Centrum voor Wiskunde en Informatica

REPORT RAPPORT

MAS

Modelling, Analysis and Simulation

Modelling, Analysis and Simulation

On the convergence analysis of advection-diffusion schemes on non-uniform grids

W.H. Hundsdorfer, J.G. Verwer

REPORT MAS-R0118 DECEMBER 31, 2001

CWI is the National Research Institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organization for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

On the Convergence Analysis of Advection-Diffusion Schemes on Non-Uniform Grids

W. Hundsdorfer, J.G. Verwer
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
Willem.Hundsdorfer@cwi.nl , Jan.Verwer@cwi.nl

Abstract

Numerical schemes for advection-diffusion problems are often used with non-uniform grids. Non-uniform grids are known to greatly complicate the convergence analysis and their use therefore is much less straightforward than for uniform grids. For example, it is possible that a scheme which is inconsistent at the level of the local truncation error truly converges with order two. The purpose of this paper is to contribute to the theory of spatial discretizations on non-uniform grids. We shall present spatial convergence results for a number of vertex and cell centered schemes for the linear 1D time-dependent advection-diffusion problem. The focus hereby lies on the discrepancy between local and global order.

2000 Mathematics Subject Classification: 65M06, 65M12, 65M50, 65N06, 65N12, 65N50.
Keywords and Phrases: Advection-diffusion problems, non-uniform grids, error analysis.
Note: Work carried out under project MAS1.2 - Analysis, Asymptotics and Computing.

1 Introduction

To resolve strongly varying solutions, advection-diffusion schemes are often used with non-uniform grids. In particular for multi-dimensional problems this may lead to significantly fewer grid points than with uniform grids. On the other hand, non-uniform grids are known to greatly complicate the convergence analysis and their use therefore is much less straightforward than for uniform grids. For example, a scheme that is inconsistent at the level of the local truncation error still may be convergent. The purpose of this paper is to contribute to the theory of spatial discretizations on non-uniform grids. We shall present spatial convergence results for a number of vertex and cell centered schemes for the linear 1D time-dependent advection-diffusion problem. The focus hereby lies on the discrepancy between local and global order. This discrepancy is known in the numerical PDE literature. However, an essential part of our error analysis is new and based on simple criteria, and therefore believed to further improve insight in the use of non-uniform grids for advection-diffusion problems.

The discrepancy between local and global order observed on non-uniform grids already becomes clear with linear, one-dimensional problems, see e.g. Samarskij [8], Manteuffel &

White [5] and Weiser & Wheeler [12]. In this paper we consider the scalar advection-diffusion problem in conservation form,

$$u_t + (a(x,t)u)_x = (d(x,t)u_x)_x + s(x,t)\,. \tag{1.1}$$

We restrict ourselves to this linear model problem for the sake of analysis and in the actual convergence proofs we will even take a and d constant. The schemes we consider are semi-discrete finite difference or finite volume schemes as derived in the method of lines approach. We consider both vertex and cell centered schemes. Let $\Omega_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ be a grid cell of width h_j associated to a grid point x_j. The semi-discrete schemes then fit in the general conservation form

$$w_j'(t) = \frac{1}{h_j}\Big(f_{j-\frac{1}{2}}(t,w(t)) - f_{j+\frac{1}{2}}(t,w(t))\Big) + g_j(t) \tag{1.2}$$

with linear fluxes $f_{j+\frac{1}{2}}$ defined at the cell vertices $x_{j+\frac{1}{2}}$. The approximations $w_j(t)$ can be interpreted as semi-discrete approximations to the cell average values

$$\bar{u}(x_j,t) = \frac{1}{h_j}\int_{\Omega_j} u(s,t)\,ds$$

or to the point values $u(x_j,t)$. The difference between cell centered and vertex centered discretizations lies in the location of the cell vertices with respect to the grid points x_j. The source term can be taken averaged over Ω_j or pointwise. These schemes are standard and they do have a wider applicability, e.g. to cater for nonlinear terms.

In our convergence analysis we focus on the discrepancy between local and global spatial orders. For this purpose we first present, in Section 2, a general convergence theorem for semi-discrete approximations through which this discrepancy can be clarified. We note that the scope of applicability of this theorem goes beyond non-uniform grids. The theorem is also useful to deal with boundary discretizations of lower order than discretizations used in the interior. However, here we focus entirely on the non-uniform grid issue. The theorem will be applied to a central vertex centered scheme in Section 3 and to a central cell centered scheme in Section 4. In both sections also some alternatives are discussed, including upwind schemes. The theoretical findings will be illustrated numerically in Section 5 and the paper is concluded with a brief practical summary in Section 6.

2 A general convergence theorem

Consider a linear PDE problem like (1.1) with a sufficiently differentiable solution $u(x,t)$ for $t \geq 0$ and $x \in \Omega$ with Ω a bounded closed interval in $\mathbb{R}$. Without loss of generality we may take $\Omega = [0,1]$. The problem can have periodic boundary conditions or given conditions at $x = 0, 1$ and is subjected to an initial condition at $t = 0$. Spatial discretization on any grid Ω_h, with h representing a suitable measure of the grid, is supposed to yield a semi-discrete finite-volume or finite-difference scheme taking the form of a linear ODE initial value problem in $\mathbb{R}^m$,

$$w'(t) = A(t)w(t) + g(t)\,, \tag{2.1}$$

with $w(0)$ given. The matrix $A(t)$ contains negative powers of grid sizes and $w(t)$ is a continuous time grid function on Ω_h with components $w_j(t), j = 1, \ldots, m$, which can be interpreted as approximating $\bar{u}(x_j, t)$ or $u(x_j, t)$. In our analysis we shall mainly work with the point values, but this is not essential for the analysis. Note that boundary values are supposed to be included in (2.1). Obviously, when written in system form the linear conservation scheme (1.2) fits in class (2.1).

The spatial (global) discretization error is defined by $\varepsilon(t) = u_h(t) - w(t)$ where $u_h(t)$ is the restriction of the exact PDE solution to Ω_h. The spatial (local) truncation error is defined by

$$\sigma_h(t) = u_h'(t) - \big(A(t)u_h(t) + g(t)\big) \,,$$

being the residual left by substituting the PDE solution in (2.1). Let $\|\cdot\|$ denote the norm in $\mathbb{R}^m$ used in the convergence analysis. We will consider the convergence problem on a fixed time interval $[0, T]$ and, as common, call the scheme (2.1) *consistent* of order q if, for $h \to 0$,

$$\|\sigma_h(t)\| = \mathcal{O}(h^q) \quad \text{uniformly for } 0 \le t \le T \,,$$

and *convergent* of order p if, for $h \to 0$,

$$\|\varepsilon(t)\| = \mathcal{O}(h^p) \quad \text{uniformly for } 0 \le t \le T \,.$$

Of course, the exact PDE solution is assumed to be sufficiently differentiable and the order constants involved independent of h and of modest size. On uniform grids, and neglecting boundary conditions, the standard case is $p = q$. With non-uniform grids we can have $p > q$ and even $q = 0$ and yet convergence.

The solution of the semi-discrete system (2.1) is given by the variation of constants formula

$$w(t) = V(t)w(0) + \int_0^t V(t)V(s)^{-1} g(s)\, ds \,,$$

where $V(t) \in \mathbb{R}^{m\times m}$ is the fundamental matrix solution of the homogeneous problem $V'(t) = A(t)V(t)$, $V(0) = I$. The semi-discrete scheme (2.1) is called *stable* if on all grids Ω_h we have

$$\|V(t)V(s)^{-1}\| \le K \quad \text{for } 0 \le s \le t \le T$$

in the induced matrix norm, with constant $K \ge 1$ independent of h. Note that for constant matrix A we have $V(t) = \exp(tA)$. Transparent sufficient conditions for stability based on logarithmic matrix norms can be found for instance in [1, 2]. With stability at hand, convergence can be concluded from consistency. By subtracting $w(t)$ from $u_h(t)$ we get the global error equation $\varepsilon'(t) = A(t)\varepsilon(t) + \sigma_h(t)$ and through the variation of constants formula we obtain

$$\|\varepsilon(t)\| \le K\,\|\varepsilon(0)\| + Kt \max_{0\le s\le t} \|\sigma_h(s)\| \,. \tag{2.2}$$

Thus, assuming stability and consistency of order q, convergence of order $p = q$ follows. However, this standard way of reasoning is often inadequate since the local truncation error estimates may be too pessimistic. The following more refined result serves to remedy this.

