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ABSTRACT
Pickands constants play an important role in the exact asymptotic of extreme values for Gaussian stochastic processes. By the *generalized Pickands constant* $\mathcal{H}_\eta$ we mean the limit

$$\mathcal{H}_\eta = \lim_{T\to\infty} \frac{\mathcal{H}_\eta(T)}{T},$$

where $\mathcal{H}_\eta(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]}\left(\sqrt{2}\eta(t) - \sigma_\eta^2(t)\right)\right)$ and $\eta(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_\eta^2(t)$.

Under some mild conditions on $\sigma_\eta^2(t)$ we prove that $\mathcal{H}_\eta$ is well defined and we give a comparison criterion for the generalized Pickands constants. Moreover we prove a theorem result of Pickands for certain stationary Gaussian processes.

As an application we obtain the exact asymptotic behavior of $\psi(u) = \mathbb{P}(\sup_{t\geq 0}\zeta(t) - ct > u)$ as $u\to\infty$, where $\zeta(x) = \int_0^x Z(s)\,ds$ and $Z(s)$ is a stationary centered Gaussian process with covariance function $R(t)$ fulfilling some integrability conditions.

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1 Introduction

J. Pickands III [11], [12] found an elegant way of computing the exact asymptotics of the probability $\mathbb{P}(\max_{t\in[0,T]} X(t) > u)$ for a centered stationary Gaussian process $X(t)$ with covariance function $R(t) = 1 - |t|^\alpha + o(|t|^\alpha)$ as $t \to 0$, $\alpha \in (0, 2]$ and $R(t) < 1$ for all $t > 0$. For such a process he proved

$$\mathbb{P}(\max_{t\in[0,T]} X(t) > u) = \mathcal{H}_{B_{\alpha/2}} T u^{2/\alpha} \Psi(u)(1 + o(1)) \text{ as } u \to \infty, \tag{1.1}$$

where $\mathcal{H}_{B_{\alpha/2}}$ is the *Pickands constant* and $\Psi(u)$ is the tail distribution of standard normal law. Recall that $\mathcal{H}_{B_{\alpha/2}}$ is defined by the following limit

$$\mathcal{H}_{B_{\alpha/2}} = \lim_{T\to\infty} \frac{\mathbb{E}\exp\left(\max_{t\in[0,T]} \sqrt{2}B_{\alpha/2}(t) - \mathrm{Var}(B_{\alpha/2}(t))\right)}{T}, \tag{1.2}$$

where $B_{\alpha/2}(t)$ is a fractional Brownian motion (**FBM**) with Hurst parameter $\alpha/2$, that is a centered Gaussian process with stationary increments, continuous sample paths and variance function $\mathrm{Var}(B_{\alpha/2}(t)) = t^\alpha$. Pickands proved (1.1) using *the double sum method*, that is by breaking the interval $[0, T]$ into several subintervals on which the following asymptotics may be applied: for each $T > 0$

$$\mathbb{P}(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u) = \mathcal{H}_{B_{\alpha/2}}(T)\Psi(u)(1 + o(1)) \tag{1.3}$$

as $u \to \infty$, where

$$\mathcal{H}_{B_{\alpha/2}}(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]}\left(\sqrt{2}B_{\alpha/2}(t) - \mathrm{Var}(B_{\alpha/2}(t))\right)\right). \tag{1.4}$$

Asymptotics (1.3) is a useful tool for computing the exact asymptotics in extreme value theory for a wide class of Gaussian processes (see Piterbarg [13]). Unfortunately it does not cover all the cases interesting in applications (see for example the class of Gaussian integrated processes considered in Dębicki [2]). In particular the stationarity assumption seem to be too strong. We present an extension of (1.3) in Section 2 (Theorem 2.1).

It turns out that the asymptotics obtained in Theorem 2.1 yields a natural extension of Pickands constants. Namely instead of **FBM** $B_{\alpha/2}(t)$ in (1.2) there appear more general centered Gaussian processes $\eta(t)$ with stationary increments.

Throughout this article $\eta(t)$ is a centered Gaussian process with stationary increments, a.s. continuous sample paths, $\eta(0) = 0$ and such that the variance function $\mathrm{Var}(\eta(t)) = \sigma_\eta^2(t)$ satisfies

C1 $\sigma_\eta^2(t) \in C^1([0,\infty))$ is strictly increasing and there exists $\epsilon > 0$ such that

$$\frac{\dot\sigma_\eta^2(t)}{\sigma_\eta^2(t)} \le \frac{\epsilon}{t} \text{ as } t \to \infty; \tag{1.5}$$

C2 $\sigma_\eta^2(t)$ is regularly varying at 0 with index $\alpha_0 \in (0, 2]$ and $\sigma_\eta^2(t)$ is regularly varying at ∞ with index $\alpha_\infty \in (0, 2)$.

In the paper we use the notation $\dot\sigma^2(t)$ or $\ddot\sigma^2(t)$ for the derivatives of $\sigma^2(t)$.

Note that **C1** is strongly related to **C2**. In fact if $\sigma_\eta^2(t)$ satisfies **C1** in such a way that $\lim_{t\to\infty}\sigma_\eta^2(t)=\infty$ and $\lim_{t\to\infty}\frac{t\dot\sigma_\eta^2(t)}{\sigma_\eta^2(t)}=\epsilon$, then $\sigma_\eta^2(t)$ is regularly varying at ∞ and $\alpha_\infty=\epsilon$ (see [1], p 59). Conversely if $\sigma_\eta^2(t)$ is regularly varying at ∞ with $\alpha_\infty=\epsilon$ and $\dot\sigma_\eta^2(t)$ is ultimately monotone, then (1.5) holds.

For $\eta(t)$ that satisfies **C1-C2** define

$$\mathcal{H}_\eta(T)=\mathbb{E}\exp\left(\max_{t\in[0,T]}\left(\sqrt{2}\eta(t)-\sigma_\eta^2(t)\right)\right). \tag{1.6}$$

More generally for independent centered Gaussian processes with stationary increments $\eta_1(t),....,\eta_N(t)$ that satisfy **C1-C2**, where the indices of regularity of variance functions may differ for each process, we define

$$\mathcal{H}_{\eta_1,\ldots,\eta_N}(T)=\mathbb{E}\exp\left(\max_{(t_1,\ldots,t_N)\in[0,T]^N}\left(\sqrt{\frac{2}{N}}\sum_{i=1}^N\eta_i(t_i)-\frac{1}{N}\sum_{i=1}^N\sigma_{\eta_i}^2(t_i)\right)\right). \tag{1.7}$$

Note that in a special case, when $\eta(t)=B_{\alpha/2}(t)$ and $N=1$, we obtain the constants $\mathcal{H}_{B_{\alpha/2}}(T)$ defined in (1.4). We analyze properties of $\mathcal{H}_{\eta_1,\ldots,\eta_N}(T)$ in Section 3.

By *the generalized Pickands constant* $\mathcal{H}_\eta$ we understand

$$\lim_{T\to\infty}\frac{\mathcal{H}_\eta(T)}{T}=\mathcal{H}_\eta,$$

provided that the limit exists. In Section 3 (see Theorem 3.1) we prove that under conditions **C1-C2** this limit exists, is positive and finite. Moreover in Theorem 3.2 we give a comparison criterion for generalized Pickands constants.

With $\eta(t)$ we associate a family $\{X_{\eta;u}(t),u>0\}$ (indexed by $u>0$) of centered Gaussian processes, where the relation between $\eta(t)$ and $X_{\eta;u}(t)$ is given by assumption **D0** presented in Section 2. By the attached bar we always mean the standardized process, that is $\overline{X}(t)=X(t)/\sigma_X(t)$.

In Section 2 (Theorem 2.1) we extend asymptotics (1.3) to a standardized family of Gaussian fields $\{\overline{X}_{\eta;u}(t),u>0\}$ that satisfy condition **D0**.

Combination of Theorem 2.1 with the double sum method yields new exact asymptotics in extreme value theory. In particular in Section 4 we present Theorem 4.1 which extends results of Piterbarg [13] and enables us to obtain exact asymptotics for some families of Gaussian processes $\{X_{\eta;u}(t),u>0\}$, where for sufficiently large u the variance function $\sigma^2_{X_{\eta;u}}(t)$ attains maximum at a unique point t_u.

Recently the asymptotics of

$$\psi(u)=\mathbb{P}(\sup_{t\geq 0}\zeta(t)-ct>u)$$

for a centered Gaussian process $\zeta(t)$ with stationary increments and $c>0$ was studied in many papers; see e.g. [10, 3, 4, 7]. The problem of analyzing $\psi(u)$ stemmed from the theory of Gaussian fluid models, where the following cases are of special interest:

- $\zeta(x)=\int_0^x Z(s)\,ds$, where $Z(s)$ is a stationary centered Gaussian process with covariance function $R(t)=\mathbb{E}Z(0)Z(t)$ fulfilling some integrability conditions; we call such the case *integrated Gaussian* (**IG**),
- $\zeta(x)=B_{\alpha/2}(t)$ being a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$, where $\alpha\in(0,2)$.

