Centrum voor Wiskunde en Informatica

REPORT RAPPORT

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Probability, Networks and Algorithms (PNA)

PNA-R0017 December 31, 2000

Report PNA-R0017
ISSN 1386-3711

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Given an affine system on a simplex, the problem of reaching a particular facet of the simplex, using affine state feedback is studied. Necessary and sufficient conditions for the existence of a solution are derived in terms of linear inequalities on the input vectors at the vertices of the simplex. If these conditions are met, a constructive procedure yields an affine feedback control law, that solves this reachability problem.

2000 Mathematics Subject Classification: 93C05, 93C15, 93B05.
Keywords and Phrases: Piecewise-linear hybrid system, polyhedral set, simplex, facet, reachability, control law.
Note: Research is supported in part by the Project Verification of Hybrid Systems (VHS, Esprit project 26270) of the European Commission.

1 Introduction

The purpose of this paper is to present a result on reachability of a facet for an affine dynamical system on a full-dimensional simplex. This problem is motivated by the control of piecewise-linear hybrid systems.

Control of engineering systems is often carried out by computers. This implementation induces an interaction between the discrete dynamics of a computer program on the one hand, and the continuous dynamics of an engineering system on the other. Research in hybrid systems aims at the modeling of this interaction, and the development of systems and control theory for this class of systems. An overview of some of the current research in this area can be found in the conference proceedings [10, 15].

In [12, 13, 14], E.D. Sontag has introduced the class of piecewise-linear hybrid systems. A system of this class consists of an automaton and, for each discrete state, of an affine system on a polyhedral set. A simple case of a polyhedral set is a simplex, the n-dimensional generalization of the triangle in $\mathbb{R}^2$. The class of piecewise-linear hybrid systems has been analyzed by several authors, see e.g. [2, 3]. Also the reachability of general hybrid systems has received considerable interest, see e.g. [1, 8, 9]. A particular approach to the reachability problem was developed by the second co-author in [16]. In case of piecewise-linear hybrid systems, this method requires the solution of a reachability problem of an affine system, by steering the state to a particular facet of a polyhedral set. The latter problem is treated in this paper, under the additional assumption that the polyhedral set under consideration is a simplex.

Given an affine system on a full-dimensional simplex, the main problem can be formulated as follows: determine necessary and sufficient conditions for the existence of an affine control law such that, independent of the initial state, all state-trajectories of the closed-loop system reach a particular facet of the simplex in finite time. In the solution of this problem, convexity arguments play an important role. The necessary condition may be derived for affine systems on arbitrary polytopes, using continuous (i.e. not necessarily affine) feedback, and consists of a set of linear inequalities for the input vectors at the vertices of the polytope. The sufficient condition for an affine system on a simplex using affine state feedback is based on an analysis of the dynamics of the corresponding closed-loop system. The necessary and sufficient conditions are identical in case of a simplex. For general polytopes the situation is more complicated; this problem will be treated in a future paper.

Once the linear inequalities on the input vectors at the vertices of the simplex are obtained, existing algorithms may be used to check the existence of a solution. For this purpose, computer programs have been developed, for example in the research groups of Verimag (N. Halbwachs, B. Jeannet) and of IRISA (D.K. Wilde), see [7, 18]. The final step is the computation of the affine control law. For this problem, a simple procedure is provided.

In the literature there are several publications on the invariance of linear systems on polyhedral sets, see [5, 17] and on invariance of piecewise-linear hybrid systems, see [4]. The problem treated in this paper differs from that of those references in that the trajectories of the system concerned need to reach a particular facet in finite time, hence the system is not invariant, and in that the conditions are more explicit than those in the literature.

The paper is organized as follows. The next section contains the problem formulation and terminology on polyhedral sets and simplices. Necessary conditions for existence of a continuous feedback law realizing the control objective are stated in Section 3. A sufficient condition for existence of an affine feedback meeting the control objective is stated in Section 4. In Section 5 computational issues for the control laws are discussed. Concluding remarks are stated in Section 6. Appendix A provides several technical results on simplices.

2 Problem formulation

Let $N \in \mathbb{N}$, and consider the N-dimensional space $\mathbb{R}^N$. Let $v_1, \ldots, v_{N+1}$ be $N+1$ affinely independent points in $\mathbb{R}^N$, which means that there exists no hyperplane of $\mathbb{R}^N$, containing all these $N+1$ points. The full-dimensional simplex S_N is defined as the convex hull of $v_1, \ldots, v_{N+1}$. The points $v_1, \ldots, v_{N+1}$ are called the *vertices* of S_N.

For every $i \in \{1, \ldots, N+1\}$, the convex hull of the points $\{v_1, \ldots, v_{N+1}\} \backslash \{v_i\}$ is a *facet* of S_N, that will be denoted by F_i. Let n_i be the normal vector of the corresponding hyperplane. By convention, n_i points outward of the simplex S_N, and $\|n_i\| = 1$. There exists an $\alpha_i \in \mathbb{R}$ such that the hyperplane containing the points $\{v_1, \ldots, v_{N+1}\} \backslash \{v_i\}$ (and thus the facet F_i) is described by

$$\{x \in \mathbb{R}^N \mid n_i^T x = \alpha_i\}.$$

This indicates that the simplex S_N is located inside the half-space $\{x \in \mathbb{R}^N \mid n_i^T x \leq \alpha_i\}$. Since the same observation holds for every $i \in \{1, \ldots, N+1\}$, we obtain the following *implicit description* of the simplex S_N:

$$S_N = \{x \in \mathbb{R}^N \mid \ \forall i = 1, \ldots, N+1 : \ n_i^T x \leq \alpha_i\}.$$

Some technical results needed in this paper regarding the sets of vertices and normal vectors of a full-dimensional simplex are collected in Appendix A.

On the full-dimensional simplex S_N, we consider an affine control system

$$\dot{x} = Ax + Bu + a, \qquad x(0) = x_0, \tag{1}$$

with $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$, and $a \in \mathbb{R}^N$. So, on every time instant $t \in T$, the state $x \in \mathbb{R}^N$ is assumed to be contained in the simplex S_N. Also the input u is assumed to take values in a polyhedral set $U \subset \mathbb{R}^m$ only. Note that the affine differential equation (1) only remains valid, as long as the state x is contained in the simplex S_N. In a hybrid system, as soon as the state-trajectory reaches one of the facets of S_N, a discrete event occurs, together with a resetting of the state x. In general the state x will leave the simplex S_N, and proceed to a different simplex, where it is ruled by a different type of affine dynamics. In this paper, we consider the control problem of steering the state of system (1) in finite time to one particular facet of the simplex S_N.

Problem 2.1 For any initial state $x_0 \in S_N$, find a time-instant $T_0 \geq 0$ and an input function $u : [0, T_0] \longrightarrow U$, such that

(i) $\forall t \in [0, T_0] : \ x(t) \in S_N$,

(ii) $x(T_0) \in F_1$, where F_1 is the facet of the simplex S_N, not containing the vertex v_1,

(iii) $n_1^T \dot{x}(T_0) > 0$, i.e. the velocity vector $\dot{x}(T_0)$ at the point $x(T_0) \in F_1$ has a positive component in the direction of n_1. This implies that in the point $x(T_0)$, the velocity vector $\dot{x}(T_0)$ points out of the simplex S_N.

Furthermore, if possible this input function u should be realized by the application of an affine feedback law

$$u(t) = Fx(t) + g, \tag{2}$$

with $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$, that is independent of the initial state x_0.

Note that in Problem 2.1 the choice of the exit facet F_1 is completely arbitrary. Without loss of generality the facet F_1 may be replaced by any other facet of S_N.

