Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

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R. Helmers, I W. Mangku, R. Zitikis

Probability, Networks and Algorithms (PNA)

PNA-R0102 February 28, 2001

Report PNA-R0102
ISSN 1386-3711

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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Roelof Helmers
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
E-mail: R.Helmers@cwi.nl

I Wayan Mangku[1]
Department of Mathematics, Bogor Agricultural University
Jl. Raya Pajajaran Bogor, 16144 Indonesia

Ričardas Zitikis[2]
Department of Statistics, University of Manitoba
Winnipeg, Canada R3T 2N2
E-mail: Ricardas_Zitikis@umanitoba.ca

ABSTRACT
We consider a kernel-type nonparametric estimator of the intensity function of a cyclic Poisson process when the period is unknown. We assume that only a single realization of the Poisson process is observed in a bounded window which expands in time. We compute the asymptotic bias, variance, and the mean squared error of the estimator when the window indefinitely expands.

2000 Mathematics Subject Classification: 60G55, 62G05, 62G20
Keywords and Phrases: Poisson process, point process, intensity function, period, nonparametric estimation, consistency, bias, variance, mean-squared error
Note: Carried out under project PNA3.2

1. Introduction

In Helmers, Mangku and Zitikis [HMZ] (2000) we constructed a consistent estimator of a cyclic Poisson intensity function λ under the following circumstances:

a) The period (i.e., cycle) of the intensity function λ is unknown.

b) Only one observation of the Poisson process X is available in a bounded window $W_n \subset \mathbf{R}$.

c) The window W_n depends on time n and expands when n increases.

The estimator and the main result of HMZ (2000) are recorded, respectively, in definition (1.5) and Theorem 1.1 below.

There are many applications where estimating cyclic Poisson intensity functions is of importance. For some of them, we refer to the monographs by Cox and Lewis (1966), Lewis (1972), Daley and Vere-Jones (1988), Karr (1991), Snyder and Miller (1991), Reiss (1993), and Kutoyants (1998).

Before formulating our main results, Theorems 1.2–1.5 below, we first introduce some necessary definitions and assumptions.

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[1]Supported by a cooperation project between The Netherlands and Indonesia on “Applied Mathematics and Computational Methods” of the Royal Netherlands Academy of Sciences (KNAW).

[2]Partially supported by an NSERC of Canada grant, and by the Netherlands Organization for Scientific Research (NWO).

Let X be a Poisson point process on the real line $\mathbf{R}$ with (unknown) locally integrable intensity function λ. We assume throughout that λ is periodic with (unknown) period

$$\tau > 0, \tag{1.1}$$

that is, $\lambda(z + k\tau) = \lambda(z)$ for any real $z \in \mathbf{R}$ and integer $k \in \mathbf{Z}$. Note that assumption (1.1) excludes the trivial case $\lambda(s) \equiv c$ from our consideration, since in this case we would have $\tau = 0$. In fact, in this paper we implicitely exclude even a larger class of intensity function λ, namely, all those λ that are constant almost everywhere with respect to Lebesgue measure on $\mathbf{R}$. To demonstrate that the latter exclusion is necessary for constructing consistent estimators of the period τ, we argue as follows. Let λ_τ be a periodic Poisson intensity function having period $\tau > 0$ and such that, for a constant c, $\lambda_\tau(x) = c$ for every $x \in \mathbf{R} \setminus N$, where $N \subseteq \mathbf{R}$ is a set of Lebesgue measure 0. Then the mean measure $\mu_\tau(B) := \int_B \lambda_\tau(x)dx$ of X is such that $\mu_\tau(B) = c|B|$ for any Borel set $B \subseteq \mathbf{R}$, where $|B|$ stands for the Lebesgue measure of B. Since the distribution of any Poisson process is completely specified by the corresponding mean measure, we conclude that the Poisson process X can not be distinguished (as far as distributions are concerned) from the (homogeneous) Poisson process X_0 having the intensity function $\lambda_0(x) := c$, for all $x \in \mathbf{R}$. In view of this, no consistent estimator of $\tau > 0$ can be constructed from X, and thus assumptions of the theorems below are not satisfied.

We assume that $W_1,\ W_2,\ ... \subset \mathbf{R}$, called windows, are intervals of finite length $|W_n|$ that indefinitely increases when $n \to \infty$, that is,

$$|W_n| \to \infty.$$

(Unless confusion is possible, we shall always suppress $n \to \infty$ to make the presentation shorter.) Note that without restriction of generality we can and thus do assume that $W_1, W_2, \dots$ is an increasing sequence of intervals, that is,

$$W_1 \subset \cdots \subset W_n \subset W_{n+1} \subset \cdots \subset \mathbf{R}. \tag{1.2}$$

Indeed, the inclusion $W_n \subset W_{n+1}$ means that we "update" the information about X as the time progresses. Finally, we assume that

$$0 \in W_1, \tag{1.3}$$

which means that we "start" at 0 or, in other words, denote the starting point by 0.

Suppose that the Poisson process X has been observed in the window W_n and a consistent estimator $\hat{\tau}_n \geq 0$ of the period τ has been constructed, that is, we have

$$\hat{\tau}_n \to_P \tau, \tag{1.4}$$

where $\to_P$ stands for the convergence in probability. For example, one may use the estimators constructed by Vere-Jones (1982), Mangku (2001). Using the estimator $\hat{\tau}_n$ of τ, in HMZ (2000) we constructed the following estimator

$$\hat{\lambda}_{n,K}(s) := \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) X(dx) \tag{1.5}$$

of $\lambda(s)$. In order to demonstrate that $\hat{\lambda}_{n,K}(s)$ is a consistent estimator of $\lambda(s)$, in HMZ (2000) we assumed several assumptions that we also assume in this paper and thus record them now. Namely, we assume that s is a Lebesgue point of the intensity function λ. Furthermore, we assume that the sequence $h_1, h_2, \dots$ of positive real numbers h_n converges to 0 in such a way that

$$h_n|W_n| \to \infty. \tag{1.6}$$

We also assume that the kernel function K is a bounded probability density function with the support, supp(K), being a subset of the interval $[-1,1]$. If it is not stated otherwise, we also assume that K has only a finite number of discontinuities. The later assumption is is a technical and very mild one needed in the proofs to control the fluctuations of the function

$$x \mapsto K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)$$

depending on the fluctuations of $\hat{\tau}_n$ around τ. Under the assumptions above, in HMZ (2000) we proved weak and strong consistency of the estimator $\hat{\lambda}_{n,K}(s)$, as well as obtained a rate of consistency. In particular, we proved the following theorem.

Theorem 1.1 (HMZ, 2000) *Let the following assumption*

$$\mathbf{P}\left\{\frac{|W_n|}{h_n}\,|\hat{\tau}_n-\tau|\geq\delta\right\}=o(1) \tag{1.7}$$

hold for any (fixed) $\delta>0$. *Then the estimator* $\hat{\lambda}_{n,K}(s)$ *is weakly consistent.*

Assumption (1.7) is, certainly, an explicit way to state that

$$\frac{|W_n|}{h_n}\,|\hat{\tau}_n-\tau| \;\xrightarrow{p}\; 0. \tag{1.8}$$

We prefer using (1.7), instead of (1.8), since in results below we impose assumption which can be compared with (1.7) in an easier and more straightforward way than with (1.8).

In the present paper we focus on further statistical properties of the estimator $\hat{\lambda}_{n,K}(s)$, such as asymptotic unbiasedness, asymptotic behaviour of the variance and the mean-squared error. Actually, we use the slight modification

$$\hat{\lambda}^{\diamond}_{n,K}(s):=\mathbf{I}\{\hat{\lambda}_{n,K}(s)\leq D_n\}\hat{\lambda}_{n,K}(s) \tag{1.9}$$

of the estimator $\hat{\lambda}_{n,K}$ of HMZ (2000), where $D_n\to\infty$ is a (non-random) sequence. We note at the outset that the use of the "truncated" estimator $\hat{\lambda}^{\diamond}_{n,K}(s)$, instead of the original one $\hat{\lambda}_{n,K}(s)$ of HMZ (2000), should be as natural in the context of the present paper as the use of the original one $\hat{\lambda}_{n,K}(s)$, since we are estimating bounded (periodic) intensity functions $\lambda(s)$. The intuition behind the need of having the truncated estimator in this paper will be explained below.

In what follows, we aim at deriving results under minimal assumptions on the intensity function λ, the estimator $\hat{\tau}_n$ of τ, and other parameters involved. As to the assumptions on $\hat{\tau}_n$, we aim at imposing "in-probability" type assumptions which, on the one hand, are along the lines of assumption (1.7) and, on the other hand, are convenient to verify in practical situations.

Our first main result is concerned with the asymptotic unbiasedness of the estimator $\hat{\lambda}^{\diamond}_{n,K}(s)$ and is formulated as follows.

Theorem 1.2 *Assuming that, for any* $\delta>0$,

$$\mathbf{P}\left\{\frac{|W_n|}{h_n}\,|\hat{\tau}_n-\tau|\geq\delta\right\}=o\left(\frac{1}{D_n}\right), \tag{1.10}$$

we have that

$$\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s)\to\lambda(s). \tag{1.11}$$

Assumption (1.10) connects the truncation level D_n in the definition of $\hat{\lambda}^{\diamond}_{n,K}$ with the rate of convergence of $\hat{\tau}_n$ to τ. Namely, it says that the faster the random variable $\{|W_n|/h_n\}\,|\hat{\tau}_n - \tau|$ converges to 0 in probability, the higher the truncation level D_n can be chosen so that statement (1.11) would still hold true. This is natural since errors made when estimating the period τ are then accumulated and enlarged a number of times when estimating $\lambda(s)$ itself, depending on the number of non-zero summands in the sum on the right-hand side of (1.5). This may naturally result into the situation when $\hat{\lambda}^{\diamond}_{n,K}(s)$ stays too far away from $\lambda(s)$, and with a too large probability. This is a situation we avoid by using the truncated estimator $\hat{\lambda}^{\diamond}_{n,K}(s)$.

In the next two paragraphs we discuss two interesting cases when one can choose $D_n = +\infty$ and thus have the equality $\hat{\lambda}^{\diamond}_{n,K} = \hat{\lambda}_{n,K}$, the original estimator of HMZ (2000).

When the period τ is known, then $\hat{\tau}_n \equiv \tau$ and thus the left-hand-side of (1.10) equals 0. Therefore, we can (though somewhat formally) choose $D_n = +\infty$ in (1.10). In this case, we can convince ourselves in the validity of the result of Theorem 1.2 in the following, more direct way:

$$
\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \\
&\approx \int_{\mathbf{R}} K(x)\lambda(h_n x + s)dx \\
&\to \lambda(s),
\end{aligned}
\tag{1.12}
$$

where convergence to $\lambda(s)$ in (1.12) is due to the assumptions that K is a probability density function and s is a Lebesgue point of λ (for more detail, we refer to the proof of Statement 3.4 below). We conclude the paragraph with the note that the case of known period τ, though in more complicated than periodic situations, was investigated by Helmers and Zitikis (1999).

The sequence D_n can also be choosen to be $+\infty$ in the case when, for any $\delta > 0$, we can find an $n_0 := n_0(\delta)$ such that, for all $n \geq n_0$,

$$
\delta_n := \frac{|W_n|}{h_n}\,|\hat{\tau}_n - \tau| \leq \delta \quad \text{a.s.} \tag{1.13}
$$

Note that the just introduced assumption requires n_0 to be the same for almost all points ω of the sample space. Assumption (1.13) is, therefore, stronger than the almost sure convergence of δ_n to 0. For further detail we refer to Mangku (2001, p.101-107).

In Theorem 1.3 below we derive the first two terms in the asymptotic expansion of $\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s)$. Naturally, the result requires additional assumptions on λ, K and other quantities involved, in order to obtain the required bound of the remainder term.

Theorem 1.3 *Let the second derivative λ'' of the intensity function λ exist and and be finite at the point s. Let the kernel K be symmetric and satisfy the Lipschitz condition between the (finite number of) discontinuity points. Furthermore, let the sequence D_n be such that, for some $c > 0$ and $\epsilon > 0$, the bound $D_n \geq c h_n^{-\epsilon}$ holds for all sufficiently large n. Assuming that, for any $\delta > 0$,*

$$
\mathbf{P}\left\{\frac{|W_n|}{h_n^3}\,|\hat{\tau}_n - \tau| \geq \delta\right\} = o\left(\frac{h_n^2}{D_n}\right) \tag{1.14}
$$

and $h_n^2|W_n| \to \infty$, we have that

$$
\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s) = \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2\int_{-1}^{1} x^2 K(x)dx + o(h_n^2). \tag{1.15}
$$

Note that, contrary to Theorem 1.2, in Theorem 1.3 we require that the truncation level D_n should not be too low, depending on the bandwidth h_n. This is so in order to be able to extract the term $0.5\lambda''(s)h_n^2\int_{-1}^1 x^2K(x)dx$ out of the estimator $\hat{\lambda}_{n,K}^{\diamond}(s)$, with the error $o(h_n^2)$. Note also that, given the constraints of Theorem 1.3, if we take the lowest truncation level $D_n = c/h_n^{\epsilon}$, it will give us the weakest assumption (1.14), which is

$$\mathbf{P}\left\{\frac{|W_n|}{h_n^3}\,|\hat{\tau}_n - \tau| \geq \delta\right\} = o(h_n^{2+\epsilon}).$$

The main reason for formulating a result like Theorem 1.3 with general D_n is to allow some needed flexibility when combining results with different sequences D_n. We employ this observation, for example, in Corollary 1.2 below, which is a consequence of two results: Theorems 1.3 and 1.5.

In Theorem 1.3 we assume that $h_n^2|W_n| \to \infty$, which is a stronger assumption than (1.6). In fact, without assuming $h_n^2|W_n| \to \infty$, we prove that the remainder term on the right-hand side of (1.15) is of the order $o(h_n^2) + O(|W_n|^{-1})$. Since the second term on the right-hand side of (1.15) is exactly of the order $O(h_n^2)$, we thus have to require $o(h_n^2) + O(|W_n|^{-1})$ to be of the order $o(h_n^2)$, in order to have a meaningful statement. Thus, the assumption $h_n^2|W_n| \to \infty$ appears in Theorem 1.3 above.

We conclude the discussion concerning Theorem 1.3 with the note that the right hand side of (1.15) is of the form that is usual in the context of kernel-type density estimation form.

In the following two theorems we consider the convergence of variance $\mathbf{Var}\{\hat{\lambda}_{n,K}^{\diamond}(s)\}$ to 0, as well as the rate of convergence.

