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Juggling polynomials

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Juggling Polynomials

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ABSTRACT
A mathematical model for juggling has previously been described by Buhler, Eisenbud, Graham and Wright. This paper uses this model and observes that juggling patterns have different 'states' at different times. States can be represented by polynomials. This representation is exploited to give a new proof of an enumeration theorem on juggling patterns by Buhler et al. The paper concludes by discussing state graphs and a generalization of the juggling model. Both lead to new enumeration problems.

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Keywords and Phrases: Juggling pattern, siteswap, state sequence, Stirling numbers of the second kind, state graph, cycle.
Note: The results in this paper are based on the author's master's thesis [4] written at the Rijksuniversiteit Groningen under the guidance of Jaap Top.

1. Introduction

Consider the juggling pattern that is shown in Figure 1. The horizontal axis is the time axis. Balls are caught and then thrown at integer points in time. At time points $\ldots, -9, -6, -3, 0, 3, 6, \ldots$ each ball is thrown high enough so that it lands four time units later. Similarly, at time points $\ldots, -8, -5, -2, 1, 4, 7, \ldots$ each ball is thrown so that it lands five time units later, while at time points $\ldots, -7, -4, -1, 2, 5, 8, \ldots$ no balls are caught or thrown. Thus the pattern is periodic with period 3. In this pattern there are three balls since the arcs describe three infinite paths.

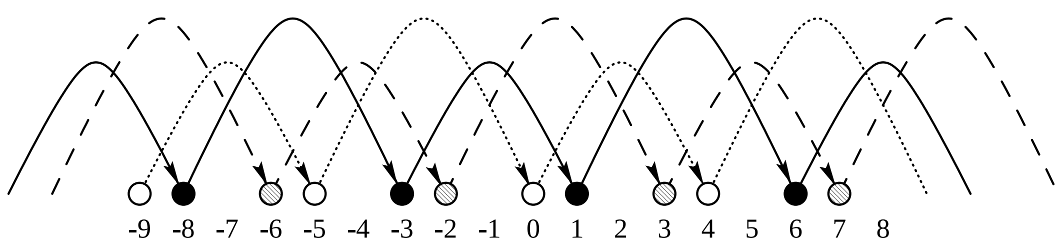

Figure 1: The juggling pattern induced by 4 5 0.

In this paper, we will consider juggling patterns where the juggler can catch and throw at most one ball at a time. We can represent such juggling patterns by a permutation $f : \mathbb{Z} \to \mathbb{Z}$ in the following manner.

$$f(u) := \begin{cases} u & \text{if } \textit{no} \text{ ball is thrown at time point } u \\ v & \text{if the ball thrown at time point } u \text{ lands at time point } v \end{cases}$$

So the pattern in Figure 1 is represented by

$$f(t) := \begin{cases} t+4 & \text{if } t \equiv 0 \bmod 3 \\ t+5 & \text{if } t \equiv 1 \bmod 3 \\ t & \text{if } t \equiv 2 \bmod 3. \end{cases}$$

Buhler, Eisenbud, Graham and Wright [1] put this notion of a juggling pattern into mathematical terms.

Definition 1.1. A *juggling pattern* is a permutation $f : \mathbb{Z} \to \mathbb{Z}$ with $f(t) \geq t$ for all $t \in \mathbb{Z}$.

A juggling pattern f partitions the integers into orbits. Since $f(t) \geq t$, the orbits are either infinite or singletons. The number of infinite orbits is the *number of balls* of f and is denoted by balls(f).

We will concern ourselves mainly with periodic patterns, like the pattern of Figure 1.

Definition 1.2. Let $n \in \mathbb{N}$. A *period-n juggling pattern* is a juggling pattern f with $f(t+n) = f(t)+n$ for all $t \in \mathbb{Z}$.

A periodic juggling pattern f is not periodic in the mathematical sense. However, $t \mapsto f(t) - t$ is. A periodic pattern can be denoted by a sequence of non-negative integers.

Definition 1.3. A sequence $a_0 \ \cdots \ a_{n-1}$ of non-negative integers is called a *siteswap* if $t \mapsto t + a_{t \bmod n}$ is a period-n juggling pattern. It is called the juggling pattern *induced* by the siteswap $a_0 \ \cdots \ a_{n-1}$.