The last model was recently studied by Hüsler and Piterbarg [6] who obtained exact asymptotic of $\psi(u)$ for $\zeta(x)$ being a fractional Brownian motion; see also Narayan [9]. Theorem 4.1, presented in Section 4, enables us to obtain the exact asymptotics of $\psi(u)$ for a class of **IG** processes that play an important role in the fluid model theory and is not covered by the results of Hüsler and Piterbarg [6]. Namely we focus on the case where $\zeta(x) = \int_0^x Z(s)\,ds$ possesses the *short range dependence* (**SRD**) property, that is the covariance function $R(t)$ of $Z(t)$ fulfills

SRD.1 $R(t) \in C([0,\infty))$, $\lim_{t\to\infty} tR(t) = 0$;

SRD.2 $\int_0^t R(s)ds > 0$ for each $t > 0$ and $t = \infty$;

SRD.3 $\int_0^\infty s^2|R(s)|\,ds < \infty$.

We exclude from the following considerations the degenerated case $R(0) = 0$. We comment the validity of the **SRD** assumption in Remark 5.1 and give the exact asymptotic of $\psi(u)$ for $\zeta(t) \in$ **SRD** in Theorem 5.1.

2 Extension of Pickands theorem

We write $\{X_{\eta;u}(t), u > 0\}$ for the family of centered Gaussian processes $\{X_{\eta;u}(t) : t \geq 0\}$ $(u > 0)$ and assume that for each $u > 0$ the Gaussian process $X_{\eta;u}(t)$ has continuous trajectories. The family $\{X_{\eta;u}(t), u > 0\}$ is related to a Gaussian process $\eta(t)$ with stationary increments and variance function $\sigma_\eta^2(t)$ that satisfies **C1-C2** in such a way that the following assumption holds
D0 There exist functions $\Delta(u)$ and $f(u)$ such that

$$\sup_{s,t\in J(u)} \left| \frac{1 - \mathbf{Cov}(\overline{X}_{\eta;u}(t), \overline{X}_{\eta;u}(s))}{\frac{\sigma_\eta^2(|t-s|)}{f^2(u)}} - 1 \right| \to 0$$

as $u \to \infty$, where $J(u) = [-\Delta(u), \Delta(u)]$ and $\eta(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_\eta^2(t)$ that satisfies **C1-C2**.

Remark 2.1 The assumption that $\sigma_\eta^2(t)$ is strictly increasing ensures that asymptotically (for large u) $\mathbf{Cov}(\overline{X}_{\eta;u}(t), \overline{X}_{\eta;u}(s))$ is a decreasing function of $|t-s|$ for $s,t \in J(u)$. It plays a crucial role in the technique of the proof of Theorems 3.1, 4.1 (Lemmas 6.1, 6.2).

In the sequel we present families of Gaussian processes that satisfy **D0**.

Example 2.1 *Let $X(t)$ be a stationary centered Gaussian process with covariance function $R(t) = \exp(-|t|^\alpha)$ $(\alpha \in (0,2])$. Straightforward calculation shows that $X(t)$ satisfies* **D0** *with $\eta(t) = B_{\alpha/2}(t)$ (and thus $\sigma_\eta^2(t) = |t|^\alpha$), $\Delta(u)$ such that $\lim_{u\to\infty}\Delta(u) = 0$ and $f(u) = 1$. This immediately implies that, for a given function $h(u) > 0$, the family $\{X_{\eta;u}(t) = X\left(\frac{t}{h(u)}\right), u\}$ satisfies* **D0** *with $\eta(t) = B_{\alpha/2}(t)$, $\Delta(u)$ such that $\lim_{u\to\infty}\Delta(u)/h(u) = 0$ and $f(u) = h^{\alpha/2}(u)$.*

Example 2.2 *Consider a centered Gaussian process $\zeta(t)$ with stationary increments and the variance function $\sigma^2_\zeta(t)$ that satisfies* **C1-C2**. *Define $X_{\eta;u}(t) = \zeta(h(u)+t)$where $h(u)$ is such that $\lim_{u\to\infty} h(u) = \infty$. In the following lemma we show that $\overline{X}_{\eta;u}(t)$ appropriately satisfies* **D0**.

Lemma 2.1 *If $\zeta(t)$ is a centered Gaussian process with stationary increments that satisfies* **C1-C2**, *then for $h(u)$ such that $\lim_{u\to\infty} h(u) = \infty$, the process $X_{\eta;u}(t) = \zeta(h(u)+t)$ satisfies* **D0** *with $f(u) = \sqrt{2}\sigma_\zeta(h(u))$, $\eta(t) = \zeta(t)$ and $\Delta(u)$ such that $\lim_{u\to\infty} \frac{\Delta(u)}{h(u)} = 0$.*

Proof. From the definition of $X_{\eta;u}(t)$

$$\begin{aligned}\mathbf{Cov}(\overline{X}_{\eta;u}(t), \overline{X}_{\eta;u}(s)) - 1 &= \frac{(\sigma_\zeta(h(u)+t) - \sigma_\zeta(h(u)+s))^2}{2\sigma_\zeta(h(u)+s)\sigma_\zeta(h(u)+t)} \\ &- \frac{\sigma^2_\zeta(|t-s|)}{2\sigma_\zeta(h(u)+s)\sigma_\zeta(h(u)+t)} = W_1 - W_2. \qquad (2.1)\end{aligned}$$

Since $\sigma_\zeta(t)$ is regularly varying at ∞ with index $\alpha_\infty \in (0,2)$ it suffices to show that $\frac{W_1}{W_2} \to 0$ uniformly for $s,t \in [-\Delta(u), \Delta(u)]$ as $u \to \infty$. This follows from the following chain of algebraic manipulations:

$$\begin{aligned}\frac{W_1}{W_2} &= \frac{(\sigma_\zeta(h(u)+t) - \sigma_\zeta(h(u)+s))^2}{\sigma^2_\zeta(|t-s|)} = \frac{(\sigma^2_\zeta(h(u)+t) - \sigma^2_\zeta(h(u)+s))^2}{\sigma^2_\zeta(|t-s|)(\sigma_\zeta(h(u)+t) + \sigma_\zeta(h(u)+s))^2} \\ &\le \frac{(\sigma^2_\zeta(h(u)+t) - \sigma^2_\zeta(h(u)+s))^2}{4\sigma^2_\zeta(|t-s|)\sigma^2_\zeta(h(u)-\Delta(u))} = \frac{1}{4}\left(\frac{|t-s|(\dot{\sigma}^2_\zeta(h(u)+\rho)}{\sigma_\zeta(|t-s|)\sigma_\zeta(h(u)-\Delta(u))}\right)^2 \qquad (2.2) \\ &\le \epsilon^2\left(\frac{\sigma_\zeta(h(u)+\rho)}{(h(u)+\rho)}\frac{|t-s|}{\sigma_\zeta(|t-s|)}\right)^2, \qquad (2.3)\end{aligned}$$

where (2.2) follows from the mean value theorem and (2.3) is a consequence of the fact that by **C1** there exists $\epsilon > 0$ such that $\dot{\sigma}^2_\zeta(h(u)+\rho) \le \epsilon \frac{\sigma^2_\zeta(h(u)+\rho)}{h(u)+\rho}$. Moreover

$$\sigma_\zeta(1)\sigma_\zeta(x) \ge \mathbf{Cov}(\zeta(1), \zeta(x)) = (\sigma^2_\zeta(1) + \sigma^2_\zeta(x) - \sigma^2_\zeta(|1-x|))/2.$$

Thus

$$\sigma^2_\zeta(1) - \sigma^2_\zeta(1-x) \le 2\sigma_\zeta(1)\sigma_\zeta(x) \qquad (2.4)$$

for sufficiently small $x > 0$. From the mean value theorem $\sigma^2_\zeta(1) - \sigma^2_\zeta(1-x) = x\dot{\sigma}^2_\zeta(1-\rho_x)$, which combined with **C1** and (2.4) implies $\frac{x}{\sigma_\zeta(x)} \le \frac{4\sigma_\zeta(1)}{\dot{\sigma}^2_\zeta(1)}$ for sufficiently small $x > 0$. Hence $\limsup_{x\to 0} \frac{|x|}{\sigma_\zeta(x)} < \infty$. Combining it with the fact that $\frac{\sigma_\zeta(t)}{t}$ is regularly varying at ∞ with index $\frac{\alpha_\infty}{2} - 1 < 0$ we obtain $\frac{W_1}{W_2} \to 0$ as $|t-s| \to 0$.

□

Remark 2.2 Families of Gaussian processes considered in Example 2.2 appeared in the analysis of some Gaussian fluid models (Massoulie & Simonian [8]). Logarithmic asymptotics of supremum of such families of Gaussian processes was obtained by Dębicki [2].

We need the following notation. Let $\overline{X}_{\eta_1;u}(t)$, $\overline{X}_{\eta_2;u}(t)$,..., $\overline{X}_{\eta_N;u}(t)$ be independent families of centered Gaussian processes that satisfy **D0** with common $\Delta(u) = T > 0$ and $f(u)$. Define

$$\overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{X}_{\eta_i;u}(t_i).$$

Theorem 2.1 *Let $n(u)$ be such that $\lim_{u\to\infty} n(u) = \infty$ and $\lim_{u\to\infty}\frac{f(u)}{n(u)} = 1$. Then*

$$\mathbb{P}\left(\sup_{(t_1,...,t_N)\in[0,T]^N}\overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) > n(u)\right) = \mathcal{H}_{\eta_1,...,\eta_N}(T)\Psi(n(u))(1+o(1)) \quad \text{as} \quad u\to\infty \tag{2.5}$$

Proof. We present the proof of Theorem 2.1 in Section 6.1.

