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J. van den Berg, H. Kesten

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P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

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Randomly Coalescing Random Walk in Dimension ≥ 3

J. van den Berg
CWI
P.O. Box 94079, 1090 GB Amsterdam
The Netherlands

Harry Kesten
Department of Mathematics (White Hall)
Cornell University, Ithaca, NY 14853
USA

Abstract.
Suppose at time 0 each site of $\mathbb{Z}^d$ contains one particle, which starts to perform a continuous time random walk. The particles interact only at times when a particle jumps to an already occupied site: if there are j particles present, then the jumping particle is removed from the system with probability p_j. We assume that p_j is increasing in j. In an earlier paper we proved that if the dimension d is at least 6, then $p(t) := P\{\text{there is at least one particle at the origin at time } t\} \sim C(d)/t$, with $C(d)$ an explicitly identified constant. We also conjectured that the result holds for $d \geq 3$. In the present paper we show that, under the quite natural condition that the number of particles per site is bounded, this is indeed the case.

The key step in the proof is to improve a certain variance bound, which is needed to estimate the error terms in an approximate differential equation for $p(t)$. We do this by making more refined use of coupling methods and (correlation) inequalities.

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1. Introduction.

In the basic coalescing random walk (CRW) model particles move according to continuous time (rate 1) simple random walks: a particle waits an exponentially (with mean 1) distributed time, and then jumps with equal probabilities to one of its $2d$ neighbour sites. When a particle jumps to a site which is already occupied, the two particles coalesce to a single particle, which continues its random walk until it meets again another particle etc. The coalescence of particles is their only interaction. These and related models, like the annihilating random walk where two particles which meet do not coalesce but annihilate each other, and modified models with two types of particles where only particles of different type interact, are partly motivated by the study of chemical reactions, population dynamics etc. Another motivation for the basic CRW is its dual model, the so-called voter model. See the introduction of van den Berg and Kesten (2000) (in the remainder of this paper abbreviated as [BK]) and the references there.

Let

$$p(t) := P\{\mathbf{0} \text{ is occupied at time } t\},$$

when the initial configuration has one particle at each site. Bramson and Griffeath (1980), have shown for the basic CRW with $d \geq 3$,

$$p(t) \sim \frac{1}{\gamma_d t}, \tag{1.1}$$

where

$$\gamma_d = P\{\text{simple random walk in } \mathbb{Z}^d \text{ never returns to the origin after first leaving it}\}. \tag{1.2}$$

They also give the (different) asymptotic behavior of $p(t)$ for $d = 1$ or 2, but we are only concerned with $d \geq 3$ here. Their proof made heavy use of the duality with the voter model (and in particular of a result by Sawyer (1979) for the latter model), and seems not to be very robust. In [BK] an alternative method is presented which is based on a natural and intuitive heuristics, and we think this method is, in some sense, quite robust.

We give here a short outline of the heuristics in the case of the basic CRW; see Section 1 of [BK] for more details. First of all, it is not hard to see that the forward equation for $p(t)$ is

$$\frac{d}{dt}p(t) = -P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\},$$

where e_1 denotes the site $(1, 0, \ldots, 0)$. Next, note that if $\mathbf{0}$ and e_1 are occupied at time t, then the particles at these two sites must have been at some sites x

and y, respectively, at the earlier time $t-\Delta$, and the paths of the particles from x to $\mathbf{0}$ and from y to e_1 must not have coincided during $[t-\Delta, t]$. Intuitively one has to take, on one hand, Δ large with t so that only the contributions from pairs x, y far apart play a role, and on the other hand one has to take Δ much smaller than t to make the probability of the existence of several choices for x, y negligible. Further, when x and y are far apart, it is intuitively clear that the events $\{x \text{ is occupied at time } t-\Delta\}$ and $\{y \text{ is occupied at time } t-\Delta\}$ are nearly independent.

Let $\{S_s\}_{s\ge 0}, \{S'_s\}_{s\ge 0}, \{S''_s\}_{s\ge 0}$ be independent copies of a continuous time simple random walk starting at $\mathbf{0}$. The above considerations lead to the heuristic approximation of

$$P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\}$$

by

$$\sum_{x,y} P\{x \text{ is occupied at } t-\Delta\} P\{y \text{ is occupied at } t-\Delta\}$$
$$\times P\{x + S'_\Delta = \mathbf{0}, y + S''_\Delta = e_1, x + S'_s \ne y + S''_s \text{ for } 0 \le s \le \Delta\}.$$

By reversing time in the interval $(t-\Delta, t)$, using other natural approximations, and then reversing time again we get, with $\{\widetilde{S}_s\}_{s\ge 0}, \{\widetilde{S}'_s\}_{s\ge 0}, \{\widetilde{S}''_s\}_{s\ge 0}$ independent copies of the time-reversed random walk (for simple random walks these are again simple random walks)

$$\begin{aligned}
&P\{\mathbf{0} \text{ and } e_1 \text{ are occupied at time } t\} \\
&\approx \gamma_d \sum_x P\{\widetilde{S}'_\Delta = x\} p(t-\Delta) \sum_y P\{e_1 + \widetilde{S}''_\Delta = y\} p(t-\Delta) \\
&= \gamma_d \sum_x P\{S'_\Delta = -x \text{ and } x \text{ is occupied at } t-\Delta\} \\
&\qquad\qquad \times \sum_y P\{S''_\Delta = e_1 - y \text{ and } y \text{ is occupied at } t-\Delta\} \\
&\approx \gamma_d P\{\mathbf{0} \text{ is occupied at } t\} P\{e_1 \text{ is occupied at } t\} = \gamma_d p^2(t),
\end{aligned}$$

where $A \approx B$ means that $A - B$ is negligible for our purposes. From these relations one expects $p(t)$ to behave asymptotically like the solution of the equation

$$\frac{d}{dt} y(t) = -\gamma_d y^2(t)$$

which vanishes at ∞, namely $y(t) = 1/(\gamma_d t)$. This is the heuristic reason for (1.1). The method in [BK] consists of turning this heuristic into a rigorous proof by bounding the errors in the above approximations.

To illustrate the robustness of the method, [BK] applied it to a modified CRW model, in which a particle which jumps to an occupied site does not always coalesce, but only with a probability which depends on the number of particles already present at that site. We will call this model 'randomly coalescing random walk' or RCRW for short. The method was also applied successfully to other models, by Kesten (2000) and Stephenson (2001). More precisely the RCRW model is as follows: Let $\{S_t\}_{t\geq 0}$ be a continuous time random walk starting at $\mathbf{0}$. Denote by $q(y)$ the probability that $S_.$ has a jump of size y when it jumps ; thus,

$$q(\mathbf{0}) = 0. \tag{1.3}$$

Throughout we assume that the random walk is genuinely d-dimensional, that is,

$$\text{the support of } q(\cdot) \text{ contains } d \text{ linearly independent vectors.} \tag{1.4}$$

Assume that the motion of a particle starting at x is distributed like $\{x + S_t\}$, independent of the motion of all other particles. However, if a particle jumps to a site which already contains j particles, then it colaesces with one of these j particles with a certain probability p_j. We are in this paper only interested in the number of particles in a site, not in their mass. (The mass of a particle is the number of original particles by whose coalescence it is formed; so, if two particles of mass m_1 and m_2 coalesce the new particle has mass $m_1 + m_2$). Therefore, for our purposes it is simpler to say that the particle which jumps is removed from the system and, with the exception of the proofs of a few intermediate results where the coalescence interpretation is more convenient, we shall follow this convention.

The main result in [BK] is that for the RCRW model with $d \geq 6$, which satisfies (1.3), (1.4) and

$$p_0 = 0, \quad p_1 > 0, \tag{1.5}$$

and

$$p_j \text{ is increasing in } j, \tag{1.6}$$

and

$$ES_t = t \sum_{y\in\mathbb{Z}^d} y q(y) = \mathbf{0} \text{ and } \sum_{y\in\mathbb{Z}^d} \|y\|^2 q(y) < \infty, \tag{1.7}$$

it holds that

$$p(t) \sim \frac{1}{C(d)t} \text{ as } t \to \infty, \tag{1.8}$$

with

$$C(d) = \frac{p_1\gamma}{1-(1-p_1)(1-\gamma)}, \tag{1.9}$$

and $\gamma = \gamma_d$ as in (1.2).

The drawback was that although we believed that the above mentioned method ‘has the correct structure’, and that the result holds for $d \geq 3$, the earlier mentioned error bounds in [BK] were not good enough for $d = 3, 4$ or 5. One might think that this is due to a fundamental shortcoming of the method. This motivated us to refine the techniques in order to weaken the condition on d. The key tool in [BK] was an upper bound for a certain variance. In the present paper we significantly improve (under certain extra conditions) that bound. As a result we can now prove the following theorem. Let

$$E(t) := E\{\text{the number of particles at } \mathbf{0} \text{ at time } t\},$$

when the initial configuration has one particle at each site.

Theorem. *Consider RCRW with $d \geq 3$ for which (1.3), (1.4), (1.5), (1.6) and (1.7) hold. Assume further that there exists an M for which*

$$p_j = 1 \text{ for all } j \geq M. \tag{1.10}$$

For this model

$$E(t) \sim p(t) \sim \frac{1}{C(d)t}, \tag{1.11}$$

where $C(d)$ is given by (1.9).

Remarks:

(i) In fact [BK] and the present paper prove results which are somewhat sharper than (1.8) and (1.11), namely that there exists a $\zeta = \zeta(d) > 0$ such that $p(t)$ and $E(t)$ differ from $1/(C(d)t)$ by $O\big(t^{-1-\zeta}\big)$, as $t \to \infty$.

(ii) The main improvement in the present paper is (under the extra condition of a bounded number of particles per site) a much better bound on the variance of

$$\sum_x \beta(x)\xi_t(x)$$

for suitable $\beta(\cdot)$. Here $\xi_t(x)$ denotes the number of particles at site x at time t. This new variance estimate is derived in Section 3. Its proof starts as in [BK] (using the method of bounded differences) but then deviates from the old one and is considerably longer and more refined. Some tools used for the proof of the new variance bound (Proposition 13) are given in Section 2. Once this new variance bound has been derived, we follow mainly the proof of the approximate differential equation in [BK]. This consists of a number of lemmas, some of which have (under the condition (1.10)) a stronger form and an easier proof now (by using correlation inequalities), which works for all $d \geq 3$. We do not give detailed proofs of the new versions of most lemmas. Instead we point out where and why the lemmas and their proofs differ from their analogues in [BK]. We have tried to do this in such a way that the present paper is understandable without having first to read all of [BK]. Finally, at the end of the paper we prove that the Proposition and the Lemmas imply the Theorem. Throughout we use C_i for various finite, strictly positive constants. The same symbol C_i may stand for different constants in different formulae.

2. Descriptions of the process and presentation of some tools. As pointed out in [BK] we may assume without loss of generality that

$$\text{the group generated by the support of } q(\cdot) \text{ is all of } \mathbb{Z}^d. \tag{2.1}$$

Since in our system of random walks the number of particles at each site is bounded, the standard existence theorems (see Liggett (1985), Ch. 1) can be applied to show that there exists a Markov process which corresponds to the intuitive description given just before the Theorem in Section 1. (This in contrast with [BK] where alternative arguments had to be given to prove existence). In fact, there are several ways in which the process can be described. In [BK] this is done by assigning random elements to the sites which tell when a particle jumps from this site (depending on the number of particles present), where it jumps to and (depending on the number of particles already present there) the probability that it is removed by this jump. That description does not keep track of individual particles. Later in this section (see the subsection on "Ghost particles and coupling") we show alternative ways to describe the process.

The next Lemma gives a useful comparison of chains with different finite initial states, that is, states in which the total number of particles present is finite.

Lemma 1. *Assume (1.6). Let ξ'_0, ξ''_0 and $\xi^{\#}_0$ be finite initial states which satisfy*

$$\xi'_0(x) \le \xi^{\#}_0(x) \le \xi'_0(x) + \xi''_0(x) \text{ for all } x \in \mathbb{Z}^d. \tag{2.2}$$

Then the corresponding processes ξ'_t, $t \ge 0$, ξ''_t, $t \ge 0$ and $\xi^{\#}_t$, $t \ge 0$, can be coupled in such a way that the $'$-process and the $''$-process are independent and such that, with probability 1, for all $t \ge 0$

$$\xi'_t(x) \le \xi^{\#}_t(x) \le \xi'_t(x) + \xi''_t(x) \text{ for all } x \in \mathbb{Z}^d. \tag{2.3}$$

The left hand inequality remains valid even without (1.6).

Remark: If ξ''_0 has only one particle, then, by the independence claim of the lemma, this particle moves as a 'free particle'. That is, it performs a random walk and has no interaction with the ξ'-process. The proof of Lemma 1 in [BK] shows that, in the case where $\xi^{\#}_0 = \xi'_0 + \xi''_0$ and ξ''_0 has one particle only, the above coupling can be done in such a way that a designated #-particle can be identified with the $''$-particle until the #-particle is removed. (The ξ''-particle is, of course, never removed).

Proof. See Lemma 1 in [BK].

The next lemma, a generalization of Lemma 2 in [BK], compares processes with the same initial states, but with different collections of p_j. It is intuitively clear and can be proved in a similar way as Lemma 1 in [BK].

Lemma 2. *Let ξ_t and ξ'_t be two RCRW processes which satisfy*

$$\xi_0(x) \geq \xi'_0(x) \text{ for all } x.$$

Assume that the parameters of these processes, $p_1, p_2, \dots,$ and $p'_1, p'_2, \dots,$ respectively, satisfy (1.5) and (1.6), and, in addition, $p'_j \geq p_j$. Then the two processes can be coupled in such a way that, with probability 1, for all t

$$\xi_t(x) \geq \xi'_t(x) \text{ for all } x.$$

As before, let $E(t)$ be the density of particles at time t, when we start at time 0 with one particle at every site:

$$E(t) := E\xi_t(x). \tag{2.4}$$

This is independent of x. The following lemma gives the asymptotic order of $E(t)$.

Lemma 3. *Assume (1.5) and (1.7). Then, for $d \geq 3$, there exist constants $0 < C_1 \leq C_2 < \infty$ such that*

$$\frac{C_1}{t} \leq E(t) \leq \frac{C_2}{t}, \quad t \geq 1. \tag{2.5}$$

The right hand inequality holds for any initial state ξ_0 with $\xi_0(x) \leq 1$ for all x.

Proof. These estimates basically come from Arratia (1983) and Bramson and Griffeath (1980). See [BK] Lemma 8. The uniformity of the right hand inequality for all initial states with $\xi_0(x) \leq 1$ follows from the monotonicity property given in Lemma 1.

Finally we define

$$\alpha_s(y) = P\{S_s = -y\}. \tag{2.6}$$

We then have the estimate

$$\sup_y \alpha_s(y) = \sup_y P\{y + S_s = \mathbf{0}\} \leq \frac{C_3}{(s+1)^{d/2}}, \tag{2.7}$$

which follows from the local central limit theorem (see Spitzer (1976), Proposition 7.9 and the Remark following it). It will be used several times in Section 3.

Correlation inequalities. In this subsection we formulate a correlation inequality (proved by Reimer) and apply it to a nice subclass of our RCRW models. It then turns out to be useful for *all* our RCRW models, by the use of domination arguments.