After application of the feedback law (2) to system (1) we obtain the closed-loop system

$$\dot{x} = (A + BF)x + (a + Bg), \qquad x(0) = x_0, \tag{3}$$

hence the system dynamics remain affine after application of this feedback. This type of autonomous affine systems exhibits interesting convexity properties, for which it does not make any difference whether the state x is restricted to the simplex S_N, or the whole space $\mathbb{R}^N$ is considered.

Lemma 2.2 *Consider the autonomous affine system in $\mathbb{R}^N$, given by $\dot{x} = Ax + a$, and let p_1, p_2 be two points in $\mathbb{R}^N$, and let $n \in \mathbb{R}^N$ be a nonzero vector in $\mathbb{R}^N$. If $n^T\dot{x}\mid_{p_1}= n^T(Ap_1+a) < 0$ and $n^T\dot{x}\mid_{p_2}= n^T(Ap_2+a) < 0$, then for any $\lambda \in (0,1)$ also $n^T\dot{x}\mid_{\lambda p_1+(1-\lambda)p_2}= n^T(A(\lambda p_1 + (1-\lambda)p_2) + a) < 0$.*

Proof: $n^T\dot{x}\mid_{\lambda p_1+(1-\lambda)p_2}= n^T(A(\lambda p_1+(1-\lambda)p_2)+a) = \lambda n^T(Ap_1+a)+(1-\lambda)n^T(Ap_2+a) < 0.$ ■

Of course the result of Lemma 2.2 still holds if all '$<$' signs are replaced by '$\leq$', '$>$', or '$\geq$' signs. The lemma indicates that if in two points $p_1, p_2 \in \mathbb{R}^N$ the vector field of the velocity of an autonomous affine system points into the same direction w.r.t. a normal vector n, then in all points on the straight line, joining p_1 and p_2, the velocity vector field points into the same direction. In this paper, this simple observation will play a crucial role in the solution of Control Problem 2.1.

3 Necessary conditions for feedback control to a facet

In this section we will show how the convexity result of Lemma 2.2 may be used to obtain necessary conditions for the solution of Problem 2.1, by restricting our attention to the determination of a suitable control input at the vertices $v_1, \ldots, v_{N+1}$ of the simplex S_N.

First, we consider one of the facets of the simplex S_N, with v_1 as one of its vertices, say the facet F_i ($i = 2, \ldots, N+1$), defined as the convex hull of $\{v_1, \ldots, v_{N+1}\}\backslash\{v_i\}$, with normal vector n_i. The state variable x is not allowed to leave the simplex S_N through this facet. This can only be the case if in every point of this facet the vector field of the velocity of x does not point out of the simplex S_N. So, in every point of $p \in F_i$ there should exist a $u \in U$, such that $n_i^T(Ap + Bu + a) \leq 0$. However, due to convexity it suffices to check this condition at the vertices of F_i only.

Lemma 3.1 *Let $i \in \{2, \ldots, N+1\}$. For any point $p \in F_i$ there exists an input $u \in U$ such that*

$$n_i^T(Ap + Bu + a) \leq 0, \tag{4}$$

if and only if there exist $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{N+1} \in U$, such that for all $j \in \{1, \ldots, N+1\}\backslash\{i\}$:

$$n_i^T(Av_j + Bu_j + a) \leq 0. \tag{5}$$

Proof: Necessity of condition (5) is obvious because all vertices $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{N+1}$ are points in F_i. To prove sufficiency, take $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{N+1} \in [0,1]$ such that $\sum_{j\neq i}\lambda_j = 1$ and $\sum_{j\neq i}\lambda_j v_j = p$. Choose $u = \sum_{j\neq i}\lambda_j u_j$; then $u \in U$ because U is a polyhedral set. Furthermore, $n_i^T(Ap + Bu + a) = n_i^T\sum_{i\neq j}\lambda_j(Av_j + Bu_j + a) = \sum_{i\neq j}\lambda_j n_i^T(Av_j + Bu_j + a) \leq 0$. ■

Note that Lemma 3.1 still holds if in (4) and (5) the '$\leq$' signs are replaced by '$<$' signs.

Next we consider the exit facet F_1. For this facet, the direction of the vector field is not completely determined by the description of Problem 2.1. It might be possible to solve this problem by steering the state into the simplex S_N at one part of the facet F_1, and out of the simplex S_N at another part. In this paper we want to avoid this situation by a slight modification of the problem description.

Problem 3.2 Consider the same problem as described in Problem 2.1, and add the additional constraint that

(iv) $T_0 \geq 0$ is the smallest time-instant in the interval $[0, T_0]$ for which the state x reaches the exit facet F_1, i.e. $T_0 = \min\{t \mid t \geq 0 \text{ and } x(t) \in F_1\}$.

In combination with condition *(iii)*, condition *(iv)* implies that in any point of the facet F_1 the input $u \in U$ should be chosen in such a way that the vector field of the velocity of x has a positive component in the n_1-direction, so is pointing out of the simplex S_N. Using the same arguments as in Lemma 3.1, it suffices to verify this condition at the vertices of F_1 only.

Lemma 3.3 *For any point $p \in F_1$ there exists an input $u \in U$ such that*

$$n_1^T(Ap + Bu + a) > 0,$$

if and only if there exist $u_2, \dots, u_{N+1} \in U$, such that for all $j \in \{2, \dots, N+1\}$:

$$n_1^T(Av_j + Bu_j + a) > 0. \tag{6}$$

In Lemmas 3.1 and 3.3 conditions were derived on the existence of a set of inputs, that guarantee that the vector field of the velocity of x can be put in the right direction at the boundaries of the simplex S_N. Note that for every single facet F_i ($i = 1, \dots, N+1$) this gives a different set of conditions on the inputs u_j ($j \neq i$) at the vertices $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{N+1}$ of F_i. So, for solving Problem 3.2 it seems sufficient that for every facet F_i a different N-tuple of inputs $u_j \in U$ ($j \neq i$) is found, satisfying the inequalities (5) and (6), respectively. However, as soon as we apply state feedback, the input at every vertex of S_N is fixed, and inequalities (5) and (6) have to be solved simultaneously. This leads to the following set of necessary conditions for the existence of a closed-loop solution to Problem 3.2.

Proposition 3.4 *Consider the affine dynamical system $\dot{x}(t) = Ax(t) + Bu(t) + a$, with $x \in S_N$ and $u \in U$. Assume that there exists a continuous function $f : S_N \longrightarrow U$, such that the state feedback control law $u(t) = f(x(t))$ solves Problem 3.2, i.e. irrespective of the initial state $x_0 \in S_N$, the closed-loop system*

$$\dot{x} = Ax + Bf(x) + a, \qquad x(0) = x_0,$$

has a solution x, satisfying the conditions (i)—(iv) of Problems 2.1 and 3.2. Then there exist $u_1, \dots, u_{N+1} \in U$ such that

(1) $n_1^T(Av_j + Bu_j + a) > 0$ *for* $j = 2, \dots, N+1$,

(2) $n_i^T(Av_1 + Bu_1 + a) \leq 0$ *for* $i = 2, \dots, N+1$, *and there exists an* $i \in \{2, \dots, N+1\}$ *such that* $n_i^T(Av_1 + Bu_1 + a) < 0$,

(3) $n_i^T(Av_j + Bu_j + a) \leq 0$ *for all* $i, j = 2, \dots, N+1$ *with* $i \neq j$.

Proof: Suppose that the continuous function $f : S_N \longrightarrow U$ generates a feedback law $u(t) = f(x(t))$, that solves Control Problem 3.2. We show that the inputs $u_j = f(v_j)$ ($j = 1, \dots, N+1$) satisfy (1), (2), and (3).

(1): For every $j \in \{2, \dots, N+1\}$, $v_j \in F_1$. So, as soon as the vertex v_j is reached, the state trajectory should leave the simplex S_N with a positive velocity in the n_1-direction. This implies that $n_1^T(Av_j + Bf(v_j) + a) > 0$, so by the definition of u_j: $n_1^T(Av_j + Bu_j + a) > 0$ for $j = 2, \dots, N+1$.