□

3 Generalized Pickands constants

In this section we define and study properties of *generalized Pickands constants.* We begin with a subadditivity property of $\mathcal{H}_{\eta_1,...,\eta_N}(T)$.

Lemma 3.1 *If $\eta_1(t),....,\eta_N(t)$ are independent centered Gaussian processes with stationary increments that satisfy* **C1-C2***, then for all $T\in\mathbb{N}$*

$$\mathcal{H}_{\eta_1,...,\eta_N}(T) \leq T^N\mathcal{H}_{\eta_1,...,\eta_N}(1). \tag{3.1}$$

Proof. The complete proof is presented in Section 6.

□

In the rest of this section we concentrate on the one-dimensional case of $\mathcal{H}_\eta(T)$. Note that the same argument as in the proof of Lemma 3.1 yields $\mathcal{H}_\eta(x+y) \leq \mathcal{H}_\eta(x)+\mathcal{H}_\eta(y)$ for all $x, y > 0$.

The main result of this section is given in the following theorem.

Theorem 3.1 *If the variance function $\sigma^2_\eta(t)$ of a centered Gaussian process $\eta(t)$ with stationary increments satisfies* **C1-C2***, then*

$$\lim_{T\to\infty}\frac{\mathcal{H}_\eta(T)}{T} = \mathcal{H}_\eta, \tag{3.2}$$

where $\mathcal{H}_\eta > 0$ and is finite.

Proof. The proof of Theorem 3.1 is given in Section 6.2.

□

If $\eta(t) = B_{\alpha/2}(t)$ is a fractional Brownian motion with Hurst parameter $\alpha/2$ ($\alpha \in (0,2)$), then it is known that Theorem 3.1 holds (see Piterbarg [13], page 16, Theorem D.2). $\mathcal{H}_{B_{\alpha/2}}$ are known in the literature as the *Pickands constants.*

By the *generalized Pickands constants* we mean the constants $\mathcal{H}_\eta$ introduced in Theorem 3.1.

In the following theorem we give a criterion that enables us to compare the generalized Pickands constants $\mathcal{H}_\eta$.

Theorem 3.2 *Let $\eta_1(t), \eta_2(t)$ be centered Gaussian processes with stationary increments that satisfy* **C1-C2**. *If for all $t \geq 0$*

$$\sigma^2_{\eta_1}(t) \leq \sigma^2_{\eta_2}(t), \tag{3.3}$$

then

$$\mathcal{H}_{\eta_1} \leq \mathcal{H}_{\eta_2}. \tag{3.4}$$

Proof. The complete proof is presented in Section 6.3.

□

Remark 3.1 Observe that the conclusion of Theorem 3.2 holds also for $\eta_2 = B_1(t)$ (that is for $\sigma^2_\eta(t) = t^2$). The proof of this fact is analogous to the proof of Theorem 3.2 with the exception that instead of $\overline{X}^{(\delta)}_{\eta_2;u}(t)$ we take $X((1+\delta)t/(\sqrt{2}u))$, where $X(t)$ is a stationary Gaussian process with covariance function $R(t) = \exp(-|t|^2)$.

Corollary 3.1 *If the variance function $\sigma^2_\eta(t)$ of $\eta(t) = \int_0^t Z(s)ds$ satisfies* **C1-C2**, *where $Z(s)$ is a stationary centered Gaussian process with covariance function $R(t)$, then*

$$\mathcal{H}_\eta \leq \sqrt{\frac{R(0)}{\pi}}.$$

Proof. Note that

$$\sigma^2_\eta(t) = \int_0^t \int_0^t \mathbf{Cov}(Z(v), Z(w))dvdw \leq R(0)t^2 = R(0)\sigma^2_{B_1}(t).$$

Thus from Theorem 3.2 and Remark 3.1 $\mathcal{H}_\eta \leq \mathcal{H}_{\sqrt{R(0)}B_1}$. Since $\mathcal{H}_{\sqrt{R(0)}B_1}(T) = \mathcal{H}_{B_1}(\sqrt{R(0)}T)$, then $\mathcal{H}_{\sqrt{R(0)}B_1} = \sqrt{R(0)}\mathcal{H}_{B_1}$. Hence

$$\begin{aligned}\mathcal{H}_\eta &\leq \mathcal{H}_{\sqrt{R(0)}B_1} = \sqrt{R(0)}\mathcal{H}_{B_1} \\ &= \sqrt{\frac{R(0)}{\pi}}, \end{aligned} \tag{3.5}$$

where (3.5) follows from the fact that $\mathcal{H}_{B_1} = 1/\sqrt{\pi}$. This completes the proof.

□

In the following corollary we find an upper bound for $\mathcal{H}_\eta$ in the case of $\eta(t)$ with covariance function $\sigma^2_\eta(t)$ fulfilling some integrability conditions.

Corollary 3.2 *If $\eta(t) = \int_0^t Z(s)ds$ satisfies* **SRD.1**, **SRD.3** *and $R(t) \geq 0$ for each $t \geq 0$, where $Z(t)$ is a centered stationary Gaussian process with covariance function $R(t)$, then*

$$\mathcal{H}_\eta \leq 2\int_0^\infty R(s)ds.$$

Proof. Let $\Upsilon = 2\int_0^\infty R(v)\,dv$. From **SRD.1**,**SRD.3** and the fact that $R(t) \geq 0$ for each $t \geq 0$ we infer that

$$\sigma_\eta^2(t) = 2\int_0^t \int_0^s R(v)\,dv\,ds \tag{3.6}$$

$$= \Upsilon t - 2\int_0^\infty vR(v)\,dv + 2\int_t^\infty (v-t)R(v)\,dv$$

$$\leq \Upsilon t = \sigma^2_{\sqrt{\Upsilon}B_{1/2}}(t) \tag{3.7}$$

and $\eta(t)$ satisfies **C1-C2** with $\alpha_0 = 2$ and $\alpha_\infty = 1$.

Analogous considerations as in the proof of Corollary 3.1 yield

$$\mathcal{H}_{\sqrt{\Upsilon}B_{1/2}}(T) = \mathcal{H}_{B_{1/2}}(\Upsilon T). \tag{3.8}$$

Since $\mathcal{H}_{B_{1/2}} = 1$, then combining (3.8) with (3.7) and Theorem 3.2 we complete the proof.

□

4 Double sum method

Theorem 2.1 enables us to obtain exact asymptotics for some families of Gaussian processes with variance function that attains its maximum at a unique point.

For the introduced in Section 2 family $\{X_{\eta;u}(t); u > 0\}$ of centered Gaussian processes we additionally assume that for sufficiently large $u > 0$ the function $\sigma_{X_{\eta;u}}(t)$ attains its maximum at a unique point t_u with $0 < t_u < \infty$. Without loss of generality we assume $\sigma^2_{X_{\eta;u}}(t_u) = 1$. Furthermore we claim that $\{X_{\eta;u}(t); u > 0\}$ satisfies the following conditions.

D1 Condition **D0** is fulfilled for $(t,s) := (t + t_u, s + t_u)$.

D2 There exist $\beta > 0$ and a function $g(u)$ such that

$$\sup_{s,t\in J(u)} \left| \frac{1 - \sigma_{X_{\eta;u}}(t+t_u)}{\frac{|t|^\beta}{g^2(u)}} - 1 \right| \to 0 \quad \text{as } u \to \infty.$$

D3 $\frac{f(u)}{g(u)} \to 0$ as $u \to \infty$.

Theorem 4.1 *If the family* $\{X_{\eta;u}(t)\}$ *satisfies* **D1-D3** *with* $\Delta(u) = \left(\frac{g(u)\log(n(u))}{n(u)}\right)^{2/\beta}$, *where* $n(u)$ *is such that* $\lim_{u\to\infty} n(u) = \infty$ *and* $\lim_{u\to\infty} \frac{f(u)}{n(u)} = 1$, *then*

$$\mathbb{P}\left(\sup_{t\in J(u)} X_{\eta;u}(t+t_u) > n(u)\right) = \frac{2\mathcal{H}_\eta\Gamma(1/\beta)}{\beta}(A(u))^{-1}\Psi(n(u))(1+o(1)) \tag{4.1}$$

as $u \to \infty$ *and* $A(u) = \left(\frac{n(u)}{g(u)}\right)^{2/\beta}$.

Proof. The proof is given in Section 6.4.

□

Remark 4.1 Note that, under conditions of Theorem 4.1, if $J(u) = \left[0, \left(\frac{g(u)\log(n(u))}{n(u)}\right)^{2/\beta}\right]$, then $\mathbb{P}\left(\sup_{t\in J(u)} X_{\eta;u}(t+t_u) > n(u)\right) = \frac{\mathcal{H}_\eta \Gamma(1/\beta)}{\beta}(A(u))^{-1}\Psi(n(u))(1+o(1))$ as $u \to \infty$.

In the rest of this section we discuss the special case of Theorem 4.1, where we assume that in condition **D1** we have $\eta(t) = B_{\alpha/2}(t)$ for $\alpha \in (0,2]$. The property of multiplicativity of the variance function $\sigma^2_{B_{\alpha/2}}(t) = t^\alpha$ of fractional Brownian motion $B_{\alpha/2}(t)$ enables us to relax the assumption that $f(u)$ in **D1** is of the same order as $n(u)$.