Let V be a finite set. Also, let S_i be a finite set, for each $i \in V$, and let $\Omega = \prod_{i \in V} S_i$. For $\omega \in \Omega$ and $K \subset V$, $[\omega]_K$ denotes the set of all $\omega' \in \Omega$ which agree with ω on K (that is, for which $\omega'_i = \omega_i, i \in K$). We shall call $[\omega]_K$ a *cylinder.* For $A, B \subset \Omega$, $A \square B$ is defined as the set of all $\omega \in \Omega$ for which there exist disjoint $K, L \subset V$ with $[\omega]_K \subset A$ and $[\omega]_L \subset B$. Let μ be a product measure on Ω. Reimer (2000) proved that

$$\mu(A \,\square\, B) \leq \mu(A)\mu(B), \text{ for all } A, B \subset \Omega. \tag{2.8}$$

(For our purposes we do not need the full generality of Reimer's inequality. An earlier special case of van den Berg and Fiebig (1987) would suffice.)

We will now give a suitable space-time diagram description of certain special RCRW models which makes it possible to apply the Reimer's inequality. The special models we mean here are those for which

$$\text{there is a positive integer } M \text{ such that } p_j = j/M \wedge 1, \; j = 1, 2, \ldots \tag{2.9}$$

In particular, $p_j = 1$ for $j \geq M$, so that (if we start with at most M particles per site) there will never be more than M particles at a site. The space-time diagram is as follows: Introduce M different colours. Let $\mathcal{C}$ be the set of these colours. At time 0 we assign to each vertex randomly (and uniformly) one of these M colours. This will be the initial colour of the particle starting in that vertex. Further we have for each vertex x a time axis (a copy of the infinite half-line $[0, \infty)$). On this time axis we consider, for each pair of colours c, c' and each v with $q(v) > 0$, a Poisson point process with intensity $q(v)/M$. For each such Poisson point we draw an arrow from x to $x + v$, and we colour the tail of the arrow with colour c and the tip with colour c'. All these Poisson point processes are taken independent, and also independent of the initial colours of the particles. The dynamics, in terms of the above processes, are now as follows: A particle with initial colour c_1 stays in his initial position until there is an outgoing arrow from that position, with tail colour c_1. Then the particle jumps to the other endpoint of that arrow and takes on the colour of the tip of the arrow, say c_2 (which may be equal to c_1). Then it stays in its new position until there is an outgoing arrow from that position with tail colour c_2, jumps to the tip of that arrow, etc. Note that if two particles are in the same location at the same time and have the same colour, then they stay together forever, and this is how coalescence is described by this space-time diagram. Accordingly, in this description, the state of the process at position x and time t is the number of different colours present at x at time t. We will not formally prove here that this description gives the correct dynamics but only make the following remark: When a particle (or more precisely, a class of particles of the same colour) jumps to a site, occupied by k different colours, then, given all information until that time, the probability that the jumping particle will coalesce is exactly the probability that the tip of the corresponding arrow has a colour equal to one of those k colours. Since the distribution of the colour at the tip is always the uniform distribution (independent of the position and colour of its tail) the coalescence probability is k/M, as it should be.

We now apply inequality (2.8) to these special RCRW models.

Lemma 4. *Let x and y be two different vertices. For RCRW satisfying (2.9),*

$$P\{\xi_t(x) \geq 1 \text{ and } \xi_t(y) \geq 1\} \leq P\{\xi_t(x) \geq 1\}\, P\{\xi_t(y) \geq 1\}.$$

Remark: For the case where $M = 1$ (that is, for the basic CRW model) this lemma was proved by Arratia (1981), Lemma 1, by a different method.

Proof. Let

$$A = \{\xi_t(x) \geq 1\} \text{ and } B = \{\xi_t(y) \geq 1\}.$$

We use the space-time description with colours discussed above, and for this proof only interpret $\xi_s(z)$ as the number of colours present at z at time s. To make matters suitable for application of (2.8) we first make a discrete-time approximation: Fix $\delta > 0$ and partition the time axes in intervals $[\ell\delta, (\ell+1)\delta)$, $\ell = 0, 1, \ldots$. The discrete-time dynamics is similar to that given above, except that the particle postpones its jump until the end of the time interval in which the corresponding arrow is located. Moreover, we will decide that if that time interval has more than one arrow going out of the current location of the particle, with tail colour equal to the colour of the particle, the particle will stay in that location forever. In this way the dynamics is completely determined by what kind of outgoing arrows there are in the intervals, not in which order they appear. In the limit as $\delta \to 0$ the effect of this somewhat strange rule becomes negligible. For the time being we also restrict to finite space: We fix a positive integer $N > \|x - y\|$ and only consider particles which up till time t always are within distance N from x and from y. For similar reasons we ignore jumps of size larger than $2N$.

The time-discretization and finite-space restriction allow us to work with a finite space-time diagram which is more suitable for application of (2.8) Let $A_{\delta,N}$ and $B_{\delta,N}$ be the analogues of A and B respectively after the above modifications. Later we will first let $\delta \to 0$ and then $N \to \infty$. To translate to the language of (2.8), let V be the set $V = X \cup V'$, where X is the set of all vertices which have distance $\leq N$ from x and from y, and V' the set of possible (multi-) indices of arrows: $V' = X \times T \times Q_N \times \mathcal{C} \times \mathcal{C}$, where T is the set of all positive integers k with $k\delta \leq t$, and Q_N the set of all v with $q(v) > 0$ and $\|v\| \leq 2N$. Further we take $\Omega = \mathcal{C}^X \times \{0,1\}^{V'}$. This is the space of all initial colourings of X and possible choices of (discretized) arrow locations and colours for arrows with tail in X. The connection of this set with the (discretized) space-time diagram is as follows: for $\omega \in \Omega$, $u, z \in X$, $k \in T$, $c, c' \in \mathcal{C}$ and $v \in Q_N$, we take ω_u equal to the initial colour of the particle starting in u, and

$$\omega_{(z,k,v,c,c')} = I\{\text{there is an outgoing arrow from vertex } z \text{ to vertex } z+v \\ \text{in the time interval } [k\delta, (k+1)\delta) \text{ with tail colour } c \text{ and colour of the tip } c'\}.$$

This correspondence naturally induces a product measure on Ω. The marginal distributions of this measure are

$$P\{\omega_x = c\} = \frac{1}{M} \text{ for } x \in X, c \in \mathcal{C} \text{ and } P\{\omega_{(z,k,v,c,c')} = 1\} = (1 - e^{-q(v)\delta/M}).$$

Let $D = D_\delta$ be the event that for all $k \in T$ and $x \in X$ there is at most one arrow in the time interval $[k\delta, (k+1)\delta)$ going out from x. Note that if δ goes to 0, the probability of D tends to 1. We now consider an event $\widetilde{A}_{\delta,N} \subset \Omega$ which agrees with $A_{\delta,N}$ on D, that is

$$\widetilde{A}_{\delta,N} \cap D = A_{\delta,N} \cap D. \tag{2.10}$$

$\widetilde{A}_{\delta,N}$ is a union of cylinders. Each such cylinder is decribed by listing the conditions on the system of colours and arrows which force a particle to move from some site u_0 at time 0 to x at time t (provided D occurs). Let us call a *coloured path* an initial site with a colour and a sequence of arrows with their colours such that the location and colour of the tail of the j-th arrow are the same as the location and colour of the tip of the $(j-1)$-th arrow, and such that the j-th arrow occurs later in time than the $(j-1)$-th arrow. (The location and colour of the tip of the zeroeth arrow are taken to be the location and colour of the initial particle). To find which cylinders are included in $\widetilde{A}_{\delta,N}$ we list all coloured paths in (discretized) space-time by which a particle can reach x at time t (again assuming that D occurs). Suppose such a particle starts in position u_0 with colour c_0, stays until time 3δ and then moves to $u_0 + v_1$ and changes colour to c_1, stays there until time 8δ and then moves to $u_0 + v_1 + v_2$, changing colour to c_2, etc. We then have the sequence $\omega_{u_0} = c_0$; $\omega_{u_0,k,v,c_0,c} = 0$ for all $k \in \{0,1,2\}$, $v \in Q$, $c \in \mathcal{C}$; $\omega_{u_0,3,v_1,c_0,c_1} = 1$; $\omega_{u_0,3,v,c_0,c} = 0$ for all tuples $(v,c) \neq (v_1,c_1)$; $\omega_{u_0+v_1,k,v,c_1,c} = 0$ for all $4 \le k \le 7$, $v \in Q$ and $c \in \mathcal{C}$, etc. So the (multi-)indices occurring in this sequence are u_0; $(u_0,k,v,c_0,c), 0 \le k \le 2$, $v \in Q$, $c \in \mathcal{C}$; $(u_0,3,v_1,c_0,c_1)$, etc. Thus, for such a coloured path the value of ω_{u_0} and of certain $\omega_{z,k,v,c,c'}$ have to be prescribed in order that $A_{\delta,N}$ occurs (provided D occurs). In other words for such a path there is a certain cylinder $[\omega(K)]_K$ (for a suitable $\omega(K)$ and $K \subset V$) whose intersection with D is contained in $A_{\delta,N}$. We take $\widetilde{A}_{\delta,N}$ to be the union of these cylinders over all possible coloured paths. It is then clear that (2.10) holds. In the same way we take $\widetilde{B}_{\delta,N}$ to be a union of cylinders $[\omega'(L)]_L$ such that (2.10) with A replaced by B holds. These constructions yield

$$A_\delta \cap B_\delta \cap D_\delta = \widetilde{A}_\delta \cap \widetilde{B}_\delta \cap D_\delta = \bigcup_{[\omega(K)]_K \subset \widetilde{A}_\delta, [\omega'(L)]_L \subset \widetilde{B}_\delta} [\omega(K)]_K \cap [\omega'(L)]_L, \quad (2.11)$$

where we have dropped the N from the notation for brevity. We claim that we may restrict the last union to only those pairs $[\omega(K)]_K, [\omega'(L)]_L$ with K and L disjoint. To see this assume that $\omega \in [\omega(K)]_K$ requires $\omega_{(z,k,v,c,c')} = \varepsilon'$ and that $\omega \in [\omega'(L)]_L$ requires $\omega_{(z,k,v,c,c')} = \varepsilon''$. If $\varepsilon' \neq \varepsilon''$, then $[\omega(K)]_K \cap [\omega'(L)]_L = \emptyset$. If $\varepsilon' = \varepsilon''$, then both $[\omega(K)]_K$ and $[\omega'(L)]_L$ correspond to a coloured path which is at z at time $k\delta$ and which has colour c at that time. As observed before, two particles following these paths must have coalesced by time $k\delta$ and therefore cannot end up at the different sites x and y at time t. This proves our claim. It follows that

$$A_\delta \cap B_\delta \cap D_\delta = \widetilde{A}_\delta \cap \widetilde{B}_\delta \cap D_\delta = \bigcup_{\substack{[\omega(K)]_K \subset \widetilde{A}_\delta, [\omega'(L)]_L \subset \widetilde{B}_\delta \\ K \cap L = \emptyset}} [\omega(K)]_K \cap [\omega'(L)]_L. \quad (2.12)$$

Consequently

$$A_\delta \cap B_\delta \cap D \subset \widetilde{A}_\delta \,\square\, \widetilde{B}_\delta,$$

and $P\{\widetilde{A}_\delta \cap \widetilde{B}_\delta\} \le P\{\widetilde{A}_\delta \,\square\, \widetilde{B}_\delta\} + P\{D_\delta^c\}$. By (2.8) this is at most

$$P\{\widetilde{A}_\delta\}\, P\{\widetilde{B}_\delta\} + P\{D^c\} \le [P\{A_\delta\} + P\{D^c\}][P\{B_\delta\} + P\{D^c\}] + P\{D^c\}.$$

The lemma follows by first taking $\delta \to 0$ and then $N \to \infty$. ■

In a similar way the following lemmas 5-7 can be proved:

Lemma 5. *Let $x_1, \dots, x_k$ be distinct vertices, t a nonnegative real and $n_1, \dots n_k$ non-negative integers. Then, for RCRW satisfying (2.9),*

$$P\{\xi_t(x_1) \ge n_1, \dots \xi_t(x_k) \ge n_k\} \le \prod_{i=1}^k P\{\xi_t(x_i) \ge n_i\}.$$

Lemma 6. *Let $t > 0$, x a vertex, and n and m nonnegative integers. Then, for RCRW satisfying (2.9),*

$$P\{\xi_t(x) \ge n + m\} \le P\{\xi_t(x) \ge n\}\, P\{\xi_t(x) \ge m\}.$$

The following lemma needs some explanation. As we said before we are in this paper only interested in how many particles there are in a vertex, not in the 'mass' of the particles. But for our analysis it is sometimes convenient to consider a particle π as a set, namely, the set of the original particles which coalesced to form π. For RCRW systems satisfying (2.9) this set is well-defined (by using the space-time diagram described just before Lemma 4). We call two particles *disjoint* if the corresponding two sets are.

Lemma 7. *Let $x_1, \dots x_n$ be vertices and $t_1, \dots t_n$ be non-negative reals. Let $\mathcal{S}(x,t)$ denote the set of particles present at x at time t. According to the remark above each particle in $\mathcal{S}(x,t)$ is itself considered as a set. Let $\mathcal{D}$ be the size of the largest subset of $\mathcal{S}(x_1,t_1) \cup \dots \cup \mathcal{S}(x_n,t_n)$ in which the elements are pairwise disjoint. Then, for RCRW satisfying (2.9),*

$$P\{\mathcal{D} \geq n+m\} \leq P\{\mathcal{D} \geq n\}\, P\{\mathcal{D} \geq m\},\ n,m \geq 0. \tag{2.13}$$

The above results are stated for RCRW satisfying (2.9). Combined with domination arguments they imply useful results which hold for all RCRW models which satisfy the conditions of our Theorem.

Lemma 8. *Let $d \geq 3$. If (1.5)–(1.7) hold, then there exists a constant $C > 0$ such that for all $t > 0$, all positive integers k and $n_1, \dots n_k$, and all vertices $x_1, \dots, x_k$,*

$$P\{\xi_t(x_1) \geq n_1, \dots \xi_t(x_k) \geq n_k\} \leq \left(\frac{C}{t}\right)^{n_1+\dots+n_k}. \tag{2.14}$$

Proof. For RCRW satisfying (2.9), (2.14) follows immediately from Lemma 5, Lemma 6 and (2.5). If the p_j's don't satisfy (2.9) we can always find a suitable M such that $p_j \geq p'_j := j/M \wedge 1$. (Note that this uses (1.5) and (1.6).) So our RCRW model is dominated by the RCRW model with parameters p'_j (by virtue of Lemma 2), and the result follows.

■

Let $u_1, \dots, u_p \in \mathbb{Z}^d$ (not necessarily distinct). Define

$$\sum_{i=1}^{p}{}^{*}\xi_t(u_i) \tag{2.15}$$

to be the sum of the $\xi_t(u_i)$ only over the distinct u_i in $\{u_1, \dots, u_p\}$. Thus if a given u appears several times among the u_i, there is still only one summand $\xi_t(u)$ in (2.15). Define further

$$\begin{aligned} &\Lambda_t(u_1, u_2, \dots, u_p) \\ &= \Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i)\Big)\Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i) - 1\Big)\dots\Big(\sum_{i=1}^{p}{}^{*}\xi_t(u_i) - p + 1\Big). \end{aligned} \tag{2.16}$$

$\Lambda_t(u_1, \dots, u_p)$ represents the number of ordered p-tuples of distinct particles which we can select from the $\sum{}^{*}\xi_t(u_i)$ particles present at the sites $u_1, \dots, u_p$ at time t.