Theorem 2.1 *Suppose the scheme is stable and the truncation error can be decomposed as*

$$\sigma_h(t) = A(t)\xi(t) + \eta(t) \qquad \textit{with} \qquad \|\xi(t)\|,\ \|\xi'(t)\|,\ \|\eta(t)\| \le Ch^r, \tag{2.3}$$

for $0 \le t \le T$. We then have convergence of order $p = r$ with the error bound

$$\|\varepsilon(t)\| \le K\|\varepsilon(0)\| + \big(1 + K + Kt\big)Ch^r \,, \qquad 0 \le t \le T \,.$$

Proof. The global error $\varepsilon(t)$ satisfies

$$\varepsilon'(t) = A(t)\varepsilon(t) + \sigma_h(t) = A(t)\big(\varepsilon(t) + \xi(t)\big) + \eta(t) \ .$$

Hence $\tilde{\varepsilon}(t) = \varepsilon(t) + \xi(t)$ satisfies

$$\tilde{\varepsilon}'(t) = A(t)\tilde{\varepsilon}(t) + \xi'(t) + \eta(t) \ , \quad \tilde{\varepsilon}(0) = \varepsilon(0) + \xi(0) \ .$$

We can now proceed in the same way as above to obtain

$$\|\varepsilon(t)\| \le \|\xi(t)\| + K\,\|\varepsilon(0) + \xi(0)\| + Kt \max_{0\le s\le t} \|\xi'(s) + \eta(s)\| \ .$$

The stated result follows immediately from this error bound. □

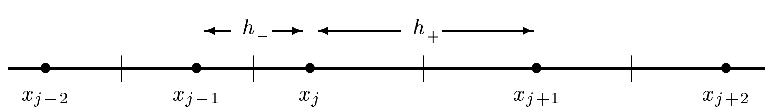
The simple but crucial idea here is the decomposition (2.3) of the local truncation error. Since $A(t)$ will contain negative powers of the grid sizes, we can have $r > q$. Together with the stability assumption, the local error decomposition enables us to conclude convergence with order r rather than with the consistency order q. Such behaviour was called *supra-convergence* in [4]. To exploit this theorem for actual non-uniform grid discretizations, we thus have first to verify stability and second to find decompositions (2.3) with $r > q$. In this paper most of the attention will be given to the second part.

3 Vertex centered schemes

Consider a non-uniform grid defined by unequally spaced grid points $\{x_j\}$. For the moment boundaries of the domain are not taken along in our considerations. For a conservative discretization we need to prescribe the location of cell boundaries, or vertices, for flux computations. Here we consider the vertex centered grid where the cell vertices $x_{j+\frac{1}{2}}$ are centered between the grid points x_j and x_{j+1}, thus giving

$$x_{j+\frac{1}{2}} = \tfrac{1}{2}(x_j + x_{j+1}) \ , \quad h_j = \tfrac{1}{2}(x_{j+1} - x_{j-1}) \ . \tag{3.1}$$

Let for convenience of notation $h_- = x_j - x_{j-1}$, $h_+ = x_{j+1} - x_j$ when a specified grid point x_j is considered.



For the variable coefficient advection-diffusion problem (1.1) the most simple central flux form on the vertex centered non-uniform grid is

$$f_{j+\frac{1}{2}}(t,w) \ = \ a_{j+\frac{1}{2}}\,\frac{w_j + w_{j+1}}{2} \ + \ d_{j+\frac{1}{2}}\,\frac{w_j - w_{j+1}}{h_+} \ , \tag{3.2}$$

where $a_{j+1/2}$, $d_{j+1/2}$ are exact or approximate values at time t at the cell boundary $x_{j+1/2}$, for example linear averages. For uniform grids this flux gives the standard 2-nd order central discretization. With the flux (3.2) and denoting $w_j = w_j(t)$, the central non-uniform vertex centered discretization thus becomes

$$\begin{aligned} w_j' = \frac{1}{2h_j}\Big(a_{j-\frac12} w_{j-1} \;+\; \big(a_{j-\frac12} - a_{j+\frac12}\big)\, w_j \;-\; a_{j+\frac12} w_{j+1} \Big) \\ +\; \frac{1}{h_j}\Big(\frac{d_{j-\frac12}}{h_-}\, w_{j-1} - \Big(\frac{d_{j-\frac12}}{h_-} + \frac{d_{j+\frac12}}{h_+}\Big)\, w_j + \frac{d_{j+\frac12}}{h_+}\, w_{j+1} \Big) \;+\; g_j \;. \end{aligned} \tag{3.3}$$

Upwinding can be introduced by an artificial increase of the diffusion coefficients to

$$\tilde d_{j\pm\frac12} \;=\; d_{j\pm\frac12} \;+\; \tfrac12\, \kappa_\pm h_\pm a_{j\pm\frac12} \;,$$

where the 1-st order upwind scheme corresponds to the choice $\kappa_\pm = \mathrm{sign}(a_{j\pm1/2})$. Intermediate values $|\kappa_\pm| < 1$ can be chosen such that the upwinding becomes exponentially fitted as in the well-known Il'in scheme [6, 7]. Here we focus mainly on the central scheme assuming that the grid has been properly placed to avoid oscillations, which could be controlled for instance through the cell Péclet numbers.

3.1 Consistency, stability and convergence properties

The theoretical complications that are introduced by non-uniform grids are best illustrated by the central scheme (3.3) with constant coefficients. Unless indicated otherwise, it is assumed in the following that a and d are constant.

3.1.1 Consistency

Using Taylor expansions we obtain for the spatial truncation error $\sigma_h(t) = \big(\sigma_{h,j}(t)\big)$,

$$\begin{aligned} \sigma_{h,j}(t) = \frac{a}{2}\Big(h_+ - h_-\Big)\, u_{xx}(x_j,t) + \frac{a}{6}\Big(h_+^2 - h_+h_- + h_-^2\Big)\, u_{xxx}(x_j,t) \\ -\; \frac{d}{3}\Big(h_+ - h_-\Big)\, u_{xxx}(x_j,t) - \frac{d}{12}\Big(h_+^2 - h_+h_- + h_-^2\Big)\, u_{xxxx}(x_j,t) + \cdots \;. \end{aligned} \tag{3.4}$$

The leading error terms are proportional to $h_+ - h_-$ both for the advection and diffusion contributions. Hence for arbitrary grid spacings we have only a *first* order truncation error for $h_-, h_+ \to 0$. If we assume the grid to be *smooth* in the sense that $h_+ - h_- = \mathcal{O}(h^2)$ where h denotes a maximal mesh width, the usual *second* order behaviour in h is recovered. For example, if the grid is based on a smooth transformation $x = x(\xi)$ with a uniform grid for the underlying variable ξ, we get such a smooth grid and then (3.4) can be further expanded to reveal second order consistency. Omitting arguments we then can write

$$\sigma_h = a\bar h^2\big(\tfrac12 x_{\xi\xi}\, u_{xx} + \tfrac16 x_\xi^{\,2}\, u_{xxx}\big) - d\bar h^2\big(\tfrac13 x_{\xi\xi}\, u_{xxx} + \tfrac1{12} x_\xi^{\,2}\, u_{xxxx}\big) + \mathcal{O}(\bar h^4) \tag{3.5}$$

where $\bar h$ is now the uniform mesh width used for ξ.

3.1.2 Stability

We next examine the stability properties of (3.3), with a, d constant, on the spatial interval $[0,1]$ with Dirichlet boundary conditions at $x_0 = 0$, $x_{m+1} = 1$. Since stability is concerned with differences between solutions, we can consider for our linear problem homogeneous boundary conditions and source term $s = 0$. Then (3.3) can be written in the following linear system form in $\mathbb{R}^m$,

$$w'(t) = Aw(t)\,, \qquad A = H^{-1}\big(B_1 + B_2\big)\,, \tag{3.6}$$

where $H = \mathrm{diag}\,(h_j)$ and B_1, B_2 are the contributions of advection and diffusion. The matrices B_1 and B_2 are easily seen to be skew-symmetric and symmetric non-positive definite, respectively. The discrete L_2 inner product and corresponding norm on the non-uniform grid are defined in a natural way by

$$\langle u, v\rangle \;=\; \sum_{j=1}^{m} h_j u_j v_j \;=\; u^T H v\;, \qquad \|v\|^2 = \langle v, v\rangle\;.$$

With this inner product we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 \;&=\; \langle w(t)\,,\, w'(t)\rangle \;=\; \langle w(t)\,,\, H^{-1}\big(B_1 + B_2\big)w(t)\rangle \\ &=\; w(t)^T\big(B_1 + B_2\big)\,w(t) \;=\; w(t)^T B_2\, w(t) \;\le\; 0\;.\end{aligned}$$

It follows that $\|w(t)\|$ is non-increasing in t and thus we have stability in the L_2-norm.