Theorem 1.4 *Assuming that, for any $\delta > 0$,*

$$p_n := \mathbf{P}\left\{\frac{|W_n|}{h_n}\,|\hat{\tau}_n - \tau| \geq \delta\right\} = o\left(\frac{1}{D_n^2}\right), \tag{1.16}$$

we have that

$$\mathbf{Var}\{\hat{\lambda}_{n,K}^{\diamond}(s)\} \to 0. \tag{1.17}$$

In view of the discussion immediately after Theorem 1.2, it should not be surprising to see the rate $p_n = o(D_n^{-2})$ in Theorem 1.4, if compared to $p_n = o(D_n^{-1})$ in Theorem 1.2. Indeed, since in Theorem 1.4 we consider the variance of $\hat{\lambda}_{n,K}^{\diamond}(s)$, instead of the mean, even moderate errors when estimating τ may enlarge the variance of $\hat{\lambda}_{n,K}^{\diamond}(s)$ in a more profound way than in the case of the mean. To controle the errors, in Theorem 1.4 we therefore impose the requirement that the probability p_n converges to 0 at least twice as fast as in Theorem 1.2.

Using Theorems 1.2 and 1.4, we immediately obtain that the mean-squared error

$$\mathbf{MSE}\{\hat{\lambda}_{n,K}^{\diamond}(s)\} = \mathbf{Var}\{\hat{\lambda}_{n,K}^{\diamond}(s)\} + \left(\mathbf{Bias}\{\hat{\lambda}_{n,K}^{\diamond}(s)\}\right)^2$$

converges to 0, the result of following Corollary 1.1.

Corollary 1.1 *Assuming that, for any $\delta > 0$, assumption* (1.16) *holds, we have that*

$$\mathbf{MSE}\{\hat{\lambda}_{n,K}^{\diamond}(s)\} \to 0. \tag{1.18}$$

In Theorem 1.5 below we derive the first term in the asymptotic expansion of the variance $\mathbf{Var}\{\hat{\lambda}_{n,K}^{\diamond}(s)\}$ and in this way demonstrate that the variance is of order $O(1/\{|W_n|h_n\})$. Naturally, the result requires stronger assumptions than those of Theorem 1.4, in order to obtain the needed bound of the remainder term.

Theorem 1.5 *Let the kernel K satisfy the Lipschitz condition between the (finite number of) discontinuity points. Furthermore, let the sequence D_n be such that, for some $c > 0$ and $\epsilon > 0$, the bound $D_n \geq c(h_n|W_n|)^\epsilon$ holds for all sufficiently large n. Assuming that, for any $\delta > 0$,*

$$\mathbf{P}\left\{ \frac{|W_n|^{3/2}}{h_n^{1/2}} |\hat{\tau}_n - \tau| \geq \delta \right\} = o\left(\frac{1}{D_n^2 |W_n| h_n} \right), \tag{1.19}$$

we have that

$$\mathbf{Var}\{\hat{\lambda}^{\diamond}_{n,K}(s)\} = \frac{\tau\lambda(s)}{|W_n|h_n} \int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right). \tag{1.20}$$

Using Theorems 1.3 and 1.5, in following Corollary 1.2 we derive an asymptotic formula for the mean-squared error $\mathbf{MSE}\{\hat{\lambda}^{\diamond}_{n,K}(s)\}$.

Corollary 1.2 *Let the second derivative λ'' of the intensity function λ exist and be finite at the point s. Let the kernel K be symmetric and satisfy the Lipschitz condition between the (finite number of) discontinuity points. Furthermore, let the sequence D_n be such that, for some $c > 0$ and $\epsilon > 0$, the bound $D_n \geq c \max\{h_n^{-1}, h_n|W_n|\}^\epsilon$ holds for all sufficiently large n. Assuming that, for any $\delta > 0$, assumption (1.19) holds, we obtain that*

$$\mathbf{MSE}\{\hat{\lambda}^{\diamond}_{n,K}(s)\} = \frac{\tau\lambda(s)}{|W_n|h_n} \int_{-1}^{1} K^2(x)dx + \frac{1}{4}\left(\lambda''(s)\int_{-1}^{1} x^2 K(x)dx\right)^2 h_n^4 + o\left(\frac{1}{|W_n|h_n}\right) + o\left(h_n^4\right). \tag{1.21}$$

Minimizing the sum of the first two terms on the right-hand side of (1.21), we obtain the following (optimal) choice for the bandwidth h_n:

$$h_n = \left\{ \frac{\tau\lambda(s)\int_{-1}^{1} K^2(x)dx}{\left(\lambda''(s)\int_{-1}^{1} x^2 K(x)dx\right)^2} \right\}^{1/5} \frac{1}{|W_n|^{1/5}}. \tag{1.22}$$

With this h_n, the optimal rate of decrease of $\mathbf{MSE}\{\hat{\lambda}^{\diamond}_{n,K}(s)\}$ is of the order $O(|W_n|^{-4/5})$. More precisely, under the assumptions of Corollary 1.2, and with a sequence D_n such that, for some $c > 0$ and $\epsilon > 0$, the bound $D_n \geq ch_n^{-\epsilon}$ holds for all sufficiently large n, we have that

$$\mathbf{MSE}\{\hat{\lambda}^{\diamond}_{n,K}(s)\} = \frac{5}{4}\left\{\tau\lambda(s)\int_{-1}^{1} K^2(x)dx\right\}^{4/5} \left\{\lambda''(s)\int_{-1}^{1} x^2 K(x)dx\right\}^{2/5} \frac{1}{|W_n|^{4/5}} + o\left(\frac{1}{|W_n|^{4/5}}\right). \tag{1.23}$$

2. Discussion: a connection with the classical density estimation

The formulas (1.15), (1.20), (1.22) and (1.23) closely resemble the corresponding ones in the classical kernel-type density estimation. To demonstrate this we now construct an artificial density function f as follows:

$$f(s) := \begin{cases} \frac{1}{\theta\tau}\lambda(s), & s \in [0,\tau]; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\theta := \frac{1}{\tau}\int_0^\tau \lambda(s)ds.$$

For the sake of argument, we assume that both the period τ and the parameter θ are known. (This is an unrealistic assumption from the practical point of view but convenient to demonstrate the connection between the results of this paper and those in the classical area of kernel-type density estimation.) Under these assumptions, the quantity

$$\hat{f}_{n,K}(s) := \frac{1}{\theta\tau}\hat{\lambda}^{\diamond}_{n,K}(s)$$

can be viewed as an estimate of $f(s)$.

Allying (1.15) in the just described situation, we obtain

$$\begin{aligned}\mathbf{E}\hat{f}_{n,K}(s) &= \frac{1}{\theta\tau}\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s)\\ &= \frac{1}{\theta\tau}\lambda(s) + \frac{f''(s)\theta\tau}{2\theta\tau}h_n^2\int_{-1}^{1}x^2K(x)dx + o(h_n^2) + O\left(\frac{1}{|W_n|}\right)\\ &= f(s) + \left[\frac{f''(s)}{2}h_n^2\int_{-1}^{1}x^2K(x)dx\right] + o(h_n^2) + O\left(\frac{1}{|W_n|}\right). \qquad (2.1)\end{aligned}$$

Note that the term in brackets $[\cdot]$ on the right-hand side of (2.1) is the same as the well-known formula for the asymptotic bias in the classical kernel-type density estimation.

Applying (1.20) in the above described situation, we obtain the following formula

$$\begin{aligned}\mathbf{Var}\{\hat{f}_{n,K}(s)\} &= \mathbf{Var}\{\frac{1}{\theta\tau}\hat{\lambda}^{\diamond}_{n,K}(s)\}\\ &= \frac{1}{(\theta\tau)^2}\frac{\tau f(s)(\theta\tau)}{|W_n|h_n}\int_{-1}^{1}K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right)\\ &= \frac{f(s)}{\theta|W_n|h_n}\int_{-1}^{1}K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right). \qquad (2.2)\end{aligned}$$

Note that since λ is periodic, $\mathbf{E}X(W_n)$ is approximately $\theta|W_n|$. Hence, it is appropriate to compare $\theta|W_n|$ in the context of the current paper with the sample size N in the context of kernel-type density estimation. Therefore, replacing $\theta|W_n|$ on the right-hand side of (2.2) by N, we reduce the right-hand side of (2.2) to the following well-known expression for the variance in the kernel density estimation:

$$\mathbf{Var}\{\hat{f}_{n,K}(s)\} = \frac{1}{Nh_n}f(s)\int_{-1}^{1}K^2(x)dx + o\left(\frac{1}{Nh_n}\right). \qquad (2.3)$$

Combining (2.1) and (2.3), we obtain the corresponding formulas for $\mathbf{MSE}\{\hat{f}_{n,K}(s)\}$, which are in parallel to the corresponding ones in the classical area of the kernel density estimation.

3. Proofs

We note at the outset that, instead of the assumption of Section 1 requiring the kernel K to have only a finite number of discontinuities, in the current section we assume the following, somewhat weaker assumption.

Assumption 3.1 *For any $\alpha > 0$, there exists a finite collection of disjoint compact intervals B_1,..., B_{M_α} and a continuous function $K_\alpha : \mathbf{R} \to \mathbf{R}$ such that*

i) the Lebesgue measure of the set $[-1,1] \setminus \cup_{i=1}^{M_\alpha} B_i$ does not exceed α, and *ii)* $|K(u) - K_\alpha(u)| \le \alpha$ *for all* $u \in \cup_{i=1}^{M_\alpha} B_i$.

We note that it is easy to construct a kernel K such that Assumption 3.1 is satisfied but the original one on K of Section 1 is not. Assumption 3.1, just like the original one on K, is intended for controlling the fluctuations of the function

$$x \mapsto K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)$$

depending on the fluctuations of $\hat{\tau}_n$ around τ.

In what follows, we prove Theorems 1.2–1.5. The technical tools we are using for proving the four theorems are similar, and so are the proofs. To avoid repetition as much as possible, we thus give a very detail proof of Theorem 1.2. The proofs of the remaining three Theorems 1.3–1.5 are therefore sketchy, often referring to the proof of Theorem 1.2 for hints and further detail. In order to make the hints and other detail more useful and transparent, we thus have presented more detail in the proof of Theorem 1.2 than it would otherwise be necessary for the sake of proving only the theorem itself.

3.1 Proof of Theorem 1.2

Denote

$$A_n := \left\{ |\hat{\tau}_n - \tau| \le \frac{\delta h_n}{|W_n|} \right\}. \tag{3.1}$$

With this notation, we have the following representation

$$\begin{aligned} \mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s) &= \mathbf{E}\big(\mathbf{I}\{\hat{\lambda}_{n,K}(s) \le D_n\}\hat{\lambda}_{n,K}(s)\big) \\ &= \Gamma_n(1) - \Gamma_n(2) + \Gamma_n(3), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \Gamma_n(1) &:= \mathbf{E}\big((1-\mathbf{I}\{A_n\})\mathbf{I}\{\hat{\lambda}_{n,K}(s) \le D_n\}\hat{\lambda}_{n,K}(s)\big), \\ \Gamma_n(2) &:= \mathbf{E}\big(\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}\hat{\lambda}_{n,K}(s)\big), \\ \Gamma_n(3) &:= \mathbf{E}\big(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)\big). \end{aligned}$$

Obviously, Theorem 1.2 follows if we demonstrate that $\Gamma_n(1)$ and $\Gamma_n(2)$ can be made as small as desired, and $\Gamma_n(3)$ can be made as close to $\lambda(s)$ as desired, by taking n sufficiently large and/or $\delta > 0$ sufficiently small. Before proving these results, we note in passing that if assumption (1.13) holds (which is a stronger requirement than assumed in Theorem 1.2), then the set A_n has probability 1. In this case, the quantity $\Gamma_n(1)$ equals to 0, $\Gamma_n(2)$ can also be made 0 by choosing $D_n = +\infty$. Therefore, $\Gamma_n(3) = \mathbf{E}\hat{\lambda}_{n,K}(s)$, and we thus only have to verify the statement $\Gamma_n(3) \to \lambda(s)$ in order to prove Theorem 1.2.

The quantity $\Gamma_n(1)$ can obviously be estimated as follows:

$$\Gamma_n(1) \le D_n \mathbf{P}\left\{ \frac{|W_n|}{h_n} |\hat{\tau}_n - \tau| \ge \delta \right\}. \tag{3.3}$$

Due to assumption (1.10), the right-hand side of (3.3) converges to 0. This proves that $\lim_{n\to\infty} \Gamma_n(1) = 0$ for any fixed $\delta > 0$. The same statement holds for the quantity $\Gamma_n(2)$, as we will now demonstrate. We start with the elementary bound:

$$\Gamma_n(2) \le \frac{1}{D_n}\mathbf{E}\big(\mathbf{I}\{A_n\}\{\hat{\lambda}_{n,K}(s)\}^2\big). \tag{3.4}$$

Since $D_n \to \infty$, the desired result follows if the expectation $\mathbf{E}(\mathbf{I}\{A_n\}\{\hat{\lambda}_{n,K}(s)\}^2)$ is asymptotically bounded. The latter statement follows from statement (3.88) below, and we thus take it now for granted. (We note that the proof of (3.88) does not require assumption (1.16), which is stronger than (1.10) assumed in the current proof.) In view of the observations above, we complete the proof of Theorem 1.2 if we demonstrate that the quantity

$$\limsup_{n\to\infty} |\Gamma_n(3) - \lambda(s)| \tag{3.5}$$

can be made as small as desired by taking $\delta > 0$ sufficiently small. We start the proof of this result with the following elementary representation:

$$\Gamma_n(3) = \Lambda_n(1) + \Lambda_n(2) + \Lambda_n(3), \tag{3.6}$$

where

$$\Lambda_n(1) := \mathbf{E}\Big\{\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s) - \frac{\tau}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\Big\},$$

$$\Lambda_n(2) := \frac{\tau}{|W_n|h_n}\mathbf{E}\Big\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\Big\},$$

$$\Lambda_n(3) := \mathbf{E}\Big\{\mathbf{I}\{A_n\}\frac{\tau}{|W_n|h_n}\int_{W_n}\sum_{k=-\infty}^{\infty} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\Big\}.$$

In Lemmas 3.1–3.3 below, we prove that $\Lambda_n(1)$ and $\Lambda_n(2)$ can be made as small as desired, and also $\Lambda_n(3)$ can be made as close to $\lambda(s)$ as desired, by taking n sufficiently large and the parameters $\alpha > 0$ (cf. Assumption 3.1) and $\delta > 0$ sufficiently small.

Lemma 3.1 *We have that* $\lim_{n\to\infty}\Lambda_n(1) = 0$ *for any fixed* $\delta > 0$.