Lemma 9. *Assume (1.5)–(1.7) and $d \geq 3$. Then for any $p \geq 2$ and $u_1, \dots, u_p \in \mathbb{Z}^d$,*

$$E\Lambda_t(u_1, \dots, u_p) \leq C_3(p)t^{-p}. \tag{2.17}$$

Proof. This follows from Lemma 8. ■

Remark: This is (for our processes) a considerable improvement on Lemma 10 in [BK]. There $d \geq 5$ was required, and we had for $p = 2$ the same result as here, but for $p \geq 3$ we had, instead of (2.17), for each $0 < \varepsilon < 1/2$, a bound of the form $C_3(\varepsilon, p)[t^{-p} \vee t^{-d(1-\varepsilon)/2}]$. Our present, improved form uses the boundedness of the number of particles per site, and plays an important role in the weakening of the dimension condition ($d \geq 3$ instead of $d \geq 6$) in our main theorem.

The next lemma gives another consequence of the above inequalities.

Lemma 10. *Assume (1.5)–(1.7). Then for $d \geq 3$,*

$$0 \leq E(t) - p(t) \leq E(t) - P\{\xi_t(\mathbf{0}) = 1\} \leq \frac{C_4}{t^2}. \tag{2.18}$$

Proof.

$$E(t) - P\{\xi_t(\mathbf{0}) = 1\} = \sum_{\ell \geq 2} \ell P\{\xi_t(\mathbf{0}) = \ell\}.$$

For $t \leq 2$ (2.18) is obvious. For $t \geq 2$ apply Lemma 8 to each term (with $k = 1$ and $n_1 = \ell$). ■

Remark: The preceding lemma is the analog of Lemma 11 in [BK]. However, there Lemma 10 of [BK] was used which led to the requirement $d \geq 5$ (see the Remark after Proposition 7 in [BK]).

Ghost particles and coupling. In this subsection we describe techniques which use so-called ghost particles. These techniques are useful when we want to compare the future evolution of two RCRW systems with the same dynamics but whose initial configurations differ only at one or two vertices.

For these methods it is convenient to formulate the dynamics in a way which keeps track of individual particles. We will define (and use) these new dynamics only for finite particle systems. When we look at *numbers* of particles at each site only, these dynamics are equivalent to the 'old' dynamics. The new dynamics are as follows: Assign to each particle π a 'Poisson clock', so we have a sequence of i.i.d. exponentially (mean 1) distributed random variables $\tau_1(\pi), \tau_2(\pi), \dots$. The clock rings at times $\tau_1(\pi)$, $\tau_1(\pi) + \tau_2(\pi), \dots$, and π jumps exactly at those times. Also assign to π a sequence $Y_1(\pi), Y_2(\pi), \dots$ of i.i.d. random variables with distribution q. $Y_n(\pi)$ denotes the jump π makes at time $\tau_1(\pi) + \dots + \tau_n(\pi)$. The coalescence (or, rather, removal of particles) is described as follows: assign to each particle π

a sequence $U_\pi(1), U_\pi(2), \dots$ of i.i.d. random variables, each uniformly distributed in the interval $(0,1)$. Now suppose π makes its n-th jump and this jump brings it to a vertex where already k particles are present. Then π is removed if and only if $U_\pi(n) < p_k$. We take all the above sequences of random variables independent of each other.

We need the notion of a particle being 'pivotal' for the removal of some other particle. Suppose a particle π makes its n-th jump and this brings it to a vertex x where k particles are already present. If $p_{k-1} < U_\pi(n) < p_k$, then π is removed but would not be removed if there had been one particle less in x. We say that each of the k particles already present in x is *pivotal* for the removal of π.

In the remainder of this section, $\pi(x)$ will denote the particle which started in x at time 0. In the above given representation of the RCRW process, the particle which is removed when particles meet is always the jumping particle. Since we are eventually only interested in the *number* of particles at every vertex (and not their identities) we can change the above rule and instead remove one of the particles already present. (Of course the rule to select the particle which has to be removed should not use any future information of the system). This observation motivates us to introduce the notion of 'ghost particles'.

Systems with one ghost particle

A system with one ghost particle (and all other particles 'normal') is described in the same way as after Lemma 10, except for the following change: One of the particles is special. It has the property that when the situation arises that (according to the earlier description) it would be pivotal for the removal of some other particle which just jumped, then the special particle is removed instead of the particle which just jumped. If the special particle has not been removed at time t and is at position x at that time, then it is counted in $\xi_t(x)$. Apart from this, everything proceeds exactly as before. In particular, until the moment that the special particle becomes pivotal for the removal of some particle, it behaves exactly as a normal particle. It is easy to see that in a system with one such special particle, the other particles behave exactly as they would in the corresponding system without that special particle. In other words, they don't 'feel' the special particle. Therefore we call the special particle a ghost particle. The introduction of a ghost particle is very convenient for comparing two systems whose initial configurations are the same except that at one vertex one of the configurations has one more particle than the other.

Since we will need to compare systems whose initial configurations differ in two vertices (the first configuration having one more particle than the second configuration in one vertex and and one less in another vertex) we will also discuss systems with two ghost particles. Before we do this, we briefly discuss certain ways to couple two random walks.

Coupling of two random walks

There are several natural ways to couple two copies $\{S'_s\}$ and $\{z + S''_s\}$ of our continuous time random walk on $\mathbb{Z}^d$ (with different starting positions). The simple coupling method which we shall describe works well for continuous time random walk on $\mathbb{Z}^d$, but it needs modification if one wants to couple two discrete time random walks on $\mathbb{Z}^d$. The way we use to couple $\{S'_s\}$ and $\{z + S''_s\}$ is as follows. First assume that

$$q(z) > 0 \text{ or } q(-z) > 0. \tag{2.19}$$

For the sake of argument assume $q(z) > 0$. Then for each $y \in \mathbb{Z}^d$ with $q(y) > 0$ and $y \neq z$ let $\tau_1(y) < \tau_2(y) < \dots$ be the jump times of a Poisson process of rate $q(y)$. Also let $\tau'_1(z) < \tau'_2(z) < \dots$ and $\tau''_1(z) < \tau''_2(z) < \dots$ be the jump times of two Poisson processes of rate $q(z)$. All these Poisson processes are taken independent of each other. At time $\tau_k(y)$ both processes $\{S'\}$ and $\{z + S''\}$ make a jump of value y. At time $\tau'_k(z)$ $\{S'\}$ makes a jump of value z, but $\{z + S''\}$ does not jump, while at time $\tau''_k(z)$ only the $\{z + S''\}$-process makes a jump of value z. It is clear that $\{S'_s - (z + S''_s)\}$ performs a continuous time random walk whose jumps occur at rate $2q(z)$ and have the values z or $-z$, each with probability $1/2$. Thus the two processes have the same value at the first time $\tau'_k(z)$ or $\tau''_k(z)$ at which there has been one more jump of the sequence $\tau'(z)$ than of the sequence $\tau''(z)$. Call this time ϕ. From that time on we do not use the procedure described above, but 'glue the random walks together'. It follows from Spitzer (1976), Proposition 32.3, that

$$P\{\phi \geq t\} \leq C_1 \frac{1}{\sqrt{(t+1)}}, \tag{2.20}$$

for some constant C_1 which depends on z only.

We can use the same argument to couple $\{S'_s\}$ and $\{nz + S''_s\}$ for any integer $n \geq 0$. If we still denote the coupling time by ϕ we have to replace the estimate (2.20) by

$$P\{\phi \geq t\} \leq C_1 n \frac{1}{\sqrt{(t+1)}}. \tag{2.21}$$

To see this note the following estimate for a symmetric simple random walk $\{T_k\}$ on $\mathbb{Z}$:

$$\begin{aligned} &P\{T_\cdot \text{ first returns to } \mathbf{0} \text{ at a time } \geq \ell\} \\ &\geq P\{T_\cdot \text{ hits } n \text{ before it returns to } \mathbf{0}\} \\ &\qquad \times P\{n + T_\cdot \text{ first hits } \mathbf{0} \text{ at a time } \geq \ell\} \\ &= \frac{1}{2n} P\{n + T_\cdot \text{ first hits } \mathbf{0} \text{ at a time } \geq \ell\}. \end{aligned}$$

The last equality is just the gambler's ruin formula (see Feller (1968), equation XIV.2.5). Combined with Proposition 32.3 in Spitzer (1976) this gives

$$P\{n + T_\cdot \text{ first hits } \mathbf{0} \text{ at a time } \geq \ell\} \leq C_2 n \frac{1}{\sqrt{\ell + 1}}.$$

It is not hard to derive (2.21) from this.

The same argument works if $q(-z) > 0$ instead of $q(z) > 0$. To obtain a similar estimate for all z (even without the restriction in (2.19)) and to get a better handle on the dependence of the coupling time on z we now pick M and independent vectors $z_1, \dots, z_M \in \mathbb{Z}^d$ such that the additive group generated by them is all of $\mathbb{Z}^d$ and such that

$$q(z_i) > 0,\ 1 \le i \le M.$$

Such vectors exist by virtue of (2.1); one merely has to pick the z_i such that each coordinate vector is an integral linear combination of the z_i. There then exists a constant $C_3 = C_3(z_1, \dots, z_M)$ such that each $z \in \mathbb{Z}^d$ can be written as $z = \sum_{i=1}^M \varepsilon_i n_i(z) z_i$ for some $\varepsilon_i = \pm 1$ and some nonnegative integers n_i which satisfy

$$\sum_{i=1}^M n_i(z) \le C_3 \|z\|.$$

We can now successively couple $\{\sum_{i=1}^\ell \varepsilon_i n_i z_i + S'_s\}$ with $\{\sum_{i=1}^{\ell+1} \varepsilon_i n_i z_i + S''_s\}$ for $0 \le \ell < M$ by the method just described. This leads to the following lemma.

Lemma 11. *Two particles which move according to $\{S'_s\}$ and $\{z + S''_s\}$ can be coupled such that*

$$P\{\text{the two particles don't meet before time } t\} \le C \frac{\|z\|}{\sqrt{(t+1)}}, \tag{2.22}$$

where C is a constant depending on the jump distribution q and the dimension d only.

A system with two coupled ghost particles.

A system with two coupled ghost particles, say g and g', (and all other particles 'normal') is described as follows: Each of the two ghost particles behaves as in the single-ghost description and, as long as neither of the two has been removed, their random-walk paths are coupled as described above. Once they meet they stay together and behave as one ghost particle. It is easy to check that if we only observe the normal particles and g, we 'see' a system with one ghost as in the single-ghost description. And similarly for g'. In particular, the normal particles behave exactly as they would without the two ghosts: they don't 'feel' the presence of the ghosts. This construction therefore provides a natural coupling of the time evolution of two systems whose initial configurations differ only at two vertices, in which the first configuration has one more particle in the first vertex and the other configuration has one more particle in the second vertex. The usefulness of this coupling is shown in the following situation: Let σ be a configuration, and let x and y be two vertices. Let $\sigma^{(x)}$ be the configuration obtained from σ by adding one particle at x and $\sigma^{(x,y)}$ the configuration obtained from σ by adding one particle at x and one particle at y. For a configuration σ in which both x and y are occupied, denote by $\widetilde{P} = \widetilde{P}_{s,\sigma,x,y}$ the conditional distribution governing a RCRW system on the time interval $[s, \infty)$, with one of the particles in x and one of the particles in y coupled as ghost particles in the sense described above, given that the configuration at time s is σ. Let $\widetilde{E} = \widetilde{E}_{s,\sigma,x,y}$ denote the corresponding expectation operator. For a RCRW with only normal particles we use the notation $E_{s,\sigma}$ for the conditional expectation on the time interval $[s, \infty)$ when the configuration at time s is σ. The next lemma is based on the above coupling of two ghost particles.

Lemma 12. *For any finite state σ and any function β on $\mathbb{Z}^d$ which satisfies*

$$\sum_{z \in \mathbb{Z}^d} |\beta(z)| < \infty,$$

we have for $s \le t$ that

$$\begin{aligned} &E_{s,\sigma^{(x)}}\Big[\sum_{z\in\mathbb{Z}^d} \beta(z)\xi_t(z)\Big] - E_{s,\sigma^{(y)}}\Big[\sum_{z\in\mathbb{Z}^d} \beta(z)\xi_t(z)\Big] \\ &= \widetilde{E}_{s,\sigma^{(x,y)},x,y}\Big[\sum_{z\in\mathbb{Z}^d} \beta(z)\big(I[g_x \text{ is in } z \text{ at time } t] - I[g_y \text{ is in } z \text{ at time } t]\big)\Big], \end{aligned} \tag{2.23}$$

where g_x and g_y are the ghost particles in x and in y at time s, respectively.

3. Improved variance estimate. As in [BK] we shall write $\{\xi_t(\mathbb{1})\}$ and $\{\xi_t(\mathbb{1}^{(N)})\}$ for the processes $\{\xi_t\}$ with initial states $\mathbb{1}$ and $\mathbb{1}^{(N)}$, respectively, where

$$\mathbb{1}(x) = 1 \text{ for all } x \in \mathbb{Z}^d,$$

and

$$\mathbb{1}^{(N)}(x) = \begin{cases} 1 & \text{if } \|x\| \le N \\ 0 & \text{if } \|x\| > N. \end{cases}$$

We write $\xi_{N,t}$ for $\xi_t(\mathbb{1}^{(N)})$.
In the following Proposition, which plays a key role in the proof of our Theorem, we take the initial state to be $\xi_0 = \mathbb{1}$.