(2): Consider the closed-loop system $\dot{x}(t) = Ax(t) + Bf(x(t)) + a$, with initial value $x(0) = v_1$. The corresponding solution x can only leave the simplex S_N through the facet F_1. So in v_1, the velocity vector does not point out of the simplex S_N. Therefore $n_i^T(Av_1 + Bu_1 + a) = n_i^T(Av_1 + Bf(v_1) + a) \leq 0$ for $i = 2, \dots, N+1$. Furthermore, if $n_i^T(Av_1 + Bu_1 + a) = 0$

for $i = 2, \dots, N+1$, then $Av_1 + Bu_1 + a = 0$ because the vectors $n_2, \dots, n_{N+1}$ constitute a basis of $\mathbb{R}^N$ (see Lemma A.2). This implies that v_1 is a fixed point of the closed-loop system $\dot{x} = Ax + Bf(x) + a$, and the corresponding solution $x(t) \equiv v_1$ will never leave the simplex S_N through the facet F_1. This contradicts the assumption that $u(t) = f(x(t))$ solves Control Problem 3.2.

(3): Since for $N = 1$ Condition (3) is void, we assume, without loss of generality, that $N > 1$. We prove (3) by contradiction. Suppose that there exist $i, j \in \{2, \dots, N+1\}$, $i \neq j$, such that $n_i^T(Av_j + Bu_j + a) > 0$. Define the function

$$h : S_N \longrightarrow \mathbb{R} : \; h(x) = n_i^T(Ax + Bf(x) + a).$$

Then h is continuous, and by assumption $h(v_j) > 0$. So, there exists $\delta > 0$, such that for all $x \in S_N$, with $\|x - v_j\| < \delta$: $h(x) > 0$.

Let $0 < \varepsilon < \min\left(\frac{\delta}{\sum_{k=1}^{N+1} \|v_k\|}, 1\right)$, and define

$$p := (1-\varepsilon)v_j + \frac{\varepsilon}{N-1} \sum_{k=1, k\neq i,j}^{N+1} v_k.$$

Then p is a convex combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{N+1}$, with only strictly positive coefficients. So $p \in F_i$, but p does not belong to any of the other facets of S_N. Furthermore,

$$\begin{aligned} \|p - v_j\| &= \left\| (1-\varepsilon)v_j + \left(\frac{\varepsilon}{N-1} \sum_{k=1, k\neq i,j}^{N+1} v_k \right) - v_j \right\| \\ &= \left\| -\varepsilon v_j + \frac{\varepsilon}{N-1} \sum_{k=1, k\neq i,j}^{N+1} v_k \right\| \leq \varepsilon \cdot \sum_{k=1, k\neq i}^{N+1} \|v_k\| < \delta, \end{aligned}$$

so $h(p) > 0$. Therefore, the trajectory of the closed-loop system $\dot{x} = Ax + Bf(x) + a$, with initial value $x(0) = p \in F_i$, will immediately leave the simplex S_N through the facet F_i because $n_i^T \dot{x}(0) = h(p) > 0$. This contradicts the fact that the feedback law $u(t) = f(x(t))$ is a solution to Problem 3.2, irrespective of the initial state $x_0 \in S_N$. We conclude that $n_i^T(Av_j + Bu_j + a) \leq 0$ for all $i, j \in \{2, \dots, N+1\}$ with $i \neq j$. ∎

The necessary conditions derived in Proposition 3.4 may be generalized from full-dimensional simplices S_N in $\mathbb{R}^N$ to arbitrary full-dimensional polytopes P_N in $\mathbb{R}^N$. The main observation is that conditions (1) to (3) characterize the direction of the vector field of $\dot{x}$, the derivative of the state of the closed-loop system, at the vertices of S_N. So, after a suitable reformulation the same conditions remain valid for general full-dimensional polytopes.

Proposition 3.5 *Let P_N be a full-dimensional polytope in $\mathbb{R}^N$ with vertices $v_1, \dots, v_M$, $(M \geq N+1)$. Let $F_1, \dots, F_L$ denote the facets of P_N, with normal vectors $n_1, \dots, n_L$, respectively, pointing out of the polytope P_N. For $i \in \{1, \dots, L\}$, let $V_i \subset \{1, \dots, M\}$ be the index set such that $\{v_j \mid j \in V_i\}$ is the set of vertices of the facet F_i. Conversely, for every $j \in \{1, \dots, M\}$, the set $W_j \subset \{1, \dots, L\}$ contains the indices of all facets of which v_j is a vertex. Assume that F_1 is the exit facet of P_N. If Control Problem 3.2, with S_N replaced by P_N, is solvable by a continuous state feedback f, then there exist inputs $u_1, \dots, u_M \in U$ such that*

(1*) *$\forall j \in V_1$: $n_1^T(Av_j + Bu_j + a) > 0$,* (2*) $\forall i \in \{2, \ldots, L\}\ \forall j \in V_i$: $n_i^T(Av_j + Bu_j + a) \leq 0$,

(3*) $\forall j \in \{1, \ldots, M\} \backslash V_1\ \exists i \in W_j$: $n_i^T(Av_j + Bu_j + a) < 0$.

In comparison with the original formulation of Proposition 3.4, conditions (1) and (1*) are identical, condition (3) and the first part of condition (2) are replaced by (2*), and the second part of (2) is now condition (3*). The proofs of (1*) and (2*) may be carried out analogously to the proof of Proposition 3.4, because these proofs do not rely on the fact that S_N is a simplex. Only the proof of (3*) requires a slight modification, based on the following result.

Lemma 3.6 *Let $v \in \mathbb{R}^N$ be a vertex of a full-dimensional polytope in $\mathbb{R}^N$. Let $F_1, \ldots, F_K$ denote all facets of P_N, containing v. Then the normal vectors $n_1, \ldots, n_K$ of $F_1, \ldots, F_K$ generate $\mathbb{R}^N$.*

Proof: Let $F_{K+1}, \ldots, F_L$ denote the other facets of P_N, with normal vectors $n_{K+1}, \ldots, n_L$, and assume without loss of generality that all normal vectors are pointing out of the polytope P_N. Then there exist $\alpha_1, \ldots, \alpha_L \in \mathbb{R}$ such that

$$P_N = \{x \in \mathbb{R}^N \mid \forall i = 1, \ldots, L:\ n_i^T x \leq \alpha_i\}$$

is the implicit description of P_N. At the vertex v we know that $n_i^T v = \alpha_i$ for $i = 1, \ldots, K$, and $n_i^T v < \alpha_i$ for $i = K+1, \ldots, L$.

Suppose that $n_1, \ldots, n_K$ do not generate $\mathbb{R}^N$. Then there exists a nonzero vector $n \in \mathbb{R}^N$ such that $n_i^T n = 0$ for $i = 1, \ldots, K$. Hence, there exists a $\delta > 0$ such that for all $\varepsilon \in (-\delta, \delta)$ the point $v + \varepsilon n \in P_N$. This indicates that v is a convex combination of two other points in P_N, and thus contradicts the fact that v is a vertex of P_N (see e.g. [11, p. 162]). ■

Using Lemma 3.6, the proof of Condition (3*) becomes obvious. If there exists a $j \in \{1, \ldots, M\} \backslash V_1$ such that $n_i^T(Av_j + Bu_j + a) = 0$ for all $i \in W_j$, then $Av_j + Bu_j + a = 0$ because the set $\{n_i \mid i \in W_j\}$ generates $\mathbb{R}^N$. This implies that the vertex $v_j \notin F_1$ is a fixed point of the closed-loop system, which contradicts the assumption that the feedback $u(t) = f(x(t))$ solves Control Problem 3.2.