We note that this L_2-stability result can be easily extended to variable coefficients provided $a(x)$ is smooth in x. Then the diagonal contributions of the advective terms in (3.3) are $\mathcal{O}(1)$ uniformly in h, and stability with moderate growth can be proven by considering the diagonal as a perturbation on the skew-symmetric case. Variable diffusion coefficients can be dealt with without change. With Neumann boundary conditions a similar result can be obtained by considering an inner product that is slightly modified at the boundaries. In the maximum norm we can establish stability if the cell Péclet numbers

$$\mu_{j+\frac{1}{2}} = a_{j+\frac{1}{2}}(x_{j+1} - x_j)/d_{j+\frac{1}{2}}$$

are at most 2 in modulus. Also this follows easily by considering the logarithmic maximum norm of the matrix A, see [1, 2] for instance. For advection dominated problems such a restriction on the cell Péclet numbers is impractical to impose over the whole spatial domain, but by local adaptation of the mesh we can impose this in those regions where the variation of the solutions is large and where maximum norm stability will matter most.

3.1.3 Convergence

Having stability and an estimate for $\|\sigma_h(t)\|$, a global error bound for $\|\varepsilon(t)\|$ can be obtained by the standard inequality (2.2). If the grid is smooth, 2-nd order convergence will then hold similar as for the uniform grid. However, the truncation error suggests that on *arbitrary* grids only 1-st order convergence will hold. This is too pessimistic, as will be shown next. Earlier results of this type for stationary advection-diffusion problems are e.g. found in [5, 8]. The approach here is simpler.

As above we consider the spatial interval $[0, 1]$ with Dirichlet conditions at the boundaries $x_0 = 0$ and $x_{m+1} = 1$ and denote

$$\Delta_j = x_j - x_{j-1}\;, \quad j = 1, \ldots, m+1\,.$$

To derive correct global estimates with 2-nd order convergence Theorem 2.1 is applied. The essential point then is to find a vector $\xi \in \mathbb{R}^m$ such that

$$A\xi = \rho\,, \qquad \|\xi\| = \mathcal{O}(h^2)\,,$$

where h is the maximal grid size and $\rho \in \mathbb{R}^m$ is the leading term of the truncation error $\sigma_h(t)$ for t fixed. From (3.4) we have

$$\rho_j = \big(\Delta_{j+1} - \Delta_j\big) v(x_j)\,, \qquad v(x_j) = \tfrac{1}{2} a u_{xx}(x_j,t) - \tfrac{1}{3} d u_{xxx}(x_j,t)\,.$$

Note that the higher order terms of the truncation error are incorporated in the function η of Theorem 2.1. These will give an $\mathcal{O}(h^2)$ contribution to the global error.

First consider the pure diffusion case, $a = 0$, and set for convenience $d = 1$.[1)] Then $A\xi = \rho$ reads

$$\frac{1}{\Delta_j}\big(\xi_{j-1} - \xi_j\big) - \frac{1}{\Delta_{j+1}}\big(\xi_j - \xi_{j+1}\big) = h_j \rho_j\,, \quad j = 1,\ldots,m\,, \tag{3.7}$$

with $\xi_0 = \xi_{m+1} = 0$. Define

$$p_j = \sum_{k=1}^{j-1} h_k \rho_k\,, \quad q_j = \sum_{k=1}^{j} \Delta_k p_k\,, \quad j = 0,1,\ldots,m{+}1\,, \tag{3.8}$$

where empty sums are taken equal to zero. Then it can be verified directly that the solution of the recursion (3.7) with $\xi_0 = \xi_{m+1} = 0$ is given by

$$\xi_j \;=\; q_j - x_j\, q_{m+1}\,, \quad j = 0,1,\ldots,m{+}1\,. \tag{3.9}$$

Now we can estimate $|p_j|$ and then use $|q_j| \le \max_k |p_k|$. We have

$$\begin{aligned} p_j \;&=\; \sum_{k=1}^{j-1} h_k(\Delta_{k+1} - \Delta_k) v(x_k) \;=\; \sum_{k=1}^{j-1} \tfrac{1}{2}(\Delta_{k+1}^2 - \Delta_k^2) v(x_k) \\ &=\; \tfrac{1}{2}\Delta_j^2 v(x_{j-1}) \;-\; \tfrac{1}{2}\sum_{k=2}^{j-1} \Delta_k^2\big(v(x_k) - v(x_{k-1})\big) \;-\; \tfrac{1}{2}\Delta_1^2 v(x_1)\,. \end{aligned}$$

Thus we obtain $|p_j|, |q_j| \le \frac{1}{2}Ch^2$ for all j, with a constant $C > 0$ determined by bounds on $|v|, |v_x|$, [2)] and from (3.9) it follows that $|\xi_j| \le Ch^2$ for $j = 1,\ldots,m$. Hence $\|\xi\| \le Ch^2$ in any L_p-norm establishing the 2-nd order convergence.

With advection terms the proof becomes somewhat more technical. Then the relation $A\xi = \rho$ reads

$$\frac{1}{\Delta_j}\big(d + \tfrac{1}{2}a\Delta_j\big)\big(\xi_{j-1} - \xi_j\big) - \frac{1}{\Delta_{j+1}}\big(d - \tfrac{1}{2}a\Delta_{j+1}\big)\big(\xi_j - \xi_{j+1}\big) = h_j \rho_j \tag{3.10}$$

for $j = 1,\ldots,m$ and $\xi_0 = \xi_{m+1} = 0$. Define

$$c_j = \prod_{k=1}^{j} \Big(\frac{d - \frac{1}{2}a\Delta_k}{d + \frac{1}{2}a\Delta_k}\Big)\,, \qquad c_{j+\frac{1}{2}} = \big(d - \tfrac{1}{2}a\Delta_{j+1}\big) c_j\,. \tag{3.11}$$

1) Taking $d = 1$ is without loss of generality, since d multiplies A as well as ρ if $a = 0$.

2) More precisely, we can set $C = \max\limits_{1\le j\le m} \Big(\theta_j^2 \max(|2v(x_j)|, |v_x(x_j)|)\Big)$ with $\theta_j = h^{-1}\Delta_j$.

Then (3.10) can be written as

$$\frac{c_{j-\frac12}}{\Delta_j}\big(\xi_{j-1}-\xi_j\big)-\frac{c_{j+\frac12}}{\Delta_{j+1}}\big(\xi_j-\xi_{j+1}\big)=c_jh_j\rho_j\ .$$

This is of the same form as (3.7) only with $\tilde\Delta_j=\Delta_j/c_{j-\frac12}$ and $\tilde h_j=h_jc_j$ replacing Δ_j, h_j. Defining likewise

$$\tilde p_j=\sum_{k=1}^{j-1}\tilde h_k r_k\ ,\qquad \tilde q_j=\sum_{k=1}^{j}\tilde\Delta_k\tilde p_k\ ,$$

we obtain

$$\xi_j=\tilde q_j-\tilde x_j\tilde q_{m+1}\ ,\qquad \tilde x_j=\sum_{k=1}^{j}\tilde\Delta_k\Big/\sum_{k=1}^{m+1}\tilde\Delta_k\ .$$

For fixed $a\in\mathbb{R}$, $d>0$ we have $c_j=e^{-ax_j/d}+\mathcal{O}(h^2)$ and we can estimate the terms $|\xi_j|$ in a similar way as for the pure diffusion case, to arrive at a bound $|\xi_j|\le Ch^2$ for all j.

The above proof breaks down if we allow $|a|/d\to\infty$, say $d\to0$ with a fixed, since then $\{c_j\}$ will no longer be a smooth grid sequence. Moreover, then also boundedness assumptions on derivatives of u are no longer justified since a steep layer should be expected at the outflow boundary. For this limit case different types of error bounds are required. Some numerical results and remarks relevant to this case will be presented in Section 5.2.