Theorem 4.2 *If the family $X_{\eta;u}(t)$ satisfies* **D1-D2** *with $\eta(t) = B_{\alpha/2}(t)$ for $\alpha \in (0,2]$ and $\Delta(u) = \left(\frac{g(u)\log(n(u))}{n(u)}\right)^{2/\beta}$, where $n(u)$ is such that $\lim_{u\to\infty} n(u) = \infty$, then*

$$\mathbb{P}(\sup_{t\in I(u)} X_{\eta;u}(t+t_u) > n(u)) = \frac{2\mathcal{H}_{B_{\alpha/2}}\Gamma(1/\beta)}{\beta}\left(\frac{g(u)}{n(u)}\right)^{2/\beta}\left(\frac{n(u)}{f(u)}\right)^{2/\alpha}\Psi(n(u))(1+o(1))$$

as $u \to \infty$.

Proof. The proof is presented in Section 6.5. □

5 Exact asymptotics of $\mathbb{P}(\sup_{t\geq 0}\int_0^t Z(s)\,ds - ct > u)$

In this section we find the exact asymptotics of $\psi(u) = \mathbb{P}(\sup_{t\geq 0}\zeta(t) - ct > u)$ for the **SRD** model. Let $G = \frac{1}{\int_0^\infty R(t)\,dt}$ and $B = \int_0^\infty tR(t)\,dt$.

Theorem 5.1 *If $\zeta(t)$ possesses the* **SRD** *property, then*

$$\psi(u) = \frac{\mathcal{H}_{\frac{cG}{\sqrt{2}}\zeta}}{Gc^2} e^{-c^2G^2B} e^{-Gcu}(1+o(1)) \tag{5.1}$$

Proof. The proof of Theorem 5.1 is presented in Section 6.6. □

Remark 5.1 Since $\dot{\sigma}^2_\zeta(t) = 2\int_0^t R(s)ds$, then **SRD.2** is equivalent to the fact that $\sigma^2_\zeta(t)$ is strictly increasing. It ensures that $\mathcal{H}_{\frac{cG}{\sqrt{2}}\zeta}$ exists (Theorem 3.1) and assumption **D1** of Theorem 4.1 is satisfied. In the language of the spectral density function $f_R(t)$ of the covariance function $R(t)$ we have

$$\begin{aligned}
\int_0^t R(s)ds &= 2\int_0^t\int_0^\infty cos(xs)f_R(x)dxds = \\
&= 2\int_0^\infty \frac{sin(xt)}{x} f_R(x)dx && (5.2)\\
&= 2\int_0^\infty \frac{sin(y)}{y} f_R\left(\frac{y}{t}\right)dy. && (5.3)
\end{aligned}$$

Hence if $0 < f_R(0) < \infty$ and $\frac{f_R(x)}{x}$ is nonincreasing for $x \geq 0$, then from (5.2) we have $\int_0^t R(s)ds > 0$ for each $t > 0$. Moreover from (5.3) we have $\int_0^\infty R(s)ds = \lim_{t\to\infty}\int_0^t R(s)ds = \pi f_R(0)$. In this case $G = \frac{1}{\pi f_R(0)}$.

Remark 5.2 Using Corollary 3.1 we are able to give an asymptotical upper bound:

$$\limsup_{u\to\infty} \frac{\mathbb{P}(\sup_{t\geq 0}\zeta(t)-t>u)}{\sqrt{\frac{R(0)}{2\pi}}e^{-G^2B}e^{-Gu}} \leq 1. \tag{5.4}$$

This result is consistent with the asymptotical upper bound obtained by Dębicki & Rolski [3].

6 Proofs

In this section we prove theorems presented in Sections 2-5.

6.1 Proof of Theorem 2.1

The idea of the proof is analogous to the proof of Pickands lemma presented in Piterbarg [13] (Lemma D.1) and is based on the fact that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(t_1,...,t_N)\in[0,T]^N} \overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) > n(u)\right) = \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \exp(-v^2/2)\mathbb{P}\left(\sup_{(t_1,...,t_N)\in[0,T]^N} \overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) > n(u)|\overline{X}_{\eta_1,...,\eta_N;u}(0,...,0)=v\right)dv \\
&= \Psi(n(u))(1+o(1))\int_{\mathbf{R}} \exp\left(\omega - \frac{\omega^2}{2n^2(u)}\right)\times \\
&\quad \times\mathbb{P}\left(\sup_{(t_1,...,t_N)\in[0,T]^N} \xi_u(t_1,...,t_N) > \omega|\overline{X}_{\eta_1,...,\eta_N;u}(0,...,0) = \frac{n^2(u)-\omega}{n(u)}\right)d\omega,
\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}
&\lim_{u\to\infty}\int_{\mathbf{R}} \exp(\omega - \frac{\omega^2}{2n^2(u)})\times \\
&\quad \times\mathbb{P}(\sup_{(t_1,...,t_N)\in[0,T]^N} \xi_u(t_1,...,t_N) > \omega|\overline{X}_{\eta_1,...,\eta_N;u}(0,...,0) = n(u) - \frac{\omega}{n(u)})\,d\omega = \\
&= \mathcal{H}_{\eta_1,...,\eta_N}(T),
\end{aligned} \tag{6.2}$$

where (6.1) is a consequence of changing of variables $v = n(u) - \frac{\omega}{n(u)}$ and the notation $\xi_u(t_1,...,t_N) = n(u)(\overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) - n(u)) + \omega$. Equality (6.2) is a consequence of the fact that the family of processes

$$\chi_u(t_1,...,t_N) = \xi_u(t_1,...,t_N)\Bigg|\left(\overline{X}_{\eta_1,...,\eta_N;u}(0,...,0) = n(u) - \frac{\omega}{n(u)}\right), \qquad 0\leq t_1,...,t_N\leq T$$

converges weakly in $C[0,T]^N$ to the Gaussian process

$$\chi(t_1,...,t_N) = \sqrt{\frac{2}{N}}\sum_{i=1}^N \eta_i(t_i) - \frac{1}{N}\sum_{i=1}^N \sigma^2_{\eta_i}(t_i).$$

The proof of the weak convergence is analogous to the relevant part of the proof of Lemma D.1 in [13] and is based on the suspection of the convergence of finite dimensional distributions of the appropriate processes and tightness of family $\chi_u(t_1,...,t_N)$. In the sequel we argue that $\chi_u(t_1,...,t_N)$ is tight.

In order to prove the tightness of $\chi_u(t_1,...,t_N)$ it suffices to show that the sequence of centered processes $\chi_u^{(0)}(t_1,...,t_N) = \chi_u(t_1,...,t_N) - \mathbb{E}\chi_u(t_1,...,t_N)$ is tight. Since $\chi_u^{(0)}(0,...,0) = 0$ for all $u > 0$, then a straightforward consequence of Straf's criterion for tightness of Gaussian fields [16] implies that it suffices to show that for any $\mu, \varrho > 0$ there exists $\delta \in (0,1)$ and $u_0 > 0$ such that

$$\mathbb{P}\left(\sup_{\{(s_1,...,s_N):\|(s_1,...,s_N)-(t_1,...,t_N)\|\le\delta\}} |\chi_u^{(0)}(s_1,...,s_N) - \chi_u^{(0)}(t_1,...,t_N)| \ge \mu\right) \le \varrho\delta^N \quad (6.3)$$

for each $(t_1,...,t_N) \in [0,T]^N$ and $u > u_0$, where $||(t_1,...,t_N)|| = \max_{i\in\{1,...,N\}} |t_i|$.

Note that for sufficiently large u

$$\mathbb{E}(\chi_u^{(0)}(t_1,...,t_N) - \chi_u^{(0)}(s_1,...,s_N))^2 \le \sum_{i=1}^N \sigma_{\eta_i}^2(|t_i - s_i|) \le C^2 \sum_{i=1}^N |t_i - s_i|^{\alpha_{i,0}}$$

for all $(t_1,...,t_N), (s_1,...,s_N) \in [0,T]^N$, some constant $C > 0$ and $\alpha_{i,0}$ being the indices of regularity at 0 of $\sigma_{\eta_i}^2(t)$ respectively. Thus

$$\max_{\{(s_1,...,s_N),(t_1,...,t_N):\|(s_1,...,s_N)-(t_1,...,t_N)\|\le\delta\}} \mathrm{Var}(\chi_u^{(0)}(s_1,...,s_N) - \chi_u^{(0)}(t_1,...,t_N)) \le C^2 \sum_{i=1}^N |\delta|^{\alpha_{i,0}}.$$

which combined with Borell's theorem gives (6.3).

□

6.2 Proof of Theorem 3.1

Before the proof of Theorem 3.1 we need some technical lemmas that are also of independent interest. We begin with the proof of Lemma 3.1.

Proof of Lemma 3.1. It suffices to note that under notation of Theorem 2.1, for sufficiently large u,

$$\mathbb{P}\left(\sup_{(t_1,...,t_N)\in[0,T]^N} \overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) > n(u)\right) \le$$
$$\le \sum_{k_1=1}^T ... \sum_{k_N=1}^T \mathbb{P}\left(\sup_{(t_1,...,t_N)\in\prod_{i=1}^N[k_i-1,k_i]} \overline{X}_{\eta_1,...,\eta_N;u}(t_1,...,t_N) > n(u)\right).$$

Now applying Theorem 2.1 to both sides of the above inequality we complete the proof.

□

The following lemmas play a crucial role in sequel.