Proposition 13. *Assume (1.6), (1.7). Then there exist constants $C_0 > 0$ and $\kappa > 0$, which are independent of β, K, t and the p_j, such that for $d \ge 3$ and $\beta(x) \in \mathbb{R}$ and $K < \infty$ it holds that*

$$\mathrm{Var}\Big\{\sum_{\|x\|\le K} \beta(x)\xi_t(x)\Big\} \le C_0 t^{-1/2-\kappa} \sum_{x\in\mathbb{Z}^d} \beta^2(x). \tag{3.1}$$

If

$$\sum_{x\in\mathbb{Z}^d} |\beta(x)| < \infty, \tag{3.2}$$

then also

$$\mathrm{Var}\Big\{\sum_{x\in\mathbb{Z}^d} \beta(x)\xi_t(x)\Big\} \le C_0 t^{-1/2-\kappa} \sum_{x\in\mathbb{Z}^d} \beta^2(x). \tag{3.3}$$

Proof of Proposition 13. Fix $K < \infty$ and let

$$Z = \sum_{\|x\|\le K} \beta(x)\xi_t(x),$$
$$Z_N = \sum_{\|x\|\le K} \beta(x)\xi_{N,t}(x).$$

As pointed out in the proof of Proposition 7 of [BK], it is easy to see (using monotone convergence and Fatou's lemma, or even bounded convergence in the present situation with bounded ξ) that it suffices for (3.1) to prove

$$\liminf_{N\to\infty} \mathrm{Var}(Z_N) = \liminf_{N\to\infty} \mathrm{Var}\Big\{\sum_{\|x\|\le K} \beta(x)\xi_{N,t}(x)\Big\} \le C_0 t^{-1/2-\kappa} \sum_{x\in\mathbb{Z}^d} \beta^2(x). \tag{3.4}$$

Now let $\mathcal{F}_s$ be the σ-field containing all information up to time s, and define

$$\Delta_\ell = \Delta_\ell(p) = \Delta_\ell(p,N,t) = E\{Z_N|\mathcal{F}_{\ell t/p}\} - E\{Z_N|\mathcal{F}_{(\ell-1)t/p}\}.$$

Then for each integer $p \ge 1$,

$$Z_N - EZ_N = \sum_1^p \Delta_\ell$$

and

$$\mathrm{Var}(Z_N) = \sum_1^p E\Delta_\ell^2(p) = \liminf_{p\to\infty} \sum_1^p E\Delta_\ell^2(p) = \liminf_{p\to\infty} \sum_1^p E\big\{E\{\Delta_\ell^2(p)|\mathcal{F}_{(\ell-1)t/p}\}\big\}.$$

We fix N and unless otherwise indicated, the initial state in the remainder of this section is $\xi_0 = \mathbb{1}^{(N)}$. We write $W_\ell = W_\ell(p,N)$ for the random elements which summarize all the information which becomes available between time $(\ell-1)t/p$ and $\ell t/p$. We have

$$\mathcal{F}_{\ell t/p} = \sigma\{W_1,\dots,W_\ell\}$$

and the W_ℓ for different ℓ are independent. We denote the distribution of W_ℓ by μ_ℓ (i.e., $\mu_\ell(dw) = P\{W_\ell \in dw\}$). $Z_N = f(W_1, W_2, \dots, W_p)$ for a suitably measurable function $f = f_N$ and therefore

$$\begin{aligned} &E\{Z_N | \mathcal{F}_{\ell t/p}\} \\ &= \int \prod_{i=\ell+1}^{p} \mu_i(dw_i) f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p) \\ &= \int \prod_{i=\ell}^{p} \mu_i(dw_i) f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p). \end{aligned}$$

Hence

$$\begin{aligned} \Delta_\ell = \int \prod_{i=\ell}^{p} \mu_i(dw_i) \big[& f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p) \\ & - f(W_1, \dots, W_{\ell-1}, w_\ell, w_{\ell+1}, \dots, w_p) \big]. \end{aligned} \tag{3.5}$$

Note that Δ_ℓ is a function of $W_1, \dots, W_\ell$, and that therefore

$$E\{\Delta_\ell^2 | \mathcal{F}_{(\ell-1)t/p}\} = \int \mu_\ell(dW_\ell) \Delta_\ell^2$$

and

$$E\Delta_\ell^2 = \int \prod_{j \le \ell} \mu_j(dW_j) \Delta_\ell^2.$$

Now define

$$\begin{aligned} I'_\ell(x,y) &= I'_\ell(x,y)(W_1, \dots, W_\ell, w_\ell) \\ &= I\big[\text{there is a single jump during } ((\ell-1)t/p, \ell t/p] \\ &\qquad \text{in configuration } (W_1, \dots, W_{\ell-1}, W_\ell), \text{ and this jump is from } x \text{ to } y, \\ &\qquad \text{but there is no jump during } ((\ell-1)t/p, \ell t/p], \\ &\qquad \text{in configuration } (W_1, \dots, W_{\ell-1}, w_\ell). \end{aligned}$$

We have

$$\begin{aligned} \Delta_\ell = \int \prod_{i=\ell}^{p} \mu_i(dw_i) \big[& f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p) \\ & - f(W_1, \dots, W_{\ell-1}, w_\ell, \dots, w_p) \big] \sum_{x,y} I'_\ell(x,y) \\ & + \text{ negligible terms.} \end{aligned}$$

These negligible terms come from cases where at least two particles jump during $((\ell-1)t/p, \ell t/p]$ in configuration $(W_1, \dots, W_{\ell-1}, W_\ell)$, no particle jumps in configuration $(W_1, \dots, W_{\ell-1}, W_\ell)$, or at least one particle jumps in $(W_1, \dots, W_{\ell-1}, w_\ell)$. The first type of cases contribute only $O(1/p^2)$ to $E\Delta_\ell^2$ because the probability of two or more jumps during $((\ell-1)t/p, \ell t/p]$ is $O(1/p^2)$ (compare [BK], pp. 329, 330). In the second type of cases we have no jump in $(W_1, \dots, W_\ell)$. Then $f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p) - f(W_1, \dots, W_{\ell-1}, w_\ell, \dots, w_p) \neq 0$ will occur only when there is a jump in configuration $(W_1, \dots, W_{\ell-1}, w_\ell)$. But this occurs only with probability $O(1/p)$, and hence $\Delta_\ell = O(1/p)$ on the event that there is no jump in $(W_1, \dots, W_\ell)$. Thus the second (and also the third) type of cases give a contribution $O\big(1/p^2\big)$ to $E[\Delta_\ell^2]$, and hence (after summing over ℓ) of $O\big(1/p\big) = o(1)$ to the variance and hence are indeed negligible.

Now, since $I'_\ell(x,y) = 1$ for at most one pair x, y, and is independent of $w_{\ell+1}, \dots, w_p$, we have, apart from negligible terms,

$$\Delta_\ell^2 \le \int \mu_\ell(dw_\ell) \sum_{x,y} I'_\ell(x,y) \Big[\int \prod_{i=\ell+1}^{p} \mu_i(dw_i) \big(f(W_1, \dots, W_\ell, w_{\ell+1}, \dots, w_p) - f(W_1, \dots, W_{\ell-1}, w_\ell, \dots, w_p) \big) \Big]^2.$$

Now we use the ghost particle ideas of Section 2, in particular (2.23), and bound (the absolute value of) the inner integral above by

$$\widetilde{E}\big[\sum_z |\beta(z)| \big(J(x,y,z) + J'(x,y,z) \big)\big],$$

where

$$J(x,y,z) = I[g_x \text{ but not } g_y \text{ ends at } z \text{ at time } t],$$

$$J'(x,y,z) = I[g_y \text{ but not } g_x \text{ ends at } z \text{ at time } t],$$

and where (in the notation of Lemma 12)

$$\widetilde{E} = \widetilde{E}_{s,\sigma^{(y)},x,y}$$

with σ the configuration at time $(\ell-1)t/p$ and $s = \ell t/p$. To see this note that $I'_\ell(x,y) = 1$ can occur only if there is a particle at x in configuration σ and this particle jumps to y during $((\ell-1)t/p, \ell t/p]$ in $(W_1, \dots, W_\ell)$. However, this particle does not jump in $(W_1, \dots, W_{\ell-1}, w_\ell)$. Thus, if $W_1, \dots, W_\ell$ occurs, then at time $\ell t/p$ the state of the system is described by $\sigma^{(y)}$ minus one particle at x, and if $W_1, \dots, W_{\ell-1}, w_\ell$ occurs this state is σ. A small additional remark is needed here. To account in the simplest way for the possibility that the jumping particle is removed during the jump, we let g_y disappear immediately with probability $p_{\sigma(y)}$

(with σ as above). So the event describing J' above includes the requirement that this immediate disappearance of g_y does not happen. So we have

$$\begin{aligned}\Delta_\ell^2 \le 2\int \mu_\ell(dw_\ell)\sum_{x,y} I'_\ell(x,y)\Big[\widetilde{E}[\sum_z |\beta(z)|J(x,y,z)]\Big]^2 \\ + 2\int \mu_\ell(dw_\ell)\sum_{x,y} I'_\ell(x,y)\Big[\widetilde{E}[\sum_z |\beta(z)|J'(x,y,z)]\Big]^2, \qquad (3.6)\end{aligned}$$

plus negligible terms. We replace the factor $I'_\ell(x,y)$ by the larger factor

$$\begin{aligned}I_\ell(x,y) :&= I_\ell(x,y)(W_1,\dots,W_\ell) \\ &= I\big[\text{there is a single jump during } ((\ell-1)t/p, \ell t/p] \\ &\qquad \text{in configuration } (W_1,\dots,W_{\ell-1},W_\ell), \text{ and this jump is from } x \text{ to } y.\end{aligned}$$

I_ℓ does not depend on w_ℓ so we can then carry out the integral over w_ℓ. In addition we only look at the first sum in the right hand side of (3.6); the treatment of the second sum is similar. By an application of Schwarz (but a somewhat more careful one than in [BK]) and by the facts that $J(x,y,z) \ne 0$ for at most one z and $J(x,y,z)$ equals 0 or 1, the integral over w_ℓ of this first sum is at most

$$\sum_{x,y} I_\ell(x,y)\widetilde{E}\Big[\sum_z |\beta^2(z)|J(x,y,z)\Big]\widetilde{P}\Big[\sum_z |\beta(z)|J(x,y,z) \ne 0\Big]. \qquad (3.7)$$

Now write s for $\ell t/p$ and define (for some $A>0$, $B_1>0$ and $B>B_1$ to be chosen later such that $s+A\le t$ and $s+B\le t$)

$$\begin{aligned}J_1(x,y,z) &= I\big[\text{the random walk paths of } g_x \text{ and } g_y \text{ do not meet} \\ &\qquad\qquad \text{during } [s,s+A] \text{ and } g_x \text{ ends at } z \text{ at time } t\big], \\ J_2(x,y,z) &= I\big[g_y \text{ is removed during } [s,s+A] \text{ and} \\ &\qquad\qquad g_x \text{ ends at } z \text{ at time } t\big], \\ K_1(x,y) &= I\big[\text{the random walk paths of } g_x \text{ and } g_y \text{ do not meet} \\ &\qquad\qquad \text{during } [s,s+B]\big], \\ K_{2,a}(x,y) &= I\big[g_y \text{ is removed during } [s,s+B_1]\big], \\ K_{2,b}(x,y) &= I\big[g_x \text{ and } g_y \text{ do not meet during } [s,s+B_1] \text{ and} \\ &\qquad\qquad g_y \text{ is removed during } [s+B_1,s+B]\big].\end{aligned}$$

Note that in the description of J_1 and K_1 we require that the *random walk paths* of g_x and g_y do not meet. In accordance with our present description of our system, there is attached to each particle a random walk path which describes its motion until the particle is removed. However, the random walk path exists for all times, even after the removal of the particle. The requirement in the description of J_1 and K_1 is that the coupled random walk paths of g_x and g_y do not meet in the appropriate time interval, even when g_x or g_y is removed. We then have

$$I\Big[\sum_z \beta(z)J(x,y,z) \neq 0\Big] \leq K_1(x,y) + K_{2,a}(x,y) + K_{2,b}(x,y),$$

and

$$J(x,y,z) \leq J_1(x,y,z) + J_2(x,y,z).$$

The right hand side of (3.7) is therefore bounded by

$$\sum_{x,y} I_\ell(x,y)\widetilde{E}\Big[\sum_z \beta^2(z)\big(J_1(x,y,z)+J_2(x,y,z)\big)\Big] \times \widetilde{E}\Big[K_1(x,y)+K_{2,a}(x,y)+K_{2,b}(x,y)\Big]. \tag{3.8}$$

This is the estimate which we shall use for $s \geq t^\alpha$ for a suitable α. For $s < t^\alpha$ we shall use a different estimate.

Each of the six combinations of J's and K's in (3.8) leads to a contribution to $E[\Delta_\ell^2]$ and we estimate each of these contributions. For the time being we consider values of $s \geq t^\alpha$ for an $\alpha \in (0,1)$ to be chosen later.

Contribution of J_1K_1. By Lemma 11 (see (2.22)) we have

$$\widetilde{E}[K_1] \leq \frac{C_1\|y-x\|}{\sqrt{B+1}} \tag{3.9}$$

on the event $\{I_\ell(x,y)=1\}$. We also have

$$\begin{aligned}&\widetilde{E}[J_1(x,y,z)]\\ &\leq P\{x+S' \text{ and } y+S'' \text{ do not meet during } [0,A] \text{ and } x+S' \text{ is at } z \text{ at time } t-s\}\\ &= P\{S' \text{ and } y-x+S'' \text{ do not meet during } [0,A] \text{ and } S' \text{ is at } z-x \text{ at time } t-s\},\end{aligned} \tag{3.10}$$

where S' and S'' are two coupled copies of S (in the sense that $x+S'$ and $y+S''$ are coupled as described in Section 2, in the subsection preceding Lemma 11). S' and S'' are independent of $W_1,\dots,W_\ell$ and w_ℓ.

By means of (3.9) and (3.10) it is easy to estimate the integral of J_1K_1. Indeed, by replacing y by $x+v$ we obtain

$$\begin{aligned}
&\int \prod_{i\le\ell} \mu_i(dW_i) \sum_{x,y} I_\ell(x,y)\widetilde{E}\Big[\sum_z \beta^2(z) J_1\Big]\widetilde{E}\Big[K_1\Big] \\
&\qquad \le \frac{C_1}{\sqrt{B+1}} \int \prod_{i\le\ell} \mu_i(dW_i) \sum_{x,v} \|v\| I_\ell(x,x+v) \sum_z \beta^2(z) \\
&\qquad \times P\{S' \text{ and } v+S'' \text{ do not meet during } [0,A] \text{ and } S' \text{ is at } z-x \text{ at time } t-s\}.
\end{aligned} \tag{3.11}$$

Because $I_\ell(x,x+v)=1$ can occur only if $\xi_{(\ell-1)t/p}(x)\ge 1$, we have by (2.5)

$$\int \prod_{i\le\ell} \mu_i(dW_i) I_\ell(x,x+v) \le \frac{C_2 t q(v)}{sp}. \tag{3.12}$$

Substituting this estimate in (3.11) and summing over x shows that (3.11) is at most

$$\frac{C_3 t}{sp\sqrt{B+1}} \sum_z \beta^2(z) \sum_v \|v\| q(v) P\{S' \text{ and } v+S'' \text{ do not meet during } [0,A]\}.$$

Finally, by applying (2.22) to the probability here, and using (1.7) we get that (3.11) (the contribution of J_1K_1) is bounded by

$$\frac{C_4 t}{sp\sqrt{(A+1)(B+1)}} \sum_z \beta^2(z) \sum_v \|v\|^2 q(v) \le \frac{C_5 t}{sp\sqrt{(A+1)(B+1)}} \sum_z \beta^2(z). \tag{3.13}$$

Contribution of $J_1K_{2,a}$.

Note that $K_{2,a}(x,y)(W_1,\dots,W_\ell,w_{\ell+1},\dots,w_p)=1$ is possible only if there is a particle at some site $u\in\mathbb{Z}^d$ at time s which meets g_y during $[s,s+B_1]$. This particle which meets g_y has to be different from g_x and g_y. Therefore,

$$\begin{aligned}
&\widetilde{E}[K_{2,a}] \\
&\quad \le \sum_u [\xi_{(\ell-1)t/p}(u)-\delta_{u,x}]^+ P\{y+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,B_1]\},
\end{aligned} \tag{3.14}$$

where S'' and S''' are independent copies of the random walk S.

We now consider

$$\int \prod_{i\le\ell} \mu_i(dW_i) \sum_{x,y} I_\ell(x,y)\widetilde{E}\Big[\sum_z \beta^2(z) J_1\Big]\widetilde{E}\Big[K_{2,a}\Big]. \tag{3.15}$$

We substitute the right hand side of (3.14) for $\widetilde{E}[K_{2,a}]$ and again use (3.10) and write $x+v$ for y. We then see that the expression (3.15) is bounded by

$$\int \prod_{i\le \ell} \mu_i(dW_i) \sum_{x,v} I_\ell(x,x+v) \sum_z \beta^2(z)$$
$$\times P\{S' \text{ and } v+S'' \text{ do not meet during } [0,A] \text{ and } S' \text{ is at } z-x \text{ at time } t-s\}$$
$$\times \sum_u [\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+ P\{x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,B_1]\}. \tag{3.16}$$

Here the pair S', S'' is as in (3.10) and S''' is a copy of S which is independent of S' and S''.