4 Control to a facet by affine state feedback

In the previous section we derived some necessary conditions for the solution of Control Problem 3.2 by continuous static state feedback. Nevertheless, the conditions that were obtained in Propositions 3.4 and 3.5 seem to have a strong open-loop control character: they are formulated as a set of inequalities on the inputs to the system at the vertices of the simplex S_N or the polytope P_N. In this section we restrict our attention to simplices again, and show that if an $N+1$-tuple of inputs exists, satisfying inequalities (1), (2), and (3) of Proposition 3.4, then it may be realized by the application of an affine static state feedback law. Moreover, such an affine state feedback law is also a solution to Control Problem 3.2. The proof of this result consists of two parts. First it is shown that every trajectory of the closed-loop system cannot leave the simplex S_N through one of the facets $F_2, \ldots, F_{N+1}$. Subsequently it is proven that every trajectory will reach the exit facet F_1 in finite time. These observations indicate that for the solution of Problem 3.2 it is not necessary to consider the whole class of continuous static state feedback laws. Instead it is sufficient to confine ourselves to affine static state feedback.

Remark 4.1 The results of this section are stated and proved for full-dimensional simplices only. Generalization of these results to general full-dimensional polytopes seems difficult, because the design of the static state feedback and also several of the proofs in this section rely on the fact that the state set S_N is assumed to be a full-dimensional simplex.

Assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that conditions (1), (2), and (3) of Proposition 3.4 are satisfied. Since input u_i should be applied at the moment that the state vector reaches vertex v_i, we first search for an affine feedback control law $u = Fx + g$, with $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$ such that

$$u_i = Fv_i + g, \qquad (i = 1, \ldots, N+1). \tag{7}$$

Transposition of (7) yields the equalities $v_i^T F^T + g^T = u_i^T$, $(i = 1, \ldots, N+1)$, and by collecting all equalities in a matrix, the following linear equation has to be solved for F and g:

$$\begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_{N+1}^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ \hline g^T \end{pmatrix} = \begin{pmatrix} u_1^T \\ \vdots \\ u_{N+1}^T \end{pmatrix}. \tag{8}$$

Note however that this equation has a *unique* solution. Indeed, because of Lemma A.1, the vectors $v_2 - v_1, v_3 - v_1, \ldots, v_{N+1} - v_1$ are linearly independent and thus

$$\det \begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_{N+1}^T & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ v_2^T - v_1^T & 0 \\ \vdots & \vdots \\ v_{N+1}^T - v_1^T & 0 \end{pmatrix} = (-1)^N \det \begin{pmatrix} v_2^T - v_1^T \\ \vdots \\ v_{N+1}^T - v_1^T \end{pmatrix} \neq 0,$$

so this square $(N+1) \times (N+1)$ matrix is invertible. Furthermore, since the input set U is a polyhedral set, the input determined by the feedback control law $u = Fx + g$ belongs to U for all $x \in S_N$:

Lemma 4.2 *Let $u_1, \ldots, u_{N+1} \in U$, and let $F \in \mathbb{R}^{m \times N}, g \in \mathbb{R}^m$, such that for all $i = 1, \ldots, N+1$: $u_i = Fv_i + g$. Then*

$$\forall x \in S_N: \; u = Fx + g \in U. \tag{9}$$

Proof: Let $x \in S_N$. Then there exist $\lambda_1, \ldots, \lambda_{N+1} \in [0,1]$, with $\sum_{i=1}^{N+1} \lambda_i = 1$ such that $x = \sum_{i=1}^{N+1} \lambda_i v_i$. Then

$$u = Fx + g = F \sum_{i=1}^{N+1} \lambda_i v_i + \sum_{i=1}^{N+1} \lambda_i g = \sum_{i=1}^{N+1} \lambda_i (Fv_i + g) = \sum_{i=1}^{N+1} \lambda_i u_i \in U. \qquad \blacksquare$$

Proposition 4.3 *Consider the dynamical system $\dot{x}(t) = Ax(t) + Bu(t) + a$, with $x \in S_N$ and $u \in U$, and assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that*

(2') $n_i^T(Av_1 + Bu_1 + a) \leq 0$ *for* $i = 2, \ldots, N+1$,

(3) $n_i^T(Av_j + Bu_j + a) \leq 0$ *for all* $i, j = 2, \ldots, N+1$ *with* $i \neq j$.

Let F, g be the corresponding solution of (8), and apply the feedback law $u(t) = Fx(t) + g$. Then $u(t) \in U$, and for any initial state $x_0 \in S_N$, the state x of the closed-loop system

$$\dot{x} = (A + BF)x + (a + Bg), \qquad x(0) = x_0, \tag{10}$$

can only leave the simplex S_N through the facet F_1.

Note that conditions (2') and (3) are not only sufficient for the construction of an affine feedback law $u = Fx + g$, but also necessary; condition (3) literally occurs in Proposition 3.4 and (2') is just slightly weaker than condition (2) of Proposition 3.4. Condition (1) of Proposition 3.4 does not occur in Proposition 4.3, because no information on the component of the velocity vector field in the direction of n_1 at the exit facet F_1 is required to guarantee that a solution of the closed-loop dynamics cannot leave the simplex S_N through one of the facets $F_2, \ldots, F_{N+1}$. Whether or not the state trajectory leaves the simplex through the facet F_1 will be discussed in Theorem 4.6.

In the proof of Proposition 4.3 we need the following version of Gronwall's Lemma.

Lemma 4.4 ([6, p. 19]) *Let $\varphi \in C^+[0, b]$, where $C^+[0, b]$ denotes the set of continuous nonnegative functions on the interval $[0, b]$. If a function $f \in C^+[0, b]$ satisfies*

$$f(t) \leq \varphi(t) + K \int_0^t f(s)\, ds, \qquad (t \in [0, b]), \tag{11}$$

for some fixed $K > 0$, then

$$f(t) \leq \varphi(t) + K \int_0^t e^{K(t-s)} \varphi(s)\, ds, \qquad (t \in [0, b]). \tag{12}$$

In particular, if there exist $K > 0$ and $\beta > 0$ such that $f \in C^+[0, b]$ satisfies (11) with $\varphi(t) = \beta t$, then

$$f(t) \leq \frac{\beta}{K}(e^{Kt} - 1), \qquad (t \in [0, b]). \tag{13}$$

Proof of Proposition 4.3: Let $w \in \mathbb{R}^N$ be the vector $w = \sum_{j=2}^{N+1}(v_j - v_1)$, and define $n := \frac{1}{\|w\|} w$. Then $\|n\| = 1$, and according to Lemma A.4, $n_i^T n < 0$ for $i = 2, \ldots, N+1$.

Let $\varepsilon > 0$, and consider the following perturbed closed-loop dynamics

$$\dot{x}_\varepsilon = (A + BF)x_\varepsilon + (a + Bg) + \varepsilon n. \tag{14}$$

We first show that a trajectory $x_\varepsilon(t)$, with $x_\varepsilon(0) \in S_N$ and satisfying (14), cannot leave the simplex S_N through one of the facets $F_2, \ldots, F_{N+1}$. Indeed, let $i \in \{2, \ldots, N+1\}$, and $p \in F_i$. Since p is a convex combination of the vertices $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{N+1}$, there exist $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{N+1} \in [0, 1]$ such that $\sum_{j \neq i} \lambda_j = 1$ and $p = \sum_{j \neq i} \lambda_j v_j$. Then

$$\begin{aligned}
n_i^T \dot{x}_\varepsilon \mid_p &= n_i^T((A + BF)p + (a + Bg) + \varepsilon n) \\
&= n_i^T \left((A + BF) \sum_{j \neq i} \lambda_j v_j + \sum_{j \neq i} \lambda_j (Bg + a) + \varepsilon n \right) \\
&= n_i^T \sum_{j \neq i} \lambda_j ((A + BF)v_j + Bg + a) + \varepsilon n_i^T n \\
&= \sum_{j \neq i} \lambda_j n_i^T (Av_j + Bu_j + a) + \varepsilon n_i^T n < 0,
\end{aligned}$$

because $\sum_{j \neq i} \lambda_j n_i^T(Av_j + Bu_j + a) \leq 0$ by conditions (2') and (3) and $n_i^T n < 0$. So, unless $p \in F_1$, the velocity vector field of (14) is pointing strictly into the simplex S_N. This implies that any solution x_ε of the perturbed closed-loop system (14) can only leave the simplex S_N through the facet F_1. By contradiction we will prove that the same is true for the unperturbed closed-loop system.