Lemma 6.1 *If the variance function $\sigma_\eta^2(t)$ of a centered Gaussian process $\eta(t)$ with stationary increments satisfies* **C1-C2**, *then for each $C > 1$ there exists $\varepsilon > 0$ such that*

$$\inf_{t>0} \frac{\sigma_\eta^2(Ct)}{\sigma_\eta^2(t)} \ge 1 + \varepsilon.$$

Moreover for each $\varepsilon \in (0,1)$ there exists $C > 1$ such that

$$\sup_{t>0} \frac{\sigma_\eta^2(t)}{\sigma_\eta^2(Ct)} \le 1 - \varepsilon.$$

Proof. The proof of Lemma 6.1 follows from assumption **C2** that $\sigma_\eta^2(t)$ is regularly varying at 0 and at ∞ and the fact that $\sigma_\eta^2(t)$ is strictly increasing.

$\square$

Lemma 6.2 *If family $\{\overline{X}_{\eta;u}(t); u > 0\}$ of centered Gaussian processes with continuous sample paths satisfies* **D0** *with $\Delta(u)$ such that $\lim_{u\to\infty}\Delta(u) = \infty$ and*

$$\lim_{u\to\infty} \frac{\sigma_\eta^2(\Delta(u))}{f^2(u)} < 1/2, \tag{6.4}$$

then for each $T > 0$, $\delta > 0$ and $n(u)$ such that $\lim_{u\to\infty} f(u)/n(u) = 1$

$$\mathbb{P}\left(\sup_{s\in[0,T]} \overline{X}_{\eta;u}(s) > n(u); \sup_{t\in[\delta+T,\delta+2T]} \overline{X}_{\eta;u}(t) > n(u)\right) \le C_2 T^2 \exp(-C_1\sigma_\eta^2(\delta))\Psi(n(u))(1+o(1)) \tag{6.5}$$

as $u \to \infty$. Inequality (6.5) holds uniformly with respect to u for $\delta \le \Delta(u) - 2T$.

Proof. The idea of the proof is analogous to the proof of Lemma 6.3 in [13] and thus we present only the main steps of the argumentation.

Consider the Gaussian field $\boldsymbol{Y}_u(s,t) = \overline{X}_{\eta;u}(s) + \overline{X}_{\eta;u}(t)$ and let $A_0 = [0,T]$, $A_{\delta+T} = [\delta+T, \delta+2T]$ for $0 \le \delta \le \Delta(u) - 2T$. We have

$$\mathbb{P}\left(\sup_{t\in[0,T]} \overline{X}_{\eta;u}(t) > n(u); \sup_{t\in[\delta+T,\delta+2T]} \overline{X}_{\eta;u}(t) > n(u)\right) \le \mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \boldsymbol{Y}_u(s,t) > 2n(u)\right) \tag{6.6}$$

Note that for each $s \in A_0$, $t \in A_{\delta+T}$ and sufficiently large u

$$\mathrm{Var}(\boldsymbol{Y}_u(s,t)) \ge 4 - 4\frac{\sigma_\eta^2(|t-s|)}{f^2(u)} \ge 2 \tag{6.7}$$

and

$$\mathrm{Var}(\boldsymbol{Y}_u(s,t)) \le 4 - \frac{\sigma_\eta^2(|t-s|)}{f^2(u)} \le 4 - \frac{\sigma_\eta^2(\delta)}{f^2(u)}, \tag{6.8}$$

where (6.7) follows from (6.4). Let $\overline{\mathbf{Y}}_u(s,t) = \frac{\mathbf{Y}_u(s,t)}{\sqrt{\mathrm{Var}(\mathbf{Y}_u(s,t))}}$ and observe that

$$\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \mathbf{Y}_u(s,t) > 2n(u)\right) \le \mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \overline{\mathbf{Y}}_u(s,t) > \frac{2n(u)}{\sqrt{4 - \frac{\sigma_\eta^2(\delta)}{f^2(u)}}}\right). \tag{6.9}$$

Moreover for each $s, s_1 \in A_0$ and $t, t_1 \in A_{\delta+T}$

$$\begin{aligned}
\mathbb{E}(\overline{\mathbf{Y}}_u(s,t) - \overline{\mathbf{Y}}_u(s_1,t_1))^2 &\le \frac{4}{\mathrm{Var}(\boldsymbol{Y}_u(s,t))}\mathbb{E}(\boldsymbol{Y}_u(s,t) - \boldsymbol{Y}_u(s_1,t_1))^2 \\
&\le 4(\mathbb{E}(\overline{X}_{\eta;u}(s) - \overline{X}_{\eta;u}(s_1))^2 + \mathbb{E}(\overline{X}_{\eta;u}(t) - \overline{X}_{\eta;u}(t_1))^2) \\
&\le \frac{1}{2}(\mathbb{E}(\overline{X}_{\eta;u}(C_0 s) - \overline{X}_{\eta;u}(C_0 s_1))^2 + \mathbb{E}(\overline{X}_{\eta;u}(C_0 t) - \overline{X}_{\eta;u}(C_0 t_1))^2),
\end{aligned} \tag{6.10}$$

where the existence of constant C_0 in (6.10) follows from Lemma 6.1. Hence for $X^{(1)}_{\eta;u}(t), X^{(2)}_{\eta;u}(t)$ being independent copies of the process $\overline{X}_{\eta;u}(t)$ the covariance function of the process $\frac{1}{\sqrt{2}}(X^{(1)}_{\eta;u}(C_0 s) + X^{(2)}_{\eta;u}(C_0 t))$ is dominated by the covariance function of $\overline{\mathbf{Y}}_u(s,t)$. Thus from Slepian's inequality (see [13], Theorem C.1)

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \overline{\mathbf{Y}}_u(s,t) > \frac{2n(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{f^2(u)}}}\right) \leq \\
&\leq \mathbb{P}\left(\sup_{(s,t)\in A_0^2} \frac{1}{\sqrt{2}}(X^{(1)}_{\eta;u}(C_0 s) + X^{(2)}_{\eta;u}(C_0 t)) > \frac{2n(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{f^2(u)}}}\right) \qquad (6.11)\\
&= \mathcal{H}_{\eta,\eta}(C_0 T)\Psi\left(\frac{2n(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{f^2(u)}}}\right)(1+o(1)) \qquad (6.12)\\
&\leq C_2 T^2 \exp(-C_1\sigma_\eta^2(\delta))\Psi(n(u))(1+o(1)), \qquad (6.13)
\end{aligned}$$

where (6.11) holds uniformly with respect to u for $\delta \leq \Delta(u) - 2T$ and (6.12) follows from the combination of Theorem 2.1 and Lemma 3.1. Thus the assertion of Lemma 6.2 follows by combining (6.6), (6.9) and (6.13).

□

Proof of Theorem 3.1: Since $\mathcal{H}_\eta(T)$ is subadditive, it suffices to prove that

$$\liminf_{T\to\infty} \frac{\mathcal{H}_\eta(T)}{T} > 0.$$

The above follows from the same argumentation, as in the proof of the existence of classical *Pickands constants* presented in [13] (the proof of Theorem D.2 in [13]), applied to the family $X_{\eta;u}(t) = \eta(u+t)$.

□

6.3 Proof of Theorem 3.2

Let $\delta > 0$ be given. Define

$$\overline{X}^{(\delta)}_{\eta_1;u}(t) = \frac{\eta_1(\sigma_{\eta_1}^{-1}(u) + (1+\delta)t)}{\sigma_{\eta_1}(\sigma_{\eta_1}^{-1}(u) + (1+\delta)t)}$$

$$\overline{X}^{(\delta)}_{\eta_2;u}(t) = \frac{\eta_2(\sigma_{\eta_2}^{-1}(u) + (1+\delta)t)}{\sigma_{\eta_2}(\sigma_{\eta_2}^{-1}(u) + (1+\delta)t)}$$

and observe that from **C1-C2** the inverse functions $\sigma_{\eta_1}^{-1}(u), \sigma_{\eta_2}^{-1}(u)$ are well defined.

From Lemma 6.1 there exists $\epsilon > 0$ such that

$$\sigma_{\eta_2}^2((1+\delta)t) \geq (1+\epsilon)^2\sigma_{\eta_2}^2(t) \qquad (6.14)$$

for each $t \geq 0$.

Let $T>0$ be given. From Lemma 2.1 processes $\overline{X}^{(\delta)}_{\eta_1;u}(t)$, $\overline{X}^{(\delta)}_{\eta_2;u}(t)$ satisfy **D0** with $f(u)=\sqrt{2}u$, $\Delta(u)=T$ and $\eta=\eta_1$ or $\eta=\eta_2$ respectively. Thus for $s,t\in[0,T]$ and sufficiently large u

$$\begin{aligned}
1-\mathbf{Cov}\left(\overline{X}^{(\delta)}_{\eta_2;u}(t),\overline{X}^{(\delta)}_{\eta_2;u}(s)\right) &\geq \frac{1}{1+\epsilon}\frac{\sigma^2_{\eta_2}((1+\delta)|t-s|)}{2u^2}\\
&\geq (1+\epsilon)\frac{\sigma^2_{\eta_2}(|t-s|)}{2u^2} \qquad (6.15)\\
&\geq (1+\epsilon)\frac{\sigma^2_{\eta_1}(|t-s|)}{2u^2},\\
&\geq 1-\mathbf{Cov}(\overline{X}^{(0)}_{\eta_1;u}(t),\overline{X}^{(0)}_{\eta_2;u}(s)), \qquad (6.16)
\end{aligned}$$

where (6.15) follows from (6.14) and (6.16) follows from the fact that $\sigma^2_{\eta_1}(t)\leq\sigma^2_{\eta_2}(t)$.

