leq u_3 \leq 1\). So every \( u_3 \in [-1, -\frac{1}{2}] \) is a solution.

To obtain an affine feedback, we fix the inputs at the vertices by choosing \( u_1 = \frac{1}{2} \), \( u_2 = 0 \), and \( u_3 = -\frac{3}{4} \), and compute \( F = ( f_1 \ f_2 ) \) and \( g \) using formula (8):

\[
\begin{pmatrix}
-1 & 0 & 1 \\
1 & 1 & 1 \\
1 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
g \\
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} \\
0 \\
-\frac{3}{4} \\
\end{pmatrix}.
\]

This yields the following affine feedback solution for Problem 3.2:

\[
u = \begin{pmatrix}
\frac{-7}{16} \\
\frac{3}{8} \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
+ \frac{1}{16}.
\]

(20)

It guarantees that for all \( x \in S_2 \), the corresponding input \( u \) satisfies \( |u| \leq 1 \), and that the state of the closed-loop system leaves the simplex in finite time through the facet \( F_1 \).

In Figure 1, solution (20) is depicted graphically. Without input, the vector field \( \dot{x} = Ax + a \) at the vertices \( v_1 \) and \( v_3 \) does not point into the right direction; in \( v_1 \) it is required that \( \dot{x} \) points into the simplex \( S_2 \), and in \( v_3 \), \( \dot{x} \) should point into the cone generated by the inequalities \( x_1 \geq 1 \) and \( x_1 + 2x_2 \geq -1 \). This can be resolved by the choice of suitable inputs \( u \). At \( v_1 \) the input \( u_1 = \frac{1}{2} \) changes \( \dot{x} \) to the required direction, and in \( v_3 \), \( u_3 = -\frac{3}{4} \) is a suitable input. In vertex \( v_2 \) the vector \( \dot{x} = Ax + a \) is already pointing in the right direction, without applying any input. This justifies the choice of \( u_2 = 0 \). The affine feedback (20) realizes these inputs \( u_i \), \( (i = 1, 2, 3) \), at the vertices \( v_i \). Finally, the convexity of the problem may be used to show that this affine feedback also solves Problem 3.2 on the whole simplex \( S_2 \). A typical state-trajectory of the closed-loop system is depicted in Figure 2.

6 Concluding remarks

In this paper, a reachability problem for affine systems on simplices and polytopes was considered. First, necessary conditions were derived for the existence of a continuous feedback law, that realizes the control objective of steering the state to a particular facet of the polytope. Next, sufficient conditions were obtained for solving the same control problem by affine feedback, in case that the polytope is a simplex. It turned out that for affine systems on simplices the necessary conditions on the one hand, and the sufficient conditions on the other, are identical. Furthermore, a procedure has been described for the computation of the control law.

Further research is in progress on the extension of the sufficient conditions to affine systems on general polyhedral sets. The results will be used in reachability analysis of piecewise-linear hybrid systems.
Appendix A

In this appendix some technical results on full-dimensional simplices are collected, that are needed throughout the paper. For this purpose we first recall some definitions regarding simplices.

Let $N \in \mathbb{N}$. The full-dimensional simplex $S_N$ in $\mathbb{R}^N$ is defined as the convex hull of $N + 1$ points $v_1, \ldots, v_{N+1} \in \mathbb{R}^N$, called the vertices of $S_N$, with the property that there exists no hyperplane of $\mathbb{R}^N$, containing $v_1, \ldots, v_{N+1}$. For $i \in \{1, \ldots, N+1\}$, the normal vector of the hyperplane in $\mathbb{R}^N$, generated by the points $\{v_1, \ldots, v_{N+1}\} \setminus \{v_i\}$ is denoted by $n_i$. Without loss of generality it is assumed that $\|n_i\| = 1$, and that $n_i$ is pointing out of the simplex $S_N$. Let $\alpha_i \in \mathbb{R}$ be chosen in such a way that the hyperplane generated by $\{v_1, \ldots, v_{N+1}\} \setminus \{v_i\}$ is described by the equation $n_i^T x = \alpha_i$. Then the simplex $S_N$ is characterized by

$$S_N = \{ x \in \mathbb{R}^N \mid \forall i = 1, \ldots, N : n_i^T x \leq \alpha_i \}.$$ 

Moreover, for the vertices $v_1, \ldots, v_{N+1}$ we know that

$$n_i^T v_j = \alpha_i \text{ for } i, j \in \{1, \ldots, N + 1\} \text{ and } i \neq j,$$

$$n_i^T v_i < \alpha_i \text{ for } i \in \{1, \ldots, N + 1\}. \quad (21)$$

Lemma A.1 The vectors $v_2 - v_1, v_3 - v_1, \ldots, v_{N+1} - v_1$ constitute a basis of $\mathbb{R}^N$.