Remark 3.1 For the pure advection problem $u_t+au_x=0$, $a>0$ on $\Omega=[0,\infty)$ with $u(0,t)$ given,[3)] the central scheme (3.3) may be convergent with order one rather than with order two if the grid is non-smooth. To see why there is no favourable propagation of the truncation error, in contrast to the advection-diffusion case, consider once more the relation $A\xi=\rho$ with leading truncation error terms $\rho_j=\frac12 a(\Delta_{j+1}-\Delta_j)u_{xx}(x_j,t)$. Then we get

$$\xi_{j-1}-\xi_{j+1}=\big(\Delta_{j+1}^2-\Delta_j^2\big)v(x_j)\ ,\qquad v(x_j)=\tfrac12 u_{xx}(x_j,t)\ ,$$

for $j\ge1$ with $\xi_0=0$, see (3.10). Here ξ_1 is free to choose, reflecting singularity of A, and we can take for instance $\xi_1=0$. It follows that

$$\xi_j=(\Delta_1^2-\Delta_2^2)v(x_1)+(\Delta_3^2-\Delta_4^2)v(x_3)+\cdots+(\Delta_{j-1}^2-\Delta_j^2)v(x_{j-1})$$

if j is even, and a similar expression is found for j odd. If the grid is unfavourable there will be no error cancellation. For example, consider a theoretical sequence with $\Delta_{2k-1}=h$, $\Delta_{2k}=\frac12h$. Then, for j even,

$$\xi_j=\tfrac34h^2\big(v(x_1)+v(x_3)+\cdots+v(x_{j-1})\big)=\mathcal{O}(h)\qquad\text{for } x_j \text{ fixed, } h\to0\ .$$

Of course, this choice of grid is only of theoretical interest, but it does show that for 2-nd order convergence with the central scheme (3.3) we either need $d>0$ or some smoothness of the grid. Numerical illustrations are given in Section 5.1. ◇

3) Here Ω is chosen unbounded to the right to avoid additional numerical boundary conditions. For a complete convergence proof in L_1 or L_2-norm it can then be assumed that the solution has compact support, say $u(x,t)=0$ for $x\ge x^*$, $0\le t\le T$.

3.2 A finite difference alternative

With three arbitrarily spaced grid points it is possible to discretize the advection equation $u_t + (a(x)u)_x = 0$ to obtain a finite difference formula with a 2-nd order truncation error.[4)] An elementary calculation shows that this is achieved by the discretization

$$w_j' = \frac{1}{2h_j}\Big(\frac{h_+}{h_-}a_{j-1}w_{j-1} - \Big(\frac{h_+}{h_-} - \frac{h_-}{h_+}\Big)a_j w_j - \frac{h_-}{h_+}a_{j+1}w_{j+1}\Big)\,. \tag{3.12}$$

If a is constant, the spatial truncation error satisfies

$$\sigma_{h,j}(t) \;=\; \tfrac{1}{6}\,ah_+h_-u_{xxx}(x_j,t) + \cdots \;=\; \mathcal{O}(h^2)\,.$$

If the grid is smooth, the truncation errors of (3.12) and the conservative advection discretization used in (3.3) are similar. For arbitrary grid spacings scheme (3.12) seems preferable in view of its smaller local truncation error. However, (3.12) is not conservative, and this may lead to a wrong qualitative behaviour for problems where mass conservation is essential. Moreover there is a question of stability. When brought in the linear system form $w'(t) = Aw(t)$, $A = H^{-1}B_1$, the matrix B_1 for this scheme is not skew-symmetric. Theoretical stability results are lacking, but it was found experimentally that the scheme is (weakly) L_2-stable on finite time intervals $[0,T]$, provided the ratios between largest and smallest mesh widths are not too large. Such grids are often called *quasi-uniform.* On arbitrary grids the scheme can become unstable. For problems with spatial periodicity a relatively wide range of grids are admissible for stability. Instabilities arise more easily for advection-diffusion equations with small diffusion coefficients and Dirichlet conditions at outflow boundaries. This observation was made already in [10]. A related case for a cell centered scheme will be discussed more extensively in Section 4.1. Numerical comparisons between (3.12) and (3.3) will be given in Section 5.

3.3 First order upwind advection

Standard upwinding is achieved by taking the advective fluxes as

$$f_{j+\frac{1}{2}}(t,w) = \begin{cases} a_{j+\frac{1}{2}}w_j & \text{if } a_{j+\frac{1}{2}} \ge 0\,, \\ a_{j+\frac{1}{2}}w_{j+1} & \text{if } a_{j+\frac{1}{2}} \le 0\,. \end{cases} \tag{3.13}$$

If a is constant, stability for this scheme in the L_1, L_2 and max-norm is easily established. In fact, stability in the L_1-norm $\|v\|_1 = \sum_j h_j|v_j|$ can be shown to hold for arbitrary variable $a(x)$ by considering the logarithmic matrix norm. Convergence is again more complicated. For constant $a > 0$ the truncation error is found to be

$$\sigma_{h,j}(t) = \frac{a}{2h_j}(\Delta_j - \Delta_{j+1})u_x(x_j,t) - \frac{a}{2h_j}\Delta_j^2 u_{xx}(x_j,t) + \cdots$$

and a similar expression is obtained for $a < 0$. Hence on non-smooth grids the local truncation error gives an inconsistency. However, here we again have a favourable propagation of the truncation error, leading to 1-st order convergence.

[4)] With 3 arbitrarily spaced grid points a 2-nd order truncation error for the diffusion equation $u_t = du_{xx}$ is impossible. Therefore we consider here only the advection equation.

Consider, as in Remark 3.1, $u_t + au_x = 0$ with $a > 0$ on the spatial domain $\Omega = [0, \infty)$ with a Dirichlet condition at $x = 0$. Then setting $A\xi = \sigma_h$ gives

$$\xi_{j-1} - \xi_j = \tfrac{1}{2}\big(\Delta_j - \Delta_{j+1}\big) u_x(x_j, t) + \mathcal{O}(h^2) \ , \quad j \geq 1 \ , \qquad \xi_0 = 0 \ ,$$

which is satisfied for instance with

$$\xi_j = \tfrac{1}{2}\Delta_{j+1} u_x(x_{j+1}, t) - \tfrac{1}{2}\Delta_1 u_x(x_1, t) \ ,$$

showing 1-st order convergence by Theorem 2.1.

4 Cell centered schemes

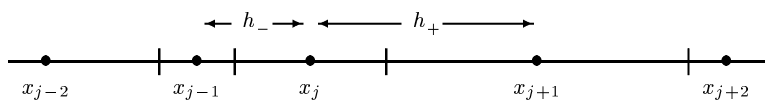
Next consider a partitioning of the spatial domain into cells $\Omega_j = [x_{j-1/2}, x_{j+1/2}]$ and let the points x_j be the cell centers. So here the grid is primarily defined by the cells, that is by the sequence $\{x_{j+1/2}\}$, and we have

$$x_j = \tfrac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}) \ , \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \ . \tag{4.1}$$

As before, we denote shortly

$$h_- = x_j - x_{j-1} = \tfrac{1}{2}\big(h_{j-1} + h_j\big) \ , \quad h_+ = x_{j+1} - x_j = \tfrac{1}{2}\big(h_j + h_{j+1}\big)$$

when we are considering a specified grid point x_j.



On this grid the standard choice for a central flux is

$$f_{j+\frac{1}{2}}(t, w) \ = \ a_{j+\frac{1}{2}} \left(\frac{h_{j+1}}{2h_+} w_j + \frac{h_j}{2h_+} w_{j+1} \right) \ + \ d_{j+\frac{1}{2}} \frac{w_j - w_{j+1}}{h_+} \ . \tag{4.2}$$

Note that the expression for the diffusive part has a similar form as with the vertex centered scheme, but the location of x_j and $x_{j\pm1/2}$ is different here. The advective fluxes are obtained by linear interpolation. With this expression for the fluxes the central non-uniform cell centered advection-diffusion discretization becomes

$$\begin{aligned} w_j' \ = \ & \frac{1}{2h_j}\Big(\frac{a_{j-\frac{1}{2}} h_j}{h_-} w_{j-1} + \Big(\frac{a_{j-\frac{1}{2}} h_{j-1}}{h_-} - \frac{a_{j+\frac{1}{2}} h_{j+1}}{h_+} \Big) w_j - \frac{a_{j+\frac{1}{2}} h_j}{h_+} w_{j+1} \Big) \\ & + \ \frac{1}{h_j}\Big(\frac{d_{j-\frac{1}{2}}}{h_-} w_{j-1} - \Big(\frac{d_{j-\frac{1}{2}}}{h_-} + \frac{d_{j+\frac{1}{2}}}{h_+} \Big) w_j + \frac{d_{j+\frac{1}{2}}}{h_+} w_{j+1} \Big) \ + \ g_j \ . \end{aligned} \tag{4.3}$$

Upwinding can again be introduced by artificially increasing the diffusion coefficients.