Hence for each $\delta>0$, $t>0$ and sufficiently large u we can apply Slepian's inequality

$$\begin{aligned}
\mathbb{P}(\sup_{t\in[0,(1+\delta)T]}\overline{X}^{(0)}_{\eta_2;u}(t)>\sqrt{2}u) &= \mathbb{P}(\sup_{t\in[0,T]}\overline{X}^{(\delta)}_{\eta_2;u}(t)>\sqrt{2}u)\\
&\geq \mathbb{P}(\sup_{t\in[0,T]}\overline{X}^{(0)}_{\eta_1;u}(t)>\sqrt{2}u). \qquad (6.17)
\end{aligned}$$

To complete the proof it is enough to note that from Theorem 2.1

$$\mathbb{P}(\sup_{t\in[0,(1+\delta)T]}\overline{X}^{(0)}_{\eta_2;u}(t)>\sqrt{2}u) = \mathcal{H}_{\eta_2}((1+\delta)T)\Psi(\sqrt{2}u)(1+o(1))$$

and

$$\mathbb{P}(\sup_{t\in[0,T]}\overline{X}^{(0)}_{\eta_1;u}(t)>\sqrt{2}u) = \mathcal{H}_{\eta_1}(T)\Psi(\sqrt{2}u)(1+o(1))$$

as $u\to\infty$. Combining this with (6.17) we obtain that $\mathcal{H}_{\eta_2}((1+\delta)T)\geq\mathcal{H}_{\eta_1}(T)$ for each $\delta>0$. Having in mind that $\mathcal{H}_{\eta_1}=\lim_{T\to\infty}\frac{\mathcal{H}_{\eta_1}(T)}{T}$ and $\mathcal{H}_{\eta_2}=\lim_{T\to\infty}\frac{\mathcal{H}_{\eta_2}(T)}{T}$ the proof is completed.

□

6.4 Proof of Theorem 4.1

The idea of the proof is analogous to the proof of Theorem D.3 [13] and thus we present only the main steps of the argumentation.

In the proof we denote for short $\theta(u)=\mathbb{P}(\sup_{t\in J(u)}X_{\eta;u}(t+t_u)>n(u))$. From **D2** for each $\epsilon\in(0,1)$ there exists u_0 such that for $u\geq u_0$ and $t\in J(u)$

$$\theta(u) \leq \mathbb{P}\left(\sup_{t\in J(u)}\overline{X}_{\eta;u}(t+t_u)\frac{1}{1+(1-\epsilon)\frac{|t|^\beta}{g^2(u)}}>n(u)\right)=\theta_1(u)$$

and

$$\theta(u) \geq \mathbb{P}\left(\sup_{t\in J_u}\overline{X}_{\eta;u}(t+t_u)\frac{1}{1+(1+\epsilon)\frac{|t|^\beta}{g^2(u)}}>n(u)\right)=\theta_2(u).$$

The rest of the proof consists of two parts, where an upper and lower bound for $\theta(u)$ is derived.

1°. (Upper bound.) Our goal is to prove that

$$\limsup_{u\to\infty} \frac{\theta(u)}{\frac{2\mathcal{H}_\eta\Gamma(1/\beta)}{\beta}(A(u))^{-1}\Psi(n(u))} \le 1\ . \tag{6.18}$$

Since $\theta(u) \le \theta_1(u)$, we focus on the asymptotics of $\theta_1(u)$. Let $T > 0$ be given and let $\Delta(u) = \left(\frac{g(u)\log(n(u))}{n(u)}\right)^{2/\beta}$. Note that $\Delta(u) \to \infty$ as $u \to \infty$. We consider a skeleton $J_k = [kT, (k+1)T]$ of $\mathbb{R}$ and define events

$$C_k(u) = \begin{cases} \max_{t\in J_k}\{\overline{X}_{\eta;u}(t+t_u) > n(u)(1+(1-\epsilon)\frac{|(k+1)T|^\beta}{g^2(u)})\} & k=-1,-2,\dots \\ \max_{t\in J_k}\{\overline{X}_{\eta;u}(t+t_u) > n(u)(1+(1-\epsilon)\frac{|kT|^\beta}{g^2(u)})\} & k=0,1,\dots \end{cases} \tag{6.19}$$

Now using the Bonferroni's inequality and Theorem 2.1 we get

$$\begin{aligned}
\theta_1(u) &\le \sum_{-\frac{\Delta(u)}{T}-1\le k\le \frac{\Delta(u)}{T}} \mathbb{P}(C_k(u)) \\
&= \sum_{-\frac{\Delta(u)}{T}-1\le k\le 0} \mathcal{H}_\eta(T)\Psi\left(n(u)(1+(1-\epsilon)\frac{|(k+1)T|^\beta}{g^2(u)})\right)(1+o(1)) \\
&+ \sum_{0<k\le \frac{\Delta(u)}{T}} \mathcal{H}_\eta(T)\Psi\left(n(u)(1+(1-\epsilon)\frac{|kT|^\beta}{g^2(u)})\right)(1+o(1)). \\
&\le \frac{\mathcal{H}_\eta(T)\Psi(n(u))}{TA(u)} \sum_{-\frac{\Delta(u)}{T}-1\le k\le 0} TA(u)\exp\left(-(1-\epsilon)\left(TA(u)|(k+1)|\right)^\beta\right)(1+o(1)) \\
&+ \frac{\mathcal{H}_\eta()\Psi(n(u))}{TA(u)} \sum_{0<k\le \frac{\Delta(u)}{T}} TA(u)\exp\left(-(1-\epsilon)\left(TA(u)k\right)^\beta\right)(1+o(1))
\end{aligned} \tag{6.20}$$

as $u \to \infty$, where $A(u) = (\frac{n(u)}{g(u)})^{2/\beta}$ Since $\lim_{u\to\infty} A(u) = 0$ (see **D3** and assumption that $\lim_{u\to\infty}\frac{f(u)}{n(u)} = 1$), then letting $u \to \infty$ and $T \to \infty$ in such a way that $TA(u) \to 0$, we obtain

$$\limsup_{u\to\infty} \frac{\theta_1(u)}{2\mathcal{H}_\eta\frac{\Psi(n(u))}{A(u)}\int_0^\infty e^{-(1-\epsilon)x^\beta}\,dx.} \le 1.$$

Using that $\beta\int_0^\infty e^{-x^\beta}\,dx = \Gamma(1/\beta)$ and letting $\epsilon \to 0$ we obtain (6.18).
2°. (Lower bound) To get

$$\liminf_{u\to\infty} \frac{\theta(u)}{\frac{2\mathcal{H}_\alpha\Gamma(1/\beta)}{\beta}(A(u))^{-1}\Psi(n(u))} \ge 1 \tag{6.21}$$

we have to adapt the preceding proof as follows.
Define events

$$C'_k(u) = \begin{cases} \max_{t\in J_k}\{\overline{X}_{\eta;u}(t+t_u) > n(u)(1+(1+\epsilon)\frac{|kT|^\beta}{g^2(u)})\} & k=-1,-2,\dots \\ \max_{t\in J_k}\{\overline{X}_{\eta;u}(t+t_u) > n(u)(1+(1+\epsilon)\frac{|(k+1)T|^\beta}{g^2(u)})\} & k=0,1,\dots \end{cases} \tag{6.22}$$

Using Bonferroni's inequality

$$\theta_2(u) \geq \sum_{-\frac{\Delta(u)}{T} \leq k \leq \frac{\Delta(u)}{T}-1} \mathbb{P}(C'_k(u)) - 2 \sum_{-\frac{\Delta(u)}{T} \leq k < l \leq \frac{\Delta(u)}{T}-1} \mathbb{P}(C'_k(u) \cap C'_l(u)).$$

Thus it suffices to prove that

$$\lim_{u\to\infty} \frac{\sum_{-\frac{\Delta(u)}{T} \leq k < l \leq \frac{\Delta(u)}{T}-1} \mathbb{P}(C'_k(u) \cap C'_l(u))}{(A(u))^{-1}\Psi(n(u))} = 0,$$

which, using Lemma 6.2, follows by the same argumentation as the estimation of the double sum in the proof of Theorem D.1 in [13]. This completes the proof.

□

6.5 Proof of Theorem 4.2

The proof follows from the straightforward application of Theorem 4.1 to the family

$$Y_{\eta;u}(t+t_u) = X_{\eta;u}\left(\left(\frac{f(u)}{n(u)}\right)^{2/\alpha} t + t_u\right).$$

□

6.6 Proof of Theorem 5.1

We give the proof of Theorem 5.1 after a sequence of lemmas.

The idea of the proof of Theorem 5.1 is based on an appropriate application of Theorem 4.1. Namely since

$$\psi(u) \quad = \quad \mathbb{P}\Big(\sup_{t\geq 0} \zeta(t) - ct > u\Big) = \mathbb{P}\left(\sup_{t\geq 0} \frac{\zeta(t)}{c} - t > \frac{u}{c}\right)$$

it suffices to give the proof for $c = 1$. Thus without loss of generality we assume that $c = 1$.