We now use for the first time in this section that our system satisfies (1.10). *This is one of the principal new steps in this variance estimate.* We use (2.14) to estimate the following quantity arising in (3.16).

$$\int \prod_{i\le \ell} \mu_i(dW_i) I_\ell(x,x+v)[\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+. \tag{3.17}$$

First assume $u \neq x$. Then

$$I_\ell(x,x+v)[\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+$$
$$\le MI[x \text{ and } u \text{ are occupied at time } (\ell-1)t/p]$$
$$\times I\big[\text{a particle jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p, \ell t/p]\big].$$

In this case (2.14) (see also (3.12)) shows that for $\ell \ge 2$, (3.17) is at most

$$C_6 \frac{tq(v)}{p[(\ell-1)t/p]^2} \le 4C_6 \frac{tq(v)}{p}\frac{1}{s^2}. \tag{3.18}$$

If $u = x$, then

$$I_\ell(x,x+v)[\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+$$
$$\le MI[\text{ there are at least 2 particles at } x \text{ at time } (\ell-1)t/p]$$
$$\times I\big[\text{a particle jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p, \ell t/p]\big],$$

and we again get (3.18) for $\ell \ge 2$.

By using (3.17) and (3.18) in (3.16) we find that (3.15) is at most

$$\sum_z \beta^2(z) \sum_{x,v} 4C_6 \frac{tq(v)}{p}\frac{1}{s^2}$$
$$\times P\{S' \text{ and } v+S'' \text{ do not meet during } [0,A] \text{ and } S' \text{ is at } z-x \text{ at time } t-s\}$$
$$\times \sum_u P\{x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,B_1]\}. \tag{3.19}$$

We first deal with

$$\sum_u P\{x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,B_1]\}. \tag{3.20}$$

Clearly this sum is independent of x and v. In fact this sum equals the expected number of u for which $u+S'''_r = S''_r$ for some $r \le B_1$, and this is at most

$$\begin{aligned} &C_7 \sum_u E[\text{Lebesgue measure of } \{r \le B_1+1 : u+S'''_r = S'_r\}] \\ &= C_7 \int_0^{B_1+1} \sum_u P\{u+S'''_r = S'_r\}\, \mathrm{d}r = C_7(B_1+1). \end{aligned} \tag{3.21}$$

We substitute this estimate in (3.19) and sum over x. This, together with (2.22) shows that (3.15), the contribution of $J_1K_{2,a}$, is bounded by

$$\begin{aligned} &\sum_z \beta^2(z) C_8(B_1+1) \sum_v \frac{tq(v)}{p}\frac{1}{s^2} P\{S' \text{ and } v+S'' \text{ do not meet during } [0,A]\} \\ &\le C_9 \sum_z \beta^2(z) \frac{t}{p} \frac{B_1+1}{s^2\sqrt{A+1}} \end{aligned} \tag{3.22}$$

(compare (3.13)).

Contribution of $J_1K_{2,b}$. We will now handle the estimate of $J_1K_{2,b}$. In fact, we will merely point out which adjustments have to be made in the estimate of $J_1K_{2,a}$ and what the result is. First of all the probability in the right hand side of (3.14) is replaced by

$$P\{x+S' \text{ and } y+S'' \text{ do not meet in } [0,B_1] \text{ and } y+S'' \text{ and } u+S''' \text{ meet during } [B_1,B]\}. \tag{3.23}$$

Here S', S'' and S''' are as in (3.16). A similar change is made in (3.16), and the summation over u in (3.19) is replaced by

$$\begin{aligned} \sum_u P\{&x+S' \text{ and } x+v+S'' \text{ do not meet in } [0,B_1] \\ &\text{and } x+v+S'' \text{ and } u+S''' \text{ meet during } [B_1,B]\}. \end{aligned} \tag{3.24}$$

Now let, for each vertex u, S^u be an independent copy of S. The S^u are also taken independent of S' and S''. Let R denote the number of u such that $x+v+S''$ and $u+S^u$ meet in $[B_1,B]$, and V the event that $x+S'$ and $x+v+S''$ do not meet during $[0,B_1]$. Then the summation (3.24) is equal to

$$P(V)\, E[R \,|\, V].$$

However (using translation invariance) it is clear that R is independent of $(S''_\tau, 0 \le \tau \le B_1)$, and hence of V. As in (3.21) we find that $E[R|V] \le C_7(B - B_1 + 1) \le C_7(B+1)$. Further, the probability of V is (by (2.22)), at most $C_1\|v\|/\sqrt{B_1+1}$. So the sum (3.24) is bounded by $C_7C_1(B+1)\|v\|/\sqrt{B_1+1}$. This takes the place of the estimate (3.21) for (3.20). As in (3.22) we conclude that the contribution of $J_1K_{2,b}$ is at most

$$C_{10}\sum_z \beta^2(z)\frac{t}{p}\frac{B+1}{s^2\sqrt{A+1}}\frac{1}{\sqrt{B_1+1}}. \tag{3.25}$$

Contribution of J_2K_1. This case is very similar to that of $J_1K_{2,a}$. We leave it to the reader to check that the contribution of J_2K_1 is at most

$$C_{11}\sum_z \beta^2(z)\frac{t}{p}\frac{A+1}{s^2\sqrt{B+1}}. \tag{3.26}$$

Contribution of $J_2K_{2,a}$. Analogously to (3.14) we have

$$\begin{aligned}\widetilde{E}[J_2] \le \sum_{\widetilde{u}}[\xi_{(\ell-1)t/p}(\widetilde{u}) - \delta_{\widetilde{u},x}]^+ P\{y + S''_{\cdot} \text{ and } \widetilde{u} + S'''_{\cdot} \text{ meet during } [0,A] \\ \text{and } x + S'_{t-s} = z\},\end{aligned} \tag{3.27}$$

with S', S'' and S''' as in (3.16). Now $I_\ell(x, x+v)\widetilde{E}J_2\widetilde{E}K_{2,a}$ contains the product

$$\begin{aligned}&I[x \text{ is occupied at time } (\ell-1)t/p] \\ &\times I\big[\text{a particle jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p, \ell t/p]\big] \\ &\times [\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+[\xi_{(\ell-1)t/p}(\widetilde{u}) - \delta_{\widetilde{u},x}]^+.\end{aligned}$$

The integral of this product with respect to $\prod_{j\le\ell}\mu_j(dW_j)$ is at most

$$\begin{cases} \dfrac{C_{12}tq(v)}{ps^3} & \text{if } u \ne \widetilde{u} \\ \dfrac{C_{12}tq(v)}{ps^2} & \text{if } u = \widetilde{u}, \end{cases} \tag{3.28}$$

by virtue of (2.14). First we estimate the contribution to

$$\int \prod_{j\le\ell}\mu_i(dW_j)\sum_{x,v} I_\ell(x,x+v)\widetilde{E}\Big[\sum_z \beta^2(z)J_2\Big]\widetilde{E}[K_{2,a}] \tag{3.29}$$

of the terms with $u \neq \widetilde{u}$. These terms with $u \neq \widetilde{u}$ contribute at most

$$\frac{C_{12}t}{ps^3}\sum_z \beta^2(z)\sum_{x,v} q(v)$$
$$\times \sum_{\widetilde{u}} P\{x+v+S''_\cdot \text{ and } \widetilde{u}+S'''_\cdot \text{ meet during } [0,A] \text{ and } x+S'_{t-s}=z\}$$
$$\times \sum_u P\{x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,B_1]\}$$
$$\leq \frac{C_{12}C_7(B_1+1)t}{ps^3}\sum_z \beta^2(z)\sum_{x,v} q(v)$$
$$\times \sum_{\widetilde{u}} P\{x+v+S''_\cdot \text{ and } \widetilde{u}+S'''_\cdot \text{ meet during } [0,A] \text{ and } x+S'_{t-s}=z\}$$
$$= \frac{C_{12}C_7(B_1+1)t}{ps^3}\sum_z \beta^2(z)\sum_{x,v} q(v)$$
$$\times \sum_{u'} P\{x+v+S''_\cdot \text{ and } x+u'+S'''_\cdot \text{ meet during } [0,A] \text{ and } x+S'_{t-s}=z\}$$
$$= \frac{C_{12}C_7(B_1+1)t}{ps^3}\sum_z \beta^2(z)\sum_{x,v} q(v)$$
$$\times \sum_{u'} P\{v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [0,A] \text{ and } S'_{t-s}=z-x\}, \tag{3.30}$$

where in the first equality we substituted $x+u'$ for $\widetilde{u}$. We now sum over x, u' and v (in this order) to obtain the bound

$$\frac{C_{13}t}{ps^3}\sum_z \beta^2(z)(A+1)(B_1+1) \tag{3.31}$$

for the contribution to (3.29) of the terms with $u \neq \widetilde{u}$.

For the terms with $u = \widetilde{u}$ one obtains similarly (again by replacing u by $u'+x$ and then summing over x) the bound

$$\frac{C_{12}t}{ps^2}\sum_z \beta^2(z)\sum_v q(v)\sum_{u'} P\{v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [0,A]\}$$
$$\times P\{v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [0,B_1]\}. \tag{3.32}$$

Now, by well known estimates for the Green function (see Spitzer (1976), Proposition 26.1) we have for $d \geq 3$ and $u \in \mathbb{Z}^d$

$$P\{S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,A]\}$$
$$\leq P\{S''_r - S'''_r = u \text{ for some } r \geq 0\}$$
$$\leq \frac{C_{14}}{(\|u\|+1)}. \tag{3.33}$$

Actually the estimate in Spitzer only holds for 3-dimensional random walk (see Uchiyama (1998)) and therefore should be applied to a triple of coordinates of the random walks $\{S''\}$ and $\{S'''\}$. It is further well known that for $d \geq 3$ and any $n \geq 0$ and z with $2^k \leq \|z\| < 2^{k+1}$

$$\begin{aligned} &P\{\|S''_r - S'''_r\| > 2^{k+1} \text{ for all } r \geq n + 2^{2k} \big| S''_n - S'''_n = z\} \\ &\geq P\{\|S''_{n+2^{2k}} - S'''_{n+2^{2k}}\| \geq 2^{k+3} \big| S''_n - S'''_n = z\} \\ &\qquad \times \left[1 - P\{\|S''_r - S'''_r\| \leq 2^{k+1} \text{ for some } r \geq 0 \big| \|S''_0 - S'''_0\| \geq 2^{k+3}\}\right] \end{aligned} \tag{3.34}$$

is bounded away from zero. Note that the last factor can be bounded from below by looking only at three coordinates, so that we only need the asymptotic behavior of the Green function in dimension 3. (3.34) easily implies

$$E\{\text{number of } z \text{ with } 2^k \leq \|z\| < 2^{k+1} \text{ ever visited by } \{S'' - S'''\}\} \leq C_{15} 2^{2k}.$$

Together with (3.33) this shows that, for fixed v,

$$\begin{aligned} &\sum_{u' : 2^k \leq \|u'-v\| < 2^{k+1}} [P\{v + S''_{\cdot} \text{ and } u' + S'''_{\cdot} \text{ meet during } [0, A]\}]^2 \\ &\leq C_{14} 2^{-k} \sum_{u : 2^k \leq \|u\| < 2^{k+1}} P\{S''_r - S'''_r = u \text{ for some } r \geq 0\} \\ &\leq C_{16} 2^k. \end{aligned} \tag{3.35}$$

Combined with the estimates (3.33) and (3.21) this gives

$$\begin{aligned} &\sum_{u'} \left[P\{v + S''_{\cdot} \text{ and } u' + S'''_{\cdot} \text{ meet during } [0, A]\}\right]^2 \\ &\leq 1 + \sum_{k: 2^k \leq \sqrt{(A+1)}} C_{16} 2^k \\ &\qquad + \frac{C_{14}}{\sqrt{(A+1)}} \sum_{u'} P\{v + S''_{\cdot} \text{ and } u' + S'''_{\cdot} \text{ meet during } [0, A]\} \\ &\leq C_{17} (A+1)^{1/2}. \end{aligned} \tag{3.36}$$

Thus, by Schwarz' inequality, the expression (3.32) is at most

$$\frac{C_{18} t}{p s^2} \sum_z \beta^2(z) [(A+1)(B_1+1)]^{1/4}.$$

Consequently, the contribution of $J_2K_{2,a}$ is

$$\int \prod_{i\le \ell} \mu_i(dW_i) \sum_{x,y} I_\ell(x,y) \widetilde{E}\Big[\sum_z \beta^2(z) J_2\Big] \widetilde{E}[K_{2,a}]$$
$$\le \frac{C_{13}t}{ps^3} \sum_z \beta^2(z)(A+1)(B_1+1) + \frac{C_{18}t}{ps^2} \sum_z \beta^2(z)[(A+1)(B_1+1)]^{1/4}. \tag{3.37}$$

Contribution of $J_2K_{2,b}$. Now we estimate

$$\int \prod_{i\le \ell} \mu_i(dW_i) \sum_{x,y} I_\ell(x,y) \widetilde{E}\Big[\sum_z \beta^2(z) J_2\Big] \widetilde{E}[K_{2,b}]. \tag{3.38}$$

Just as in the estimate of the contribution of $J_1K_{2,b}$ we have for $I_\ell(x,y)=1$,

$$\widetilde{E}[K_{2,b}]$$
$$\le \sum_u [\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+ P\{x+S'_\cdot \text{ and } y+S''_\cdot \text{ do not meet in } [0,B_1]$$
$$\text{and } y+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [B_1,B]\}.$$

We combine this with (3.27) and set $y = x+v$ as before. We also use the estimate (3.28). Analogously to (3.30), (3.31), this time using (3.24), (3.25), we then find that the contribution of the terms with $u \ne \widetilde{u}$ to (3.38) is at most

$$C_{12} \sum_x \sum_v \sum_z \frac{\beta^2(z) t q(v)}{ps^3}$$
$$\times \sum_{u,\widetilde{u}} P\{x+v+S''_\cdot \text{ and } \widetilde{u}+S'''_\cdot \text{ meet during } [0,A] \text{ and } x+S'_{t-s}=z\}$$
$$\times P\{x+S'_\cdot \text{ and } x+v+S''_\cdot \text{ do not meet in } [0,B_1] \text{ but}$$
$$x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [B_1,B]\}$$
$$\le C_{12}C_7C_1 \frac{t}{ps^3} \sum_x \sum_v q(v) \sum_z \beta^2(z) \frac{(B+1)\|v\|}{\sqrt{(B_1+1)}}$$
$$\times \sum_{u'} P\{v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [0,A] \text{ and } x+S'_{t-s}=z\}$$
$$\le \frac{C_{19}t(A+1)(B+1)}{ps^3\sqrt{(B_1+1)}} \sum_z \beta^2(z). \tag{3.39}$$

Now we estimate the contribution to (3.38) of the terms with $u = \widetilde{u}$. These terms contribute at most

$$\frac{C_{20}t}{ps^2}\sum_z \beta^2(z)\sum_{x,v} q(v)$$
$$\times \sum_u P\{x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [0,A] \text{ and } x+S'_{t-s}=z\}$$
$$\times P\{x+S'_\cdot \text{ and } x+v+S''_\cdot \text{ do not meet in } [0,B_1] \text{ but}$$
$$x+v+S''_\cdot \text{ and } u+S'''_\cdot \text{ meet during } [B_1,B]\}.$$

After replacing u by $x+u'$ this equals

$$\frac{C_{20}t}{ps^2}\sum_z \beta^2(z)\sum_{x,v} q(v)$$
$$\times \sum_{u'} P\{v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [0,A] \text{ and } S'_{t-s}=z-x\}$$
$$\times P\{S'_\cdot \text{ and } v+S''_\cdot \text{ do not meet in } [0,B_1] \text{ but } v+S''_\cdot \text{ and}$$
$$u'+S'''_\cdot \text{ meet during } [B_1,B]\}. \tag{3.40}$$