The advective fluxes in (4.2) are obtained by linear interpolation. As an alternative these fluxes can simply be based on averaging

$$f_{j+\frac{1}{2}}(t, w) = \tfrac{1}{2}\, a_{j+\frac{1}{2}} (w_j + w_{j+1}) \ , \tag{4.4}$$

with diffusive fluxes taken as in (4.2). This leads to the same formula as in (3.3); again it is the location of x_j with respect to the points $x_{j\pm1/2}$ which is different than with the vertex centered scheme. As we shall see this modification with averaged fluxes has more favourable stability properties than (4.3), and for that reason it was considered in [11] for stationary problems. However for time-dependent problems with dominating advection the accuracy will turn out to be insufficient.

4.1 Consistency, stability and convergence properties

Similar as for the vertex centered scheme, the cell centered scheme (4.3) and the modification (4.4) will be analyzed under the assumption that a and d are constant.

4.1.1 Consistency

Insertion of the exact solution into (4.3) yields for constant coefficients the truncation error

$$\begin{aligned} \sigma_{h,j}(t) = -\frac{d}{4h_j}\Big(h_{j+1} - 2h_j + h_{j-1}\Big) u_{xx}(x_j,t) \\ -\frac{d}{6h_j}\Big(h_+^2 - h_-^2\Big) u_{xxx}(x_j,t) + \frac{a}{8}\Big(h_{j+1} - h_{j-1}\Big) u_{xx}(x_j,t) + \mathcal{O}(h^2)\ , \end{aligned} \tag{4.5}$$

with h being the maximal mesh width. Note that for arbitrary grids, without smoothness, the diffusion discretization now even leads to $\sigma_h = \mathcal{O}(h^0)$, that is *inconsistency*. For smooth grids generated by a transformation $x = x(\xi)$, with underlying mesh width $\bar{h}$, we reobtain 2-nd order consistency comparable with (3.5),

$$\begin{aligned} \sigma_h \ &= \ a\bar{h}^2\big(\tfrac{1}{4}\, x_{\xi\xi}\, u_{xx} + \tfrac{1}{6}\, x_\xi^{\,2}\, u_{xxx}\big) \\ &- d\bar{h}^2\, \big(\tfrac{1}{4}\, (x_\xi)^{-1} x_{\xi\xi\xi}\, u_{xx} - \tfrac{1}{6}\, x_{\xi\xi}\, u_{xxx} + \tfrac{1}{12}\, x_\xi^{\,2}\, u_{xxxx}\big) \ + \ \mathcal{O}(\bar{h}^3)\,. \end{aligned} \tag{4.6}$$

Remark 4.1 Since with this cell centered scheme the grid is primarily defined by the cells, and not by the points x_j, it might be argued that consistency should be regarded with respect to cell average values $\bar{u}(x_j,t)$ instead of point values $u(x_j,t)$. However, also upon inserting these cell average values into the scheme, a similar expression for the truncation error is obtained, again with inconsistency for the diffusion term. As far as 2-nd order convergence is concerned it makes little difference whether point values or cell averages are regarded. Since

$$\bar{u}(x_j,t) = u(x_j,t) + \tfrac{1}{24} h_j^2\, u_{xx}(x_j,t) + \mathcal{O}(h_j^4)\ ,$$

the difference between the two is $\mathcal{O}(h^2)$. $\diamond$

If we consider the scheme with averaged advective fluxes (4.4) for constant a, the leading advection contributions in the truncation error become

$$\begin{aligned} \sigma_{h,j}(t) &= a\Big(\frac{h_+ + h_-}{2h_j} - 1\Big) u_x(x_j,t) + a\,\frac{h_+^2 - h_-^2}{4h_j}\, u_{xx}(x_j,t) \\ &= \frac{a}{4}\Big(\frac{h_{j-1} - 2h_j + h_{j+1}}{h_j}\Big) u_x(x_j,t) + a\Big(\frac{h_{j+1} - h_{j-1}}{8} + \frac{h_{j+1}^2 - h_{j-1}^2}{16h_j}\Big) u_{xx}(x_j,t)\ , \end{aligned}$$

so here we get an inconsistency on non-smooth grids for the advection problem. The terms due to diffusion remain the same as in (4.5) of course. Therefore, the leading term in the truncation error for the advection-diffusion problem $u_t + au_x = du_{xx}$ becomes

$$\sigma_{h,j}(t) = \Big(\frac{h_{j-1} - 2h_j + h_{j+1}}{4h_j}\Big)\big(au_x(x_j,t) - du_{xx}(x_j,t)\big) + \cdots .$$

This leading term vanishes for steady state problems, but otherwise we may have inconsistency. As we shall see later on, convergence will still be 2-nd order if $d > 0$, also on non-smooth grids. However, for pure advection problems the inconsistency may prevent convergence.

4.1.2 Stability

For the vertex centered scheme (3.3) L_2-stability for constant coefficients was easily found to hold due to the skew-symmetry present in the advective terms. With the cell centered scheme (4.3) this property is lost and there are diagonal contributions in the advective terms. Therefore, L_2-stability of this scheme can only be demonstrated easily if the grid is smooth, in which case these diagonal contributions are $\mathcal{O}(1)$ uniformly in h. Extensive numerical tests on various non-uniform grids have shown that for problems with spatial periodicity the scheme is L_2-stable on finite time intervals $[0,T]$ if the ratios between largest and smallest mesh widths are bounded. For advection-diffusion problems with outflow Dirichlet conditions instabilities can easily occur if the grid is not smooth. This instability can be avoided to some extent by using upwinding in the advection discretization at the outflow boundary. These results are comparable to those of the finite difference scheme (3.12), but it was observed experimentally that the cell centered scheme (4.3) does allow for larger irregularities in the grids.

In this respect the modification (4.4) has an advantage since then again L_2-stability will hold for Dirichlet conditions without restriction on a, d or the mesh. However, as mentioned already, this modification may lead to non-convergence for time-dependent problems on non-smooth grids, see also Remark 4.3 below.

Finally we note that similar as for the vertex centered scheme, it easily follows that (4.3) will be stable in the max-norm if the cell Péclet numbers $a_{j\pm1/2}h_j/d_{j\pm1/2}$ are bounded by 2 in modulus.

Remark 4.2 The observed difference in stability for (4.3) between the periodic case and the Dirichlet case can be understood to some extent by an eigenvalue analysis for a constant, say $a = 1$, and $d = 0$.

Then with spatial periodicity it follows by some calculations that the matrix A of (4.3) can be written as

$$A \;=\; (I + E^T)\,D^{-1}\,(E - I)$$

with shift operator $E(v_1, v_2, \ldots, v_m)^T = (v_m, v_1, \ldots, v_{m-1})^T$ and $D = 2\mathrm{diag}(\Delta_1, \ldots, \Delta_m)$, $\Delta_j = x_j - x_{j-1}$. For any two square matrices M, N the eigenvalues of the product MN are the same as for NM, see [3, Thm. 1.3.20]. Therefore the eigenvalues of A are the same as for

$$\tilde{A} \;=\; D^{-1}\,(E - I)\,(I + E^T) \;=\; D^{-1}\,(E - E^T)\,.$$

This matrix however is similar to the skew-symmetric matrix $D^{1/2}\tilde{A}D^{-1/2}$, and thus we see that the eigenvalues of A are purely imaginary.