We rewrite

$$\mathbb{P}\Big(\sup_{t\geq 0}(\zeta(t) - t) > u\Big) = \mathbb{P}(\sup_{t\geq 0} X_{\zeta;u}(t) > m(u)),$$

where $X_{\zeta;u}(t) = \frac{\zeta(t)}{u+t} m(u)$ and $m(u) = \min_{t\geq 0} \frac{(u+t)}{\sigma_\zeta(t)}$.

Remark 6.1 Condition **SRD** yields

$$\sigma_\zeta^2(t) = 2\int_0^t \int_0^s R(v)\, dv\, ds = \frac{2}{G} t - 2B + r(t), \tag{6.23}$$

where

$$r(t) = 2\int_t^\infty (s-t)R(s)\, ds = o(t^{-1})$$

(see Dębicki and Rolski [3] or Dębicki [2]). This shows for example that $\sigma_\zeta^2, \sigma_\zeta \in C^2$. From the above we immediately conclude

$$\dot{\sigma}_\zeta^2(t) = \frac{2}{G} + r_1(t), \tag{6.24}$$

with $r_1(t) = o(1)$. Note also that from **SRD.2** $\dot{\sigma}^2_\zeta(t) = 2\int_0^t R(s)ds > 0$ for each $t > 0$ and hence $\sigma^2_\zeta(t)$ is strictly increasing.

Lemma 6.3 *If the variance function* $\sigma^2_\zeta(t)$ *of the process* $\zeta(t)$ *satisfies* **C1-C2***, then for*

$$\overline{X}_{\zeta;u}(t) = \frac{\zeta(h(u)+t)}{\sigma_\zeta(h(u)+t)}$$

where $h(u)$ *is such that* $\lim_{u\to\infty} h(u) = \infty$*, there exists constant* $C > 0$ *such that for each* $I_{\delta,T} = [\delta, \delta+T] \subset [-h(u)/2, h(u)]$ *and sufficiently large* u

$$\mathbb{P}(\sup_{t\in I_{\delta,T}} \overline{X}_{\zeta;u}(t) > w) \le \mathbb{P}(\sup_{t\in I_{0,T}} \overline{X}_{\zeta;u}(Ct) > w) \tag{6.25}$$

for all $w > 0$.

Proof. The idea of the proof is based on Slepian's inequality (see Piterbarg [13], Theorem C.1). Let $s, t \in I_{0,T}$. Hence for sufficiently large u we have

$$s+\delta, t+\delta \in I_{\delta,T} \subset [-h(u)/2, h(u)]. \tag{6.26}$$

From the definition of $\overline{X}_{\zeta;u}(t)$, for sufficiently large u we have

$$\begin{aligned}
&\mathbb{E}(\overline{X}_{\zeta;u}(t+\delta) - \overline{X}_{\zeta;u}(s+\delta))^2 = \\
&= 2(1 - \mathbf{Cov}(\overline{X}_{\zeta;u}(t+\delta), \overline{X}_{\zeta;u}(s+\delta))) = \\
&= \frac{\sigma^2_\zeta(|t-s|)}{\sigma_\zeta(h(u)+s+\delta)\sigma_\zeta(h(u)+t+\delta)} - \\
&\quad - \frac{(\sigma_\zeta(h(u)+t+\delta) - \sigma_\zeta(h(u)+s+\delta))^2}{\sigma_\zeta(h(u)+s+\delta)\sigma_\zeta(h(u)+t+\delta)}.
\end{aligned} \tag{6.27}$$

From (6.26) it follows that $h(u)+s+\delta, h(u)+t+\delta > \frac{h(u)}{2}$ and since $\sigma_\zeta(t)$ is increasing, the expression in (6.27) is less or equal than $\frac{\sigma^2_\zeta(|t-s|)}{\sigma_\zeta(h(u)/2)\sigma_\zeta(h(u)/2)}$. Now by Remark 6.1 and Lemma 6.1, there exist constants $C_1, C_2 > 0$ such that

$$\frac{\sigma^2_\zeta(|t-s|)}{\sigma_\zeta(h(u)/2)\sigma_\zeta(h(u)/2)} \le C_1 \frac{\sigma^2_\zeta(|t-s|)}{\sigma_\zeta(h(u))\sigma_\zeta(h(u))} \le \frac{\sigma^2_\zeta(C_2|t-s|)}{\sigma_\zeta(h(u))\sigma_\zeta(h(u))}.$$

Furthermore, by Lemma 6.1 and Lemma 2.1, there exists constant $C > 0$ such that the above is less or equal to

$$2(1 - \mathbf{Cov}(\overline{X}_{\zeta;u}(Ct), \overline{X}_{\zeta;u}(Cs))) = \mathbb{E}(\overline{X}_{\zeta;u}(Ct) - \overline{X}_{\zeta;u}(Cs))^2.$$

Now it is enough to use Slepian's inequality to complete the proof.

□

Lemma 6.4 *Let* $\zeta(t)$ *possesses* **SRD** *property.*
(a) For sufficiently large u*, there exists a unique* $t = t_u$ *such that* $t_u \to \infty$ *as* $u \to \infty$ *and*

$$\frac{d}{dt}\frac{\sigma^2_\zeta(t)}{(u+t)^2} = 0\ .$$

Furthermore

$$\dot{\sigma}_\zeta(t_u)(u+t_u) = \sigma_\zeta(t_u). \tag{6.28}$$

and

$$\dot{\sigma}^2_\zeta(t_u)(u+t_u) = 2\sigma^2_\zeta(t_u). \tag{6.29}$$

(b)

$$t_u = u(1+o(1)) \quad \text{as} \quad u \to \infty. \tag{6.30}$$

Proof. (a) Differentiating and equating to zero we obtain

$$\frac{d}{dt}\frac{\sigma^2_\zeta(t)}{(u+t)^2} = 0 \quad \text{iff} \quad (\dot{\sigma}^2_\zeta(t))(u+t) = 2\sigma^2_\zeta(t)\,. \tag{6.31}$$

Hence

$$u+t = \frac{2\sigma^2_\zeta(t)}{\dot{\sigma}^2_\zeta(t)}. \tag{6.32}$$

It suffices to show that the function $\phi(t) = \frac{2\sigma^2_\zeta(t)}{\dot{\sigma}^2_\zeta(t)} - t$ is ultimately strictly monotone and converging to ∞. The first derivative of $\phi(t)$ is

$$2\frac{(\dot{\sigma}^2_\zeta(t))^2 - \sigma^2_\zeta(t)\ddot{\sigma}^2_\zeta(t)}{(\dot{\sigma}^2_\zeta(t))^2} - 1\,.$$

The above is strictly positive if and only if

$$\frac{\dot{\sigma}^2_\zeta(t)}{\sigma^2_\zeta(t)} > \frac{2\ddot{\sigma}^2_\zeta(t)}{\dot{\sigma}^2_\zeta(t)}\,.$$

However, since $\zeta(t)$ possesses **SRD** property, using (6.23) and that $\dot{\sigma}^2_\zeta(t)$ is converging to a constant, and in view of **SRD.1**, the inequality holds for t sufficiently big. We now prove that $\phi(t)$ tends to ∞. By (6.23) and (6.24), $\phi(t)$ is ultimately bounded below by a linear functions with a positive slope.
(b) Equality (6.30) is a consequence of applying (6.23) and (6.24) to (6.29).

□

In the sequel, t_u will denote the point at which $\sigma_\zeta(t_u)/(u+t_u)$ attains its maximum.

Proposition 6.1 *If $\zeta(t)$ possesses* **SRD** *property, then*

$$m^2(u) = \frac{(u+t_u)^2}{\sigma^2_\zeta(t_u)} = 2Gu + 2G^2B + o(1) \quad \text{as } u \to \infty. \tag{6.33}$$

Proof. By Lemma 6.4 (b) we can choose $u_0 > 0$ such that for $u \geq u_0$

$$m^2(u) = \min_{t\geq 0} \frac{(u+t)^2}{\frac{2}{G}t - 2B + r(t)} = \min_{t \geq t_0} \frac{(u+t)^2}{\frac{2}{G}t - 2B + r(t)},$$

where t_0 is such that $r(t) \geq -2\epsilon$ for $\epsilon > 0$ and $t \geq t_0$. Hence

$$m^2(u) \leq \min_{t\geq t_0} \frac{(u+t)^2}{\frac{2}{G}t - 2(B+\epsilon)}\,.$$

Differentiating and equating to zero we obtain that the minimum of the r.h.s is achieved at $t = u + 2(B+\epsilon)G$ and equals to $2Gu + 2(B+\epsilon)G^2$.

On the other hand, for $\epsilon > 0$, we can choose t_0 such that $r(t) \le 2\epsilon$ for $t \ge t_0$. Thus

$$m^2(u) \ge \min_{t \ge t_0} \frac{(u+t)^2}{\frac{2}{G}t - 2(B-\epsilon)} = 2Gu + 2(B-\epsilon)G^2 ,$$

and hence (6.33) follows.

□

Lemma 6.5 *If $\zeta(t)$ possesses* **SRD** *property, then $\zeta(t)$ satisfies* **C1-C2** *with $\alpha_\infty = 1$ and $\alpha_0 = 2$.*

Proof. The proof follows in a straightforward way by using **SRD** and Remark 6.1.