Consider the affine dynamical system

$$\dot{x} = (A + BF)x + (a + Bg), \tag{15}$$

with $x \in \mathbb{R}^N$. So for a moment we assume that differential equation (15) does not only hold on the simplex S_N, but on the whole space $\mathbb{R}^N$. Suppose that there exists $x_0 \in S_N$ such that the solution $x(t)$ of (15) with initial value $x(0) = x_0$ leaves S_N through the facet F_i with $i \in \{2, \ldots, N+1\}$ before it has reached the facet F_1. Then there exists $t_0 > 0$ such that

(i) $\beta := n_i^T x(t_0) > \alpha_i$,

(ii) $\forall t \in [0, t_0] : \; n_1^T x(t) < \alpha_1$.

Define $\gamma := \max\{n_1^T x(t) \mid t \in [0, t_0]\}$ and $K := \|A + BF\| = \max\{\|(A + BF)x\| \mid \|x\| = 1\}$. Let $0 < \varepsilon < \frac{K}{(e^{Kt_0} - 1)} \cdot \min\left(\frac{1}{2}(\beta - \alpha_i), \frac{1}{2}(\alpha_1 - \gamma)\right)$. We will compare the solutions $x_\varepsilon(t)$ of the perturbed system (14) and $x(t)$ of the unperturbed system (15), both with initial value $x_0 \in S_N$, on the interval $[0, t_0]$. For all $t \in [0, t_0]$ we have

$$\begin{aligned}
x(t) - x_\varepsilon(t) &= \int_0^t \dot{x}(s) - \dot{x}_\varepsilon(s)\, ds \\
&= \int_0^t (A + BF)x(s) + (a + Bg) - ((A + BF)x_\varepsilon(s) + (a + Bg) + \varepsilon n)\, ds \\
&= \int_0^t (A + BF)(x(s) - x_\varepsilon(s))\, ds - \int_0^t \varepsilon n\, ds,
\end{aligned}$$

hence

$$\begin{aligned}
\|x(t) - x_\varepsilon(t)\| &\leq \int_0^t \|A + BF\| \|x(s) - x_\varepsilon(s)\|\, ds + \int_0^t \varepsilon \cdot \|n\|\, ds \\
&\leq \varepsilon \cdot t + K \int_0^t \|x(s) - x_\varepsilon(s)\|\, ds.
\end{aligned}$$

Next we apply Lemma 4.4 with $f(t) = \|x(t) - x_\varepsilon(t)\|$ and $\varphi(t) = \varepsilon t$, and find that for all $t \in [0, t_0]$:

$$\|x(t) - x_\varepsilon(t)\| \leq \frac{\varepsilon}{K}(e^{Kt} - 1).$$

So, in particular

$$\|x(t) - x_\varepsilon(t)\| \leq \frac{\varepsilon}{K}(e^{Kt_0} - 1) < \min\left(\frac{1}{2}(\beta - \alpha_i), \frac{1}{2}(\alpha_1 - \gamma)\right), \qquad (t \in [0, t_0]).$$

This implies that for every $t \in [0, t_0]$ the solution x_ε satisfies

$$\begin{aligned}
n_1^T x_\varepsilon(t) &= n_1^T(x_\varepsilon(t) - x(t)) + n_1^T x(t) \leq \|n_1\| \cdot \|x_\varepsilon(t) - x(t)\| + \gamma \\
&< 1 \cdot \frac{1}{2}(\alpha_1 - \gamma) + \gamma < \alpha_1,
\end{aligned}$$

i.e., on the interval $[0, t_0]$, the solution x_ε does not reach the facet F_1. Since x_ε can only leave the simplex S_N through this facet, this indicates that $x_\varepsilon(t) \in S_N$ for all $t \in [0, t_0]$. In combination with the observation that

$$\begin{aligned} n_i^T x_\varepsilon(t_0) &= n_i^T(x_\varepsilon(t_0) - x(t_0)) + n_i^T x(t_0) = n_i^T(x_\varepsilon(t_0) - x(t_0)) + \beta \\ &\geq \beta - \|n_i\| \cdot \|x_\varepsilon(t_0) - x(t_0)\| > \beta - 1 \cdot \frac{1}{2}(\beta - \alpha_i) > \alpha_i, \end{aligned}$$

we obtain a contradiction. We conclude that every solution of the closed-loop system (15), starting in a point $x_0 \in S_N$ can only leave the simplex S_N through the exit facet F_1. ■

Proposition 4.3 does not yet describe a complete solution to Problem 3.2, using affine state feedback. Although this result guarantees that the state of the closed-loop system cannot leave the simplex S_N through one of the facets $F_2, \ldots, F_{N+1}$, we still have to check that the state reaches the exit facet F_1 in finite time. For this purpose we need conditions (1) and (2) of Proposition 3.4:

Lemma 4.5 *Consider the system $\dot{x} = Ax + Bu + a$ with $x \in S_N$ and $u \in U$, and assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that conditions (1) and (2) of Proposition 3.4 are satisfied, i.e.*

(1) $n_1^T(Av_j + Bu_j + a) > 0$ *for* $j = 2, \ldots, N+1$,

(2) $n_i^T(Av_1 + Bu_1 + a) \leq 0$ *for* $i = 2, \ldots, N+1$, *and there exists an* $i \in \{2, \ldots, N+1\}$ *such that* $n_i^T(Av_1 + Bu_1 + a) < 0$.

Let F, g be the corresponding solution of (8), and apply the feedback law $u = Fx + g$. Then $u \in U$ and in every $x_0 \in S_N$, the closed-loop dynamics satisfy

$$n_1^T \dot{x} \mid_{x_0} = n_1^T((A + BF)x_0 + (a + Bg)) > 0. \tag{16}$$

So in every point of the simplex S_N, the state of the closed-loop system is moving with a strictly positive speed in the direction of the exit facet F_1.