Proof: Since we consider $N$ vectors in $N$-dimensional space, it suffices to prove that these vectors are linearly independent. Assume that $\sum_{j=2}^{N+1} \lambda_j (v_j - v_1) = 0$. Then for every $i \in \{2, \ldots, N + 1\}$ we have

$$0 = n_i^T \sum_{j=2}^{N+1} \lambda_j (v_j - v_1) = \sum_{j=2}^{N+1} \lambda_j (n_i^T v_j - n_i^T v_1) = \sum_{j=2}^{N+1} \lambda_j (n_i^T v_j - \alpha_i) = \lambda_i (n_i^T v_i - \alpha_i),$$

where we used (21) in the last equality. Finally, $n_i^T v_i - \alpha_i < 0$, so $\lambda_i = 0$. \hfill \blacksquare

Lemma A.2 The vectors $n_2, \ldots, n_{N+1}$ constitute a basis of $\mathbb{R}^N$.

Proof: Again, it suffices to prove that these vectors are linearly independent. Assume that $\sum_{j=2}^{N+1} \lambda_j n_j = 0$. Then for every $i \in \{2, \ldots, N + 1\}$:

$$0 = \sum_{j=2}^{N+1} \lambda_j n_j^T (v_i - v_1) = \sum_{j=2}^{N+1} \lambda_j (n_i^T v_j - n_j^T v_1) = \sum_{j=2}^{N+1} \lambda_j (n_i^T v_j - \alpha_j) = \lambda_i (n_i^T v_i - \alpha_i).$$

Since $n_i^T v_i - \alpha_i < 0$, we have $\lambda_i = 0$ for $i = 2, \ldots, N + 1$. \hfill \blacksquare

Lemma A.3 The vector $n_1$ is a negative linear combination of the vectors $n_2, \ldots, n_{N+1}$, i.e. there exist $\lambda_2, \ldots, \lambda_{N+1} < 0$ such that

$$n_1 = \sum_{j=2}^{N+1} \lambda_j n_j.$$
Proof: According to Lemma A.2, the vectors $n_2, \ldots, n_{N+1}$ constitute a basis of $\mathbb{R}^N$, so there exist $\lambda_2, \ldots, \lambda_{N+1} \in \mathbb{R}$, such that

$$n_1 = \sum_{j=2}^{N+1} \lambda_j n_j.$$ 

Let $i \in \{2, \ldots, N+1\}$. Then

$$n_1^T(v_i - v_1) = \alpha_1 - n_1^Tv_1 > 0,$$

and therefore

$$0 < n_1^T(v_i - v_1) = \sum_{j=2}^{N+1} \lambda_j n_j^T(v_i - v_1) = \sum_{j=2}^{N+1} \lambda_j (n_j^Tv_i - \alpha_j) = \lambda_i (n_i^Tv_i - \alpha_i).$$

Since $n_i^Tv_i - \alpha_i < 0$, we must have $\lambda_i < 0$. ■

Lemma A.4 Let $w \in \mathbb{R}^N$ be the vector defined by $w := \sum_{j=2}^{N+1} (v_j - v_1)$. Then for all $i = 2, \ldots, N+1$: $n_i^Tw < 0$.

Proof: For $i \in \{2, \ldots, N+1\}$ we have

$$n_i^T \sum_{j=2}^{N+1} (v_j - v_1) = \sum_{j=2}^{N+1} (n_i^Tv_j - n_i^Tv_1) = \sum_{j=2}^{N+1} (n_i^Tv_j - \alpha_i) = (n_i^Tv_i - \alpha_i) < 0.$$ ■

References


Figure 1: Control of the vector field $\dot{x}$ at the vertices of $S_2$

Figure 2: A state-trajectory of the closed-loop system