Proof. We start the proof with the representation

$$|\Lambda_n(1)| = \mathbf{E}\Big(\mathbf{I}\{A_n\}\{1 - \frac{\tau}{\hat{\tau}_n}\}\hat{\lambda}_{n,K}(s)\Big). \tag{3.7}$$

We shall now estimate $\hat{\lambda}_{n,K}(s)$ from above and then use the obtained bound on the right-hand side of (3.7) to finish the proof of Lemma 3.1. Note first that if $\hat{\tau}_n = 0$, then $\hat{\lambda}_{n,K}(s) = 0$ and, in turn, $\hat{\lambda}^{\diamond}_{n,K}(s) = 0$ only. Thus, we can always restrict our considerations to the event $\hat{\tau}_n > 0$. With this observation at hand, and using the fact that the kernel K is bounded and has support in $[-1,1]$, we obtain that

$$\begin{aligned}
\hat{\lambda}_{n,K}(s) &\le c\frac{\hat{\tau}_n}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}\mathbf{I}\Big\{\frac{x-(s+k\hat{\tau}_n)}{h_n}\in[-1,1])\Big\}X(dx)\\
&= c\frac{\hat{\tau}_n}{|W_n|h_n}\int_{W_n}\sum_{k=-\infty}^{\infty}\mathbf{I}\Big\{\frac{x-s}{\hat{\tau}_n}+k\in\frac{h_n}{\hat{\tau}_n}[-1,1])\Big\}X(dx)\\
&\le c\frac{\hat{\tau}_n}{|W_n|h_n}\sup_{z\in\mathbf{R}}\Big(\sum_{k=-\infty}^{\infty}\mathbf{I}\Big\{z+k\in\frac{h_n}{\hat{\tau}_n}[-1,1])\Big\}\Big)X(W_n). \qquad (3.8)
\end{aligned}$$

Note the following easy-to-check bound

$$\sup_{z\in\mathbf{R}}\Big(\sum_{k=-\infty}^{\infty}\mathbf{I}\Big\{z+k\in\rho[-1,1])\Big\}\Big) \le 2|\rho|+1 \tag{3.9}$$

that holds for any real number ρ. Applying (3.9) on the right-hand side of (3.8), we obtain that

$$\hat{\lambda}_{n,K}(s) \le c\bigg\{\frac{\hat{\tau}_n}{h_n} + 1\bigg\}\frac{X(W_n)}{|W_n|}. \tag{3.10}$$

Using the bound $|\hat{\tau}_n - \tau| \le (\delta h_n)/|W_n|$ together with the assumptions $|W_n| \to \infty$ and $h_n \to 0$, we obtain from (3.10) that, for sufficiently large n,

$$\begin{aligned}\hat{\lambda}_{n,K}(s) &\le c\bigg\{\frac{\tau}{h_n} + \frac{\delta}{|W_n|} + 1\bigg\}\frac{X(W_n)}{|W_n|} \\ &\le c\bigg\{\frac{\tau}{h_n} + 1\bigg\}\frac{X(W_n)}{|W_n|} \\ &\le \frac{c}{h_n}\frac{X(W_n)}{|W_n|},\end{aligned} \tag{3.11}$$

where the value of constant c may differ from place to place. Applying now (3.11) on the right-hand side of (3.7), we obtain, for all sufficiently large n,

$$\begin{aligned}|\Lambda_n(1)| &\le \frac{c}{h_n}\mathbf{E}\bigg(\mathbf{I}\{A_n\}\{1 - \frac{\tau}{\hat{\tau}_n}\}\frac{X(W_n)}{|W_n|}\bigg) \\ &= \frac{c}{h_n}\mathbf{E}\bigg(\mathbf{I}\{A_n\}\frac{\hat{\tau}_n - \tau}{\tau + (\hat{\tau}_n - \tau)}\frac{X(W_n)}{|W_n|}\bigg) \\ &\le c\frac{\delta}{|W_n|}\bigg(\frac{1}{\tau+1}\bigg)\mathbf{E}\bigg(\frac{X(W_n)}{|W_n|}\bigg) \\ &\le c\frac{\delta}{|W_n|},\end{aligned} \tag{3.12}$$

where the last inequality of (3.12) was obtained using, with $p = 1$, the following statement

$$\lim_{n\to\infty} \mathbf{E}\bigg\{\frac{X(W_n)}{|W_n|}\bigg\}^p < \infty \tag{3.13}$$

that holds for any $p \ge 1$. (We shall frequently use the latter bound with different values of p in proofs below.) Since $|W_n| \to \infty$ by assumption, inequality (3.12) completes the proof of Lemma 3.1. □

Lemma 3.2 *By choosing $\alpha > 0$ and $\delta > 0$ sufficiently small, we can make the quantity* $\limsup_{n\to\infty} \Lambda_n(2)$ *as small as desired.*

Proof. Fix an $\alpha > 0$ and denote

$$A_\alpha := \bigcup_{i=1}^{M_\alpha} B_i, \tag{3.14}$$

where B_1,..., $B_{M_\alpha} \subset [-1, 1]$ are the disjoint compact intervals defined in Assumption 3.1. By the Weierstrass theorem, there exists a Lipschitz function L_α, defined on the whole real line $\mathbf{R}$, such that the bound

$$|K(u) - L_\alpha(u)| \le \alpha \tag{3.15}$$

holds for all $u \in A_\alpha$. Using $L_\alpha(u)$, we decompose $K(u)$ for any $u \in \mathbf{R}$ as follows:

$$\begin{aligned}K(u) &= \{K(u) - L_\alpha(u)\} + L_\alpha(u) \\ &= \{K(u) - L_\alpha(u)\}\mathbf{I}(u \in \mathbf{R} \setminus A_\alpha) + \{K(u) - L_\alpha(u)\}\mathbf{I}(u \in A_\alpha) + L_\alpha(u).\end{aligned} \tag{3.16}$$

Since supp$(K) \subset [-1,1]$, we assume without loss of generality that supp$(L_\alpha) \subset [-1,1]$. Consequently, decomposition (3.16) reduces to the following one

$$K(u) = \{K(u) - L_\alpha(u)\}\mathbf{I}(u \in [-1,1] \setminus A_\alpha) + \{K(u) - L_\alpha(u)\}\mathbf{I}(u \in A_\alpha) + L_\alpha(u). \quad (3.17)$$

Using decomposition (3.17), we obtain that

$$\Lambda_n(2) = \tau\big\{\Lambda_n^*(2) + \Lambda_n^{**}(2) + \Lambda_n^{**}(2)\big\}, \quad (3.18)$$

where

$$\Lambda_n^*(2) := \frac{1}{|W_n|h_n}\mathbf{E}\bigg\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} \bigg[(K-L_\alpha)\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\setminus A_\alpha\right\} - (K-L_\alpha)\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\setminus A_\alpha\right\}\bigg]X(dx)\bigg\},$$

$$\Lambda_n^{**}(2) := \frac{1}{|W_n|h_n}\mathbf{E}\bigg\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} \bigg[(K-L_\alpha)\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in A_\alpha\right\} - (K-L_\alpha)\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in A_\alpha\right\}\bigg]X(dx)\bigg\},$$

$$\Lambda_n^{***}(2) := \frac{1}{|W_n|h_n}\mathbf{E}\bigg\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\bigg[L_\alpha\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right)\bigg]X(dx)\bigg\}.$$

In the following three statements we prove that the quantities $\Lambda_n^*(2)$, $\Lambda_n^{**}(2)$, and $\Lambda_n^{***}(2)$ can be made as small as desired by appropriately choosing n, δ and α.

Statement 3.1 *By choosing $\alpha > 0$ and $\delta > 0$ sufficiently small, we can make the quantity* $\limsup_{n\to\infty} \Lambda_n^*(2)$ *as small as desired.*

Proof. We start the proof of Statement 3.1 with the note that both functions K and L_α are bounded by a finite constant c that does not depend on α. Therefore,

$$\Lambda_n^*(2) \le c\{\Psi_n^\circ + \Psi_n^{\circ\circ}\}, \quad (3.19)$$

where

$$\Psi_n^\circ := \frac{1}{|W_n|h_n}\mathbf{E}\bigg(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\setminus A_\alpha\right\}X(dx)\bigg),$$

$$\Psi_n^{\circ\circ} := \frac{1}{|W_n|h_n}\mathbf{E}\bigg(\sum_{k=-\infty}^{\infty}\int_{W_n}\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\setminus A_\alpha\right\}X(dx)\bigg).$$

The following proof of Statement 3.1 is subdivided into two Propositions 3.1 and 3.2 where we prove that both $\Psi_n^{\circ\circ}$ and Ψ_n° can be made as small as desired by appropriately choosing n, δ and α.

Proposition 3.1 *By choosing the parameter $\alpha > 0$ sufficiently small, we can make the quantity $\sup_{\delta>0} \limsup_{n\to\infty} \Psi_n^{\circ\circ}$ as small as desired.*

Proof. We rewrite $\Psi_n^{\circ\circ}$ in the following, equivalent form:

$$\Psi_n^{\circ\circ} = \frac{1}{|W_n| h_n} \sum_{k=-\infty}^{\infty} \mathbf{E}X\left(\{s + k\tau + h_n([-1,1] \setminus A_\alpha)\} \cap W_n\right). \tag{3.20}$$

Let $\mathcal{K}$ be the set of $k \in \mathbf{N}$ such that $\{s + k\tau + h_n([-1,1])\} \cap W_n \neq \emptyset$. Obviously, the number $\kappa_n := \mathrm{card}\{\mathcal{K}\}$ of elements in the set $\mathcal{K}$ satisfies the following, approximate equality:

$$\kappa_n \approx |W_n|, \tag{3.21}$$

when $n \to \infty$. Since $X(\emptyset) = 0$, there are therefore approximately κ_n non-zero summands on the right-hand side of (3.20). With the observations above, we proceed with the estimation of $\Psi_n^{\circ\circ}$ as follows:

$$\begin{aligned}
\Psi_n^{\circ\circ} &= \frac{1}{|W_n| h_n} \sum_{k\in\mathcal{K}} \mathbf{E}X\left(\{s + k\tau + h_n([-1,1] \setminus A_\alpha)\} \cap W_n\right) && (3.22)\\
&\le \frac{1}{|W_n| h_n} \sum_{k\in\mathcal{K}} \mathbf{E}X\left(s + k\tau + h_n([-1,1] \setminus A_\alpha)\right)\\
&= \frac{\kappa_n}{|W_n| h_n} \mathbf{E}X\left(s + h_n([-1,1] \setminus A_\alpha)\right)\\
&\le \frac{c\kappa_n}{|W_n| h_n}\ \mathrm{Leb}(h_n([-1,1] \setminus A_\alpha)), && (3.23)
\end{aligned}$$

where $\mathrm{Leb}(\mathcal{B})$ denotes the Lebesgue measure of a set $\mathcal{B} \subset \mathbf{R}$. By assumption, the Lebesgue measure of the set $[-1,1] \setminus A_\alpha$ does not exceed α. Thus, we have the bound

$$\mathrm{Leb}(h_n([-1,1] \setminus A_\alpha) \le h_n\alpha. \tag{3.24}$$

Due to (3.21), (3.23) and (3.24), there exists a constant c (not depending on n, δ and α) such that the bound

$$\Psi_n^{\circ\circ} \le c\alpha \tag{3.25}$$

holds. This completes the proof of Proposition 3.1. □

Before proceeding with Proposition 3.2 below, we note that the main difference between the quantities Ψ_n° and $\Psi_n^{\circ\circ}$ is the presence of the estimator $\hat{\tau}_n$, instead of τ, in each summand of Ψ_n°. Since we restrict ourselves to the event A_n only, we can therefore replace $\hat{\tau}_n$ by τ in each summand of Ψ_n° and, in this way, reduce the proof of Proposition 3.2 above to that of Proposition 3.1 below.

Proposition 3.2 *By choosing $\alpha > 0$ and $\delta > 0$ sufficiently small, we can make the quantity $\limsup_{n\to\infty} \Psi_n^{\circ}$ as small as desired.*

Proof. We start the proof with the following bound:

$$\begin{aligned}
\Psi_n^{\circ} &= \frac{1}{|W_n| h_n} \mathbf{E}\Bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} X\left(\{s + k\hat{\tau}_n + h_n([-1,1] \setminus A_\alpha)\} \cap W_n\right)\Bigg)\\
&\le \frac{1}{|W_n| h_n} \sum_{k=-\infty}^{\infty} \mathbf{E}X\left(\left\{s + k\tau + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n([-1,1] \setminus A_\alpha)\right\} \cap W_n\right). && (3.26)
\end{aligned}$$

Let $\mathcal{K}$ be the set of $k \in \mathbf{N}$ such that $\{s + k\tau + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n([-1,1] \setminus A_\alpha)\} \cap W_n \neq \emptyset$. Note that the number $\kappa_n = \text{card}\{\mathcal{K}\}$, which may be different from that in the proof of Proposition 3.1, is such that the asymptotic relationship (3.21) holds. Applying these facts on the right-hand side of (3.26), we obtain the bounds:

$$\begin{aligned} \Psi_n^\circ &\leq \frac{1}{|W_n|h_n} \sum_{k\in\mathcal{K}} \mathbf{E}X\left(\left\{s + k\tau + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n([-1,1] \setminus A_\alpha)\right\} \cap W_n\right) \\ &\leq \frac{1}{|W_n|h_n} \sum_{k\in\mathcal{K}} \mathbf{E}X\left(s + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n([-1,1] \setminus A_\alpha)\right) \\ &\leq \frac{\kappa_n}{|W_n|h_n} \mathbf{E}X\big(s + ch_n\delta[-1,1] + h_n([-1,1] \setminus A_\alpha)\big). \end{aligned} \tag{3.27}$$

(We note that the constant c on the right-hand side of (3.27) may depend on s.) The right-hand side of (3.27) does not exceed

$$\frac{c}{h_n}\text{Leb}\big(ch_n\delta[-1,1] + h_n([-1,1] \setminus A_\alpha)\big), \tag{3.28}$$

where the two constants c of (3.28) may be different, but both of them do not depend on n, α, and δ. Using the definition of the set A_α, we easily derive that there exist $c_1(\alpha)$ (possibly depending on α) and c_2 (not depending on α) such that the quantity of (3.28) does not exceed

$$\frac{c}{h_n}\left\{c_1(\alpha)\text{Leb}\big(ch_n\delta[-1,1]\big) + c_2\text{Leb}\big(h_n([-1,1] \setminus A_\alpha)\big)\right\}. \tag{3.29}$$

The quantity of (3.29) is asymptotically of order $c_1(\alpha)\delta + c_2\alpha$, which proves the following bound:

$$\Psi_n^\circ \leq c\big\{c_1(\alpha)\delta + c_2\alpha\big\}. \tag{3.30}$$

The right-hand side of (3.30) can be made as small as desired by first choosing α sufficiently small (this may result in the increase of $c_1(\alpha)$) and then choosing δ sufficiently small. This completes the proof of Proposition 3.2. □

Propositions 3.1 and 3.2 conclude the proof of Statement 3.1. □

Statement 3.2 *By choosing the parameter $\alpha > 0$ sufficiently small, we can make the quantity $\sup_{\delta>0} \limsup_{n\to\infty} \Lambda_n^{**}(2)$ as small as desired.*

Proof. Using bound (3.15), we obtain that

$$\Lambda_n^{**}(2) \leq c\alpha\big\{\Psi_n^* + \Psi_n^{**}\big\}, \tag{3.31}$$

where the constant c does not depend on n, α, and δ, and

$$\begin{aligned} \Psi_n^* &:= \frac{1}{|W_n|h_n}\mathbf{E}\left\{\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} X\left(\{s + k\hat{\tau}_n + h_nA_\alpha\} \cap W_n\right)\right\}, \\ \Psi_n^{**} &:= \frac{1}{|W_n|h_n}\mathbf{E}\left\{\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_nA_\alpha\} \cap W_n\right)\right\}. \end{aligned}$$

Following the lines of the proof of Proposition 3.1, but this time with the set A_α instead of $[-1,1]\setminus A_\alpha$, we obtain that

$$\Psi_n^{**} \leq c, \tag{3.32}$$

where the constant c does not depend on n, α, and δ. If we follow the lines of the proof of Proposition 3.2 with A_α instead of $[-1,1]\setminus A_\alpha$, we obtain that

$$\Psi_n^* \le c, \tag{3.33}$$

where c does not depend on n, α, and δ. Taking now (3.31), (3.32), and (3.33) together, we complete the proof of Statement 3.2. □

Statement 3.3 *We have that* $\limsup_{n\to\infty}\Lambda_n^{***}(2)=0$ *for any* $\alpha>0$ *and* $\delta>0$.