There is a one-to-one correspondence between siteswaps of length n and period-n juggling patterns. The pattern of Figure 1 is induced by the siteswap 4 5 0.

Now consider Figure 2. Like Figure 1, it depicts the pattern induced by 4 5 0. Again, the horizontal axis is the time axis. The vertical axis we have added this time is the ‘height’ axis.

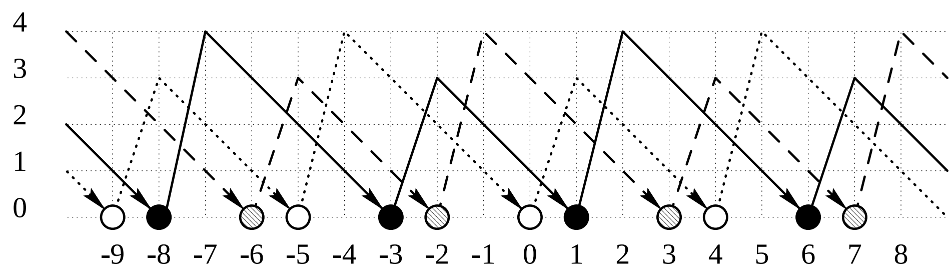

Figure 2: The states of the juggling pattern induced by 4 5 0.

Every dotted vertical line through an integer point in time intersects every infinite path precisely once. At time points $\ldots, -9, -6, -3, 0, 3, 6, \ldots$ the vertical line intersects arcs of an infinite path at ‘heights’ 0, 1 and 3. This means that there are three balls in the air of which one will land 0 time units later, another will land 1 time unit later and the third will land 3 time units later. Similarly, at time points $\ldots, -8, -5, -2, 1, 4, 7, \ldots$ the three balls in the air will land again 0, 2 and 3 time units later, while at time points $\ldots, -7, -4, -1, 2, 5, 8, \ldots$ the balls in the air will land 1, 2 and 4 time units later.

We call the set of ‘heights’ of intersection at an integer point in time the *state* at that point in time. States are represented by polynomials of $\mathbb{Z}[X]$. Thus the state at time points 0 modulo 3 is $X^0 + X^1 + X^3$, the state at time points 1 modulo 3 is $X^0 + X^2 + X^3$ and the state at time points 2 modulo 3 is $X^1 + X^2 + X^4$.

Suppose the state of a pattern at time t is equal to

$$X^{a_1} + X^{a_2} + \cdots + X^{a_{m-1}} + X^{a_m}$$

and $0 \leq a_1 < a_2 < \cdots < a_{m-1} < a_m < \infty$. Then this juggling pattern has m balls. These m balls in the air at time t will land again at time points $t+a_1$, $t+a_2$, ..., $t+a_{m-1}$ and $t+a_m$.

We will now define a state sequence formally. The condition on the number of balls is to assure that the number of terms of a state is finite. This is satisfied if the pattern is periodic.

Definition 1.4. The *state sequence* $\{S_t\}_{t\in\mathbb{Z}}$ of a juggling pattern f with $\text{balls}(f) < \infty$ is defined by

$$S_t := \sum_{\substack{i\geq 0\\ f^{-1}(t+i)<t}} X^i.$$

We call S_t the *state at time* t.

Notice that the coefficients of every state S_t are either 0 or 1. One easily verifies that a state sequence of a period-n juggling pattern is periodic with period n.

Oudshoorn and Van Rijnswou [5] used this state approach to juggling patterns and conjectured the following theorem.

Theorem 1.1. *A sequence $a_0 \cdots a_{n-1}$ of non-negative integers is a siteswap if and only if a $D \in \mathbb{Z}[X]$ exists with*

$$D\big(X^n - 1\big) = \sum_{i=0}^{n-1}\big(X^{i+a_i} - X^i\big). \tag{1.1}$$

In Section 2, we will prove this theorem and use it in Section 3 to enumerate period-n juggling patterns with m balls. Section 4 discusses an application of states that is useful for jugglers and an extension of the juggling model.