Now sum over x to get the bound

$$\frac{C_{20}t}{ps^2}\sum_z \beta^2(z)\sum_v q(v)\sum_{u'} P\{v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [0,A]\}$$
$$\times P\{S'_\cdot \text{ and } v+S''_\cdot \text{ do not meet in } [0,B_1] \text{ but } v+S''_\cdot$$
$$\text{and } u'+S'''_\cdot \text{ meet during } [B_1,B]\}. \tag{3.41}$$

Now, by using the argument between (3.24) and (3.25), we get similar to (3.36), for $d \geq 3$

$$\sum_{u'} \big[P\{S'_\cdot \text{ and } v+S''_\cdot \text{ do not meet in } [0,B_1] \text{ but } v+S''_\cdot$$
$$\text{and } u'+S'''_\cdot \text{ meet during } [B_1,B]\}\big]^2$$
$$\leq 1 + \sum_{k:2^k\leq B^{1/3}} C_{16}2^k + \frac{C_{21}}{(B+1)^{1/3}}\sum_{u'} P\{S'_\cdot \text{ and } v+S''_\cdot \text{ do not meet}$$
$$\text{in } [0,B_1] \text{ but } v+S''_\cdot \text{ and } u'+S'''_\cdot \text{ meet during } [B_1,B]\}$$
$$\leq C_{22}(B+1)^{1/3} + \frac{C_{23}}{(B+1)^{1/3}}\frac{(B+1)\|v\|}{\sqrt{(B_1+1)}}$$
$$= C_{22}(B+1)^{1/3} + \frac{C_{23}(B+1)^{2/3}\|v\|}{\sqrt{(B_1+1)}}.$$

Combining this with (3.36) and using Schwarz' inequality, the expression (3.41) is at most

$$\frac{C_{24}t}{ps^2}\sum_z \beta^2(z)(A+1)^{1/4}\big[(B+1)^{1/6}+\frac{(B+1)^{1/3}}{(B_1+1)^{1/4}}\big].$$

Together with (3.39) this shows that (3.38) is at most

$$\frac{C_{19}t(A+1)(B+1)}{ps^3\sqrt{(B_1+1)}}\sum_z \beta^2(z)+\frac{C_{24}t}{ps^2}\sum_z \beta^2(z)(A+1)^{1/4}\big[(B+1)^{1/6}+\frac{(B+1)^{1/3}}{(B_1+1)^{1/4}}\big]. \tag{3.42}$$

Contribution of terms with large ℓ to the variance. We now take the above contributions together. We have postponed the choice of A, B and B_1. We now take $A = s^{2/3}\wedge(t-s)$, and $B = s^{6/7}\wedge(t-s)$ and $B_1 = B^{2/3}$. It is straightforward to see that for $2 \le s \le t - t^{6/7}$, each of the six contributions (3.13), (3.22), (3.25), (3.26), (3.37) and (3.42) is at most of order $(t/p)\sum_z \beta^2(z)s^{-71/42}$. Hence, for ℓ with $2 \le s = \ell t/p \le t - t^{6/7}$,

$$E\big[\Delta_\ell^2\big] \le \frac{C_{25}t}{p}\sum_z \beta^2(z)\frac{1}{s^{71/42}}.$$

We sum this estimate over all ℓ for which $t^\alpha \le s = \ell t/p \le t - t^{6/7}$ and let $p\to\infty$. This yields

$$\limsup_{p\to\infty}\sum_{\ell: t^\alpha\le \ell t/p\le t-t^{6/7}} E[\Delta_\ell^2] \le C_{26}\sum_z \beta^2(z)\frac{1}{t^{29\alpha/42}}. \tag{3.43}$$

If $t - t^{6/7} \le s \le t$, then $s \sim t$. Since, in addition, $A \le s^{2/3}, B \le s^{6/7}$, one easily sees that for $t - t^{6/7} \le s \le t$ the terms of largest order among (3.13), (3.22), (3.25), (3.26), (3.37) and (3.42), are the terms

$$\frac{t}{sp\sqrt{(A+1)(B+1)}}\sum_z \beta^2(z),$$

which appears in (3.13), and

$$\frac{t}{ps^2}\sum_z \beta^2(z)(A+1)^{1/4}(B+1)^{1/6},$$

which appears in (3.37) and (3.42). Taking into account that also $A \le B \le t - s$, we see that the contribution of the terms with $t - t^{6/7} \le s = \ell t/p \le t$ is at most of order

$$\frac{t}{tp}\sum_z \beta^2(z)\sum_{\ell: t-t^{6/7}\le \ell t/p\le t}\Big[t^{-2/3}+\big[(t-\ell t/p)+1\big]^{-1}+\frac{1}{t}\big[(t-\ell t/p)+1\big]^{5/12}\Big]$$
$$= O\Big(\frac{1}{t^{7/12}}\sum_z \beta^2(z)\Big).$$

Together with (3.43) we finally obtain

$$\limsup_{p\to\infty} \sum_{\ell: t^\alpha \le \ell t/p} \Delta_\ell^2 \le C_{27} \sum_z \beta^2(z) \big[\frac{1}{t^{29\alpha/42}} + \frac{1}{t^{7/12}}\big]. \tag{3.44}$$

Contribution of terms with small ℓ to the variance.

The estimate (3.43) is not good enough for small ℓ. For ℓ with $\ell t/p \le t^\alpha$ (with α properly chosen later) we use a different estimate. First we go back to (3.7) and (3.8). We again write s for $\ell t/p$. Now we will use that $J \le L$ and $K_{2,a} + K_{2,b} \le K_2$, where

$$L = L(x,z) = I\Big[g_x \text{ is at } z \text{ at time } t\Big]$$

and

$$K_2 = K_2(y) = I\Big[g_y \text{ is removed during } [s, s+B]\Big].$$

We then want to estimate

$$\liminf_{p\to\infty} \sum_{\ell: \ell t/p \le t^\alpha} \int \prod_{i\le \ell} \mu_i(dW_i) \sum_{x,y} I_\ell(x,y) \widetilde{E}\big[\sum_z \beta^2(z) L\big] \widetilde{E}\big[K_1 + K_2\big]. \tag{3.45}$$

In this part of the proof we shall take

$$B = t^{2\alpha/3}. \tag{3.46}$$

For the contribution of LK_1 to (3.45) we do the following. First, from (3.9) we have that

$$\widetilde{E}[K_1] \le \frac{C_1 \|y-x\|}{\sqrt{B+1}} = C_1 t^{-\alpha/3} \|y-x\|. \tag{3.47}$$

To handle the rest of the expression (3.45), suppose that each particle, in addition to its ordinary clock (which, as said in the subsection on Ghost particles and coupling in Section 2, tells the particle when to jump), it has a so-called fake clock. Like the ordinary clocks, the fake clocks ring according to (rate 1) Poisson processes, independent of the ordinary clocks. They do not influence (and are not influenced by) the movement of the particles and are introduced for theoretical reasons only. We also associate with each ring of the fake clock a fake jump, which has distribution $q(\cdot)$ and is independent of all the other variables. No particle actually makes this jump, but it will appear as a label when we estimate the contribution of LK_2.

Because a ghost particle is removed easier than an ordinary particle, and by the similarity of fake and ordinary clocks, we have that

$$\begin{aligned}
&\int \prod_{i\le\ell} \mu_i(dW_i) \sum_{x,y} I_\ell(x,y) \|y-x\| \widetilde{E}\big[\sum_z \beta^2(z) L\big] \\
&\le \sum_x \sum_z \beta^2(z) P\{\text{the fake clock of a single particle rings in } ((\ell-1)t/p, \ell t/p], \\
&\qquad \text{this happens at } x \text{ and the particle is at } z \text{ at time } t\} \sum_v \|v\| q(v).
\end{aligned} \tag{3.48}$$

Instead of using a decomposition with respect to the location of the fake clock which rings, we now decompose with respect to the starting position of the particle whose fake clock rings. We then sum over ℓ with $\ell t/p \le t^\alpha$ and let p to ∞. The expression in (3.48) then becomes

$$\sum_w \sum_z \beta^2(z) E\Big[R_t(w) I\big[\text{the particle which started in } w \text{ is in } z \text{ at time } t\big]\Big] \sum_v \|v\| q(v),$$

where $R_t(w)$ denotes the numbers of rings between time 0 and t^α of the fake clock of the particle which started in w. Since the fake clocks are independent of the particle movements, this last expression is equal to

$$\begin{aligned} & t^\alpha \sum_w \sum_z \beta^2(z) P\{\text{the particle starting in } w \text{ is in } z \text{ at time } t\} \sum_v \|v\| q(v) \\ & = t^\alpha \sum_z \beta^2(z) E[\xi_t(z)] \sum_v \|v\| q(v), \end{aligned}$$

which by (2.5) is bounded by $C_2 t^\alpha / t$. Combined with the the estimate (3.47) this gives the bound

$$C_3 t^{2\alpha/3-1} \sum_z \beta^2(z) \tag{3.49}$$

for the contribution of LK_1 in (3.45) to the variance.

We shall now estimate the contribution of LK_2 to (3.45). To this end we introduce the quantity

$$\gamma(u) = P\{S'_\cdot \text{ and } u + S'''_\cdot \text{ meet during } [0, B]\}$$

(as before, S' and S''' are independent copies of S). In this notation we get the following analogue of (3.14):

$$\widetilde{E}[K_2(y)] \le \sum_u [\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+ \gamma(u-y).$$

We therefore have (with y written as $x+v$, as before)

$$\begin{aligned} & \sum_{x,v} I_\ell(x, x+v) L(x) \widetilde{E}[K_2(x+v)] \\ & \le \sum_u \sum_{x,v} \gamma(u-x-v) I\big[\text{a particle jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p, lt/p]\big] \\ & \qquad \times I[g_x \text{ is at } z \text{ at time } t]\,[\xi_{(\ell-1)t/p}(u) - \delta_{u,x}]^+. \end{aligned} \tag{3.50}$$

Next we sum this over ℓ with $\ell t/p \le t^\alpha$. We then see that

$$\begin{aligned}\sum_{\ell:\ell t/p\le t^\alpha}\int\prod_{i\le\ell}\mu_i(dW_i)\sum_{x,y}I_\ell(x,y)\widetilde{E}\big[\sum_z\beta^2(z)L\big]\widetilde{E}[K_2]\\ \le\sum_z\beta^2(z)\sum_u\sum_v\sum_{\ell:\ell t/p\le t^\alpha}\sum_x E\Big\{\gamma(u-x-v)[\xi_{(\ell-1)t/p}(u)-\delta_{u,x}]^+\\ \times E\big\{I\big[\text{a particle jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p,\ell t/p]\big]\\ \times I[g_x \text{ is at } z \text{ at time } t]\,\big|\mathcal{F}_{(\ell-1)t/p}\big\}\Big\}.\end{aligned}\tag{3.51}$$

We rewrite the inner conditional expectation by decomposing according to the starting site of the particle which jumps from x to $x+v$. This starting site can be any w with $\|w\|\le N$ (recall that the initial state is $\mathbb{1}^{(N)}$). This gives, with $\pi(w)$ denoting the particle which started at w,

$$\begin{aligned}E\big\{I\big[\text{a particle jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p,\ell t/p]\big]\\ \times I[g_x \text{ is at } z \text{ at time } t]\,\big|\mathcal{F}_{(\ell-1)t/p}\big\}\\ =\sum_{\|w\|\le N}E\big\{I[\pi(w) \text{ jumps from } x \text{ to } x+v \text{ during } ((\ell-1)t/p,\ell t/p]\big]\\ \times I[g_x \text{ is at } z \text{ at time } t]\big|\mathcal{F}_{(\ell-1)t/p}\big\}.\end{aligned}\tag{3.52}$$

This last conditional expectation will now be rewritten by 'interchanging g_x with the particle which started at w'. In the conditional expectation in (3.52) we require that $\pi(w)$ jumps from x to $x+v$ during $((\ell-1)t/p,\ell t/p]$. If this happens, this particle is then denoted as g_{x+v} and we put another particle in place of $\pi(w)$ at x, which is the ghost g_x. At time $\ell t/p$ we therefore have the particles which were present at time $(\ell-1)t/p$ plus a ghost at $x+v$ (see the definition of $\widetilde{E}$ just before Lemma 12). After this g_x has to survive till time t and to end at position z at time t. The probability of these things happening is at most the probability that the fake clock of $\pi(w)$ rings during $((\ell-1)t/p,\ell t/p]$ and that the associated fake jump equals v, and that further $\pi(w)$ survives till time t and is at z at time t. We have merely relabeled g_x as $\pi(w)$ and changed the removal rules by letting $\pi(w)$ behave as an ordinary particle and by ignoring the ghost g_{x+v}. These changes only make it more likely that $\pi(w)$ survives than that the original g_x survives. Thus, the right hand side of (3.52) is at most

$$\sum_{\|w\|\le N} E\big\{I[\pi(w) \text{ is at } x \text{ at time } (\ell-1)t/p \text{ and its fake clock rings during } ((\ell-1)t/p, \ell t/p] \text{ with associated fake jump } v] \times I[\pi(w) \text{ is at } z \text{ at time } t]\big|\mathcal{F}_{(\ell-1)t/p}\big\}$$
$$= q(v) \sum_{\|w\|\le N} E\big\{I[\pi(w) \text{ is at } x \text{ at time } (\ell-1)t/p \text{ and its fake clock rings during } ((\ell-1)t/p, \ell t/p]] \times I[\pi(w) \text{ is at } z \text{ at time } t]\big|\mathcal{F}_{(\ell-1)t/p}\big\}.$$

Note that here and in the sequel "$\pi(w)$ is at z at time t" is short for "$\pi(w)$ survives and is at z at time t". We substitute the last expression for (3.52) and find that the left hand side of (3.51) is at most

$$\sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \sum_{\|w\|\le N} E\Big\{ \sum_{\ell:\ell t/p\le t^\alpha} \sum_x [\xi_{(\ell-1)t/p}(x+u) - \delta_{u,\mathbf{0}}]^+ \times I[\pi(w) \text{ is at } x \text{ at time } (\ell-1)t/p \text{ and its fake clock rings during } ((\ell-1)t/p, \ell t/p]] \times I[\pi(w) \text{ is at } z \text{ at time } t]\Big\}. \tag{3.53}$$

Next we want to take the limit as $p \to \infty$ of this expression. To justify taking the limit inside the expectation and summations we first note that

$$\sum_{\ell:\ell t/p\le t^\alpha} \sum_x [\xi_{(\ell-1)t/p}(x+u) - \delta_{u,\mathbf{0}}]^+ \times I[\pi(w) \text{ is at } x \text{ at time } (\ell-1)t/p \text{ and its fake clock rings during } ((\ell-1)t/p, \ell t/p]] \times I[\pi(w) \text{ is at } z \text{ at time } t]$$
$$\le M\big[\text{number of rings of the fake clock of } \pi(w) \text{ during } [0, t^\alpha]\big] \times I[\pi(w) \text{ is at } z \text{ at time } t].$$