Proof. Let

$$\hat u := \frac{x-(s+k\hat\tau_n)}{h_n}, \quad u := \frac{x-(s+k\tau)}{h_n}.$$

Since the support of the function L_α is in the interval $[-1,1]$, the difference $L_\alpha(\hat u)-L_\alpha(u)$ can be decomposed in the following way:

$$\begin{aligned} L_\alpha(\hat u)-L_\alpha(u) &= \big\{L_\alpha(\hat u)-L_\alpha(u)\big\}\mathbf{I}\{\hat u\in[-1,1]\} \\ &\quad + L_\alpha(u)\big(\mathbf{I}\{\hat u\in[-1,1]\}-\mathbf{I}\{u\in[-1,1]\}\big). \end{aligned} \tag{3.34}$$

Using decomposition (3.34), we decompose $\Lambda_n^{***}(2)$ as follows:

$$\Lambda_n^{***}(2)=\Delta_n(1)+\Delta_n(2), \tag{3.35}$$

where

$$\begin{aligned} \Delta_n(1) &:= \frac{1}{|W_n|h_n}\mathbf{E}\Bigg(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\Bigg[L_\alpha\left(\frac{x-(s+k\hat\tau_n)}{h_n}\right) \\ &\qquad - L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right)\Bigg]\mathbf{I}\left\{\frac{x-(s+k\hat\tau_n)}{h_n}\in[-1,1]\right\}X(dx)\Bigg), \\ \Delta_n(2) &:= \frac{1}{|W_n|h_n}\mathbf{E}\Bigg(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right) \\ &\qquad \times\left[\mathbf{I}\left\{\frac{x-(s+k\hat\tau_n)}{h_n}\in[-1,1]\right\}-\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n}\in[-1,1]\right\}\right]X(dx)\Bigg). \end{aligned}$$

In Propositions 3.3 and 3.4 below we shall prove that the quantities $\Delta_n(1)$ and $\Delta_n(2)$ converge to 0 when $n\to\infty$. In view of decomposition (3.35), the two propositions will completes proof of Statement 3.3.

Proposition 3.3 *We have that, for any fixed* $\alpha>0$, *the quantity* $\lim_{n\to\infty}\Delta_n(1)$ *can be made as small as desired by taking* $\delta>0$ *sufficiently small.*

Proof. Since L_α is a Lipschitz function, there exists a constant $c(\alpha)$ (possibly converging to ∞ when $\alpha\to 0$) such that

$$|L_\alpha(u)-L_\alpha(v)|\le c(\alpha)|u-v| \tag{3.36}$$

for all $u,v\in\mathbf{R}$. Using (3.36), we obtain the bound

$$\begin{aligned} \Delta_n(1) &\le \frac{1}{|W_n|h_n}\mathbf{E}\Bigg\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}c(\alpha)\left|\frac{k(\hat\tau_n-\tau)}{h_n}\right|\mathbf{I}\left\{\frac{x-(s+k\hat\tau_n)}{h_n}\in[-1,1]\right\}X(dx)\Bigg\} \\ &= \frac{c(\alpha)}{|W_n|h_n}\mathbf{E}\Bigg\{\mathbf{I}\{A_n\}\frac{|\hat\tau_n-\tau|}{h_n}\sum_{k=-\infty}^{\infty}kX\left(\{s+k\hat\tau_n+h_n[-1,1]\}\cap W_n\right)\Bigg\}. \end{aligned} \tag{3.37}$$

As we did in the proof of Statement 3.1, we replace the infinite sum $\sum_{k=-\infty}^{\infty}$ on the right-hand side of (3.37) by the finite one $\sum_{k\in\mathcal{K}}$, where the number κ_n of elements in $\mathcal{K}$ satisfies the asymptotic relationship $\kappa_n \approx |W_n|$. After this replacement, we estimate k on the right-hand side of (3.37) by $c|W_n|$, where the constant c possibly depends on s but not on n, δ, or α. Furthermore, we estimate $|(\hat{\tau}_n - \tau)/h_n|$ on the right-hand side of (3.37) by $\delta/|W_n|$, which we can do because of the indicator $\mathbf{I}\{A_n\}$. Consequently, obtain the following bound:

$$\Delta_n(1) \le c(\alpha)\delta \frac{1}{|W_n|h_n}\mathbf{E}\Big\{\mathbf{I}\{A_n\}\sum_{k\in\mathcal{K}} X\left(\{s+k\hat{\tau}_n + h_n[-1,1]\}\cap W_n\right)\Big\}. \tag{3.38}$$

Following now the lines of the proof of Proposition 3.2 with $[-1,1]$ instead of $[-1,1]\setminus A_\alpha$, we obtain that

$$\frac{1}{|W_n|h_n}\mathbf{E}\Big\{\mathbf{I}\{A_n\}\sum_{k\in\mathcal{K}} X\left(\{s+k\hat{\tau}_n + h_n[-1,1]\}\cap W_n\right)\Big\} \le c, \tag{3.39}$$

where c does not depend on n, α, and δ. Applying bound (3.39) on the right-hand side of (3.38) we obtain the following one:

$$\Delta_n(1) \le c(\alpha)\delta. \tag{3.40}$$

Thus, for any fixed $\alpha > 0$, taking $\delta > 0$ sufficiently small, we make the quantity $\limsup_{n\to\infty}\Delta_n(1)$ as small as desired. This concludes the proof of Propositions 3.3. □

Proposition 3.4 *The quantity* $\sup_{\alpha>0}\limsup_{n\to\infty}\Delta_n(2)$ *can be made as small as desired by taking* $\delta > 0$ *sufficiently small.*

Proof. We first rewrite the difference

$$\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n}\in[-1,1]\right\} - \mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n}\in[-1,1]\right\}$$

in the definition of $\Delta_n(2)$ in the following equivalent, but more convenient in our subsequent considerations, form:

$$\mathbf{I}\big\{x-(s+k\tau)\in h_n[-1,1]-k(\hat{\tau}_n-\tau)\big\} - \mathbf{I}\big\{x-(s+k\tau)\in h_n[-1,1]\big\}. \tag{3.41}$$

The quantity $k(\hat{\tau}_n-\tau)$ inside the first indicator of (3.41) can be estimated as follows. First, due to the presence of the indicator $\mathbf{I}\{A_n\}$ in the definition of $\Delta_n(2)$, we have that

$$k(\hat{\tau}_n-\tau) \le \delta k h_n/|W_n|. \tag{3.42}$$

As in the proof of Proposition 3.3, we replace the sum $\sum_{k=-\infty}^{\infty}$ in the definition of $\Delta_n(2)$ by $\sum_{k\in\mathcal{K}}$, where number κ_n of elements in the set $\mathcal{K}$ satisfying the asymptotic relationship (3.21). Thus, the number k on the right-hand side of (3.42) can be estimated by $c|W_n|$, where the constant c possibly depends on s but not on n, δ, or α. This implies that the absolute value of the difference between two indicators in (3.41) does not exceed

$$\mathbf{I}\big\{x-(s+k\tau)\in h_n[-1-c\delta,-1+c\delta]\big\} + \mathbf{I}\big\{x-(s+k\tau)\in h_n[1-c\delta,1+c\delta]\big\},$$

where the constant c does not depend on n, δ and α. The latter observation implies the following bound

$$\Delta_n(2) \le \Delta_n^*(2) + \Delta_n^{**}(2). \tag{3.43}$$

where

$$\Delta_n^*(2) := \frac{\tau}{|W_n|h_n}\mathbf{E}\biggl(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right) \times \mathbf{I}\bigl\{x-(s+k\tau)\in h_n[-1-c\delta,-1+c\delta]\bigr\}X(dx)\biggr),$$

$$\Delta_n^{**}(2) := \frac{\tau}{|W_n|h_n}\mathbf{E}\biggl(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right) \times \mathbf{I}\bigl\{x-(s+k\tau)\in h_n[1-c\delta,1+c\delta]\bigr\}X(dx)\biggr).$$

The estimation of the quantities $\Delta_n^*(2)$ and $\Delta_n^{**}(2)$ is similar to each other. Thus, we only estimate one of them, say, $\Delta_n^{**}(2)$. To start with, we recall that the function L_α is bounded by a constant c that does not depend on α. Therefore, the first inequality below:

$$\begin{aligned}
\Delta_n^{**}(2) &\le c\frac{1}{|W_n|h_n}\mathbf{E}\biggl(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\mathbf{I}\bigl\{x-(s+k\tau)\in h_n[1-c\delta,1+c\delta]\bigr\}X(dx)\biggr)\\
&\le c\frac{1}{|W_n|h_n}\mathbf{E}\biggl(\mathbf{I}\{A_n\}\sum_{k\in\mathcal{K}}\int_{W_n}\mathbf{I}\bigl\{x-(s+k\tau)\in h_n[1-c\delta,1+c\delta]\bigr\}X(dx)\biggr)\\
&\le c\frac{1}{|W_n|h_n}\sum_{k\in\mathcal{K}}\mathbf{E}\biggl(\int_{W_n}\mathbf{I}\bigl\{x-(s+k\tau)\in h_n[1-c\delta,1+c\delta]\bigr\}X(dx)\biggr)\\
&\le c\frac{1}{|W_n|h_n}\sum_{k\in\mathcal{K}}\mathbf{E}X\bigl(s+k\tau+h_n[1-c\delta,1+c\delta]\bigr)\\
&\le c\frac{1}{h_n}\mathbf{E}X\bigl(s+h_n[1-c\delta,1+c\delta]\bigr)\\
&\le c\delta,
\end{aligned} \tag{3.44}$$

where the value of the constant c may differ from line to line. Thus, by taking $\delta > 0$ sufficiently small, we make the quantity $\sup_{\alpha>0}\limsup_{n\to\infty}\Delta_n^{**}(2)$ as small as desired. Obviously now, the same statement can be proved for the quantity $\sup_{\alpha>0}\limsup_{n\to\infty}\Delta_n^{**}(1)$. These facts taken together with the bound (3.43) complete the proof of Proposition 3.4. □

Due to equality (3.35) and Propositions 3.3 and 3.4, the proof of Statement 3.3 is complete. Bound (3.18) and Statements 3.1, 3.2 and 3.3 complete the proof of Lemma 3.2. □

Lemma 3.3 *The statement* $\lim_{n\to\infty}\Lambda_n(3) = \lambda(s)$ *holds.*

Proof. We decompose $\Lambda_n(3)$ in the following way

$$\Lambda_n(3) = \Xi_n^* + \Xi_n^{**}, \tag{3.45}$$

where

$$\Xi_n^* := \frac{\tau}{|W_n|}\mathbf{E}\biggl\{\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\biggr\},$$

$$\Xi_n^{**} := \frac{\tau}{|W_n|}\mathbf{E}\biggl\{\bigl(1-\mathbf{I}\{A_n\}\bigr)\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n}K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\biggr\}.$$

In Statements 3.4 and 3.5 below, we shall demonstrate that $\Xi_n^* \to \lambda(s)$ and $\Xi_n^* \to 0$, which, in view of (3.45), will complete the proof of Lemma 3.3.

Statement 3.4 *We have that* $\lim_{n\to\infty} \Xi_n^* = \lambda(s)$.

Proof. We start the proof with the following equalities

$$\begin{aligned}\Xi_n^* &= \frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{\mathbf{R}} \mathbf{I}\{x \in W_n\} K\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x)dx \\ &= \frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \int_{\mathbf{R}} \mathbf{I}\left\{h_n x + s + k\tau \in W_n\right\} K(x)\, \lambda(h_n x + s + k\tau)dx \\ &= \frac{\tau}{|W_n|} \int_{\mathbf{R}} \left[\sum_{k=-\infty}^{\infty} \mathbf{I}\left\{h_n x + s + k\tau \in W_n\right\}\right] K(x)\, \lambda(h_n x + s)dx, \end{aligned} \tag{3.46}$$

where the last inequality of (3.46) holds due to the periodicity of λ. Since W_n is an interval, we have that, for any $z \in \mathbf{R}$,

$$\sum_{k=-\infty}^{\infty} \mathbf{I}\{z + k\tau \in W_n\} \in \left[\frac{|W_n|}{\tau} - 1, \frac{|W_n|}{\tau} + 1\right]. \tag{3.47}$$

Therefore, the right-hand side of (3.46) converges to

$$\int_{\mathbf{R}} K(x)\, \lambda(h_n x + s)dx. \tag{3.48}$$

Since the kernel K is a bounded probability density function and has support in $[-1,1]$, we obtain that

$$\begin{aligned}\int_{\mathbf{R}} K(x)\, \lambda(h_n x + s)dx &= \int_{\mathbf{R}} K(x) \left\{\lambda(h_n x + s) - \lambda(s)\right\}dx + \lambda(s) \\ &= \theta\left|\int_{\mathbf{R}} K(x) \left\{\lambda(h_n x + s) - \lambda(s)\right\}dx\right| + \lambda(s) \\ &= \theta\frac{c}{h_n} \int_{-h_n}^{h_n} \left|\lambda(x+s) - \lambda(s)\right|dx + \lambda(s), \end{aligned} \tag{3.49}$$

where $\theta \in [0,1]$ is some number. Since s is a Lebesgue point of λ, the first summand on the right-hand side of (3.49) (with θ in front of it) converges to 0. Consequently, the quantity of (3.48) converges to $\lambda(s)$. This completes the proof of Statement 3.4. □

Statement 3.5 *We have that* $\Xi_n^{**} \to 0$.

Proof. Using the Cauchy-Schwarz inequality, we have that

$$(\Xi_n^{**})^2 \le \mathbf{P}\left\{\frac{|W_n|}{\delta h_n} |\hat{\tau}_n - \tau| \ge 1\right\}\Pi_n, \tag{3.50}$$

where

$$\Pi_n := \mathbf{E}\left\{\frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right\}^2.$$

By assumption (1.10), the probability $\mathbf{P}\{\cdot\}$ on the right-hand side of (3.50) converges to 0 when $n \to \infty$, for any fixed $\delta > 0$. Therefore, in order to complete the proof of Statement 3.5, we need to demonstrate that the quantity

$$\Pi_n := \mathbf{E}\left\{\frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right\}^2$$

is asymptotically bounded. In fact, we shall demonstrate that

$$\Pi_n \to \lambda^2(s). \tag{3.51}$$

We start the proof of (3.51) with the note that, since $h_n \downarrow 0$ and the kernel K has support in $[-1,1]$, the random variables

$$\xi_k := \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx), \quad i = 1, 2, \ldots, \tag{3.52}$$

are independent for sufficiently large n. Therefore,

$$\Pi_n = \Pi_n^* - \Pi_n^{**} + \Pi_n^{***}, \tag{3.53}$$

where

$$\Pi_n^* := \frac{\tau^2}{|W_n|^2 h_n^2}\left\{\sum_{k=-\infty}^{\infty} \mathbf{E}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right\}^2,$$

$$\Pi_n^{**} := \frac{\tau^2}{|W_n|^2 h_n^2}\sum_{k=-\infty}^{\infty} \left\{\mathbf{E}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right\}^2,$$

$$\Pi_n^{***} := \frac{\tau^2}{|W_n|^2 h_n^2}\sum_{k=-\infty}^{\infty} \mathbf{E}\left\{\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right\}^2.$$

The following proof is subdivided into Propositions 3.5, 3.6, and 3.7 concerning the three quantities Π_n^*, Π_n^{**}, and Π_n^{***}.