2. Properties of State Sequences

For $a, b \in \mathbb{Z}$, we will use the notation $[a,b]$ for the interval $\{a,\ a+1,\ \ldots,\ b-1,\ b\} \subset \mathbb{Z}$ and the notation $[a,b\rangle$ for the interval $\{a,\ a+1,\ \ldots,\ b-2,\ b-1\} \subset \mathbb{Z}$. Intuitively, we saw that the number of terms of every state equals the number of balls of the corresponding pattern.

Proposition 2.1. *Let f be a juggling pattern and $\{S_t\}_{t\in\mathbb{Z}}$ its state sequence. Then*

$$S_t\big|_{X=1} = \text{balls}(f), \qquad \forall t \in \mathbb{Z}.$$

Proof. Consider the set $\{u \in \mathbb{Z} |\ u \geq t \text{ and } f^{-1}(u) < t\}$. It is contained in the union $\bigcup_{i=1}^{\text{balls}(f)} O_i$ of infinite orbits determined by f. Furthermore, it contains precisely one element of every O_i, namely the smallest element of O_i greater than or equal to t. This gives us

$$S_t\big|_{X=1} = \big|\{i \in \mathbb{N}_0 |\ f^{-1}(t+i) < t\}\big| = \big|\{u \in \mathbb{Z} |\ u \geq t \text{ and } f^{-1}(u) < t\}\big| = \text{balls}(f). \qquad \square$$

Two subsequent states of a state sequence are related in the following way.

Proposition 2.2. *Let f be a juggling pattern and $\{S_t\}_{t\in\mathbb{Z}}$ its state sequence. Then*

$$\forall t \in \mathbb{Z}, \quad S_{t+1} = X^{-1}\big(S_t + X^{f(t)-t} - 1\big).$$

Proof. Observe that $\{i \geq 0 |\ f^{-1}(t+i) = t\} = \{f(t) - t\}$ and that either $f^{-1}(t) = t$ or $f^{-1}(t) < t$. We use this and obtain

$$\begin{aligned} S_t + X^{f(t)-t} &= \left[\sum_{\substack{i>0 \\ f^{-1}(t+i)<t}} X^i + \sum_{f^{-1}(t)<t} 1 \right] + \left[\sum_{\substack{i>0 \\ f^{-1}(t+i)=t}} X^i + \sum_{f^{-1}(t)=t} 1 \right] \\ &= \left[\sum_{\substack{i\geq 0 \\ f^{-1}(t+1+i)<t+1}} X^{i+1} \right] + 1 \\ &= XS_{t+1} + 1. \end{aligned}$$

□

We conclude that the mapping from a juggling pattern to its state sequence is injective, since

$$f(t) = t + \deg(XS_{t+1} - S_t + 1), \qquad \forall t \in \mathbb{Z}.$$

By applying Proposition 2.2 to a period-n juggling pattern, induced by the siteswap $a_0 \ \cdots \ a_{n-1}$, we find

$$S_0 = S_{n-1} = X^{-n} \left(S_0 + \sum_{i=0}^{n-1} \left(X^{i+a_i} - X^i \right) \right). \tag{2.1}$$

This motivated Theorem 1.1.

Proof of Theorem 1.1. If $a_0 \ \cdots \ a_{n-1}$ is a siteswap, then, according to (2.1), $D = S_0$ verifies (1.1).

Conversely, suppose that $\sum_{i=0}^{n-1}(X^{i+a_i} - X^i)$ is an element of the principal ideal $(X^n - 1)$ of $\mathbb{Z}[X]$. Let $\phi : \mathbb{Z}[X] \to \mathbb{Z}[X]/(X^n - 1)$ be the canonical homomorphism and let $\sigma : [0, n\rangle \to [0, n\rangle$ be defined by $\sigma(i) := (i + a_i) \bmod n$. Since $\phi(X^n) = \phi(1)$, we have

$$\phi \left(\sum_{i=0}^{n-1} \left(X^{i+a_i} - X^i \right) \right) = \phi \left(\sum_{i=0}^{n-1} \left(X^{\sigma(i)} - X^i \right) \right).$$

We see that, since its degree is smaller than n, $\sum_{i=0}^{n-1}(X^{\sigma(i)} - X^i)$ is the unique representation of $\phi(\sum_{i=0}^{n-1}(X^{i+a_i} - X^i))$. Our assumption implies that it must be zero, hence σ must be a permutation.