Proof: Let $x_0 \in S_N$. Then x_0 is a convex combination of the vertices of S_N, and there exist $\lambda_1, \ldots, \lambda_{N+1} \in [0, 1]$ such that $\sum_{j=1}^{N+1} \lambda_j = 1$ and $\sum_{j=1}^{N+1} \lambda_j v_j = x_0$. So

$$n_1^T((A + BF)x_0 + (a + Bg)) = n_1^T \left((A + BF) \sum_{j=1}^{N+1} \lambda_j v_j + \sum_{j=1}^{N+1} \lambda_j (a + Bg) \right)$$

$$= n_1^T \left(\sum_{j=1}^{N+1} \lambda_j ((A + BF)v_j + Bg + a) \right) = n_1^T \left(\sum_{j=1}^{N+1} \lambda_j (Av_j + Bu_j + a) \right)$$

$$= \lambda_1 n_1^T(Av_1 + Bu_1 + a) + \sum_{j=2}^{N+1} \lambda_j n_1^T(Av_j + Bu_j + a). \tag{17}$$

According to condition (1), $n_1^T(Av_j + Bu_j + a) > 0$ for $j = 2, \ldots, N+1$, so it suffices to show that $n_1^T(Av_1 + Bu_1 + a) > 0$. For this purpose we combine Lemma A.3 with condition

(2). Since n_1 is a negative linear combination of $n_2, \dots, n_{N+1}$, there exist $\mu_2, \dots, \mu_{N+1} < 0$ such that

$$n_1 = \sum_{i=2}^{N+1} \mu_i n_i.$$

So, according to condition (2):

$$n_1^T(Av_1 + Bu_1 + a) = \sum_{i=2}^{N+1} \mu_i n_i^T(Av_1 + Bu_1 + a) > 0. \qquad \blacksquare$$

Theorem 4.6 *Consider the system $\dot{x}(t) = Ax(t) + Bu(t) + a$ with $x \in S_N$ and $u \in U$, and assume that there exist inputs $u_1, \dots, u_{N+1} \in U$ such that conditions* (1), (2), *and* (3) *of Proposition 3.4 are satisfied. Let F, g be the corresponding solution of (8). Then, irrespective of the initial state $x_0 \in S_N$, the feedback control law $u(t) = Fx(t) + g$ is a solution to Control Problem 3.2.*

Proof: First note that according to Lemma 4.2 the feedback $u = Fx + g$ is admissible: for all $x \in S_N$: $u = Fx + g \in U$. Furthermore, Proposition 4.3 states that the control law $u = Fx + g$ guarantees that the state x can only leave the simplex S_N through the facet F_1. On this exit facet F_1, the velocity vector field $\dot{x} = (A + BF)x + (a + Bg)$ points out of the simplex S_N, because of Lemma 4.5. So it suffices to show that the exit facet F_1 is reached within finite time.

For this purpose, we consider the affine function

$$h: \ S_N \longrightarrow \mathbb{R}: \ h(x) = n_1^T((A + BF)x + (a + Bg)).$$

Since the simplex S_N is compact, h attains a maximum c_2 and a minimum c_1, and, according to Lemma 4.5, $c_1 > 0$. Let $x(t) \in S_N$ be a solution of the closed-loop system $\dot{x} = (A + BF)x + (a + Bg)$, and define

$$y(t) := n_1^T x(t).$$

Then $\dot{y}(t) = n_1^T \dot{x}(t) = h(x(t)) \geq c_1$. So $y(t) \geq y(0) + c_1 t$, with $y(0) = n_1^T x(0) \leq \alpha_1$, because the state is initially located in the simplex S_N. However, since y is strictly increasing at a rate of at least c_1, there exists $0 \leq T_0 < \infty$, such that $y(T_0) = \alpha_1$. At that time, $n_1^T x(T_0) = \alpha_1$, so $x(T_0) \in F_1$, and indeed the exit facet F_1 is reached within finite time. $\blacksquare$

Combining Proposition 3.4 and Theorem 4.6, we obtain

Corollary 4.7 *Consider the affine dynamical system $\dot{x}(t) = Ax(t) + Bu(t) + a$, with $x \in S_N$ and $u \in U$. Then the following statements are equivalent:*

(i) There exists a continuous function $f : S_N \longrightarrow U$ such that the feedback control $u(t) = f(x(t))$ solves Problem 3.2 for every initial state $x(0) = x_0 \in S_N$.

(ii) There exist $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$ such that the affine feedback control $u(t) = Fx(t) + g$ solves Problem 3.2 for every initial state $x(0) = x_0 \in S_N$.

(iii) There exist $u_1, \dots, u_{N+1} \in U$ such that

(1) $n_1^T(Av_j + Bu_j + a) > 0$ *for* $j = 2, \dots, N + 1$,

(2) $n_i^T(Av_1+Bu_1+a) \leq 0$ *for* $i = 2, \ldots, N+1$, *and there exists an* $i \in \{2, \ldots, N+1\}$ *such that* $n_i^T(Av_1 + Bu_1 + a) < 0$,

(3) $n_i^T(Av_j + Bu_j + a) \leq 0$ *for all* $i, j = 2, \ldots, N+1$ *with* $i \neq j$.

Corollary 4.7 indicates that for the solution of Problem 3.2 by static state feedback it is not necessary to consider the whole class of *continuous* state feedbacks. If a solution to the problem exists, it may always be realized by an *affine* static state feedback.

Note that the considerations in the proof of Theorem 4.6 also give rise to an upper bound for the time T_0 at which the exit facet is reached. The function $y(t) = n_1^T x(t)$ satisfies $n_1^T v_1 \leq y(0) \leq \alpha_1$ and $y(T_0) = \alpha_1$, and its minimal rate of increase c_1 is given by

$$\begin{aligned} c_1 &= \min\{n_1^T((A+BF)x + (a+Bg)) \mid x \in S_N\} \\ &= \min\{n_1^T((A+BF)v_i + (a+Bg)) \mid i = 1, \ldots, N+1\}. \end{aligned} \tag{18}$$

This implies that

$$T_0 \leq \frac{\alpha_1 - n_1^T v_1}{c_1}, \tag{19}$$

where c_1 is easily computed with formula (18). Note however that the upper bound on T_0 is conservative, because T_0 depends both on the initial state x_0 and on the time-varying value $n_1^T((A+BF)x + (a+Bg))$ of the growth of $n_1^T x$ along the solution trajectory $x(t) \in S_N$. Furthermore, if the solvability conditions (1), (2), and (3) of Proposition 3.4 admit some freedom in the choice of the inputs $u_1, \ldots, u_{N+1}$, this may be used to decrease the upper bound for T_0 in (19), by increasing c_1. Since $u_i = Fv_i + g$, formula (18) may be written as

$$c_1 = \min\{n_1^T(Av_i + Bu_i + a) \mid i = 1, \ldots, N+1\}.$$

This indicates that the upper bound in (19) may be optimized by solving a constrained max-min problem for the inputs $u_1, \ldots, u_{N+1}$ at the vertices of the simplex S_N.

5 Computational issues

At first sight, the necessary and sufficient conditions (1)—(3) of Proposition 3.4 and Theorem 4.6 for the solvability of Control Problem 3.2 do not seem easy to check. This is mainly due to the fact that condition (2) consists of a set of non-strict inequalities, of which at least one inequality has to be strict. However, this condition may be rewritten as follows:

(2bis) $n_i^T(Av_1 + Bu_1 + a) \leq 0$ for $i = 2, \ldots, N+1$, and
$$\sum_{i=2}^{N+1} n_i^T(Av_1 + Bu_1 + a) < 0.$$

With this reformulation of (2), the set of all solvability conditions becomes a set of strict and non-strict linear inequalities. Since also the input set U is assumed to be polyhedral, this has the advantage that the existence of a solution $u_1, \ldots, u_{N+1} \in U$ may be checked, using existing software for polyhedral sets, like e.g. [7, 18]. This verification is further facilitated by the fact that the inequalities of conditions (1)—(3) are decoupled. Indeed, by reordering these inequalities, we obtain the following two conditions, completely equivalent with (1)—(3):

(I) $n_i^T(Av_1 + Bu_1 + a) \leq 0$ for $i = 2, \ldots, N+1$, and
$$\sum_{i=2}^{N+1} n_i^T(Av_1 + Bu_1 + a) < 0.$$

(II) For $j = 2, \dots, N+1$: $n_1^T(Av_j + Bu_j + a) > 0, \quad$ and
$n_i^T(Av_j + Bu_j + a) \le 0 \quad$ for all $\quad i \in \{2, \dots, N+1\} \backslash \{j\}$.

So the existence of an $N+1$-tuple of inputs $u_1, \dots, u_{N+1} \in U$, satisfying (I) and (II) may be checked for every of the inputs u_j separately, by only considering the inequalities in which u_j occurs. In this way, the problem of verification is split up into $N+1$ considerably smaller subproblems.