Since the fake clocks do not influence the motions of the particles, the expectation of this bound is at most

$$Mt^\alpha P\{S_t = z - w\}.$$

Since further (by the estimate (3.21) for (3.20))

$$\sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \sum_w Mt^\alpha P\{S_t = z-w\} < \infty,$$

we can apply the dominated convergence theorem to obtain that the limsup of (3.51) as $p \to \infty$ is at most

$$\sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \times \sum_{\|w\| \le N} E\Big\{ \sum_{r:\sigma_r(w) \le t^\alpha} [\xi_{\sigma_r(w)}(y_r(w)+u) - \delta_{u,\mathbf{0}}]^+ \times I[\pi(w) \text{ is at } z \text{ at time } t]\Big\}, \tag{3.54}$$

where we have written $\sigma_r(w)$, $r = 1, 2, \dots$ for the successive times at which the fake clock of $\pi(w)$ rings and $y_r(w)$ for the position of $\pi(w)$ at time $\sigma_r(w)$. We define further

$$W(u,w) = \sum_{r:\sigma_r(w) \le t^\alpha} [\xi_{\sigma_r(w)}(y_r(w)+u) - \delta_{u,\mathbf{0}}]^+.$$

Then, for any choice of the constant $C_4 > 0$, we obtain that the right hand side of (3.54) equals

$$\begin{aligned}
&\sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \\
&\quad \times \sum_{\|w\| \le N} E\big\{W(u,w) I[\pi(w) \text{ is at } z \text{ at time } t]\big\} \\
&\le \sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \\
&\quad \times \sum_{\|w\| \le N} E\big\{C_4 (\log t)^4 I[\pi(w) \text{ is at } z \text{ at time } t]\big\} \\
&\quad + \sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \\
&\quad \times \sum_{\|w\| \le N} E\big\{[W(u,w) - C_4(\log t)^4]^+ I[\pi(w) \text{ is at } z \text{ at time } t]\big\}.
\end{aligned} \tag{3.55}$$

But

$$\sum_{\|w\| \le N} I[\pi(w) \text{ is at } z \text{ at time } t]$$

is the number of particles at z at time t, when the initial state is $\mathbb{1}^{(N)}$. By lemma 3 its expectation is at most $C_5/(t+1)$, uniformly in N. The first multiple sum in the right hand side of (3.55) is therefore, for $t \ge 1$, at most

$$\begin{aligned}
&\sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \frac{C_4 C_5 (\log t)^4}{t+1} \le \frac{C_6(B+1)(\log t)^4}{t+1} \sum_z \beta^2(z) \\
&\text{(by the estimate (3.21) for (3.20))} \le 2C_6 t^{2\alpha/3-1} (\log t)^4 \sum_z \beta^2(z).
\end{aligned} \tag{3.56}$$

We next bound the second multiple sum in the right hand side of (3.55), by comparison with a slightly modified system of random walks. Fix w with $\|w\| \le N$ for the moment. Suppose at time 0 we have, apart from the particles which form the ξ_t-process, an extra particle, which we call $\widetilde{\pi}(w)$, at w. Thus at time zero we start with $\mathbb{1}^{(N)}(x) + \delta_{x,w}$ particles at x. All particles except $\widetilde{\pi}(w)$ develop according to the dynamics of the ξ_t-process with initial state $\mathbb{1}^{(N)}(\cdot)$. We denote the system with these particles by $\{\widetilde{\xi}_t\}$. $\widetilde{\pi}(w)$ performs a random walk $\{\widetilde{S}_t\}$ which is a copy of $\{S_t\}$. The motion of $\widetilde{\pi}(w)$ is independent of the $\{\widetilde{\xi}\}$-system and $\widetilde{\pi}(w)$ does not interact at all with this system. There also is a fake (rate one Poisson) clock for $\widetilde{\pi}(w)$. Then Lemma 1 in Section 2 and the Remark following that lemma imply that this new system (the $\{\widetilde{\xi}\}$-system together with $\widetilde{\pi}(w)$) can be coupled with the old system starting with $\mathbb{1}^{(N)}$ such that the particle $\pi(w)$ in the old system is identified with $\widetilde{\pi}(w)$ until $\pi(w)$ is removed, and such that the new system (including $\widetilde{\pi}(w)$) is not less than the old system at each space-time point. Now denote the time of the r-th ring of the fake clock of $\widetilde{\pi}(w)$ by $\widetilde{\sigma}_r(w)$ and the position of $\widetilde{\pi}(w)$ at time $\widetilde{\sigma}_r(w)$ by $\widetilde{y}_r(w)$. Also define

$$\widetilde{W}(u,w) = \sum_{r:\widetilde{\sigma}_r(w) \le t^\alpha} \widetilde{\xi}_{\widetilde{\sigma}_r(w)}(\widetilde{y}_r(w) + u).$$

Then we have in particular that

$$[W(u,w) - C_4(\log t)^4]^+ I[\pi(w) \text{ is at } z \text{ at time } t]$$

lies stochastically below

$$[\widetilde{W}(u,w) - C_4(\log t)^4]^+ I[\widetilde{\pi}(w) \text{ is at } z \text{ at time } t]. \tag{3.57}$$

Again by Lemma 1, this stochastic domination remains true even if we increase the initial state of the $\{\widetilde{\xi}\}$-system from $\mathbb{1}^{(N)}$ to $\mathbb{1}$ (note that this change of initial condition does have no effect on the behavior of $\widetilde{\pi}(w)$). After we make this change we have by translation invariance that the distribution of (3.57) is the same as that of

$$[\widetilde{W}(u,\mathbf{0}) - C_4(\log t)^4]^+ I[\widetilde{\pi}(\mathbf{0}) \text{ is at } z - w \text{ at time } t]. \tag{3.58}$$

Using translation invariance again (and the independence of the particle $\widetilde{\pi}(\mathbf{0})$ from the other particles) it is clear that the distribution of (3.58) is the same for each u. We shall write $P^{(\infty)}$ for the distribution of the $\widetilde{\xi}$-process and the independent particle $\widetilde{\pi}(\mathbf{0})$, when the initial state $\widetilde{\xi}_0 = \mathbb{1}$, and $E^{(\infty)}$ for expectation with respect to $P^{(\infty)}$. Further, we will denote $\widetilde{W}(\mathbf{0},\mathbf{0})$ simply by $\widetilde{W}$ from now on. Then the second multiple sum in (3.55) is at most

$$\begin{aligned}
&\sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \\
&\quad \times \sum_{\|w\|\le N} E^{(\infty)}\{[\widetilde{W} - C_4(\log t)^4]^+ I[\widetilde{\pi}(\mathbf{0}) \text{ is at } z-w \text{ at time } t]\} \\
&\le \sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) E^{(\infty)}\{[\widetilde{W} - C_4(\log t)^4]^+\} \\
&\le \sum_z \beta^2(z) \sum_u \sum_v q(v)\gamma(u-v) \\
&\qquad \times \Big[E^{(\infty)}\{[\widetilde{W} - C_4(\log t)^4]^2\}\Big]^{1/2} \Big[P^{(\infty)}\{\widetilde{W} > C_4(\log t)^4\}\Big]^{1/2}.
\end{aligned} \tag{3.59}$$

We interrupt to estimate the probability in the second factor in the right hand side of (3.59) in the following lemma.

Lemma 14. *There exist constants* $0 < C_4, C_7 < \infty$ *such that*

$$P^{(\infty)}\{\widetilde{W} > C_4(\log t)^4\} \le \frac{C_7}{t^4}.$$

Proof. For convenience we assume that t is larger than e. By another domination argument, (see Lemma 2), it is sufficient to prove that the lemma holds for those cases where the parameters of the RCRW system satisfy $p_j = j/M \wedge 1$ for some positive integer M.

In this proof we adopt the point of view of the Remark preceding Lemma 7, that is, we let particles coalesce rather than remove them. We further identify any particle ρ in the $\widetilde{\xi}$-process at a space-time point (y, s) with the set of original particles (that is, particles present at time 0) which coalesced to form ρ. Accordingly two particles (at different space-time points) are called disjoint if they are formed by coalescence from two disjoint sets of original particles. For brevity we write $\widetilde{\pi}, \widetilde{\sigma}_r$ and $\widetilde{y}_r$ for $\widetilde{\pi}(\mathbf{0}), \widetilde{\sigma}_r(\mathbf{0})$ and $\widetilde{y}_r(\mathbf{0})$, respectively.

Now let F be the event that there exists a w such that $\widetilde{\pi}$ jumps at least $\sqrt{C}\,\log t$ times to a site y which contains (possibly in 'coalesced form') the $\widetilde{\xi}$-particle which was originally (at time 0) in position w. Furthermore call a particle counted in one of the $\widetilde{\xi}_{\widetilde{\sigma}_r}(\widetilde{y}_r)$ a *new particle* if it is disjoint from all the particles counted in $\widetilde{W}$ before time $\widetilde{\sigma}_r$. Let G be the event that $\widetilde{W}$ counts at least $\sqrt{C}(\log t)^3$ new particles. For each particle ρ counted in $\widetilde{W}$, say at time $\widetilde{\sigma}_k$, there is a first time $\widetilde{\sigma}_\ell \le \widetilde{\sigma}_k$ at which one of the original particles making up ρ already appeared in a particle ρ' which was counted in $\widetilde{\xi}_{\widetilde{\sigma}_\ell}$. Then ρ' was necessarily a new particle. Moreover, the whole set of original particles which made up ρ' is contained in ρ. If F does not occur, then any new particle ρ' can be associated to at most $\sqrt{C}\log t$ particles ρ in this manner. Consequently

$$P^{(\infty)}\{\widetilde{W} > C(\log t)^4\} \le P^{(\infty)}(F) + P^{(\infty)}(G). \tag{3.60}$$

It is easy to get a good bound for F. Indeed, if $\{S'\}$ and $\{S'''\}$ are two independent copies of $\{S\}$, as before, then

$$\begin{aligned} P^{(\infty)}(F) &\le \sum_w P\{S'_\cdot \text{ and } w + S'''_\cdot \text{ meet at least } \sqrt{C}\log t \text{ times in } [0,t]\} \\ &\le (1-\gamma_d)^{(\sqrt{C}\log t - 1)} \sum_w P\{S'_\cdot \text{ and } w + S'''_\cdot \text{ meet at least once in } [0,t]\} \\ &\le C_8(t+1)(1-\gamma_d)^{(\sqrt{C}\log t-1)}, \end{aligned} \tag{3.61}$$

where the last inequality follows from (3.21).

To deal with G, let k be the smallest integer such that $t' := 2^k \log t \ge t$. Clearly $k \le 2\log t$. Now consider the intervals $I_0 := [0, \log t]$, $I_j := [2^{(j-1)}\log t, 2^j \log t]$, $j = 1, \dots, k$. If G occurs, there is a j such that $\widetilde{\pi}$ meets at least $\sqrt{C}(\log t)^2/3$ new particles in I_j. So, if H is the event that in each interval $I_j, 0 \le j \le k$, the fake clock of $\widetilde{\pi}$ rings at most $C^{1/4}|I_j|$ times, then

$$\begin{aligned} &P^{(\infty)}(G) \le P^{(\infty)}(H^c) \\ &+ \sum_{j=0}^k P^{(\infty)}\{H \text{ occurs and } \widetilde{\pi} \text{ meets at least } \sqrt{C}(\log t)^2/3 \text{ new particles during } I_j\}. \end{aligned} \tag{3.62}$$

To handle the second term of (3.62) we condition on the path and the fake clock of $\widetilde{\pi}$. So let the path of $\widetilde{\pi}$ and the $\widetilde{\sigma}_r$ be fixed such that H occurs. Denote the conditional probabilities, given these data, by P^* and the corresponding expectations by E^*. We have (by the upper bound for $E(t)$ in Lemma 3 and the independence of $\widetilde{\pi}$ of the other particles) that, for some constant D (which does not depend on C),

$$E^*\{\text{the number of new particles which } \widetilde{\pi} \text{ meets during } I_j\} \le D\,C^{1/4}\log\log t.$$

(In fact, in all intervals except I_0 we can even omit the factor $\log\log t$). So, by Markov's inequality,

$$P^*\{[\text{the number of new particles met by } \widetilde{\pi} \text{ during } I_j] \ge 2DC^{1/4}\log\log t\} \le 1/2.$$

Now note that any two new particles are necessarily disjoint. Therefore, if we denote $P^*\{\widetilde{\pi} \text{ meets at least } n \text{ new particles during } I_j\}$ by $P^*(j,n)$, then we have, by (2.13), for all j and all positive integers n, m,

$$P^*(j, n+m) \le P^*(j,n)P^*(j,m).$$

Hence, for large C,

$$P^*\{\widetilde{\pi} \text{ meets } \geq \sqrt{C}(\log t)^2/3 \text{ new particles during } I_j\}$$
$$\leq (1/2)^{\frac{\sqrt{C}(\log t)^2/3}{4DC^{1/4}\log t}} = (1/2)^{\frac{C^{1/4}\log t}{12D}}.$$

Hence (using $k \leq 2\log t$) the second term of (3.62) is at most

$$3\log t\,(1/2)^{\frac{C^{1/4}\log t}{12D}}. \tag{3.63}$$

Summarizing, we have that $P^{(\infty)}\{\widetilde{W} > C(\log t)^4\}$ is bounded by the sum of (3.61), the first term in the right hand side of (3.62) and (3.63). It is clear that, for C sufficiently large, (3.61) and (3.63) are both smaller than $1/t^4$. Further, since the number of rings of the fake clock of $\widetilde{\pi}$ during I_j has a Poisson distribution with mean $|I_j|$, it is well-known that for sufficiently large C the first term of (3.62) is also smaller than $1/t^4$. The lemma follows. ■

We return to the inequality (3.59) and the estimate for the second multiple sum in the right hand side of (3.55). By the definition of $\widetilde{W}$ and the fact that $\xi_s(x) \leq M$, we have

$$\widetilde{W} \leq M \sum_{r:\widetilde{\sigma}_r \leq t^\alpha} 1.$$

Since the $\widetilde{\sigma}_r$ are the jumptimes of a rate-one Poisson process, we have

$$\Big[E^{(\infty)}\{\big[\sum_{r:\widetilde{\sigma}_r \leq t^\alpha} 1\big]^2\}\Big]^{1/2} \leq C_9 t^\alpha$$

and

$$\Big[E^{(\infty)}\{[\widetilde{W} - C_4(\log t)^4]^2\}\Big]^{1/2} \leq MC_9 t^\alpha. \tag{3.64}$$

This holds for all choices of C_4. Finally, we have from the above lemma that we can choose C_4 so large that the last factor in the right hand side of (3.59) is at most $C_7^{1/2}t^{-2}$. By combining this with (3.59) and (3.64) (also see (3.21)) we find that the second multiple sum in (3.55) is (for a suitable choice of C_4) at most $C_{10}t^{5\alpha/3-2}\sum_z \beta^2(z)$. If we also take (3.56) into account we finally see that the contribution of LK_2 to (3.45) is at most

$$\Big[2C_6 t^{2\alpha/3-1}(\log t)^4 + C_{10}t^{5\alpha/3-2}\Big]\sum_z \beta^2(z). \tag{3.65}$$

Remark. So far we have exclusively dealt with bounds for the contribution to (3.6) of the terms containing $J(x,y,z)$. As remarked before, the terms containing $J'(x,y,z)$ can be treated in essentially the same way. Indeed, almost all the estimates for the terms coming from J' can be handled by interchanging x and y in the corresponding estimates for the terms coming from J (but one should not change the x in $\delta_{u,x}$ in (3.14) to y). There is, however, one term which needs extra arguments, to wit the term involving $\widetilde{E}L(y,z)\widetilde{E}K_1(x,y)$ when $s = \ell t/p \le t^\alpha$. We need a bound on

$$\liminf_{N\to\infty}\liminf_{p\to\infty}\sum_{\ell:\ell t/p\le t^\alpha}\int\prod_{i\le\ell}\mu_i(dW_i)\sum_{x,y}\widetilde{E}\big[\sum_z\beta^2(z)L(y,z)\big]\widetilde{E}K_1(x,y). \tag{3.66}$$