We use this to prove that $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(t) := t + a_{t \bmod n}$ is a period-n juggling pattern. We easily verify that $f(t + n) = f(t) + n$. To prove that f is injective, we write $t_i = r_i + d_i n$ with $r_i \in [0, n\rangle$. We have

$$f(t_1) = f(t_2) \quad \Longrightarrow \quad f(r_1) + d_1 n = f(r_2) + d_2 n \quad \Longrightarrow \quad f(r_1) \equiv f(r_2) \bmod n.$$

However, $i \mapsto f(i) \bmod n$ is a permutation of $[0, n\rangle$, so $r_1 = r_2$. This in turn implies $d_1 = d_2$, thus $t_1 = t_2$. To prove that f is surjective, we choose $t \in \mathbb{Z}$. Since $i \mapsto f(i) \bmod n$ is a permutation on $[0, n\rangle$, there exists an $r \in [0, n\rangle$ such that $f(r) \equiv t \bmod n$. Hence a $d \in \mathbb{Z}$ with $t = f(r) + dn = f(r + dn)$ exists. □

3. Enumerating Juggling Patterns

For $A \subset \mathbb{Z}$, we use $A^{(i)}$ to denote the i-th element of A. Thus $A = \{A^{(1)}, A^{(2)}, \ldots, A^{(|A|)}\}$ and $A^{(1)} < A^{(2)} < \cdots < A^{(|A|)}$. The following proposition counts the number of period-n juggling patterns given the state at time 0.

Proposition 3.1. *Let I be a finite subset of $\mathbb{N}_0$. A period-n juggling pattern with state $\sum_{i\in I} X^i$ at time 0 exists if and only if*

$$[n,\infty\rangle \cap I \subseteq \{i+n|\ i\in I\}. \tag{3.1}$$

Moreover, if (3.1) holds, then the number of such juggling patterns equals

$$\big|[0,n\rangle \cap I\big|!\ \mathfrak{M}\big([0,n\rangle \setminus I\big),$$

where

$$\mathfrak{M}(A) := \prod_{i=1}^{|A|}\big(A^{(i)} - i + 2\big).$$

Proof. By Theorem 1.1, the period-n juggling patterns with state $\sum_{i\in I} X^i$ at time 0 are exactly those functions $t \mapsto t + a_{t \bmod t}$, for which $a_0\ \cdots\ a_{n-1}$ is a solution of

$$\sum_{i=0}^{n-1} X^{i+a_i} = \sum_{i=0}^{n-1} X^i - \sum_{i\in[0,n\rangle\cap I} X^i + X^n \sum_{i\in I} X^i - \sum_{i\in[n,\infty\rangle\cap I} X^i. \tag{3.2}$$

While the left hand side of this equation has at most n terms, the right hand side has at least n. So both sides must have precisely n terms in order to have solutions. This means that (3.1) is a necessary condition. To show that it is sufficient as well, we will count the number of solutions.

Let us assume that (3.1) holds. We try to find sequences of non-negative integers $a_0\ \cdots\ a_{n-1}$ for which

$$\big\{i + a_i|\ i\in[0,n\rangle\big\} = \underbrace{\big([0,n\rangle \setminus I\big)}_{A} \cup \underbrace{\big(\{n+i|\ i\in I\}\setminus I\big)}_{B}.$$

Thus $a_0\ \cdots\ a_{n-1}$ is a solution if and only if $a_i = \pi(i) - i$, where $\pi : [0,n\rangle \to A\cup B$ is a bijection with $\pi(i) \geq i$. So the number of solutions equals the number of such bijections π.