With Dirichlet conditions the situation is different. Suppose we have homogeneous Dirichlet conditions at $x_0 = 0$, $x_{m+1} = 1$, which is justified if the advection equation is considered as the limit for $d \to 0$ of $u_t + u_x = du_{xx}$. Then the matrix A of (4.3) will be tri-diagonal with diagonal elements

$$a_{jj} \;=\; \frac{1}{2h_j}\Big(\frac{h_{j-1}}{\Delta_j} - \frac{h_{j+1}}{\Delta_{j+1}}\Big) \;=\; -\frac{1}{2\Delta_j} + \frac{1}{2\Delta_{j+1}} \,.$$

Hence

$$\mathrm{trace}(A) \;=\; \sum_{j=1}^{m} a_{jj} \;=\; -\frac{1}{2\Delta_1} + \frac{1}{2\Delta_{m+1}} \,,$$

and since the trace of a matrix equals the sum of its eigenvalues, it is clear that A will have eigenvalues with (large) positive real part if $\Delta_{m+1} \ll \Delta_1$. This is a natural grid choice for advection-diffusion with a boundary layer at the outflow boundary point $x = 1$. To avoid instability with (4.3), upwinding near the outflow boundary will then be needed. $\diamond$

4.1.3 Convergence

As we already saw with the vertex centered schemes, the truncation error may give incorrect information about convergence. With the cell centered schemes this is even more important in view of the inconsistency of the diffusion discretization. Here we shall demonstrate 2-nd order convergence of (4.3) for the pure diffusion case, $a = 0$, with scaling $d = 1$ and with Dirichlet conditions at $x = 0, 1$. For convenience and to obtain a closer resemblance with the results in Section 3, we assume that $x_0 = 0$ and $x_{m+1} = 1$, which means that half-cells $[0, \frac{1}{2}h_0]$ and $[1 - \frac{1}{2}h_{m+1}, 1]$ are placed at the boundaries.

First, consider only the leading term of the truncation error,

$$\rho_j = \frac{1}{h_j}\big(h_{j-1} - 2h_j + h_{j+1}\big)\, v(x_j)\,, \qquad v(x_j) = -\tfrac{1}{4}\, u_{xx}(x_j, t)\,,$$

and recall that for proving 2-nd order convergence by means of Theorem 2.1, the essential point is to find a vector $\xi \in \mathbb{R}^m$ such that $A\xi = \rho$, $\|\xi\| = \mathcal{O}(h^2)$. Setting $A\xi = \rho$, we can follow (3.7), (3.8) and (3.9), to arrive again at

$$\xi_j = q_j - x_j q_{m+1}\,, \qquad q_j = \sum_{k=1}^{j} \Delta_k p_k\,, \qquad p_j = \sum_{k=1}^{j-1} h_k \rho_k$$

where $\Delta_j = x_j - x_{j-1}$. Here the estimation of $|\xi_j|$ requires some care. We have

$$p_j \;=\; \sum_{k=1}^{j-1} \big(h_{k-1} - 2h_k + h_{k+1}\big) v(x_k) \;=\; \big(h_0 - h_1\big) v(x_1)$$
$$-\;\big(h_{j-1} - h_j\big) v(x_{j-1}) \;+\; \sum_{k=2}^{j-1} \big(h_{k-1} - h_k\big)\big(v(x_k) - v(x_{k-1})\big)\,.$$

The last sum can be developed as

$$\sum_{k=2}^{j-1} \big(h_{k-1} - h_k\big)\big(\Delta_k v_x(x_k) + \mathcal{O}(h^2)\big) \;=\; \tfrac{1}{2}\sum_{k=2}^{j-1} \big(h_{k-1}^2 - h_k^2\big) v_x(x_k) + \mathcal{O}(h^2)$$
$$= \; \tfrac{1}{2} h_1^2 v_x(x_2) \; + \; \tfrac{1}{2}\sum_{k=2}^{j-2} h_k^2\big(v_x(x_{k+1}) - v_x(x_k)\big) \; - \; \tfrac{1}{2} h_{j-1}^2 v_x(x_{j-1}) \; + \; \mathcal{O}(h^2)\,,$$

which gives in total an $\mathcal{O}(h^2)$ contribution. Hence

$$p_j \;=\; \big(h_0 - h_1\big)v(x_1) - \big(h_{j-1} - h_j\big)v(x_{j-1}) + \mathcal{O}(h^2)\;.$$

It follows that

$$q_j \;=\; \sum_{k=1}^{j} \Delta_k (h_0 - h_1) v(x_1) - \sum_{k=1}^{j} \Delta_k (h_{k-1} - h_k) v(x_{k-1}) + \mathcal{O}(h^2)$$
$$=\; x_j (h_0 - h_1) v(x_1) - \frac{1}{2}\sum_{k=1}^{j} \big(h_{k-1}^2 - h_k^2\big) v(x_{k-1}) + \mathcal{O}(h^2)$$
$$=\; x_j (h_0 - h_1) v(x_1) + \mathcal{O}(h^2)\;.$$

Since all remainder terms are $\mathcal{O}(h^2)$ uniformly in j, we obtain the estimate

$$|\,\xi_j\,| \;=\; |\,q_j - x_j\, q_{m+1}\,| \;\leq\; Ch^2\,, \qquad j = 1, \ldots, m\,,$$

with a constant $C > 0$ determined by bounds on $|v|, |v_x|, |v_{xx}|$, and thus we see that this inconsistent truncation error term will give an $\mathcal{O}(h^2)$ contribution to the global error. The same holds for the $\mathcal{O}(h)$ terms in the truncation error (4.5). This can be demonstrated just as in Section 3. Consequently, assuming stability in either L_2-norm or max-norm, the scheme will be convergent with order 2 in that norm.

Advection terms can be included in the analysis by using integrating factors as in (3.10), (3.11). For the scheme with averaged advection fluxes (4.4) we can use the integrating factors (3.11). For the standard scheme (4.3) with interpolated fluxes these factors need a little modification.[5] As for the vertex centered scheme, 2-nd order convergence can then be demonstrated provided d is bounded away from 0, for both the schemes with interpolated or averaged fluxes.

Earlier results of this type for cell centered schemes with stationary problems were derived in [5]. Generalizations for self-adjoint parabolic equations (also in 2D) with Neumann boundary conditions and implicit Euler time stepping were obtained in [12] in a mixed finite element framework. Our results should be viewed as complementary, since we consider additional advection discretizations but only in one spatial dimension.

Remark 4.3 Convergence can also be considered for pure advection problems as in Remark 3.1, and then the inconsistency of (4.4) may have a large impact. Consider as before $u_t + au_x = 0$ with $a > 0$ on the spatial domain $\Omega = [0, \infty)$ with a Dirichlet condition at $x = 0$. We assume for convenience that the boundary coincides with the grid point x_0.

[5] For (4.3) one can take $c_j = \prod_{k=1}^{j} (d - \frac{1}{2}ah_k)/(d + \frac{1}{2}ah_k)$, $c_{j+1/2} = (d - \frac{1}{2}ah_j)c_j$ to arrive again at formula (3.10).

First we examine the standard scheme (4.3). Then setting $A\xi = \sigma_h$ gives

$$\frac{1}{\Delta_j}\big(\xi_{j-1} - \xi_j\big) + \frac{1}{\Delta_{j+1}}\big(\xi_j - \xi_{j+1}\big) = \tfrac{1}{4}\big(h_{j+1} - h_{j-1}\big)u_{xx}(x_j,t) + \mathcal{O}(h^2)$$

for $j \geq 0$ with $\xi_0 = 0$. A solution is given by

$$\xi_j = -\tfrac{1}{8}h_j^2 u_{xx}(x_j,t) + \tfrac{1}{8}h_0^2 u_{xx}(0,t)\ ,$$

showing 2-nd order convergence without any restriction on the grid.

For the scheme with averaged fluxes (4.4) the situation is completely different. Then $A\xi = \sigma_h$ gives

$$\xi_{j-1} - \xi_{j+1} = \tfrac{1}{2}\big(h_{j-1} - 2h_j + h_{j+1}\big)u_x(x_j,t) + \cdots$$

with $\xi_0 = 0$, and here we may set also $\xi_1 = 0$. It easily follows that unfavourable grid choices can be made such that there is no cancellation of truncation errors. For example, with an oscillatory grid $h_{2k-1} = h$, $h_{2k} = \frac{1}{2}h$ we obtain, omitting higher order terms,

$$\xi_j = \tfrac{1}{2}h\big(u_x(x_1,t) + u_x(x_3,t) + \cdots + u_x(x_{j-1},t)\big) = \mathcal{O}(h^0)$$

if j is even, and likewise for j odd. Indeed, numerical experiments show that with this grid the scheme does not converge at all, as will be illustrated in Section 5.1. ◇

4.2 First order upwind advection

Standard advection upwinding corresponds to the fluxes (3.13). As for the vertex centered scheme, stability easily follows. Assuming $a > 0$ constant the truncation error is found to be

$$\sigma_{h,j}(t) = \frac{a}{2h_j}\big(h_{j-1} - h_j\big)u_x(x_j,t) - \frac{a}{8h_j}\big(h_{j-1} + h_j\big)^2 u_{xx}(x_j,t) + \cdots$$

and a similar expression holds for $a < 0$. Therefore the truncation error has a similar form as with the vertex centered upwind scheme, and again we have a favourable error propagation leading to 1-st order convergence on any grid, even for pure advection problems. Elaboration of this is the same as in Section 3.3 for the vertex centered upwind scheme.