Proposition 3.5 *We have that* $\lim_{n\to\infty} \Pi_n^* = \lambda^2(s)$.

Proof. Note that $\Pi_n^* = \{\Xi_n^*\}^2$, where Ξ_n^* is defined below (3.45). We proved in Statement 3.4 that $\Lambda_n(3) \to \lambda(s)$. Thus, Proposition 3.5 follows. □

Proposition 3.6 *We have that* $\lim_{n\to\infty} \Pi_n^{**} = 0$.

Proof. Since the kernel K is bounded and has support in $[-1,1]$, we obtain that

$$\Pi_n^{**} \le c\frac{1}{|W_n|^2 h_n^2}\sum_{k=-\infty}^{\infty} \left\{\mathbf{E}X(\{s+k\tau+h_n[-1,1]\}\cap W_n)\right\}^2. \tag{3.54}$$

Let $\mathcal{K}$ be the set of those $k \in \mathbf{N}$ such that $\{s+k\tau+h_n([-1,1])\}\cap W_n \neq \emptyset$. Obviously, the number κ_n of elements in the set $\mathcal{K}$ is such that the asymptotic relationship $\kappa_n \approx |W_n|$ holds. In view of this observation and the fact that $X(\emptyset) = 0$, we have that there are κ_n non-zero summands on the right-hand side of (3.54). Thus, inequality (3.54) implies that

$$\begin{aligned}\Pi_n^{**} &\le c\frac{1}{|W_n|^2 h_n^2}\sum_{k\in\mathcal{K}} \left\{\mathbf{E}X(\{s+k\tau+h_n[-1,1]\}\cap W_n)\right\}^2 \\ &\le c\frac{\kappa_n}{|W_n|^2 h_n^2}\left\{\mathbf{E}X(\{s+h_n[-1,1]\})\right\}^2 \\ &\le c\frac{1}{|W_n|}. \end{aligned}\tag{3.55}$$

The right-hand side of (3.55) converges to 0 since $|W_n| \to \infty$. This completes the proof of Proposition 3.6. □

Proposition 3.7 *We have that* $\lim_{n\to\infty} \Pi_n^{***} = 0$.

Since the kernel K is bounded and has support in $[-1,1]$, we obtain the bound

$$\Pi_n^{***} \leq c\frac{1}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \mathbf{E}\big\{X(\{s+k\tau+h_n[-1,1]\}\cap W_n)\big\}^2. \tag{3.56}$$

As in the proof of Proposition 3.6, we replace the infinite sum $\sum_{k=-\infty}^{\infty}$ on the right-hand side of (3.56) by the finite one $\sum_{k\in\mathcal{K}}$. Then, we obtain the bounds:

$$\begin{aligned}\Pi_n^{***} &\leq c\frac{1}{|W_n|^2 h_n^2} \sum_{k\in\mathcal{K}} \mathbf{E}\big\{X(\{s+k\tau+h_n[-1,1]\}\cap W_n)\big\}^2\\ &\leq c\frac{\kappa_n}{|W_n|^2 h_n^2}\mathbf{E}\big\{X(\{s+h_n[-1,1]\})\big\}^2\\ &\leq c\frac{1}{|W_n| h_n^2}\big\{h_n^2+h_n\big\}\\ &\leq c\frac{1}{|W_n| h_n}.\end{aligned} \tag{3.57}$$

The right-hand side of (3.57) converges to 0 since $|W_n|h_n \to \infty$. This completes the proof of Proposition 3.7. □

Propositions 3.5, 3.6, and 3.7 complete the proof of Statement 3.5. Statements 3.4 and 3.5 complete the proof of Lemma 3.3. Consequently, the proof of Theorem 1.2 is complete. □

3.2 Proof of Theorem 1.3

We closely follow the proof of Theorem 1.2, but with the set A_n defined now as follows:

$$A_n := \left\{ |\hat{\tau}_n - \tau| \leq \frac{\delta h_n^3}{|W_n|} \right\}. \tag{3.58}$$

As in the proof of Theorem 1.2, we use the following representation

$$\mathbf{E}\hat{\lambda}_{n,K}^{\diamond}(s) = \Gamma_n(1) - \Gamma_n(2) + \Gamma_n(3), \tag{3.59}$$

where the quantities $\Gamma_n(1)$, $\Gamma_n(2)$, and $\Gamma_n(3)$ are defined in the same way as those after (3.2), but now with the set A_n of (3.58). Note that due to the bound

$$\Gamma_n(1) \leq D_n \mathbf{P}\left\{ \frac{|W_n|}{h_n^3} |\hat{\tau}_n - \tau| \geq \delta \right\} \tag{3.60}$$

and assumption (1.14), the quantity $\Gamma_n(1)$ is of order $o(h_n^2)$. In order to prove that $\Gamma_n(2)$ is also of the same order, we start with the bound

$$\Gamma_n(2) \leq \frac{1}{D_n^r}\mathbf{E}\big(\mathbf{I}\{A_n\}\{\hat{\lambda}_{n,K}(s)\}^{r+1}\big) \tag{3.61}$$

that holds for any $r \geq 0$. Since $D_n \geq c/h_n^{\epsilon}$, we can always find a large $r \geq 0$ such that $1/D_n^r \leq o(h_n^2)$. This implies that $\Gamma_n(2) = o(h_n^2)$ provided that the quantity

$$\mathbf{E}(\mathbf{I}\{A_n\}\{\hat{\lambda}_{n,K}(s)\}^{r+1}) \tag{3.62}$$

is asymptotically bounded. In order to demonstrate this, we first replace the set A_n (which is defined in (3.58)) in quantity (3.62) by the set A_n defined in (3.1). Then, with some obvious modifications, we follow the proof of (3.88) (c.f., also the discussion concerning (3.87) below) and demonstrate that (3.62) is, indeed, asymptotically bounded..

In view of the observations above, the proof of Theorem 1.2 is completed if we demonstrate that

$$\Gamma_n(3) = \lambda(s) + \frac{\lambda''(s)}{2} h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2) + O\bigg(\frac{1}{|W_n|}\bigg). \tag{3.63}$$

(Recall that we have assumed $h_n^2|W_n| \to \infty$, which implies that $O(1/|W_n|) = o(h_n^2)$.) Just like in (3.6), we decompose $\Gamma_n(3)$ into the sum of $\Lambda_n(1)$, $\Lambda_n(2)$ and $\Lambda_n(3)$ defined below (3.6). The desired asymptotic statements concerning the three quantities are formulated in Lemmas 3.4, 3.5 and 3.6 below.

Lemma 3.4 *The statement* $\Lambda_n(1) = O(|W_n|^{-1})$ *holds.*

Proof. This is a verbatim repetition of the proof of Lemma 3.1. □

Lemma 3.5 *The quantity* $\limsup_{n\to\infty}\{h_n^{-2}\Lambda_n(2)\}$ *can be made as small as desired by taking* $\delta > 0$ *sufficiently small.*

Proof. Recall that

$$\Lambda_n(2) = \frac{\tau}{|W_n|h_n} \mathbf{E}\bigg\{ \mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K\left(\frac{x-(s+k\tau)}{h_n}\right)\right] X(dx) \bigg\}$$

By assumption, the kernel K satisfies the Lipschitz condition between the discontinuity points, say, $x_1 \le \cdots \le x_M$. Since the support of K is in the interval $[-1,1]$, we have that $-1 =: x_0 \le x_1 \le \cdots \le x_{M+1} := 1$. Thus, there exists $M+1$ subintervals $I_1, \ldots, I_{M+1} \subset [-1,1]$ such that $[-1,1] = \cup_{m=1}^{M+1} I_m$, and we decompose the kernel function K as follows:

$$K(x) = \sum_{m=1}^{M+1} K_m(x), \tag{3.64}$$

where $K_m(x) := K(x)\mathbf{I}\{x \in I_m\}$. For any $m \in \{1, \ldots, M+1\}$, let

$$\Lambda_n(2,m) := \frac{\tau}{|W_n|h_n} \mathbf{E}\bigg\{ \mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \left[K_m\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) \right. \\ \left. - K_m\left(\frac{x-(s+k\tau)}{h_n}\right)\right] X(dx) \bigg\}.$$

Statement 3.6 *For any* $m \in \{1, \ldots, M+1\}$, *the quantity* $\limsup_{n\to\infty}\{h_n^{-2}\Lambda_n(2,m)\}$ *can be made as small as desired by taking* $\delta > 0$ *sufficiently small.*

Proof. Let

$$\hat{u} := \frac{x-(s+k\hat{\tau}_n)}{h_n}, \quad u := \frac{x-(s+k\tau_n)}{h_n}$$

Since the support of K_m is in the interval I_m, we have that

$$\begin{aligned} K_m(\hat{u}) - K_m(u) = {} & K(\hat{u})\mathbf{I}\{\hat{u} \in I_m\}\big(1 - \mathbf{I}\{u \in I_m\}\big) \\ & + \big\{K(\hat{u}) - K(u)\big\}\mathbf{I}\{\hat{u} \in I_m\}\mathbf{I}\{u \in I_m\} \\ & + K(u)\mathbf{I}\{u \in I_m\}\big(\mathbf{I}\{\hat{u} \in I_m\} - 1\big). \end{aligned} \tag{3.65}$$

Since the function K is bounded and satisfies the Lipshitz condition on the interval I_m, we obtain from decomposition (3.65) that

$$\begin{aligned}|K_m(\hat{u}) - K_m(u)| \le\ & c\mathbf{I}\{\hat{u} \in I_m\}\mathbf{I}\{u \notin I_m\} \\ & + c|\hat{u} - u|\mathbf{I}\{\hat{u} \in I_m\}\mathbf{I}\{u \in I_m\} \\ & + c\mathbf{I}\{u \in I_m\}\mathbf{I}\{\hat{u} \notin I_m\}. \end{aligned} \tag{3.66}$$

Due to the presence of the indicator $\mathbf{I}\{A_n\}$ in the definition of $\Lambda_n(2,m)$, when estimating the random variable inside the expectation sign in $\Lambda_n(2,m)$ we assume without loss of generality that $|\hat{\tau}_n - \tau| \le \delta h_n^3/|W_n|$. Furthermore, we replace the infinite sum $\sum_{k=-\infty}^{\infty}$ in the definition of $\Lambda_n(2,m)$ by the sum $\sum_{k\in\mathcal{K}}$ with the (non-random) number κ_n of elements in the set $\mathcal{K}$ such that $\kappa_n \approx |W_n|$. In this way, we obtain the bound

$$\begin{aligned}|\hat{u} - u| &\le k\frac{|\hat{\tau}_n - \tau|}{h_n} \\ &\le k\frac{\delta}{|W_n|}h_n^2 \\ &\le ch_n^2, \end{aligned} \tag{3.67}$$

where the constant c may depend on s. Due to the bound (3.67), we obtain that if $\hat{u}$ is in I_m and u is outside of I_m, or the other way around, then both $\hat{u}$ and u are necessarily within the distance ch_n^2 from either the left-hand or right-hand end-point of the interval I_m. Let us assume for the sake of definiteness that $\hat{u}, u \in [x_m - ch_n^2, x_m + ch_n^2]$, in which case $\mathbf{I}\{\hat{u} \in I_m\}\mathbf{I}\{u \notin I_m\}$ and $\mathbf{I}\{u \in I_m\}\mathbf{I}\{\hat{u} \notin I_m\}$ do not exceed $\mathbf{I}\{u \in [x_m - ch_n^2, x_m + ch_n^2]\}$. We also have that $\mathbf{I}\{\hat{u} \in I_m\}\mathbf{I}\{u \in I_m\}$ does not exceed $\mathbf{I}\{u \in [-1,1]\}$. Applying these bounds on the right-hand side of (3.66), we obtain that

$$\begin{aligned}|K_m(\hat{u}) - K_m(u)| \le\ & c|\hat{u} - u|\mathbf{I}\{u \in [-1,1]\} \\ & + c\mathbf{I}\{u \in [x_m - ch_n^2, x_m + ch_n^2]\}. \end{aligned} \tag{3.68}$$

Using (3.68), we obtain the bound

$$\Lambda_n(2,m) \le c\big\{\Lambda_n^*(2,m) + \Lambda_n^{**}(2,m)\big\}, \tag{3.69}$$

where

$$\Lambda_n^*(2,m) := \frac{1}{|W_n|h_n}\mathbf{E}\left(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\left|\frac{k(\hat{\tau}_n - \tau)}{h_n}\right|\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\right\}X(dx)\right),$$

$$\Lambda_n^{**}(2,m) := \frac{1}{|W_n|h_n}\mathbf{E}\left(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [x_m - ch_n^2, x_m + ch_n^2]\right\}X(dx)\right).$$

The desired asymptotic properties of $\Lambda_n^*(2,m)$ and $\Lambda_n^{**}(2,m)$ are obtained in Propositions 3.8 and 3.9 below.

Proposition 3.8 *The quantity* $\limsup_{n\to\infty}\{h_n^{-2}\Lambda_n^*(2,m)\}$ *can be made as small as desired by taking* $\delta > 0$ *sufficiently small.*

Proof. The proof follows the lines of the proof of Proposition 3.3 and using δh_n^2 instead of δ. □

Proposition 3.9 *The quantity* $\limsup_{n\to\infty}\{h_n^{-2}\Lambda_n^{**}(2,m)\}$ *can be made as small as desired by taking* $\delta > 0$ *sufficiently small.*

Proof. The proof follows the lines of the proof of Proposition 3.4 and using δh_n^2 instead of δ. □

Due to inequality (3.69) and Propositions 3.8 and 3.9, the proof of Statement 3.6 is completed. This also completes proof of Lemma 3.5. □

Lemma 3.6 *We have that*

$$\Lambda_n(3) = \lambda(s) + \frac{\lambda''(s)}{2} h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2) + O\left(\frac{1}{|W_n|}\right). \tag{3.70}$$

Proof. We decompose $\Lambda_n(3)$ in the following way

$$\Lambda_n(3) = \Xi_n^* + \Xi_n^{**}, \tag{3.71}$$

where Ξ_n^* and Ξ_n^{**} are defined in the proof of Lemma 3.3 but now with the set A_n as in (3.58). We estimate Ξ_n^* and Ξ_n^* in Statements 3.7 and 3.8 below.