We can choose a bijection π by assigning each element of $A\cup B$ to an element of $[0,n\rangle$. We start by assigning the elements of A in increasing order and prove by induction that there are $\mathfrak{M}(A)$ ways to do this. There are $\mathfrak{M}(\{A^{(1)}\}) = A^{(1)}+1$ ways to assign $A^{(1)}$, namely to $0, \ldots, A^{(1)}$. Now assume that we already assigned $A^{(1)}, \ldots, A^{(j)}$ (with $j < |A|$) and that there were $\mathfrak{M}(\{A^{(1)}, \ldots, A^{(j)}\})$ ways to do this. There are $A^{(j+1)}+1$ candidates for $A^{(j+1)}$: $0, \ldots, A^{(j+1)}$. Of these candidates, j are already taken by $A^{(1)}, \ldots, A^{(j)}$. So there are $(A^{(j+1)}+1-j)\,\mathfrak{M}(\{A^{(1)}, \ldots, A^{(j)}\}) = \mathfrak{M}(\{A^{(1)}, \ldots, A^{(j+1)}\})$ ways to assign $A^{(1)}, \ldots, A^{(j+1)}$.

Once we have assigned the elements of A, we can freely assign the elements of B, since every element of B is greater than $n-1$. So there are $|B|! = |[0,n\rangle \cap I|!$ ways to do this. So we know that there are $|[0,n\rangle\cap I|!\ \mathfrak{M}([0,n\rangle\setminus I)$ solutions of Equation (3.2). Because this number is strictly greater that 0, condition (3.1) is sufficient. ◻

We can now count the set of period-n juggling patters with less than m balls by a summation over the right set of states.

Theorem 3.2. *The number of period-n juggling patterns with fewer than m balls equals m^n.*

Proof. We write $\mathfrak{N}_<(m,n)$ to indicate the number of period-n juggling patterns with fewer than m balls. Due to Proposition 2.1 and Proposition 3.1, we have

$$\mathfrak{N}_<(m,n) = \sum_{\substack{I\subset\mathbb{N}_0 \text{ and } |I|<m\\ [n,\infty\rangle\cap I\subseteq\{i+n|\ i\in I\}}} \big|[0,n\rangle\cap I\big|!\ \mathfrak{M}\big([0,n\rangle\setminus I\big).$$

The index collection of this sum equals

$$\{\varnothing\} \cup \bigcup_{a=1}^{m-1} \left[\bigcup_{\substack{A\subseteq[0,n\rangle\\ |A|=a}} \left[\bigcup_{b=0}^{m-1-a} \left[\bigcup_{\substack{B\subseteq\{i+n|\ i\in A\cup B\}\\ |B|=b}} \{A\cup B\} \right]\right]\right].$$

This gives us

$$\begin{aligned}\mathfrak{N}_<(m,n) &= 1 + \sum_{a=1}^{m-1} \left[\sum_{\substack{A\subseteq[0,n\rangle\\ |A|=a}} \left[\sum_{b=0}^{m-1-a} \left[\sum_{\substack{B\subseteq\{i+n|\ i\in A\cup B\}\\ |B|=b}} a!\,\mathfrak{M}\big([0,n\rangle\setminus A\big)\right]\right]\right]\\ &= 1 + \sum_{a=1}^{m-1} \left[a! \sum_{\substack{A\subseteq[0,n\rangle\\ |A|=a}} \left[\mathfrak{M}\big([0,n\rangle\setminus A\big) \sum_{b=0}^{m-1-a} \left[\sum_{\substack{B\subseteq\{i+n|\ i\in A\cup B\}\\ |B|=b}} 1\right]\right]\right].\end{aligned}$$

Consider a set $B \subset \mathbb{N}_0$ with $B \subseteq \{i+n|\ i \in A\cup B\}$ and $|B| = b$. If $i \in B$, then either $i-n \in A$ or $i-n \in B$. Therefore B is of the form

$$\bigcup_{i\in A} \{i+n,\ i+2n,\ \ldots,\ i+y_i n\},$$

with $y_i \geq 0$ and $\sum_{i\in A} y_i = b$. So the number of possible sets B equals the number of $(x_1, \ldots, x_a) \in (\mathbb{N}_0)^a$ with $x_1 + \cdots + x_a = b$. It is shown in [3, p. 15–17] that the latter number is $\binom{a+b-1}{b}$. Hence