If the inputs u are unconstrained, i.e. if $U = \mathbb{R}^m$, and if the matrix B is right-invertible, then the existence of a solution to (I) and (II) is automatically guaranteed. Intuitively this is clear because in this situation we have full control, which makes it possible to prescribe the vector field of $\dot{x}$ at the vertices of S_N completely. Alternatively, the same observation can be made directly from the inequalities in (I) and (II). For this purpose we first rewrite (I) as

$$\begin{pmatrix} n_2^T \\ \vdots \\ n_{N+1}^T \end{pmatrix} Bu_1 \le - \begin{pmatrix} n_2^T \\ \vdots \\ n_{N+1}^T \end{pmatrix} (Av_1 + a), \qquad \text{and}$$

$$(1 \cdots 1) \begin{pmatrix} n_2^T \\ \vdots \\ n_{N+1}^T \end{pmatrix} Bu_1 < -(1 \cdots 1) \begin{pmatrix} n_2^T \\ \vdots \\ n_{N+1}^T \end{pmatrix} (Av_1 + a).$$

According to Lemma A.2, the matrix $(n_2 \mid \cdots \mid n_{N+1})^T$ is invertible, so if $\text{Range}(B) = \mathbb{R}^N$, it is obvious that both inequalities may be satisfied by a suitable choice of u_1. The argument for the other inputs $u_2, \dots, u_{N+1}$ is completely similar. The only differences are some modifications of the inequality signs, and the use of a different matrix of normal vectors (for the input u_j, $(j = 2, \dots, N+1)$, one has to use a matrix consisting of all normal vectors $n_1, \dots, n_{N+1}$, except n_j; Lemma A.2 indicates that this matrix is invertible).

To illustrate the role of the inequalities in (I) and (II) we end this section with a small example.

Example 5.1 Let $N = 2$, and let the simplex S_2 be the triangle in $\mathbb{R}^2$ with vertices $v_1 = (-1, 0)^T$, $v_2 = (1, 1)^T$, and $v_3 = (1, -1)^T$ (see Figure 1). The normal vectors on the three facets F_1, F_2, and F_3 of S_2 are $n_1 = (1, 0)^T$, $n_2 = \frac{1}{\sqrt{5}}(-1, -2)^T$, and $n_3 = \frac{1}{\sqrt{5}}(-1, 2)^T$, respectively. On the simplex S_2 we consider the system

$$\dot{x} = \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ -2 \end{pmatrix} u + \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

with state $x \in S_2$ and scalar input $-1 \le u \le 1$. We want to construct an affine feedback law $u = Fx + g$ such that the state of the closed-loop system can only leave the simplex S_2 through the facet F_1, the vertical line segment between the vertices v_2 and v_3.

For the existence of a solution it is necessary and sufficient that there exist an input u_1 at vertex v_1 satisfying (I), and inputs u_2, u_3 at the vertices v_2, v_3 satisfying (II). So, for u_1 the following inequalities should hold:

(1) $n_2^T B u_1 \le -n_2^T(Av_1 + a)$, so $u_1 \le 5$,

(2) $n_3^T B u_1 \le -n_3^T(Av_1 + a)$, so $u_1 \ge \frac{1}{3}$,

(3) $n_2^T B u_1 + n_3^T B u_1 < -n_2^T(Av_1 + a) - n_3^T(Av_1 + a)$, so $u_1 \geq -2$,

and, additionally, $-1 \leq u_1 \leq 1$. Therefore all conditions are satisfied for $u_1 \in [\frac{1}{3}, 1]$. For u_2 we have

(1) $n_1^T B u_2 > -n_1^T(Av_2 + a)$, so $u_2 > -\frac{1}{2}$,

(2) $n_3^T B u_2 \leq -n_3^T(Av_2 + a)$, so $u_2 \geq -\frac{1}{6}$,

with $-1 \leq u_2 \leq 1$. Hence $u_2 \in [-\frac{1}{6}, 1]$. Finally, u_3 has to satisfy

(1) $n_1^T B u_3 > -n_1^T(Av_3 + a)$, so $u_3 > -\frac{3}{2}$,

(2) $n_2^T B u_3 \leq -n_2^T(Av_3 + a)$, so $u_3 \leq -\frac{1}{2}$,

and $-1 \leq u_3 \leq 1$. So every $u_3 \in [-1, -\frac{1}{2}]$ is a solution.

To obtain an affine feedback, we fix the inputs at the vertices by choosing $u_1 = \frac{1}{2}$, $u_2 = 0$, and $u_3 = -\frac{3}{4}$, and compute $F = \begin{pmatrix} f_1 & f_2 \end{pmatrix}$ and g using formula (8):

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{4} \end{pmatrix}.$$

This yields the following affine feedback solution for Problem 3.2:

$$u = \begin{pmatrix} -\frac{7}{16} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{16}. \tag{20}$$

It guarantees that for all $x \in S_2$, the corresponding input u satisfies $|u| \leq 1$, and that the state of the closed-loop system leaves the simplex in finite time through the facet F_1.

In Figure 1, solution (20) is depicted graphically. Without input, the vector field $\dot{x} = Ax + a$ at the vertices v_1 and v_3 does not point into the right direction; in v_1 it is required that $\dot{x}$ points into the simplex S_2, and in v_3, $\dot{x}$ should point into the cone generated by the inequalities $x_1 \geq 1$ and $x_1 + 2x_2 \geq -1$. This can be resolved by the choice of suitable inputs u. At v_1 the input $u_1 = \frac{1}{2}$ changes $\dot{x}$ to the required direction, and in v_3, $u_3 = -\frac{3}{4}$ is a suitable input. In vertex v_2 the vector $\dot{x} = Ax + a$ is already pointing in the right direction, without applying any input. This justifies the choice of $u_2 = 0$. The affine feedback (20) realizes these inputs u_i, $(i = 1, 2, 3)$, at the vertices v_i. Finally, the convexity of the problem may be used to show that this affine feedback also solves Problem 3.2 on the whole simplex S_2. A typical state-trajectory of the closed-loop system is depicted in Figure 2.

6 Concluding remarks

In this paper, a reachability problem for affine systems on simplices and polytopes was considered. First, necessary conditions were derived for the existence of a continuous feedback law, that realizes the control objective of steering the state to a particular facet of the polytope. Next, sufficient conditions were obtained for solving the same control problem by affine feedback, in case that the polytope is a simplex. It turned out that for affine systems on simplices the necessary conditions on the one hand, and the sufficient conditions on the other, are identical. Furthermore, a procedure has been described for the computation of the control law.

Further research is in progress on the extension of the sufficient conditions to affine systems on general polyhedral sets. The results will be used in reachability analysis of piecewise-linear hybrid systems.

Appendix A

In this appendix some technical results on full-dimensional simplices are collected, that are needed throughout the paper. For this purpose we first recall some definitions regarding simplices.