Statement 3.7 *We have that*

$$\Xi_n^* = \lambda(s) + \frac{\lambda''(s)}{2} h_n^2 \int_{-1}^{1} x^2 K(x)dx + o(h_n^2) + O\left(\frac{1}{|W_n|}\right). \tag{3.72}$$

Proof. We start the proof with the equalities

$$\begin{aligned} &\frac{\tau}{|W_n|}\mathbf{E}\bigg\{ \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\bigg\} \\ &\quad = \frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x) I(x \in W_n) dx \\ &\quad = \frac{\tau}{|W_n| h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \sum_{k=-\infty}^{\infty} \lambda(x+s+k\tau) \mathbf{I}(x+s+k\tau \in W_n) dx \\ &\quad = \frac{\tau}{|W_n| h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(x+s) \bigg[\sum_{k=-\infty}^{\infty} \mathbf{I}(x+s+k\tau \in W_n) \bigg] dx. \end{aligned} \tag{3.73}$$

Using bound (3.47), we obtain that the right-hand side of (3.73) equals

$$\left\{1 + \theta \frac{1}{|W_n|}\right\} \frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(x+s) dx \tag{3.74}$$

for some $|\theta| \le 1$. Using the Young's form of the Taylor theorem, we have that

$$\begin{aligned} \frac{1}{h_n} \int_{-h_n}^{h_n} K\left(\frac{x}{h_n}\right) \lambda(s+x) dx &= \int_{-1}^{1} K(x) \lambda(s + x h_n) dx \\ &= \lambda(s) + \lambda'(s) h_n \int_{-1}^{1} x K(x) dx \\ &\quad + \frac{\lambda''(s)}{2} h_n^2 \int_{-1}^{1} x^2 K(x) dx + o(h_n^2). \end{aligned} \tag{3.75}$$

Because K is symmetric around zero, we have that $\int_{-1}^{1} xK(x)dx = 0$. Therefore, the second term on the right-hand side of (3.75) equals 0, and thus the quantity in (3.74) equals the right-hand side of (3.72). This completes the proof of Statement 3.7. □

Statement 3.8 *We have that* $\Xi_n^{**} = o(h_n^2)$.

Proof. We start with the inequality

$$\Xi_n^{**} \le \big\{\mathbf{E}\big(1-\mathbf{I}\{A_n\}\big)^r\big\}^{1/r}\big\{\Upsilon_n(q)\big\}^{1/q}, \tag{3.76}$$

where $r, q > 1$ are such that $r^{-1} + q^{-1} = 1$, and

$$\Upsilon_n(q) := \mathbf{E}\bigg(\frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\bigg)^q.$$

By assumption (1.14), we have that

$$\begin{aligned}\mathbf{E}\big(1-\mathbf{I}\{A_n\}\big)^r &= \mathbf{P}\bigg\{\frac{|W_n|}{\delta h_n^3}\,|\hat{\tau}_n-\tau| \ge 1\bigg\}\\ &= o(h_n^{3+\epsilon})\end{aligned}$$

Therefore, by choosing $r > 1$ sufficiently close to 1, we obtain that

$$\big\{\mathbf{E}\big(1-\mathbf{I}\{A_n\}\big)^r\big\}^{1/r} = o(h_n^2). \tag{3.77}$$

Due to (3.77), the right-hand side of (3.76) converges to 0 faster than h_n^2 if

$$\Upsilon_n(q) \le c \tag{3.78}$$

for any sufficiently large $q > 1$. We shall actually prove that (3.78) holds for any $q > 2$. We have

$$\Upsilon_n(q) \le c\big\{Q_n(1) + Q_n(2)\big\}, \tag{3.79}$$

where

$$\begin{aligned}Q_n(1) &:= \mathbf{E}\bigg\{\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\bigg(\int_{W_n} K\bigg(\frac{x-(s+k\tau)}{h_n}\bigg)X(dx)\\ &\qquad\qquad - \mathbf{E}\int_{W_n} K\bigg(\frac{x-(s+k\tau)}{h_n}\bigg)X(dx)\bigg)\bigg\}^q,\\ Q_n(2) &:= \bigg\{\mathbf{E}\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n} K\bigg(\frac{x-(s+k\tau)}{h_n}\bigg)X(dx)\bigg\}^q.\end{aligned} \tag{3.80}$$

Note that $Q_n(2) = \{\Xi_n^*\}^q$, where Ξ_n^* is defined in Statement 3.4. According to Statement 3.4, $\Xi_n^* \to \lambda(s)$, and we thus have the desired statement that $Q_n(2)$ is bounded by a constant c that does not depend on n. Consequently, to conclude the proof of (3.78) we need to demonstrate that $Q_n(1)$ is also bounded by a constant c that does not depend on n. For this reason we first employ the classical von Bahr result (cf. Von Bahr, 1965) that implies, as a special case, the following inequality $\mathbf{E}|\sum \zeta_i|^q \le \{\sum \mathbf{Var}(\zeta_i)\}^{q/2}$ that holds for any sequence of independent random variables $\zeta_1, \zeta_2, \ldots$ having mean 0. Using this inequality, we obtain that the desired boundedness of $Q_n(1)$ follows if we demonstrate that

$$\sum_{k=-\infty}^{\infty}\mathbf{Var}\bigg\{\frac{1}{|W_n|h_n}\int_{W_n} K\bigg(\frac{x-(s+k\tau)}{h_n}\bigg)X(dx)\bigg\} \le c. \tag{3.81}$$

Statement (3.81), in turn, is a consequence of the following one

$$\sum_{k=-\infty}^{\infty} \mathbf{E}\Big\{ \frac{1}{|W_n| h_n} \int_{W_n} K\Big(\frac{x-(s+k\tau)}{h_n}\Big) X(dx) \Big\}^2 \leq c. \tag{3.82}$$

We already proved in Proposition 3.7 that the quantity of (3.82) converges to 0. This completes the proof of (3.78), and thus of Statement 3.8 as well. □

Statements 3.7 and 3.8 complete the proof of Lemma 3.6. This also completes the proof of Theorem 1.3. □

3.3 Proof of Theorem 1.4

We start with the equality

$$\mathbf{Var}\left\{ \hat{\lambda}^{\diamond}_{n,K}(s) \right\} = \mathbf{E}\left\{ \hat{\lambda}^{\diamond}_{n,K}(s) \right\}^2 - \left\{ \mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s) \right\}^2 . \tag{3.83}$$

By Theorem 1.2, the quantity $\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s)$ equals $\lambda(s) + o(1)$. Thus, in order to complete the proof of Theorem 1.4 we need to demonstrate that $\mathbf{E}\{\hat{\lambda}^{\diamond}_{n,K}(s)\}^2$ equals $\lambda^2(s) + o(1)$. As in the proof of Theorem 1.2, we use the same set A_n, that is,

$$A_n := \left\{ |\hat{\tau}_n - \tau| \leq \frac{\delta h_n}{|W_n|} \right\}.$$

We proceed with the following decomposition:

$$\begin{aligned} \mathbf{E}\left\{ \hat{\lambda}^{\diamond}_{n,K}(s) \right\}^2 &= \mathbf{E}\big(\mathbf{I}\{\hat{\lambda}_{n,K}(s) \leq D_n\} \left\{ \hat{\lambda}_{n,K}(s) \right\}^2 \big) \\ &= \Gamma_n(1) - \Gamma_n(2) + \Gamma_n(3), \end{aligned} \tag{3.84}$$

where

$$\begin{aligned} \Gamma_n(1) &:= \mathbf{E}\big((1-\mathbf{I}\{A_n\})\mathbf{I}\{\hat{\lambda}_{n,K}(s) \leq D_n\} \left\{ \hat{\lambda}_{n,K}(s) \right\}^2 \big), \\ \Gamma_n(2) &:= \mathbf{E}\big(\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\} \left\{ \hat{\lambda}_{n,K}(s) \right\}^2 \big) \\ \Gamma_n(3) &:= \mathbf{E}\big(\mathbf{I}\{A_n\} \left\{ \hat{\lambda}_{n,K}(s) \right\}^2 \big). \end{aligned}$$

Obviously, Theorem 1.4 follows from (3.84) if $\Gamma_n(1) \to 0$, $\Gamma_n(2) \to 0$, and $\Gamma_n(3) \to \lambda^2(s)$ when $n \to \infty$ and/or $\delta \to 0$. The proof that $\lim_{n\to\infty} \Gamma_n(1) = 0$ for any fixed $\delta > 0$ follows from the bound

$$\Gamma_n(1) \leq D_n^2 \mathbf{P}\Big\{ \frac{|W_n|}{h_n} |\hat{\tau}_n - \tau| \geq \delta \Big\} \tag{3.85}$$

and assumption (1.16). That proof that $\lim_{n\to\infty} \Gamma_n(2) = 0$ for any fixed $\delta > 0$ follows from the bound

$$\Gamma_n(2) \leq \frac{1}{D_n^2} \mathbf{E}\big(\mathbf{I}\{A_n\} \left\{ \hat{\lambda}_{n,K}(s) \right\}^4 \big), \tag{3.86}$$

provided that the statement

$$\lim_{n\to\infty} \mathbf{E}\big(\mathbf{I}\{A_n\} \left\{ \hat{\lambda}_{n,K}(s) \right\}^4 \big) < \infty \tag{3.87}$$

holds true. The proof of (3.87) is not trivial, though it very closely resembles the proof of the statement that the quantity

$$\limsup_{n\to\infty} |\Gamma_n(3) - \lambda(s)| \tag{3.88}$$

can be made as small as desired by taking $\delta > 0$ sufficiently small, the fact that we need to verify in order to complete the proof of Theorem 1.4. In view of the letter observation, we shall omit the proof of (3.87) and proceed with the proof of the statement concerning the smallness of (3.88).

The following elementary representation

$$\Gamma_n(3) = \Lambda_n(1) + \Lambda_n(2) + \Lambda_n(3), \tag{3.89}$$

holds, where

$$\begin{aligned}
\Lambda_n(1) &:= \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \,(\hat{\tau}_n - \tau)^2 \Big(\sum_{k=-\infty}^{\infty} \int_{W_n} K\Big(\frac{x-(s+k\hat{\tau}_n)}{h_n}\Big) X(dx) \Big)^2 \Big\}, \\
\Lambda_n(2) &:= \frac{2\tau}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \,(\hat{\tau}_n - \tau) \Big(\sum_{k=-\infty}^{\infty} \int_{W_n} K\Big(\frac{x-(s+k\hat{\tau}_n)}{h_n}\Big) X(dx) \Big)^2 \Big\}, \\
\Lambda_n(3) &:= \frac{\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \Big(\sum_{k=-\infty}^{\infty} \int_{W_n} K\Big(\frac{x-(s+k\hat{\tau}_n)}{h_n}\Big) X(dx) \Big)^2 \Big\}.
\end{aligned} \tag{3.90}$$

We shall demonstrate in Lemma 3.7 below that $\Lambda_n(3) \to \lambda^2(s)$, which also implies the other two desired statements: $\Lambda_n(1) \to 0$ and $\Lambda_n(2) \to 0$. Indeed, we estimate the difference $|\hat{\tau}_n - \tau|$ in both $\Lambda_n(1)$ and $\Lambda_n(2)$ by $\delta h_n/|W_n|$, and in this way demonstrate that $\Lambda_n(1)$ does not exceed $\{\delta h_n/|W_n|\}^2 \Lambda_n(3)$, and $\Lambda_n(2)$ does not exceed $\{\delta h_n/|W_n|\}\Lambda_n(3)$. Since $\delta h_n/|W_n|$ converges to 0, and $\Lambda_n(3)$ is bounded (c.f. Lemma 3.7 below), we obtain the above claimed statements $\Lambda_n(1) \to 0$ and $\Lambda_n(2) \to 0$.

Lemma 3.7 *The quantity* $\limsup_{n\to\infty} |\Lambda_n(3) - \lambda^2(s)|$ *can be made as small as desired by taking* $\delta > 0$ *sufficiently small.*

Proof. We have that

$$\Lambda_n(3) = \Lambda_n^*(3) + \Lambda_n^{**}(3) + \Lambda_n^{**}(3), \tag{3.91}$$

where

$$\begin{aligned}
\Lambda_n^*(3) &:= \frac{\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \Big[K\Big(\frac{x-(s+k\hat{\tau}_n)}{h_n}\Big) - K\Big(\frac{x-(s+k\tau)}{h_n}\Big) \Big] X(dx) \Big\}^2 \\
\Lambda_n^{**}(3) &:= \frac{2\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \Big[K\Big(\frac{x-(s+k\hat{\tau}_n)}{h_n}\Big) - K\Big(\frac{x-(s+k\tau)}{h_n}\Big) \Big] X(dx) \\
&\qquad\qquad \times \sum_{l=-\infty}^{\infty} \int_{W_n} K\Big(\frac{x-(s+l\tau)}{h_n}\Big) X(dx) \Big\} \\
\Lambda_n^{***}(3) &:= \frac{\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} K\Big(\frac{x-(s+k\tau)}{h_n}\Big) X(dx) \Big\}^2
\end{aligned}$$

Since $\Lambda_n^{***}(3) = \Pi_n$ with Π_n as in (3.51), we have that

$$\Lambda_n^{***}(3) \to \lambda^2(s). \tag{3.92}$$

Consequently, Lemma 3.7 follows if we prove that both $\Lambda_n^*(3)$ and $\Lambda_n^{**}(3)$ can be made as small as desired. In fact, we only need to prove this for $\Lambda_n^*(3)$, as the following argument shows: Using the Cauchy-Schwarz inequality, we obtain that

$$\Lambda_n^{**}(3) \le 2\big\{\Lambda_n^*(3)\big\}^{1/2}\big\{\Lambda_n^{***}(3)\big\}^{1/2}. \tag{3.93}$$

Therefore, in view of (3.92), the quantity $\Lambda_n^{**}(3)$ can be made as small as desired if the same can be done with the quantity $\Lambda_n^*(3)$. We prove the latter fact in Lemmas 3.8 below.

Lemma 3.8 *By choosing the parameters $\alpha > 0$ and $\delta > 0$ sufficiently small, the quantity* $\limsup_{n\to\infty} \Lambda_n^*(3)$ *can be made as small as desired.*

Proof. Note that the quantity $\Lambda_n^*(3)$ is similar to $\Lambda_n(2)$ defined below (3.6). Thus, the proof of Lemma 3.8 closely follows that of Lemma 3.2. In particular, we have the following bound (compare it with (3.18)):

$$\Lambda_n^*(3) \le c\big\{\Lambda_n^*(4) + \Lambda_n^{**}(4) + \Lambda_n^{**}(4)\big\}, \tag{3.94}$$

where

$$\begin{aligned}\Lambda_n^*(4) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Bigg\{\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} &\Bigg[(K - L_\alpha)\left(\frac{x-(s+k\hat\tau_n)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\hat\tau_n)}{h_n} \in [-1,1]\setminus A_\alpha\right\} \\ &- (K - L_\alpha)\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\setminus A_\alpha\right\}\Bigg]X(dx)\Bigg\}^2,\end{aligned}$$

$$\begin{aligned}\Lambda_n^{**}(4) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Bigg\{\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} &\Bigg[(K - L_\alpha)\left(\frac{x-(s+k\hat\tau_n)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\hat\tau_n)}{h_n} \in A_\alpha\right\} \\ &- (K - L_\alpha)\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in A_\alpha\right\}\Bigg]X(dx)\Bigg\}^2,\end{aligned}$$

$$\begin{aligned}\Lambda_n^{***}(4) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Bigg\{\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \Bigg[L_\alpha\left(\frac{x-(s+k\hat\tau_n)}{h_n}\right) \\ - L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right)\Bigg]X(dx)\Bigg\}^2.\end{aligned}$$

In the following three statements we prove that the quantities $\Lambda_n^*(4)$, $\Lambda_n^{**}(4)$, and $\Lambda_n^{***}(4)$ can be made as small as desired by appropriately choosing n, δ and α.