$$\begin{aligned}\mathfrak{N}_<(m,n) &= 1 + \sum_{a=1}^{m-1} \left[a! \sum_{\substack{A\subseteq[0,n\rangle\\ |A|=a}} \left[\mathfrak{M}\big([0,n\rangle\setminus A\big) \sum_{b=0}^{m-1-a} \binom{a+b-1}{b}\right]\right]\\ &= 1 + \sum_{a=1}^{m-1} \left[a! \sum_{\substack{A\subseteq[0,n\rangle\\ |A|=a}} \mathfrak{M}\big([0,n\rangle\setminus A\big) \binom{m-1}{a}\right]\\ &= \sum_{a=0}^{m-1} \left[a! \binom{m-1}{a} \sum_{\substack{C\subseteq[0,n\rangle\\ |C|=n-a}} \mathfrak{M}(C)\right].\end{aligned}$$

Let us focus on

$$\sum_{\substack{C\subseteq[0,n\rangle\\ |C|=n-a}} \mathfrak{M}(C) = \sum_{\substack{C\subseteq[0,n\rangle\\ |C|=n-a}} \prod_{i=1}^{n-a} \big(C^{(i)} - i + 2\big).$$

We see that every factor $C^{(i)} - i + 2$ is in $[1, a+1]$ in every term of this sum. Thus every term is of the form $1^{x_1} \cdots (a+1)^{x_{a+1}}$ with $x_i \geq 0$ and $x_1 + \cdots + x_{a+1} = n-a$. As we have just seen, the number of such products is $\binom{n}{a}$. Our sum has $\binom{n}{a}$ terms as well. Moreover, since $C^{(i)} - i + 2 \leq C^{(i+1)} - (i+1) + 2$, they are all different. So we conclude

$$\sum_{\substack{C\subseteq[0,n\rangle\\ |C|=n-a}} \mathfrak{M}(C) = \sum_{\substack{x_i\geq 0\\ x_1+\cdots+x_{a+1}=n-a}} 1^{x_1} \cdots (a+1)^{x_{a+1}}.$$

Comtet proves in [3, Theorem D, p. 207] that this sum is equal to the Stirling number of the second kind $S(n+1,\ a+1)$. This gives us

$$\mathfrak{N}_<(m,n) = \sum_{a=0}^{m-1} a! \binom{m-1}{a} S(n+1,\ a+1).$$

By using the recurrence relation on Stirling numbers of the second kind, as described in [3, Theorem A, p. 208], we can rewrite this sum.

$$\mathfrak{N}_<(m,n) = \sum_{a=0}^{m} a! \binom{m}{a} S(n,a).$$

And by [3, Theorem B, p. 207], this is equal to m^n. □

Not surprisingly, Theorem 3.2 holds the same result as theorem 3 in [1]. However, the proof of Buhler et al. is quite different. It involves excedances of permutations, Eulerian numbers and Worpitzky's identity.

We are now able to calculate the number of period-n juggling patterns with precisely m balls, $\mathfrak{N}(m,n)$.

$$\mathfrak{N}(m,n) = (m+1)^n - m^n.$$

As pointed out in [1], this is not the number of most interest to jugglers. They are more interested in the number of *exact* period-n juggling patterns with m balls, $\mathfrak{N}^*(m,n)$. That is, the number of period-n patterns with m balls that are *not* period-d for some $d|n$. Since

$$\sum_{d|n} \mathfrak{N}^*(m,d) = \mathfrak{N}(m,n),$$

we can use classic Möbius inversion to find $\mathfrak{N}^*(m,n)$.

$$\mathfrak{N}^*(m,n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \mathfrak{N}(m,d).$$

Still, cyclic shifts of siteswaps are considered distinct in our model, while they look exactly the same from a jugglers point of view. To account for this, we can divide $\mathfrak{N}^*(m,n)$ by n. Table 1 shows some examples.

$m\backslash n$	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0
1	1	1	2	3	6	9	18	30	56
2	1	2	6	15	42	107	294	780	2128
3	1	3	12	42	156	554	2028	7350	26936
4	1	4	20	90	420	1910	8820	40590	187880
5	1	5	30	165	930	5155	28830	161040	902720
6	1	6	42	273	1806	11809	77658	510510	3363976
7	1	7	56	420	3192	24052	181944	1376340	10429328
8	1	8	72	612	5256	44844	383688	3283380	28133616
9	1	9	90	855	8190	78045	745290	7118730	68064360

Table 1: The number of exact period-n juggling patterns with m balls divided by n.