5 Numerical illustrations

In this section numerical tests are presented. We consider the vertex centered scheme VC_2 given by formula (3.3), the finite difference scheme FD_2 of (3.12) and the upwind vertex centered scheme VC_1. With the latter two schemes diffusion terms are taken as in (3.3). Likewise we consider the cell centered scheme CC_2 of (4.3), its modification CC_2^a with averaged fluxes (4.4) and the upwind cell centered scheme CC_1.

5.1 Advection tests

For a first numerical illustration consider the pure advection problem $u_t + u_x = 0$ for $x \in [0,1]$ with the periodicity condition $u(x \pm 1, t) = u(x,t)$ and smooth initial profile $u(x,0) = \sin^4(\pi x)$. The problem is discretized on sequences of random grids and on the oscillatory grids of the Remarks 3.1, 4.3. The results on the oscillatory grids are merely

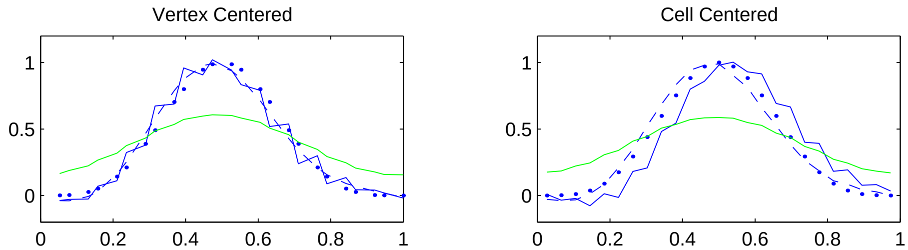


Figure 5.1: Advection test on oscillatory grids with $m = 25$. Solid lines give the results for VC_2, CC_2^a, dashed lines for FD_2, CC_2 and gray lines for VC_1, CC_1. The exact solution is indicated by dots.

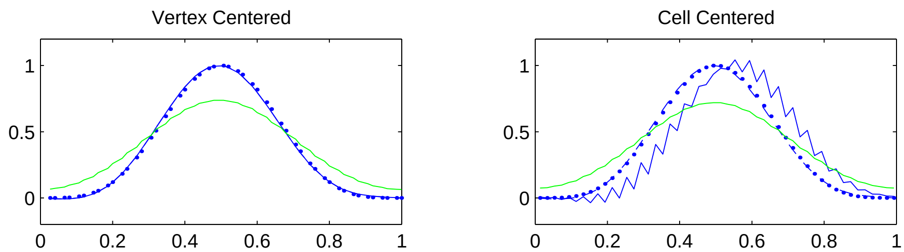


Figure 5.2: Advection test on oscillatory grids with $m = 50$. Solid lines give the results for VC_2, CC_2^a, dashed lines for FD_2, CC_2 and gray lines for VC_1, CC_1. The exact solution is indicated by dots.

intended to illustrate the theoretical behaviour discussed in these examples. Also solving this advection problem on random grids makes no practical sense in itself, but it does provide a further illustration for the sensitivity of the accuracy with respect to jumps in the grid. This sensitivity has some practical relevance for multi-dimensional problems where the choice of the grids may be dictated by complicated geometrical shapes of the spatial domain Ω.

First we consider oscillatory grids defined for the vertex centered schemes by

$$x_1 = \Delta_1\,, \quad x_m = 1\,, \qquad \Delta_j = \begin{cases} h & \text{if } j \text{ is odd,} \\ \frac{1}{2}h & \text{if } j \text{ is even,} \end{cases}$$

and for the cell centered schemes by

$$x_1 = \tfrac{1}{2}h_1\,, \quad x_m = 1 - \tfrac{1}{2}h_m\,, \qquad h_j = \begin{cases} h & \text{if } j \text{ is odd,} \\ \frac{1}{2}h & \text{if } j \text{ is even.} \end{cases}$$

The behaviour of the various schemes on these grids is illustrated in the Figures 5.1 and 5.2 for $m = 25, 50$, respectively. Although these grids are very artificial, the behaviour of the schemes is instructive nevertheless. The exact solution on the grid points x_j at time $t = 1$ is indicated by dots. We see that the vertex centered scheme VC_2 has rather large errors for $m = 25$ but these become less with more grid points.[6] On the other hand, with the

[6] In fact, the result for VC_2 in Figure 5.2 with $t = 1$ benefits from the fact that the number of grid points is even. With $m = 51$ the errors are larger, but also for m odd the scheme converges on these oscillatory grids (with order 1). If m is even and t is a multiple of $\frac{1}{2}$ there is an accidental (non-generic) cancellation of errors in time on this particular grid.

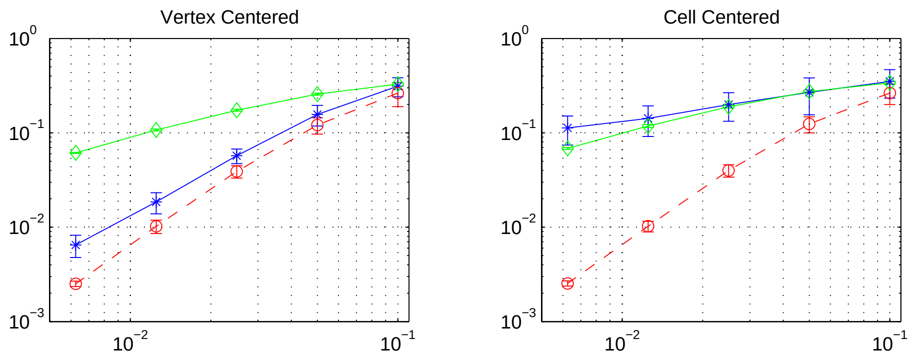


Figure 5.3: L_2-errors versus $1/m$ for the advection test on random grids at $t = 1$. Solid lines and $*$-marks for VC_2, CC_2^a. Dashed lines and $\circ$-marks for FD_2, CC_2. Gray lines and $\diamond$-marks for VC_1, CC_1.

cell centered scheme CC_2^a increasing the number of grid points does not lead to a better solution. Note that with the oscillatory cell centered grids, the cell centers x_j are in fact evenly spaced with $\Delta_j = \frac{3}{2}h$, only the mesh widths h_j are non-uniform. The behaviour of the other two schemes FD_2 and CC_2, and also the upwind schemes, is similar to that on uniform grids. These results are in agreement with the discussions in Remark 3.1 and 4.3.

Next we consider random grids. To construct the grids with a given number of grid points, random numbers $\omega_j \in (0,1)$, $j = 1, \ldots, m$ were produced by a random generator and these were normalized by $\delta_j = \omega_j / \sum_{k=1}^{m} \omega_k$. For the vertex centered case we took $x_0 = 0$ and $\Delta_j = x_j - x_{j-1} = \delta_j$. For the cell centered case $x_1 = \frac{1}{2}\delta_1$ and $h_j = \delta_j$ was used. For each value of $m = 10 \cdot 2^k$, $0 \le k \le 4$, we performed 50 runs on different random grids and the L_2-errors at $t = 1$ were measured.

In Figure 5.3 the means of these errors over the 50 runs are presented, together with the standard deviations indicated by error bars. We see that the results for the upwind schemes and for FD_2, CC_2 are not very sensitive with respect to the grid variation. In particular for the upwind schemes the error bars are very close together and therefore not well visible. For VC_2 the standard deviations are larger, which is to be expected since this scheme is convergent with order 2 on smooth grids and with order 1 only on unfavourable grids, see Remark 3.1. The behaviour of the cell centered scheme with averaged advective fluxes CC_2^a is unsatisfactory. The standard deviations are large and the convergence behaviour is worse than for the 1-st order upwind schemes, which is not surprising in view of Remark 4.3. Thus it can be concluded that CC_2^a is not suited for advection (dominated) problems if the grid is not smooth.

