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Some Properties of Generalized Pickands Constants

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ABSTRACT
We study properties of *generalized Pickands constants* $\mathcal{H}_\eta$, that appear in the extreme value theory of Gaussian processes and are defined via the limit

$$\mathcal{H}_\eta = \lim_{T\to\infty} \frac{\mathcal{H}_\eta(T)}{T},$$

where $\mathcal{H}_\eta(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]}\left(\sqrt{2}\eta(t) - \mathrm{Var}(\eta(t))\right)\right)$ and $\eta(t)$ is a centered Gaussian process with stationary increments.

We give estimates of the rate of convergence of $\frac{\mathcal{H}_\eta(T)}{T}$ to $\mathcal{H}_\eta$ and prove that if $\eta_{(n)}(t)$ weakly converges in $C([0,\infty))$ to $\eta(t)$, then under some weak conditions $\lim_{n\to\infty}\mathcal{H}_{\eta_{(n)}} = \mathcal{H}_\eta$.

As an application we prove that $\Upsilon(\alpha) = \mathcal{H}_{B_{\alpha/2}}$ is continuous on $(0,2]$, where $B_{\alpha/2}(t)$ is a fractional Brownian motion with Hurst parameter $\alpha/2$. It contradicts the conjecture that $\Upsilon(\alpha)$ is discontinuous at $\alpha = 1$.

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1 Introduction

Pickands constants $\mathcal{H}_{B_{\alpha/2}}$ play an important role in the asymptotic of supremum of Gaussian stochastic processes (see Pickands [5], Piterbarg [6]). Recall that $\mathcal{H}_{B_{\alpha/2}}$ is defined by the following limit

$$\mathcal{H}_{B_{\alpha/2}} = \lim_{T\to\infty} \frac{\mathcal{H}_{B_{\alpha/2}}(T)}{T}, \tag{1.1}$$

where

$$\mathcal{H}_{B_{\alpha/2}}(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]} \sqrt{2}B_{\alpha/2}(t) - \mathrm{Var}(B_{\alpha/2}(t))\right)$$

and $B_{\alpha/2}(t)$ is a fractional Brownian motion (**FBM**) with Hurst parameter $\alpha/2$, that is a centered Gaussian process with stationary increments, continuous sample paths, $B_{\alpha/2}(0) = 0$ and variance function $\mathrm{Var}(B_{\alpha/2}(t)) = t^{\alpha}$ $(\alpha \in (0,2])$.

It turns out that the notion of $\mathcal{H}_{B_{\alpha/2}}$ yields a natural extension (Dębicki [2]). Let $\eta(t)$ be a centered Gaussian process with stationary increments, a.s. continuous sample paths, $\eta(0) = 0$ and such that the variance function $\mathrm{Var}(\eta(t)) = \sigma_\eta^2(t)$ satisfies

C1 $\sigma_\eta^2(t) \in C^1([0,\infty))$ is strictly increasing and there exists $\epsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{t\dot{\sigma}_\eta^2(t)}{\sigma_\eta^2(t)} \leq \epsilon; \tag{1.2}$$

C2 $\sigma_\eta^2(t)$ is regularly varying at 0 with index $\alpha_0 \in (0,2]$ and $\sigma_\eta^2(t)$ is regularly varying at ∞ with index $\alpha_\infty \in (0,2)$.

In the paper we use the notation $\dot{\sigma}^2(t)$ or $\ddot{\sigma}^2(t)$ for the derivatives of $\sigma^2(t)$. Moreover, in order to short the notation, we will write that $\eta(t) \in$ **C1-C2** provided that $\eta(t)$ is a centered Gaussian stochastic process with stationary increments and variance function that satisfies **C1-C2**.

By the *generalized Pickands constant* $\mathcal{H}_\eta$ we mean

$$\mathcal{H}_\eta = \lim_{T\to\infty} \frac{\mathcal{H}_\eta(T)}{T}, \tag{1.3}$$

where

$$\mathcal{H}_\eta(T) = \mathbb{E}\exp\left(\max_{t\in[0,T]} \sqrt{2}\eta(t) - \mathrm{Var}(\eta(t))\right)$$

and $\eta(t) \in$ **C1-C2**. In Dębicki [2] it was proved that if $\eta(t) \in$ **C1-C2**, then $\mathcal{H}_\eta$ is well-defined, finite and positive. Meanwhile it turned out that generalized Pickands constants appear in the exact asymptotics of supremum of some classes of Gaussian processes (see [2],[3]).

Both from the theoretical point of view and in spite of recent attempts to find numerical approximations of $\mathcal{H}_\eta$, there is a great need of analyzing properties of $\mathcal{H}_\eta$. In the case of classical Pickands constants the afford to compute $\mathcal{H}_{B_{\alpha/2}}$ was taken by many authors, but only few fragmentary results were obtained. Namely the exact value of $\mathcal{H}_{B_{\alpha/2}}$ is known only if $\alpha = 1$ and $\alpha = 2$ (see for example Piterbarg [6] or Bräker [1]). Recently Shao [7] and Dębicki, Michna & Rolski [3] obtained some bounds of $\mathcal{H}_{B_{\alpha/2}}$. Upper and lower bounds for generalized Pickands constants were given in Dębicki [2] and Dębicki, Michna & Rolski [3].