In our juggling model, we allowed for 'zero throws' by demanding that a juggling pattern f should have $f(t) \geq t$ for all $t \in \mathbb{Z}$. One may argue that demanding $f(t) > t$ for all $t \in \mathbb{Z}$ would be better. However, the difference is not that big. Let $\mathfrak{N}_<(m,n,h)$ be the number of permutations f on $\mathbb{Z}$ with $f(t) \geq h+t$ and $f(t+n) = f(t)+n$ for all $t \in \mathbb{Z}$ that have fewer than m infinite orbits. One easily verifies that

$$\mathfrak{N}_<(m,n,h) = \begin{cases} (m-h)^n & \text{if } m \geq h \\ 0 & \text{if } m < h. \end{cases}$$

4. Concluding Remarks

In this section we will first discuss state graphs. These are a nice application of states, especially from a jugglers point of view. Then we will look at an extension of the juggling model used so far. This extension incorporates information about the hands that throw balls. See [2, 4] for more detailed discussions.

4.1 State Graph

State graphs were first described by Probert in [6, p. 172–181]. The *m-ball state graph* $\mathcal{G}_m$ is a directed graph. It consists of the vertex set

$$\mathcal{S}_m := \left\{ \sum_{i \in I} X^i \,\middle|\, I \subset \mathbb{N}_0 \text{ and } |I| = m \right\}$$

of states with m balls and the edge set

$$\left\{ (U,V) \in \mathcal{S}_m \times \mathcal{S}_m \,\middle|\, \exists a \in \mathbb{N}_0 \text{ such that } V = X^{-1}\big(U + X^a - 1\big) \right\}$$

of possible transitions from one state to another.

One can prove that this graph is strongly connected and that there is a one-to-one correspondence between the period-n juggling patterns with m balls and the closed paths of length n in $\mathcal{G}_m$. If we label the edges with the corresponding "a", then the sequence of labels of a closed path is the corresponding siteswap.

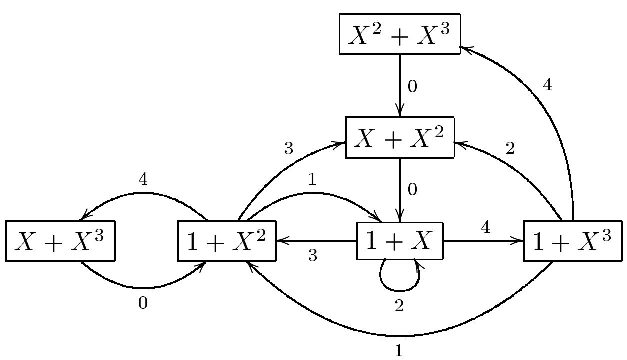

Figure 3: The subgraph of $\mathcal{G}_2$ induced by $\{S \in \mathcal{S}_2 |\ \deg(S) < 4\}$.

If we are interested in periodic patterns f with $f(t) - t \leq h$, then it suffices to consider the subgraph of $\mathcal{G}_m$ induced by the vertex set $\{S \in \mathcal{S}_m |\ \deg(S) < h\}$. Figure 3 shows an example. Please note that such graphs are *not* planar in general. If we are interested in patterns with a specific period n, then Proposition 3.1 tells us to consider the subgraph of $\mathcal{G}_m$ induced by the vertex set

$$\left\{ \sum_{i \in I} \in \mathcal{S}_m \,\middle|\, [n, \infty\rangle \cap I \subseteq \{ i+n |\ i \in I \} \right\}.$$

State graphs can be used by a juggler to find new patterns. They are especially useful for finding ways to concatenate known patterns to form a new one.

Any closed path in a state graph can be decomposed into cycles. Such a decomposition is *not* unique. Nevertheless, patterns corresponding to cycles in a state graph are interesting because they form the basic building blocks of a all periodic patterns. Enumerating these basic patterns is an unsolved problem.