5.2 Boundary layers and special grids

We next consider the standard problem $u_t + u_x = du_{xx}$, $0 \le x \le 1$, with Dirichlet conditions $u(0) = 1$, $u(1) = 0$. This problem has the stationary solution

$$u(x) = \frac{e^{1/d} - e^{x/d}}{e^{1/d} - 1}\,, \qquad 0 \le x \le 1$$

with a boundary layer at $x = 1$ if $d > 0$ is small. To resolve the boundary layer with the central schemes we use fine grids near $x = 1$. A number of special grids have been

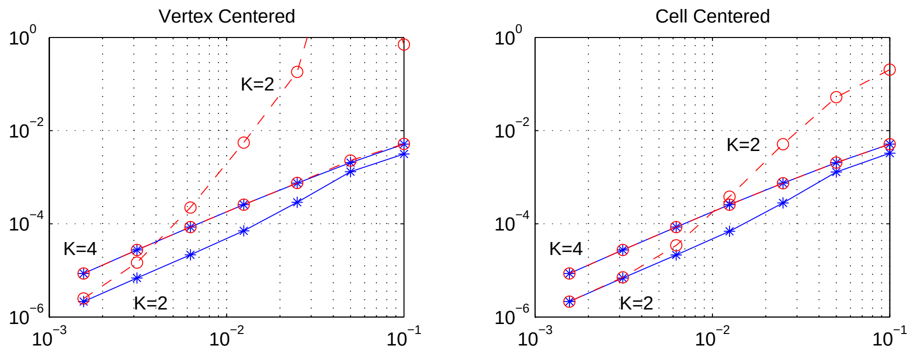


Figure 5.4: L_2-errors versus $1/m$ with Shiskin grids, $K=2,4$, for $d=10^{-3}$. Solid lines and $*$-marks for VC_2, CC_2^a. Dashed lines and $\circ$-marks for FD_2, CC_2.

constructed for such problems, see for instance [7, Sect. 2.4.2]. Here we consider a so-called *Shiskin grid* which consists of two uniform sub-grids with $m/2$ points on the intervals $[0,1-\delta]$ and $[1-\delta,1]$, where $\delta = Kd\ln m$ with positive constant K. It is assumed that $md<1$. Convergence of upwind schemes on such grids, uniformly for $d>0$, was demonstrated in [9], see also [7].

In this test we took $d=10^{-3},10^{-6}$ and $K=2,4$. The stationary solution has been approximated by central schemes. The L_2-errors for a various number of grid points m are given in the Figures 5.4 and 5.5. The schemes VC_2 and CC_2^a for which L_2-stability could be demonstrated are indicated by $*$-marks, the schemes FD_2 and CC_2 are indicated by $\circ$-marks. Since the solution is stationary, the inconsistency of the scheme CC_2^a is absent here.

Also for the two cell centered schemes we used grid points (with half-cells) at the boundaries, and therefore the results for the modified cell centered scheme CC_2^a are nearly identical to those of the vertex centered scheme VC_2. The use of virtual points and extrapolation to implement the Dirichlet conditions on a grid where the boundaries coincide with cell vertices, did lead to less accurate results for small values of d. In practice a standard cell centered grid might be used with some upwinding at outflow boundaries. In the present

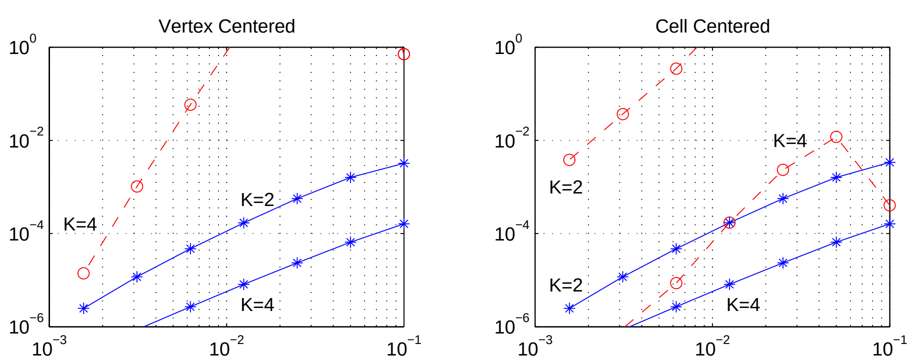


Figure 5.5: L_2-errors versus $1/m$ with Shiskin grids, $K=2,4$, for $d=10^{-6}$. Solid lines and $*$-marks for VC_2, CC_2^a. Dashed lines and $\circ$-marks for FD_2, CC_2. Results for FD_2 with $K=2$ are outside the frame (errors larger than 1).

test the results would benefit from local upwinding, but the purpose here is to test genuine central schemes.

It is obvious from the figures that the schemes VC_2 and CC_2^a, for which L_2-stability was established, produce much better results in these tests than the finite difference scheme FD_2 and the cell centered scheme CC_2. With the latter schemes strongly oscillatory solutions are obtained. If the number of grid points becomes sufficiently large, the diffusion term provides stabilization. Also for larger K the oscillations become less pronounced since then the grid interface is shifted towards the region where the solution is smooth. Further we note that the convergence rates in these tests are actually less than 2-nd order (in the range 1.6 – 1.7). In the derivation of the 2-nd order results in Sections 3 and 4 smoothness of the solutions was assumed. In the boundary layer this is not a reasonable assumption, unless the grid gets very fine ($h \ll d$). Error bounds with 1-st order convergence, uniformly in $d > 0$, were obtained in [9]. For related convergence results applicable to problems with boundary layers, we refer to the books [6, 7].

6 Practical conclusions

The above analysis and tests show that the use of schemes on non-uniform grids is far from straightforward. Among the four central schemes considered here none performs without problems. The finite difference scheme FD_2 of formula (3.12) is not mass conservative and it looses stability if there are large jumps in the grid. To a lesser extent this instability is also present with the cell centered scheme CC_2 of formula (4.3), in particular with Dirichlet conditions at the outflow. With the standard vertex centered scheme VC_2 of formula (3.3) irregularities in the grid may lead to an order of convergence less than 2 in the pure advection case, but at least 1. The inconsistency of the cell centered scheme CC_2^a of formula (4.4) with averaged advective fluxes leads to very inaccurate results for time-dependent advection (dominated) problems. In conclusion, the schemes VC_2 and CC_2 are recommended, although with the latter one large jumps in the grid should be avoided.

Acknowledgement The authors thank Kees Dekker and Karel in 't Hout for helpful discussions on the stability issues.

References

[1] K. Dekker, J.G. Verwer, *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations.* North-Holland, 1984.

[2] E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordinary Differential Equations I – Nonstiff Problems.* 2-nd ed., Springer Series in Computational Mathematics 8, Springer Verlag, 1993.

[3] R.A. Horn, C.R. Johnson, *Matrix Analysis.* Cambridge University Press, 1985.

[4] H.-O. Kreiss, T.A. Manteuffel, B. Swartz, B. Wendroff, A.B. White, *Supra-convergent schemes on irregular grids.* Math. Comp. 47 (1986), pp. 537-554.

[5] T.A. Manteuffel, A.B. White, *The Numerical Solution of Second-Order Boundary-Value Problems on Nonuniform Meshes.* Math. Comp. 47 (1986), pp. 511-535.

[6] K.W. Morton, *Numerical Solution of Convection-diffusion Problems.* Applied Mathematics and Mathematical Computation 12, Chapman & Hall, 1996.

[7] H.-G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations.* Springer, 1996.

[8] A.A. Samarskij, *Theorie der Differenzenverfahren.* Akademische Verlagsgesellschaft Geest & Portig K.-G., 1984 (published in Russian in 1977).

[9] G.I. Shiskin, *Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations* (in Russian). Second doctoral thesis, Keldysh Institute, Moscow, 1990.

[10] A.E.P. Veldman, K. Rinzema, *Playing with nonuniform grids.* J. of Engin. Math. 26 (1992), pp. 119-130.

[11] P. Wesseling, *Principles of Computational Fluid Dynamics*, Springer Series in Computational Mathematics 29, Springer, 2000.

[12] A. Weiser, M.F. Wheeler, *On convergence of block-centered finite differences for elliptic problems.* SIAM J. Numer. Anal. 25 (1988), pp. 351-375.