Statement 3.9 *By choosing the parameters $\alpha > 0$ and $\delta > 0$ sufficiently small, the quantity* $\limsup_{n\to\infty} \Lambda_n^*(4)$ *can be made as small as desired.*

Proof. We start the proof with the note that both functions K and L_α are bounded by a finite constant c that does not depend on α. Therefore,

$$\Lambda_n^*(4) \le c\{\Psi_n^\circ + \Psi_n^{\circ\circ}\}, \tag{3.95}$$

where

$$\Psi_n^{\circ} := \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\setminus A_\alpha\right\} X(dx)\Bigg)^2,$$
$$\Psi_n^{\circ\circ} := \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\setminus A_\alpha\right\} X(dx)\Bigg)^2.$$

We shall demonstrate in Propositions 3.10 and 3.11 below that, by choosing the parameters $\alpha > 0$ and $\delta > 0$ sufficiently small, we can make the quantities $\limsup_{n\to\infty} \Psi_n^{\circ}$ and $\limsup_{n\to\infty} \Psi_n^{\circ\circ}$ as small as desired. The proofs of these two statements are similar to the corresponding ones of Propositions 3.1 and 3.2, respectively.

Proposition 3.10 *By choosing the parameter $\alpha > 0$ sufficiently small, the quantity $\sup_{\delta>0} \limsup_{n\to\infty} \Psi_n^{\circ\circ}$ can be made as small as desired.*

Proof. We follow the lines of the proof of Proposition 3.1. We have

$$\begin{aligned}\Psi_n^{\circ\circ} &= \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} X\left(\{s+k\tau+h_n([-1,1]\setminus A_\alpha)\}\cap W_n\right)\Bigg)^2 \\ &= \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Bigg(\sum_{k\in\mathcal{K}} X\left(\{s+k\tau+h_n([-1,1]\setminus A_\alpha)\}\cap W_n\right)\Bigg)^2, \qquad (3.96)\end{aligned}$$

where the set $\mathcal{K}$ is the same as in the proof of Proposition 3.1. To proceed, we need a preliminary result. Namely, if $\zeta_1, \zeta_2, \ldots$ are independent Poisson random variables, then

$$\begin{aligned}\mathbf{E}\{\sum_k \zeta_k\}^2 &= \sum_{j\neq k} \mathbf{E}\zeta_j \mathbf{E}\zeta_k + \sum_k \mathbf{E}\zeta_k^2 \\ &= \Big\{\sum_k \mathbf{E}\zeta_k\Big\}^2 + \sum_k \mathbf{Var}\zeta_k \\ &= \Big\{\sum_k \mathbf{E}\zeta_k\Big\}^2 + \sum_k \mathbf{E}\zeta_k. \qquad (3.97)\end{aligned}$$

Applying equality (3.97) on the right-hand side of (3.96), which is legitimate for large n due to independence considerations, we obtain that

$$\begin{aligned}\Psi_n^{\circ\circ} &= \frac{1}{|W_n|^2 h_n^2} \Bigg(\sum_{k\in\mathcal{K}} \mathbf{E}X\left(\{s+k\tau+h_n([-1,1]\setminus A_\alpha)\}\cap W_n\right)\Bigg)^2 \\ &\quad + \frac{1}{|W_n|^2 h_n^2} \sum_{k\in\mathcal{K}} \mathbf{E}X\left(\{s+k\tau+h_n([-1,1]\setminus A_\alpha)\}\cap W_n\right) \\ &\le c\left\{\alpha^2 + \frac{1}{|W_n| h_n}\alpha\right\}, \qquad (3.98)\end{aligned}$$

where the last inequality of (3.98) was obtained using inequalities (3.23) and (3.24). This completes the proof of Proposition 3.10. □

Proposition 3.11 *By choosing the parameters $\alpha > 0$ and $\delta > 0$ sufficiently small, the quantity $\limsup_{n\to\infty} \Psi_n^{\circ}$ can be made as small as desired.*

Proof. The proof is a combination of ideas of the proofs of Propositions 3.2 and 3.10. Therefore, we omit the detail stating only the following bound

$$\Psi_n^\circ \le c\Big\{\big(c_1(\alpha)\delta + c_2\alpha\big)^2 + \big(c_1(\alpha)\delta + c_2\alpha\big)\Big\}. \tag{3.99}$$

The right-hand side of (3.99) can be made as small as desired by first choosing a sufficiently small α (this may increase $c_1(\alpha)$) and then choosing a sufficiently small δ. This concludes the proof of Proposition 3.11. □

Propositions 3.10 and 3.11 complete the proof of Statement 3.9. □

Statement 3.10 *By choosing the parameter $\alpha > 0$ sufficiently small, we can make the quantity* $\sup_{\delta>0} \limsup_{n\to\infty} \Lambda_n^{**}(4)$ *as small as desired.*

Proof. Since $|K(u) - L_\alpha(u)| \le \alpha$ for all $u \in A_\alpha$ (cf. (3.15)), we obtain that

$$\Lambda_n^{**}(4) \le c\alpha^2\big\{\Psi_n^* + \Psi_n^{**}\big\}, \tag{3.100}$$

where the constant c does not depend on n and α, and

$$\Psi_n^* := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Big\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty} X\left(\{s + k\hat{\tau}_n + h_n A_\alpha\} \cap W_n\right)\Big\}^2,$$

$$\Psi_n^{**} := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Big\{\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty} X\left(\{s + k\tau + h_n A_\alpha\} \cap W_n\right)\Big\}^2.$$

The main difference between the just defined Ψ_n^*, Ψ_n^{**} and, respectively, Ψ_n°, $\Psi_n^{\circ\circ}$ defined below (3.95) is the set A_α instead of $[-1,1] \setminus A_\alpha$. With this difference in mind, we follow the lines of the proof of Statement 3.9 (cf. also the proof of Statement 3.2 for additional detail) and obtain that both quantities Ψ_n^* and Ψ_n^{**} are asymptotically bounded. Thus, in view of (3.100), we have the bound

$$\Lambda_n^{**}(4) \le c\alpha^2, \tag{3.101}$$

where the constant c does not depend on n, δ and α. Bound (3.101) concludes the proof of Statement 3.10. □

Statement 3.11 *For any fixed $\alpha > 0$, the quantity* $\limsup_{n\to\infty} \Lambda_n^{***}(4)$ *can be made as small as desired by taking $\delta > 0$ sufficiently small.*

Proof. Using decomposition (3.34), we obtain the following one

$$\Lambda_n^{***}(4) = \Delta_n(1) + \Delta_n(2), \tag{3.102}$$

where

$$\Delta_n(1) = \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Bigg(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\left[L_\alpha\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right)\right]\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\right\}X(dx)\Bigg)^2,$$

$$\Delta_n(2) = \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\Bigg(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right) \times\left[\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\right\} - \mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\right\}\right]X(dx)\Bigg)^2.$$

In Propositions 3.12 and 3.12 below, we prove that the quantities $\Delta_n(1)$ and $\Delta_n(2)$ can be made as small as desired and, in this way, finish the proof of Statement 3.11.

Proposition 3.12 *For any fixed $\alpha > 0$, the quantity $\limsup_{n\to\infty} \Delta_n(1)$ can be made as small as desired by taking $\delta > 0$ sufficiently small.*

Proof. Following the lines of the proof of Proposition 3.3, we obtain (c.f. (3.38)) that

$$\Delta_n(1) \le c^2(\alpha)\delta^2 \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \sum_{k\in\mathcal{K}} X\left(\{s + k\hat{\tau}_n + h_n[-1,1]\} \cap W_n\right) \Big\}^2, \tag{3.103}$$

where the constant $c(\alpha)$ is possibly converging to ∞ when $\alpha \to 0$. The main difference between the quantity Ψ_n^* defined below (3.100) and the quantity

$$\frac{1}{|W_n|^2 h_n^2} \mathbf{E}\Big\{ \mathbf{I}\{A_n\} \sum_{k\in\mathcal{K}} X\left(\{s + k\hat{\tau}_n + h_n[-1,1]\} \cap W_n\right) \Big\}^2 \tag{3.104}$$

on the right-hand side of (3.103) is the interval $[-1,1]$ instead of the set A_α. We demonstrated in the proof of Statement 3.10 that Ψ_n^* is asymptotically bounded. The same arguments show that that the quantity of (3.104) is asymptotically bounded. Therefore, we obtain from bound (3.103) that

$$\Delta_n(1) \le c^2(\alpha)\delta^2, \tag{3.105}$$

which completes the proof of Proposition 3.12. □

Proposition 3.13 *For any fixed $\alpha > 0$, the quantity $\limsup_{n\to\infty} \Delta_n(2)$ can be made as small as desired by taking $\delta > 0$ sufficiently small.*

Proof. In a similar way bound (3.43) was obtained, we now obtain the following one:

$$\Delta_n(2) \le \Delta_n^*(2) + \Delta_n^{**}(2), \tag{3.106}$$

where

$$\begin{aligned}\Delta_n^*(2) := \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right) \\ \times \mathbf{I}\big\{x-(s+k\tau) \in h_n[-1-c\delta, -1+c\delta]\big\} X(dx)\bigg)^2,\\ \Delta_n^{**}(2) := \frac{1}{|W_n|^2 h_n^2} \mathbf{E}\bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} L_\alpha\left(\frac{x-(s+k\tau)}{h_n}\right) \\ \times \mathbf{I}\big\{x-(s+k\tau) \in h_n[1-c\delta, 1+c\delta]\big\} X(dx)\bigg)^2.\end{aligned}$$

The estimation of $\Delta_n^*(2)$ is similar to that of $\Delta_n^{**}(2)$, and we thus only estimate $\Delta_n^{**}(2)$. Using the fact that the function L_α, we obtain that

$$\begin{aligned}\Delta_n^{**}(2) &\le c\frac{1}{|W_n|^2 h_n^2} \mathbf{E}\bigg(\sum_{k=-\infty}^{\infty} \int_{W_n} \mathbf{I}\big\{x-(s+k\tau) \in h_n[1-c\delta, 1+c\delta]\big\} X(dx)\bigg)^2 \\ &\le c\frac{1}{|W_n|^2 h_n^2} \mathbf{E}\bigg(\sum_{k\in\mathcal{K}} X\big(\big\{s+k\tau \in h_n[1-c\delta,1+c\delta]\big\} \cap W_n\big)\bigg)^2 \\ &= c\frac{1}{|W_n|^2 h_n^2} \bigg(\sum_{k\in\mathcal{K}} \mathbf{E}X\big(\big\{s+k\tau \in h_n[1-c\delta,1+c\delta]\big\} \cap W_n\big)\bigg)^2 \\ &\quad + c\frac{1}{|W_n|^2 h_n^2} \sum_{k\in\mathcal{K}} \mathbf{E}X\big(\big\{s+k\tau \in h_n[1-c\delta,1+c\delta]\big\} \cap W_n\big),\end{aligned} \tag{3.107}$$

where the equality on the right-hand side of (3.107) was obtained using (3.97). We now enlarge W_n to the whole real line $\mathbf{R}$ in all summands on the right-hand side of (3.107) and, consequently, obtain the bound:

$$\Delta_n^{**}(2) \le c\left[\frac{1}{|W_n|h_n}\sum_{k\in\mathcal{K}}\mathbf{E}X\big(s+k\tau\in h_n[1-c\delta,1+c\delta]\big)\right]^2 \\ + c\frac{1}{|W_n|h_n}\left[\frac{1}{|W_n|h_n}\sum_{k\in\mathcal{K}}\mathbf{E}X\big(s+k\tau\in h_n[1-c\delta,1+c\delta]\big)\right]. \quad (3.108)$$

We estimate the quantity in both brackets $[\cdot]$ on the right-hand side of (3.108) by $c\delta$ (cf. the last two bounds of (3.44) for detail) and obtain the following bound:

$$\Delta_n^{**}(2) \le c\left\{\delta^2 + \frac{1}{|W_n|h_n}\delta\right\}. \quad (3.109)$$

Choosing $\delta > 0$ sufficiently small, $\Delta_n(2)$ can be made as small as desired, which completes the proof of Proposition 3.13. □

Due to equality (3.102) and Propositions 3.12 and 3.13, the proof of Statement 3.11 is complete. Bound (3.94) and Statements 3.9, 3.10 and 3.11 complete the proof of Lemma 3.8.

This completes the proof of (3.88), and thus of Theorem 1.4 as well. □

3.4 Proof of Theorem 1.5

Throughout this section we use the following definition:

$$A_n := \left\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat{\tau}_n - \tau| \le \delta\right\}. \quad (3.110)$$

The proof of Theorem 1.5 is subdivided into two main parts, Lemmas 3.9 and 3.10 below.

Lemma 3.9 *We have that*

$$\mathbf{Var}\big(\hat{\lambda}^{\diamond}_{n,K}(s)\big) = \mathbf{Var}\big(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)\big) + o\left(\frac{1}{|W_n|h_n}\right). \quad (3.111)$$

Proof. We start the proof with the inequality

$$|\mathbf{Var}\xi - \mathbf{Var}\eta\}| \le \mathbf{E}\big\{\big(|\xi-\mathbf{E}\xi| + |\eta-\mathbf{E}\eta|\big)\big|(\xi-\eta) - \mathbf{E}(\xi-\eta)\big|\big\} \\ \le \mathbf{E}\big\{(|\xi|+|\eta|)|\xi-\eta|\big\} + 3\big(\mathbf{E}|\xi| + \mathbf{E}|\eta|\big)\mathbf{E}|\xi-\eta| \quad (3.112)$$

that holds for any random variables ξ and η. Applying inequality (3.112) with

$$\xi := \hat{\lambda}^{\diamond}_{n,K}(s), \\ \eta := \mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s),$$

we obtain that (3.111) follows from Statements 3.12 and 3.13 below.