For this term it is desirable to replace the right hand side of (3.47) by $C_1\big[t^{-\alpha/3}\|y-x\|\wedge 1]$. This is clearly permissible, since $\widetilde{E}K_1$ is a probability. From this bound we see that the integral in (3.66) is at most

$$C_1\sum_z\beta^2(z)\sum_{x,y}P\{\text{a particle jumps from } x \text{ to } y \text{ during } ((\ell-1)t/p,\ell t/p] \\ \text{and ends at } z \text{ at time } t\}\Big(\frac{\|x-y\|}{t^{\alpha/3}}\wedge 1\Big).$$

After summing over ℓ such that $\ell t/p \le t^\alpha$, decomposing with respect to the original position of the particle which jumps from x to y during $((\ell-1)t/p,\ell t/p]$, we obtain the bound

$$t^{-\alpha/3}\sum_z\beta^2(z)\sum_w E\{T(w)I[\pi(w) \text{ is at } z \text{ at time } t]\}, \tag{3.67}$$

for the sum over ℓ in (3.66), where $\pi(w)$ is again the particle which starts at w, $\sigma_1 < \sigma_2 < \cdots$ the successive jumptimes of $\pi(w)$, y_r the corresponding values of these jumps, and finally

$$T(w) = C_1\sum_{\sigma_r\le t^\alpha}\big(\|y_r\|\wedge t^{\alpha/3}\big).$$

It is not difficult to see (from monotonicity arguments related to Lemma 1) that

$$\lim_{N\to\infty} E\{T(w)I[\pi(w) \text{ is at } z \text{ at time } t]\}$$

exists and is translation invariant (that is, remains the same when we replace w by $\mathbf{0}$ and z by $z-w$). In this way (and using a simple domination argument which allows to take the limit $N\to\infty$ inside the summation signs) it follows that (3.66) is at most

$$t^{-\alpha/3}\sum_z\beta^2(z)\lim_{N\to\infty} E\{T(\mathbf{0})I[\pi(\mathbf{0}) \text{ survives till time } t]\}. \tag{3.68}$$

For the remainder of this estimate we drop the argument $\mathbf{0}$ from the notation. To bound the right hand side of (3.68) we bound T itself by the sum of the following four terms:

$$T_1 := \sum_{r:\sigma_r \le t^\alpha} \big(\|y_r\| \wedge t^{\alpha/3}\big) I[\|y_r\| \ge t^{\alpha/3}],$$

$$T_2 := \sum_{r:\sigma_r \le t^\alpha} \big(\|y_r\| \wedge t^{\alpha/3}\big) I[\text{number of } r \text{ with } \sigma_r \le t^\alpha \text{ is } \ge 2t^\alpha],$$

$$T_3 := \sum_{v:\|v\|<t^{\alpha/3}} \|v\| q(v) (\text{number of } r \text{ with } \sigma_r \le t^\alpha) \\ \times I[\text{number of } r \text{ with } \sigma_r \le t^\alpha \text{ is } < 2t^\alpha],$$

$$T_4 := \Big| \sum_{r:\sigma_r \le t^\alpha} \Big(\|y_r\| I[\|y_r\| < t^{\alpha/3}] - \sum_{v:\|v\|<t^{\alpha/3}} \|v\| q(v) \Big) \Big| \\ \times I[\text{number of } r \text{ with } \sigma_r \le t^\alpha \text{ is } < 2t^\alpha].$$

It is quite straightforward to show from the finiteness of $\sum_v \|v\|^2 q(v)$ that

$$E\{T_1 I[\pi \text{ survives till time } t]\} \le t^{2\alpha/3-1} \tag{3.69}$$

for all large t, and (without using the above moment condition) that

$$E\{T_2 + T_3\} I[\pi \text{ survives till time } t]\} \le C_2 t^{\alpha-1}, \quad t \ge 1. \tag{3.70}$$

To estimate the contribution of T_4 we use that

$$\begin{aligned} &E\{T_4 I[\pi \text{ survives till time } t]\} \\ &\le t^\alpha P\{\pi \text{ survives till time } t\} + E\{T_4; T_4 > t^\alpha\} \\ &\le C_3 t^{\alpha-1} + \Big[E\{T_4^2\} P\{T_4 > t^\alpha\}\Big]^{1/2}. \end{aligned} \tag{3.71}$$

Since the sizes of the jumps of $\pi(\mathbf{0})$ are independent of the jump times, it holds that

$$E\{T_4^2\} \le 2t^\alpha \sum_v \|v\|^2 q(v). \tag{3.72}$$

Finally (again using the independence of the sizes and times of the jumps) a suitable application of Bernstein's inequality (see Chow and Teicher (1988), Exercise 4.3.14) shows that $P\{T_4 > t^\alpha\}$ is of (much) smaller order than $t^{\alpha-1}$

It follows from these estimates that, uniformly in N,

$$E\{T(\mathbf{0}) I[\pi(\mathbf{0}) \text{ survives till time } t] \le C_5 t^{\alpha-1}.$$

Consequently, the expression in (3.66) is at most

$$C_6 \sum_z \beta^2(z) t^{2\alpha/3-1},$$

which is of the same order as we found for the contribution of $L(x,z)K_1(x,y)$ in (3.49).

Final part of the proof of the Proposition. To conclude we take $\alpha = 42/57$. We then see that the contributions to the variance from the ℓ with $\ell t/p \geq t^\alpha$ (see (3.43)) and the contributions from the ℓ with $\ell t/p \leq t^\alpha$ (see (3.49) and (3.65)) are at most of order

$$t^{-29/57}(\log t)^4 \sum_z \beta^2(z),$$

so that (3.4) follows.

Once we have (3.4) we can, as pointed out in [BK], obtain (3.3) under (3.2) by using bounded convergence. This completes the proof of the proposition. ∎

4. An approximate differential equation for the expected number of particles per site.

In this section we start with one particle at each site ($\xi_0 = \mathbb{1}$) and we write ξ_t instead of $\xi_t(\mathbb{1})$. We also do not write the superscript (∞) to P and E in this section. We first derive a differential equation for $E(t)$.

Lemma 15. *$E(t)$ is differentiable and*

$$\frac{d}{dt}E(t) = -\sum_{x \in \mathbb{Z}^d} E\big\{\xi_t(\mathbf{0})q(x)p_{\xi_t(x)}\big\}. \tag{4.1}$$

Proof. This can be seen quite easily by a rather straightforward (first-order) book-keeping of the particle movements (and their effects) to and from $\mathbf{0}$ in a small time interval. See Lemma 9 in [BK] for details. ∎

The remainder of this section shows that (4.1) can be replaced by

$$\frac{d}{dt}E(t) = -C(d)(1 + o(1))E^2(t), \tag{4.2}$$

where $o(1) \to 0$ as $t \to \infty$. To this end we follow the heuristic outline of the introduction to approximate $E\{\xi_t(\mathbf{0})p_{\xi_t(x)}\}$ for $x \neq \mathbf{0}$. *Throughout we assume (1.5), (1.6), (1.7), (1.10) and $d \geq 3$.* We want the estimates to be uniform in $x \neq 0$.

Now let $\{S^{(x)}\}$, $x \in \mathbb{Z}^d$, be a collection of independent copies of $\{S\}$, and define

$$\rho(m, y) = P\{s \mapsto S_s^{(\mathbf{0})} \text{ and } s \mapsto -y + S_s^{(-y)} \text{ meet exactly } m \text{ times during } [0, \infty)\} \tag{4.3}$$

and

$$D(y) = p_1 \sum_{m=0}^{\infty} (1 - p_1)^m \rho(m, y). \tag{4.4}$$

We also define $\Lambda_t^*(u, v)$ as the number of ordered pairs of distinct particles, the first particle being present at u at time t, and the second particle at v at time t. Comparison with (2.16) shows immediately that $\Lambda_t^*(u, v) \leq \Lambda_t(u, v)$. We remind the reader that α_s was defined in (2.6).

Lemma 16. *Let $1 \leq \Delta \leq t/2$. Then for $d \geq 3$ there exists a $\delta(d) > 0$ such that uniformly in $y \neq \mathbf{0}$,*

$$\begin{aligned}
&\big|E\{\xi_t(\mathbf{0})p_{\xi_t(y)}\} - D(y) \sum_{u,v\in\mathbb{Z}^d} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y)\big| \\
&\leq C_1\Delta t^{-3} + C_1\Delta^{-\delta(d)}t^{-2}.
\end{aligned} \tag{4.5}$$

Proof. This lemma corresponds with Lemma 13 in [BK] and can be proven in a similar (but easier) way; $\delta(d)$ comes from Lemma 12 in [BK], which is a lemma about independent random walks and is valid for any $d \geq 3$. In Lemma 13 of [BK] it was required that $d \geq 5$, and instead of the first term in the right hand side of (4.5) we had $C_{25}\Delta[t^{-3} \vee t^{-d(1-\varepsilon)/2}]$. This difference is caused by the difference between our present Lemma 9 and its old analogue, Lemma 10 in [BK]. ■

Proof of Theorem. Let $d \geq 3$. Choose $\Delta = t^{1-\eta}$ with $0 < \eta < 1$ so small that

$$(1-\eta)\frac{d}{2} \geq \frac{3}{2} - \frac{\kappa}{2}, \tag{4.6}$$

with κ as in Proposition 13. Lemmas 15 and 16 then show that there exists some $\zeta = \zeta(d) \in (0, \eta \wedge \kappa/2)$ and some constant $C_3 < \infty$ such that

$$\big|\frac{d}{dt}E(t) + \sum_y q(y)D(y)\sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y)\big| \leq C_3 t^{-2-\zeta}. \tag{4.7}$$

In addition, by the definition of $\Lambda^*_{t-\Delta}(u,v)$,

$$\begin{aligned}
&\sum_{u,v} \Lambda^*_{t-\Delta}(u,v)\alpha_\Delta(u)\alpha_\Delta(v-y) \\
&= \sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v) - \sum_u \alpha_\Delta(u)\alpha_\Delta(u-y)\xi_{t-\Delta}(u).
\end{aligned}$$

Therefore, by (2.5), (3.3) and (2.7), there exists a constant C_4, independent of y

such that

$$\Big| \sum_{u,v} E\{\Lambda^*_{t-\Delta}(u,v)\}\alpha_\Delta(u)\alpha_\Delta(v-y)$$

$$- E\Big\{\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big\}E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Big|$$

$$\le \Big[\text{Var}\Big(\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big)\ \text{Var}\Big(\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big)\Big]^{1/2}$$

$$+ \frac{C_2}{t}\sum_u \alpha_\Delta(u)\alpha_\Delta(u-y)$$

$$\le C_0 t^{-1/2-\kappa}\sum_u \alpha^2_\Delta(u) + \frac{C_2}{t}\sup_u \alpha_\Delta(u)$$

$$\le C_4 \frac{t^{-1/2-\kappa}}{\Delta^{d/2}}. \tag{4.8}$$

Substitution of this estimate into (4.7) and use of (4.6) yields

$$\Big|\frac{d}{dt}E(t) + \sum_y q(y)D(y)E\Big\{\sum_u \alpha_\Delta(u)\xi_{t-\Delta}(u)\Big\}E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Big|$$

$$\le C_3 t^{-2-\zeta} + C_4 t^{-2-\kappa/2} \le C_5 t^{-2-\zeta}. \tag{4.9}$$

Moreover,

$$\sum_y q(y)D(y) = C(d). \tag{4.10}$$

Now for $\xi_t(y) \ne 0$ to occur, there must be at least one particle in the system at time $t-\Delta$ which moves to y during $[t-\Delta, t]$ without being removed. Similar arguments as for Lemma 16 (but easier) show that

$$\Big|E\xi_t(y) - E\Big\{\sum_v \alpha_\Delta(v-y)\xi_{t-\Delta}(v)\Big\}\Big|$$

$$\le \sum_v E\{\text{number of particles } \pi' \text{ which are at } v \text{ at time } t-\Delta$$

$$\text{and reach } y \text{ at time } t, \text{ but which do coincide with}$$

$$\text{some other particle } \pi \text{ during } [t-\Delta, t]\}$$

$$\le \sum_v E\{\Lambda_{t-\Delta}(v,v)\}\alpha_\Delta(v-y)$$

$$+ 2\sum_v \int_0^\Delta \sum_{z,z',w} E\{\Lambda_{t-\Delta}(v,w)\}\alpha_s(v-z)\alpha_s(w-z')q(z-z')\alpha_{\Delta-s}(z-y)ds$$

$$\le C_6 \Delta t^{-2} = C_6 t^{-1-\eta} \le C_6 t^{-1-\zeta}. \tag{4.11}$$

This estimate is uniform in $y \in \mathbb{Z}^d$, by translation invariance. Combined with (4.9), (4.10) and (2.5) this yields

$$\left|\frac{d}{dt}E(t) + C(d)E^2(t)\right| \le C_7 t^{-2-\zeta} \le C_8 t^{-\zeta} E^2(t), \quad t \ge 1.$$

Integration now gives

$$\frac{1}{E(t)} - \frac{1}{E(0)} = -\int_0^t E^{-2}(s)\frac{dE(s)}{ds}ds = C(d)t + O(t^{1-\zeta}),$$

from which (1.11) follows (see (2.18)).

References

Arratia, R. (1981), *Limiting point processes for rescalings of coalescing and annihilating random walks on* $\mathbb{Z}^d$, Ann. Probab. **9**, 909–936.

Arratia, R. (1983), *Site recurrence for annihilating random walks on* $\mathbb{Z}_d$, Ann. Probab. **11**, 706–713.

van den Berg, J. and Fiebig, U. (1987), *On a combinatorial conjecture concerning disjoint occurrences of events*, Ann. Probab. **15**, 354–374.

van den Berg, J. and Kesten, H. (2000), *Asymptotic density in a coalescing random walk model*, Ann. Probab. **28**, 303–352.

Bramson, M. and Griffeath, D., *Asymptotics for interacting particles systems on* $\mathbb{Z}^d$, Z. Wahrsch. verw. Geb. **53** (1980), 183–196.

Chow, Y. S. and Teicher, H. (1986), *Probability Theory*, 2^{nd} edition, Springer-Verlag.

Feller, W. (1968), *An Introduction to Probability Theory and its Aplications*, Vol I, 3rd ed., John Wiley & Sons.

Kesten, H. (2000), *Coalescing and annihilating random walk with'action at a distance'*, J. d'Anal. Math **80**, 183–256.

Liggett, T. M. (1985), *Interacting Particle Systems*, Springer-Verlag.

Reimer, D. (2000), *Proof of the van den Berg-Kesten conjecture*, Combin. Probab. Comput. **9**, 27–32.

Sawyer, S. (1979), *A limit theorem for patch sizes in a selectively neutral immigration model*, J. Appl. Probab. **16**, 482–495.

Spitzer, F. (1976), *Principles of Random Walk*, 2^{nd} edition, Springer-Verlag.

Stephenson, D. M. (2001), *Asymptotic density in an threshold coalescing and annihilating random walk*, Ann. Probab. **29**, 137-175.

Uchiyama, K. (1998), *Green's functions for random walks on* $\mathbb{Z}^N$, Proc. London Math. Soc. (3) **77**, 215–240.