Let $N \in \mathbb{N}$. The full-dimensional simplex S_N in $\mathbb{R}^N$ is defined as the convex hull of $N+1$ points $v_1, \ldots, v_{N+1} \in \mathbb{R}^N$, called the vertices of S_N, with the property that there exists no hyperplane of $\mathbb{R}^N$, containing $v_1, \ldots, v_{N+1}$. For $i \in \{1, \ldots, N+1\}$, the normal vector of the hyperplane in $\mathbb{R}^N$, generated by the points $\{v_1, \ldots, v_{N+1}\} \backslash \{v_i\}$ is denoted by n_i. Without loss of generality it is assumed that $\|n_i\| = 1$, and that n_i is pointing out of the simplex S_N. Let $\alpha_i \in \mathbb{R}$ be chosen in such a way that the hyperplane generated by $\{v_1, \ldots, v_{N+1}\} \backslash \{v_i\}$ is described by the equation $n_i^T x = \alpha_i$. Then the simplex S_N is characterized by

$$S_N = \{x \in \mathbb{R}^N \mid \forall i = 1, \ldots, N : \; n_i^T x \leq \alpha_i\}.$$

Moreover, for the vertices $v_1, \ldots, v_{N+1}$ we know that

$$\begin{array}{rclll} n_i^T v_j & = & \alpha_i & \text{for} & i, j \in \{1, \ldots, N+1\} \text{ and } i \neq j, \\ n_i^T v_i & < & \alpha_i & \text{for} & i \in \{1, \ldots, N+1\}. \end{array} \tag{21}$$

Lemma A.1 *The vectors $v_2 - v_1, v_3 - v_1, \ldots, v_{N+1} - v_1$ constitute a basis of $\mathbb{R}^N$.*

Proof: Since we consider N vectors in N-dimensional space, it suffices to prove that these vectors are linearly independent. Assume that $\sum_{j=2}^{N+1} \lambda_j (v_j - v_1) = 0$. Then for every $i \in \{2, \ldots, N+1\}$ we have

$$0 = n_i^T \sum_{j=2}^{N+1} \lambda_j (v_j - v_1) = \sum_{j=2}^{N+1} \lambda_j (n_i^T v_j - n_i^T v_1) = \sum_{j=2}^{N+1} \lambda_j (n_i^T v_j - \alpha_i) = \lambda_i (n_i^T v_i - \alpha_i),$$

where we used (21) in the last equality. Finally, $n_i^T v_i - \alpha_i < 0$, so $\lambda_i = 0$. ∎

Lemma A.2 *The vectors $n_2, \ldots, n_{N+1}$ constitute a basis of $\mathbb{R}^N$.*

Proof: Again, it suffices to prove that these vectors are linearly independent. Assume that $\sum_{j=2}^{N+1} \lambda_j n_j = 0$. Then for every $i \in \{2, \ldots, N+1\}$:

$$0 = \sum_{j=2}^{N+1} \lambda_j n_j^T (v_i - v_1) = \sum_{j=2}^{N+1} \lambda_j (n_j^T v_i - n_j^T v_1) = \sum_{j=2}^{N+1} \lambda_j (n_j^T v_i - \alpha_j) = \lambda_i (n_i^T v_i - \alpha_i).$$

Since $n_i^T v_i - \alpha_i < 0$, we have $\lambda_i = 0$ for $i = 2, \ldots, N+1$. ∎

Lemma A.3 *The vector n_1 is a negative linear combination of the vectors $n_2, \ldots, n_{N+1}$, i.e. there exist $\lambda_2, \ldots, \lambda_{N+1} < 0$ such that*

$$n_1 = \sum_{j=2}^{N+1} \lambda_j n_j.$$

Proof: According to Lemma A.2, the vectors $n_2, \dots, n_{N+1}$ constitute a basis of $\mathbb{R}^N$, so there exist $\lambda_2, \dots, \lambda_{N+1} \in \mathbb{R}$, such that

$$n_1 = \sum_{j=2}^{N+1} \lambda_j n_j.$$

Let $i \in \{2, \dots, N+1\}$. Then

$$n_1^T(v_i - v_1) = \alpha_1 - n_1^T v_1 > 0,$$

and therefore

$$0 < n_1^T(v_i - v_1) = \sum_{j=2}^{N+1} \lambda_j n_j^T(v_i - v_1) = \sum_{j=2}^{N+1} \lambda_j(n_j^T v_i - \alpha_j) = \lambda_i(n_i^T v_i - \alpha_i).$$

Since $n_i^T v_i - \alpha_i < 0$, we must have $\lambda_i < 0$. ■

Lemma A.4 *Let $w \in \mathbb{R}^N$ be the vector defined by $w := \sum_{j=2}^{N+1}(v_j - v_1)$. Then for all $i = 2, \dots, N+1$: $n_i^T w < 0$.*

Proof: For $i \in \{2, \dots, N+1\}$ we have

$$n_i^T \sum_{j=2}^{N+1}(v_j - v_1) = \sum_{j=2}^{N+1}(n_i^T v_j - n_i^T v_1) = \sum_{j=2}^{N+1}(n_i^T v_j - \alpha_i) = (n_i^T v_i - \alpha_i) < 0. \quad ■$$

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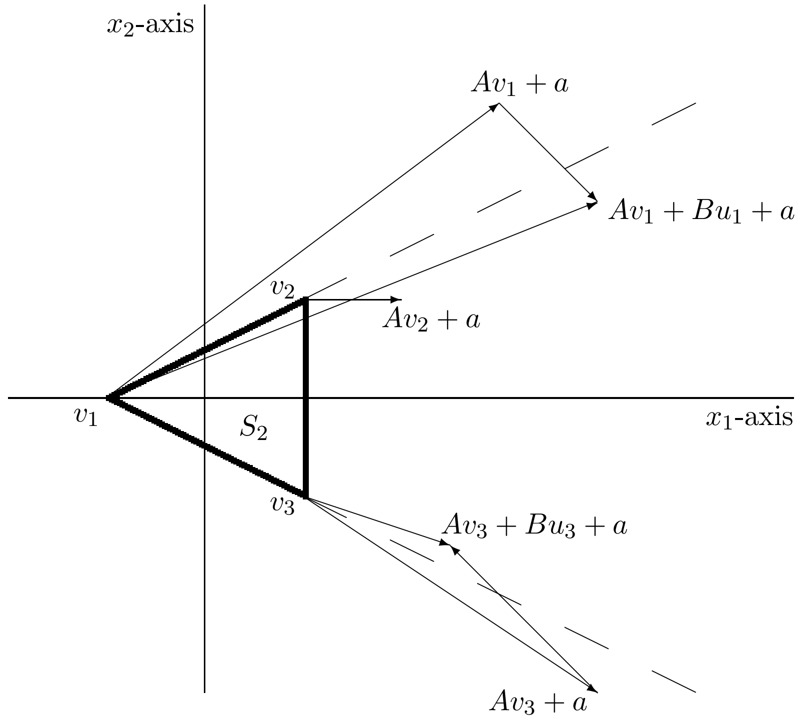

Figure 1: Control of the vector field $\dot{x}$ at the vertices of S_2

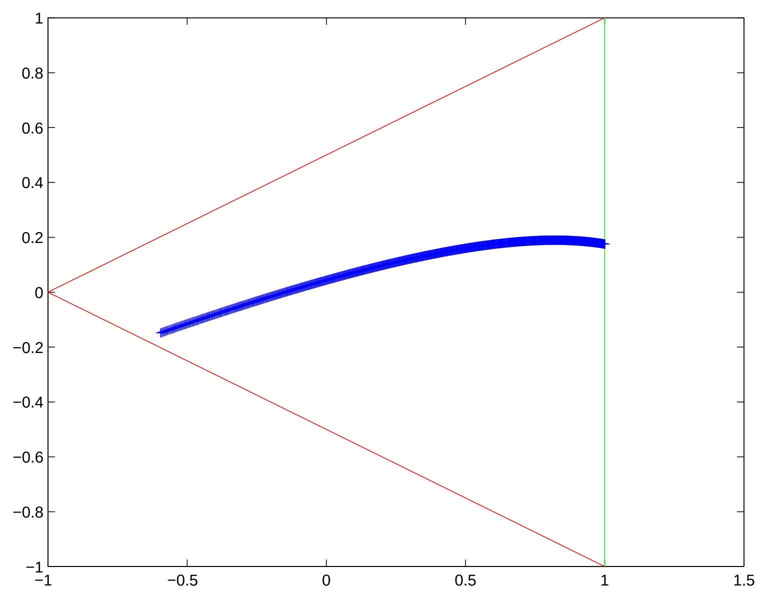

Figure 2: A state-trajectory of the closed-loop system