Statement 3.12 *We have that*

$$\big(\mathbf{E}|\xi| + \mathbf{E}|\eta|\big)\mathbf{E}|\xi-\eta| = o\left(\frac{1}{|W_n|h_n}\right). \quad (3.113)$$

Proof. The quantity $\mathbf{E}|\xi|$ does not exceed D_n. Since the set A_n defined in (3.110) is smaller than that defined in (3.1), we immediately derive from the statement of (3.5) that $\mathbf{E}|\eta|$ is bounded. Therefore, statement (3.113) follows if we show that

$$\mathbf{E}|\xi-\eta| = o\left(\frac{1}{D_n|W_n|h_n}\right). \tag{3.114}$$

We start the proof of (3.114) with the bounds:

$$\begin{aligned}\mathbf{E}|\xi-\eta| &= \mathbf{E}\big|\hat{\lambda}^{\diamond}_{n,K}(s) - \mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)\big| \\ &= \mathbf{E}\big|\big(1-\mathbf{I}\{A_n\}\big)\mathbf{I}\{\hat{\lambda}_{n,K}(s)\le D_n\}\hat{\lambda}_{n,K}(s) - \mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s)>D_n\}\hat{\lambda}_{n,K}(s)\big| \\ &\le \mathbf{E}\big\{\big(1-\mathbf{I}\{A_n\}\big)\mathbf{I}\{\hat{\lambda}_{n,K}(s)\le D_n\}\hat{\lambda}_{n,K}(s)\big\} + \mathbf{E}\big\{\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s)>D_n\}\hat{\lambda}_{n,K}(s)\big\} \\ &\le D_n\mathbf{P}\left\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat{\tau}_n-\tau|\ge\delta\right\} + \frac{1}{D_n^r}\mathbf{E}\big(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)^{r+1}\big). \end{aligned} \tag{3.115}$$

The first summand on the right-hand side of (3.115) is of order $o(1/\{|W_n|h_n\})$ due to assumption (1.19). In order to demonstrate that the second summand on the right-hand side of (3.115) is also of the same order, we proceed as follows. First, we recall the already discussed (c.f. a note below (3.62)) fact that $\mathbf{E}(\mathbf{I}\{A_n\}\{\hat{\lambda}_{n,K}(s)\}^{r+1})$ is asymptotically bounded when the set A_n is defined in (3.1). Since A_n of (3.110) is smaller than that of (3.1), we immediately obtain that the second summand on the right-hand side of (3.115) is of order $O1/D_n^r$. Since, by assumption, $D_n \ge c\{|W_n|h_n\}^{\epsilon}$, the right-hand side of (3.115) is of order $o(1/\{|W_n|h_n\})$ for a sufficiently large r. This completes the proof of (3.114), and of Statement 3.12 as well. □

Statement 3.13 *We have that*

$$\mathbf{E}\big\{(|\xi|+|\eta|)|\xi-\eta|\big\} = o\left(\frac{1}{|W_n|h_n}\right). \tag{3.116}$$

Proof. We start the proof with the representation

$$\begin{aligned}\xi-\eta &= \hat{\lambda}^{\diamond}_{n,K}(s) - \mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s) \\ &= \big(1-\mathbf{I}\{A_n\}\big)\mathbf{I}\{\hat{\lambda}_{n,K}(s)\le D_n\}\hat{\lambda}_{n,K}(s) - \mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s)>D_n\}\hat{\lambda}_{n,K}(s). \end{aligned} \tag{3.117}$$

Consequently, (3.116) follows from the following two statements:

$$\mathbf{E}\Big\{\big(1-\mathbf{I}\{A_n\}\big)\mathbf{I}\{\hat{\lambda}_{n,K}(s)\le D_n\}\{\hat{\lambda}_{n,K}(s)\}^2\Big\} = o\left(\frac{1}{|W_n|h_n}\right), \tag{3.118}$$

$$\mathbf{E}\Big\{\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s)>D_n\}\{\hat{\lambda}_{n,K}(s)\}^2\Big\} = o\left(\frac{1}{|W_n|h_n}\right). \tag{3.119}$$

In order to prove (3.118), we first estimate $\{\hat{\lambda}_{n,K}(s)\}^2$ by D_n^2. Then, it becomes obvious that statement (3.118) is implied by assumption (1.19). In order to prove (3.119), we estimate the left-hand side of (3.119) by $D_n^{-r}\mathbf{E}(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)^{r+2})$. Using an argument below (3.115), we conclude that the quantity on the right-hand side of (3.119) is of order $o(1/\{|W_n|h_n\})$ for a sufficiently large r. This completes the proof of (3.119), and thus of (3.116) as well. □

Statements 3.12 and 3.13 complete the proof of Lemma 3.9. □

Lemma 3.10 *By choosing sufficiently small $\delta>0$, the quantity*

$$\limsup_{n\to\infty}\left(|W_n|h_n\big\{\mathbf{Var}\big(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)\big) - \frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1}K^2(x)dx\big\}\right) \tag{3.120}$$

can be made as small as desired.

Proof. It is easy to see that the following representation

$$\mathbf{Var}\big(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)\big) = \tau^2 V_n(1) + V_n(2) + \theta 2\tau\sqrt{V_n(1)V_n(2)}, \tag{3.121}$$

holds, where $\theta \in [-1, 1]$ and

$$V_n(1) = \mathbf{Var}\left(\mathbf{I}\{A_n\}\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right),$$

$$V_n(2) = \mathbf{Var}\left(\mathbf{I}\{A_n\}(\hat{\tau}_n-\tau)\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right).$$

We shall show in Statement 3.15 below that $V_n(2)$ converges to 0 sufficiently fast. As to the quantity $V_n(1)$, we have the following representation

$$V_n(1) = R_n(1) + R_n(2) + \theta 2\sqrt{R_n(1)R_n(2)}, \tag{3.122}$$

where $\theta \in [-1, 1]$ (possibly different from that above) and

$$R_n(1) := \mathbf{Var}\left(\mathbf{I}\{A_n\}\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right),$$

$$R_n(2) := \mathbf{Var}\left(\mathbf{I}\{A_n\}\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} \left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right) - K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\right).$$

We shall show in Statement 3.16 below that $R_n(2)$ converges to 0 sufficiently fast. As to the quantity $R_n(1)$, we have the following representation

$$R_n(1) = Y_n(1) + Y_n(2) + \theta 2\sqrt{Y_n(1)Y_n(2)}, \tag{3.123}$$

where $\theta \in [-1, 1]$ is some number, and

$$Y_n(1) := \mathbf{Var}\left(\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right),$$

$$Y_n(2) := \mathbf{Var}\left(\big(1-\mathbf{I}\{A_n\}\big)\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right).$$

We shall show in Statement 3.17 below that $Y_n(2)$ converges to 0 sufficiently fast.

Taking now the validity of Statements 3.15–3.17 for granted, we easily see that Lemma 3.9 follows from next Statement 3.14.

Statement 3.14 *We have that*

$$\tau^2 Y_n(1) = \frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{|W_n|h_n}\right). \tag{3.124}$$

Proof. The random variables

$$\int_{W_n} K\left(\frac{x-(s+j\tau)}{h_n}\right)X(dx), \quad \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)$$

are independent for sufficiently large n, provided that $j \neq k$. Therefore,

$$Y_n(1) = \sum_{k=-\infty}^{\infty} \mathbf{Var}\bigg(\frac{1}{|W_n| h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n} \right) X(dx) \bigg). \tag{3.125}$$

Using Lemma 1.1 on p. 18 of Kutoyants (1998), we have that the right-hand side of (3.125) equals

$$\frac{1}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \int_{W_n} K^2 \bigg(\frac{x-(s+k\tau)}{h_n} \bigg) \lambda(x) dx.$$

Therefore, the following equality

$$Y_n(1) = \frac{1}{|W_n|^2 h_n^2} \int_{-\infty}^{\infty} K^2 \bigg(\frac{x}{h_n} \bigg) \lambda(x+s) \sum_{k=-\infty}^{\infty} \mathbf{I}(x+s+k\tau \in W_n) dx \tag{3.126}$$

holds. An application of (3.47) on the right-hand side of (3.126) yields the equality below:

$$\begin{aligned} \tau^2 Y_n(1) &= \bigg(\frac{\tau}{|W_n| h_n^2} + \theta \frac{\tau^2}{|W_n|^2 h_n^2} \bigg) \int_{-\infty}^{\infty} K^2 \bigg(\frac{x}{h_n} \bigg) \lambda(x+s) dx \\ &= \bigg(\frac{\tau}{|W_n| h_n^2} + \theta \frac{\tau^2}{|W_n|^2 h_n^2} \bigg) \int_{-\infty}^{\infty} K^2 \bigg(\frac{x}{h_n} \bigg) \big(\lambda(x+s) - \lambda(s) \big) dx \\ &\quad + \bigg(\frac{\tau}{|W_n| h_n^2} + \theta \frac{\tau^2}{|W_n|^2 h_n^2} \bigg) h_n \lambda(s) \int_{-1}^{1} K^2(x) dx, \end{aligned} \tag{3.127}$$

where $\theta \in [-1,1]$ is some number. Since s is a Lebesgue point of λ, and since the kernel K is bounded and has support in $[-1,1]$, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} K^2 (\frac{x}{h_n}) |\lambda(x+s) - \lambda(s)| dx &= \int_{-h_n}^{h_n} K^2 (\frac{x}{h_n}) |\lambda(x+s) - \lambda(s)| dx \\ &= o(h_n). \end{aligned} \tag{3.128}$$

Applying (3.128) on the right-hand side of (3.127), we arrive at the claim of Statement 3.14. □

The remaining proof of Lemma 3.10 consists of proving Statements 3.15, 3.16 and 3.17 where we demonstrate, respectively, that the quantities $V_n(2)$, $R_n(2)$, and $Y_n(2)$ are asymptotically of order $o(1/\{|W_n| h_n\})$.

Statement 3.15 *We have that* $V_n(2) = o\big(1/\{|W_n| h_n\}\big)$.

Proof. We have the following bounds:

$$\begin{aligned} V_n(2) &\leq \mathbf{E} \bigg(\mathbf{I}\{A_n\} (\hat{\tau}_n - \tau) \frac{1}{|W_n| h_n} \sum_{k=-\infty}^{\infty} \int_{W_n} K \bigg(\frac{x-(s+k\hat{\tau}_n)}{h_n} \bigg) X(dx) \bigg)^2 \\ &\leq \delta \frac{h_n}{|W_n|^3} \mathbf{E} \bigg(\mathbf{I}\{A_n\} \frac{1}{|W_n| h_n} \sum_{k=-\infty}^{\infty} \int_{W_n} K \bigg(\frac{x-(s+k\hat{\tau}_n)}{h_n} \bigg) X(dx) \bigg)^2 \\ &\leq c\delta \frac{h_n}{|W_n|^3} \bigg[\frac{1}{|W_n|^2 h_n^2} \mathbf{E} \bigg\{ \mathbf{I}\{A_n\} \sum_{k \in \mathcal{K}} X\left(\{s+k\hat{\tau}_n + h_n[-1,1]\} \cap W_n \right) \bigg\}^2 \bigg], \end{aligned} \tag{3.129}$$

where the set $\mathcal{K}$ of summation indices is the same as in (3.104). The quantity in brackets $[\cdot]$ on the right-hand side of (3.129) is exactly the quantity in (3.104). We noted below (3.104) that the quantity in (3.104) is bounded. Therefore, bound (3.129) and the assumption $h_n \to 0$ (or $|W_n| \to \infty$) complete the proof of Statement 3.15. □

Statement 3.16 *By choosing the parameter* δ *sufficiently small, we can make the quantity* $\limsup_{n\to\infty}\{|W_n|h_n R_n(2)\}$ *as small as desired.*

Proof. We start with the elementary bound

$$R_n(2) \le \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\biggl(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)-K\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\biggr)^2. \tag{3.130}$$

Similar to the proof of Lemma 3.5, we reduce the estimation of the right-hand side of (3.130) to that of

$$R_n(2,m) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\biggl(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\left[K_m\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)\right.$$
$$\left.-K_m\left(\frac{x-(s+k\tau)}{h_n}\right)\right]X(dx)\biggr)^2,$$

for any $m\in\{1,\ldots,M+1\}$. Following the lines of the proof of Statement 3.6, we reduce the estimation of $R_n(2,m)$ to that of the following two quantities (cf. bound (3.69) for additional detail):

$$R_n^*(2,m) := \frac{\delta^2}{|W_n|h_n}\left[\frac{1}{|W_n|^2h_n^2}\mathbf{E}\biggl(\sum_{k=-\infty}^{\infty}\int_{W_n}\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n}\in[-1,1]\right\}X(dx)\biggr)^2\right],$$

$$R_n^{**}(2,m) := \frac{1}{|W_n|^2h_n^2}\mathbf{E}\biggl(\sum_{k=-\infty}^{\infty}\int_{W_n}$$
$$\mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n}\in\left[x_m-c\frac{\delta}{\sqrt{|W_n|h_n}},x_m+c\frac{\delta}{\sqrt{|W_n|h_n}}\right]\right\}X(dx)\biggr)^2.$$

To estimate $R_n^*(2,m)$, we first note that, by statement (3.51), the quantity in brackets $[\cdot]$ in the definition of $R_n^*(2,m)$ is bounded. Thus, the bound

$$R_n^*(2,m) \le c\frac{\delta^2}{|W_n|h_n} \tag{3.131}$$

holds. Bound (3.131) holds in the case of quantity $R_n^{**}(2,m)$ as well, which can easily be verified using some ideas of the proof of Proposition 3.4. Thus, we have the bound

$$R_n(2,m) \le c\frac{\delta^2}{|W_n|h_n}, \tag{3.132}$$

which completes the proof of Statement 3.16. □

Statement 3.17 *We have that* $Y_n(2)=o\bigl(1/\{|W_n|h_n\}\bigr)$.

Proof. Using first the bound $\mathbf{Var}\{\xi\eta\}\le\mathbf{E}\{\xi^2\eta^2\}$ and then the Hölder inequality, we obtain that the following estimate

$$Y_n(2) \le \biggl(\mathbf{P}\biggl\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat{\tau}_n-\tau|\ge\delta\biggr\}\biggr)^{1/r}$$
$$\times\biggl(\mathbf{E}\biggl\{\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n}K\Bigl(\frac{x-(s+k\tau)}{h_n}\Bigr)X(dx)\biggr\}^{2q}\biggr)^{1/q} \tag{3.133}$$

holds for any numbers $r, q > 1$ such that $r^{-1} + q^{-1} = 1$. Due to assumption (1.19), and no matter what the value of $\epsilon > 0$ (cf. the assumption $D_n \geq |W_n|^{\epsilon}/h_n^{1+\epsilon}$) is, we can always find an $r > 1$ so close to 1 that the statement

$$\left(\mathbf{P}\left\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat{\tau}_n - \tau| \geq \delta\right\}\right)^{1/r} = o\left(\frac{1}{|W_n|h_n}\right), \tag{3.134}$$

holds. Consequently, using (3.134) and (3.78), we obtain from (3.133) that Statement 3.17 holds. □

The proof of Theorem 1.5 is finished. □

Acknowledgements

The work was completed while all three of us met at the Centre for Mathematics and Computer Science (CWI), Amsterdam, in January-February, 2001; we thank the CWI for the most stimulating scientific environment. The second and third authors also thank the Royal Netherlands Academy of Sciences (KNAW), the Netherlands Organization for Scientific Research (NWO), and the National Sciences and Engineering Research Council (NSERC) of Canada that made their visits at the CWI possible.

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