4.2 Hands
So far, we only considered the time point at which a ball was caught and thrown, not allowing for more than one throw at a point in time. We can extend our model by including information about the hands that throw balls and allowing simultaneous throws by distinct hands.

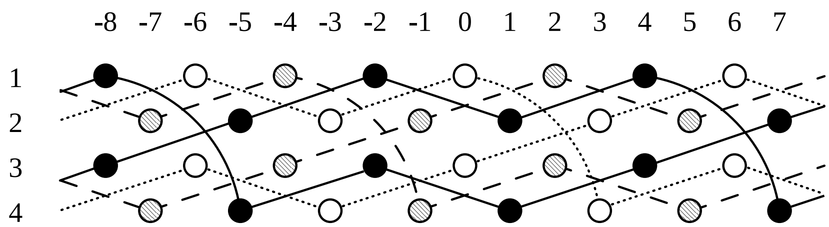

Figure 4: The standard passing pattern.

Consider the pattern in Figure 4. It depicts the standard two-juggler passing pattern. Again, the horizontal axis is the time axis. The vertical axis is the hand axis. Hands 1 and 2 belong to one juggler, hands 3 and 4 to another. If h is the number of hands, then such juggling patterns can be represented by a permutation $f : [1, h] \times \mathbb{Z} \to [1, h] \times \mathbb{Z}$ in the following manner.

$$f(i,s) := \begin{cases} (i,s) & \text{if } no \text{ ball is thrown with hand } i \text{ at time point } s \\ (j,t) & \begin{array}{l}\text{if the ball thrown with hand } i \text{ at time point } s \\ \text{lands in hand } j \text{ at time point } t.\end{array} \end{cases}$$

This means that there is a one-to-one correspondence between juggling patterns of the old model and generalized juggling patterns with $h = 1$.

Periodicity can be defined analogously. The generalized siteswap of a period-n pattern is a matrix with h rows and n columns. The elements of this matrix are vectors, consisting of a hand component and a time component. For example, the siteswap corresponding to the pattern in Figure 4 is

$$\begin{bmatrix} (4,3) & (1,0) & (2,3) & (1,0) \\ (2,0) & (1,3) & (2,0) & (1,3) \\ (2,3) & (3,0) & (4,3) & (3,0) \\ (4,0) & (3,3) & (4,0) & (3,3) \end{bmatrix}.$$

We generalize the state by making it a vector of h polynomials, one for each hand. One period of the state sequence of the example pattern is

$$S_0 = \begin{pmatrix} 1+X^2 \\ X \\ 1+X^2 \\ X \end{pmatrix}, \quad S_1 = \begin{pmatrix} X \\ 1+X^2 \\ X \\ 1+X^2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1+X^2 \\ X \\ 1+X^2 \\ X \end{pmatrix} \quad \text{and } S_3 = \begin{pmatrix} X \\ 1+X^2 \\ X \\ 1+X^2 \end{pmatrix}.$$

Unfortunately, the map that takes a juggling pattern to its state sequence is not injective anymore. Nevertheless, we can still prove a result similar to Theorem 1.1. Whether this result can be used to enumerate generalized juggling patterns is another unsolved problem.

REFERENCES

1. Joe Buhler, David Eisenbud, Ron Graham, and Colin Wright. Juggling drops and descents. *American Mathematical Monthly*, 101:507–519, June–July 1994.
2. Ed Carstens. The mathematics of juggling. Unpublished paper, November 1992.
3. Louis Comtet. *Advanced Combinatorics: The Art of Finite and Infinite Expansions.* D. Reidel Publishing Company, Dordrecht, Holland, revised and enlarged edition, 1974. Translated from the French by J.W. Nienhuys.
4. Lute Kamstra. The state of juggling. Master's thesis, Rijksuniversiteit Groningen, PO Box 800, 9700 AV, Groningen, The Netherlands, July 1999.
5. W.R. Oudshoorn. Letter to S.M. van Rijnswou.
6. Martin Probert. *Four ball juggling; from simple patterns to advanced theory.* Antony Rowe Ltd, Chippenham, Wiltshire, 1995.