□

Lemma 6.6 *If $\zeta(t)$ possesses* **SRD** *property, then the family $X_{\zeta;u}(t) = \frac{\zeta(t)}{u+t}m(u)$ fulfills conditions* **D1- D2** *with $\beta = 2$, $g(u) = \sqrt{2}(u+t_u)$, $f(u) = G\sigma_\zeta(t_u)$, $\zeta(t) = \frac{G}{\sqrt{2}}\zeta(t)$ and*

$$J(u) \quad = \quad [-\Delta(u), \Delta(u)], \tag{6.34}$$

where $\Delta(u) = \left(\frac{g(u)\log(m(u))}{m(u)}\right)^{2/\beta}$.

Proof. Note that $\overline{X}_{\zeta;u}(t+t_u) = \frac{\zeta(t+t_u)}{\sigma_\zeta(t+t_u)}$ and $\frac{\Delta(u)}{t_u} \to 0$. Thus, by suspection, **D1** is satisfied for $f(u) = G\sigma_\zeta(t_u)$ and $\zeta(t) = \frac{G}{\sqrt{2}}\zeta(t)$. Moreover

$$\sigma_{X_{\zeta;u}}(t+t_u) = m(u)\frac{\sigma_\zeta(t+t_u)}{u+t+t_u}.$$

Hence

$$\begin{aligned} \vartheta(u,t) &= \sigma_{X_{\zeta;u}}(t+t_u) - 1 = \\ &= \frac{\frac{1}{2}t^2\ddot{\sigma}_\zeta(t_u+\theta)(u+t_u)}{\sigma_\zeta(t_u)(u+t_u+t)}, \end{aligned} \tag{6.35}$$

where (6.35) follows from the expansion of $\sigma_\zeta(t+t_u)$ into a Taylor series with respect to t where $\theta \in [-\Delta(u), \Delta(u)]$. Since $\sigma_\zeta(t) \in C^2$ (see Remark 6.1) and

$$\ddot{\sigma}_\zeta(x) = \frac{\ddot{\sigma}^2_\zeta(x)}{2\sigma_\zeta(x)} - \frac{1}{4}\frac{(\dot{\sigma}^2_\zeta(x))^2}{\sigma^3_\zeta(x)},$$

then dividing (6.35) by t^2 we obtain

$$\begin{aligned} \frac{\vartheta(u,t)}{t^2} &= \frac{\ddot{\sigma}^2_\zeta(t_u+\theta)(u+t_u)}{4\sigma_\zeta(t_u+\theta)\sigma_\zeta(t_u)(u+t_u+t)} \\ &- \frac{1}{8}\frac{(\dot{\sigma}^2_\zeta(t_u+\theta))^2(u+t_u)}{\sigma^3_\zeta(t_u+\theta)\sigma_\zeta(t_u)(u+t_u+t)} \\ &= S_1 - S_2. \end{aligned}$$

Now from Remark 6.1 we immediately get that $S_2/(\sqrt{2}(u+t_u)) \to 1$ as $u \to \infty$, uniformly with respect $t, \theta \in [-\Delta(u), \Delta(u)]$. Moreover uniformly for $\theta \in [-\Delta(u), \Delta(u)]$

$$\begin{aligned} \frac{S_1}{S_2} &= \frac{2\ddot{\sigma}_\zeta^2(t_u+\theta)\sigma_\zeta^3(t_u+\theta)}{\sigma_\zeta(t_u+\theta)(\dot{\sigma}_\zeta^2(t_u+\theta))^2} \\ &= \frac{2\ddot{\sigma}_\zeta^2(t_u+\theta)\sigma_\zeta^2(t_u+\theta)}{(\dot{\sigma}_\zeta^2(t_u+\theta))^2} \to 0 \end{aligned} \tag{6.36}$$

as $u \to \infty$, where (6.36) is a consequence of **SRD**. This completes the proof.

□

Lemma 6.7 *If $\zeta(t)$ possesses* **SRD** *property, then for $J(u)$ defined by (6.34)*

$$\mathbb{P}\left(\sup_{t\in[0,\infty)} X_{\zeta;u}(t) > m(u)\right) = \mathbb{P}\left(\sup_{t\in J(u)} X_{\zeta;u}(t+t_u) > m(u)\right)(1+o(u)) \tag{6.37}$$

as $u \to \infty$.

Proof. To prove (6.37) it is sufficient to show that

$$\mathbb{P}\left(\sup_{t\in[-t_u,\infty)\setminus J(u)} X_{\zeta;u}(t+t_u) > m(u)\right) = o(\Psi(m(u))) \tag{6.38}$$

as $u \to \infty$. Let $\Delta(u)$ and $J(u)$ be the same as defined in Lemma 6.6. We have

$$\begin{aligned} \mathbb{P}\left(\sup_{t\in[-t_u,\infty)\setminus J(u)} X_{\zeta;u}(t+t_u) > m(u)\right) &\le \mathbb{P}\left(\sup_{t\in[-t_u,-t_u/2]} X_{\zeta;u}(t+t_u) > m(u)\right) \\ &+ \mathbb{P}\left(\sup_{t\in[-t_u/2,-\Delta(u)]\cup[\Delta(u),t_u]} X_{\zeta;u}(t+t_u) > m(u)\right) \\ &+ \mathbb{P}\left(\sup_{t\in[t_u,\infty)} X_{\zeta;u}(t+t_u) > m(u)\right). \end{aligned}$$

Let $\sigma_{X_{\zeta;u}}(A) = \max_{t\in[-t_u,\infty)\setminus J(u)} \sigma_{X_{\zeta;u}}(t+t_u)$. Note that from Lemma 6.4 and Lemma 6.6 for sufficiently large u

$$\sigma_{X_{\zeta;u}}(A) \le 1 - \frac{\Delta^2(u)}{2(u+t_u)^2} \le \frac{1}{1+\frac{\log^2(m(u))}{m^2(u)}},$$

From Lemma 6.3 there exists $C > 0$ such that for sufficiently large u and $[i, i+1] \subset \{[-t_u/2, -\Delta(u)] \cup [\Delta(u), t_u]\}$ we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[i,i+1]} \overline{X}_{\zeta;u}(t+t_u) > m(u)(1+\frac{\log^2(m(u))}{m^2(u)}\right) \le \\ &\le \mathbb{P}\left(\sup_{t\in[0,1]} \overline{X}_{\zeta;u}(Ct+t_u) > m(u)(1+\frac{\log^2(m(u))}{m^2(u)}\right). \end{aligned}$$

Hence

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[-t_u/2,-\Delta(u)]\cup[\Delta(u),t_u]} X_{\zeta;u}(t+t_u) > m(u)\right) \leq \\
&\leq \sum_{\Delta(u)-1\leq i\leq \frac{t_u}{2}+1} \mathbb{P}\left(\sup_{t\in[-i,-i+1]} \overline{X}_{\zeta;u}(t+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right) \\
&\quad + \sum_{\Delta(u)-1\leq i\leq t_u+1} \mathbb{P}\left(\sup_{t\in[i,i+1]} \overline{X}_{\zeta;u}(t+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right) \\
&\leq t_u\mathbb{P}\left(\sup_{t\in[0,1]} \overline{X}_{\zeta;u}(Ct+t_u) > m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right) \\
&= t_u \mathrm{Const}\Psi\left(m(u)\left(1+\frac{\log^2(m(u))}{m^2(u)}\right)\right)(1+o(1)) = o(\Psi(m(u))). \qquad (6.39)
\end{aligned}
$$

The proof of

$$
\mathbb{P}\left(\sup_{t\in[-t_u,-t_u/2]} X_{\zeta;u}(t+t_u) > m(u)\right) + \mathbb{P}\left(\sup_{t\in[t_u,\infty)} X_{\zeta;u}(t+t_u) > m(u)\right) = o(\Psi(m(u))) \qquad (6.40)
$$

follows in a straightforward way from Borell's inequality (see Piterbarg [13], Theorem D.1) and the fact that

$$
\sup_{t\in[-t_u,-t_u/2]\cup[t_u,\infty)} \sigma^2_{X_{\zeta;u}}(t+t_u) \leq 1-\mathrm{Const}_2,
$$

where $\mathrm{Const}_2 > 0$ is a constant. Thus (6.39) combined with (6.40) completes the proof.

□

Proof of Theorem 5.1. From Lemma 6.7 we have

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{t\geq 0}(\zeta(t)-t) > u\Big) &= \mathbb{P}\Big(\sup_{t\geq 0} X_{\zeta;u}(t) > m(u)\Big) \\
&= \mathbb{P}\Big(\sup_{t\in J(u)} X_{\zeta;u}(t) > m(u)\Big)(1+o(1)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{P}(\sup_{t\geq 0}(\zeta(t)-t) > u) = \\
&= \frac{2\mathcal{H}_{\frac{G}{\sqrt{2}}\zeta}\Gamma(1/2)}{2}\left(\frac{m(u)}{\sqrt{2}(u+t_u)}\right)^{-2/\beta}\Psi(m(u))(1+o(1)) \qquad (6.41)\\
&= \sqrt{\pi}\mathcal{H}_{\frac{G}{\sqrt{2}}\zeta}\frac{2\sqrt{u}}{\sqrt{G}}\Psi(m(u))(1+o(1)) \qquad (6.42)\\
&= \frac{\mathcal{H}_{\frac{G}{\sqrt{2}}\zeta}}{G}e^{-G^2B}e^{-Gu}(1+o(1)),
\end{aligned}
$$

where (6.41) and (6.42) follow from Lemma 6.6 and Theorem 4.2 and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This completes the proof.

□

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