The difficulties in analyzing $\mathcal{H}_\eta$ are manifested in the number of (sometimes opposite) conjectures that deal with properties of the function

$$\Upsilon(\alpha) = \mathcal{H}_{B_{\alpha/2}}, \qquad \alpha \in (0, 2].$$

For example in Michna [4] it was formulated a conjecture (based on some numerical approximations):

Conjecture 1 $\Upsilon(\alpha)$ is discontinuous at $\alpha = 1$.

Opposite, in some sense, conjecture is also known:

Conjecture 2 $\Upsilon(\alpha) = \frac{1}{\Gamma(1/\alpha)}$, where $\Gamma(\cdot)$ is the Gamma function.

In this paper we focus on continuity and limit properties of $\mathcal{H}_\eta$. After introducing basic notation (Section 2), we give some estimates of the rate of convergence of $\frac{\mathcal{H}_\eta(T)}{T}$ to $\mathcal{H}_\eta$, as $T \to \infty$ (Section 3). In particular, using Theorem 3.1, we prove that the asymptotic rate of convergence is almost linear (Corollary 3.1).

Theorem 3.1 enables us to analyze some properties of a sequence of generalized Pickands constants $\mathcal{H}_{\eta_{(n)}}$, where $\eta_{(n)}(t) \in$ **C1-C2** and weakly converges in $C([0,\infty))$ to a Gaussian process $\eta(t) \in$ **C1-C2**. Namely under some mild conditions on the variance functions of $\eta_{(n)}(t)$ we prove that $\lim_{n\to\infty} \mathcal{H}_{\eta_{(n)}} = \mathcal{H}_\eta$ (Theorem 4.1).

Theorem 4.1 opens a way to the analysis of properties of $\Upsilon(\alpha)$. In particular in Corollary 4.1 we contradict Conjecture 1, proving that $\Upsilon(\alpha)$ is continuous as the function of $\alpha \in (0, 2]$. Moreover, in Corollary 4.2, we obtain some limit properties of $\mathcal{H}_\eta$ for $\eta(t) = \int_0^t Z(s)ds$, where $Z(s)$ is a centered stationary Gaussian process with continuous covariance function $R(t)$ and $\eta(t) \in$ **C1-C2**.

2 Notation and preliminary results

We write $\{\eta_u(t), u > 0\}$ for the family of centered Gaussian processes $\{\eta_u(t) : t \geq 0\}$ $(u > 0)$, where $\eta_u(t) = \eta(u + t)$ and $\eta(t) \in$ **C1-C2**. This family will play the central role in the technique of proofs presented in this article.

By the attached bar we always denote the standardized process, that is $\overline{\eta}(t) = \eta(t)/\sigma_\eta(t)$.

Before we state the main results, we need to point out some properties of family $\{\eta_u(t), u > 0\}$.

Proposition 2.1 *If* $\eta(t) \in$ **C1-C2**, *then*
(i)

$$\sup_{s,t\in J(u)} \left| \frac{1 - \mathbf{Cov}(\overline{\eta}_u(t), \overline{\eta}_u(s))}{\frac{\sigma_\eta^2(|t-s|)}{2\sigma_\eta^2(u)}} - 1 \right| \to 0$$

as $u \to \infty$, *where* $J(u) = [-\Delta(u), \Delta(u)]$ *and* $\Delta(u)$ *is such that* $\lim_{u\to\infty} \frac{\Delta(u)}{u} = 0$;
(ii) there exists $\Lambda > 1$ *such that*

$$\sup_{t>0} \frac{\sigma_\eta^2(t)}{\sigma_\eta^2(\Lambda t)} \leq \frac{1}{2}. \tag{2.1}$$

Proof. (i) is a special case of Lemma 2.1 in Dębicki [2]. (ii) is a consequence of the assumption that $\sigma_\eta^2(t)$ is regularly varying at 0 and at ∞ and the fact that $\sigma_\eta^2(t)$ is strictly increasing.

□

The following constants will play an important role in further analysis. Let

$$\mathcal{H}_{\eta,\eta}(T) = \mathbb{E}\exp\left(\max_{(t_1,t_2)\in[0,T]^2}\left(\eta_{(1)}(t_1)+\eta_{(2)}(t_2)-\frac{1}{2}\left(\sigma^2_{\eta_{(1)}}(t_1)+\sigma^2_{\eta_{(2)}}(t_2)\right)\right)\right), \quad (2.2)$$

where $\eta_{(1)}(t)$, $\eta_{(2)}(t)$ are independent copies of a centered Gaussian stochastic process $\eta(t)$ such that $\eta(t) \in$ **C1-C2**. Analogously we introduce $\eta_{(1);u}(t)$, $\eta_{(2);u}(t)$ as independent copies of $\eta_u(t)$.

Proposition 2.2 *Let $n(u)$ be such that $\lim_{u\to\infty}\frac{\sqrt{2}\sigma_\eta(u)}{n(u)} = 1$. Then*

$$\mathbb{P}\left(\sup_{(t_1,t_2)\in[0,T]^2}\frac{1}{\sqrt{2}}\left(\overline{\eta}_{(1);u}(t_1)+\overline{\eta}_{(2);u}(t_2)\right) > n(u)\right) = \mathcal{H}_{\eta,\eta}(T)\Psi(n(u))(1+o(1)) \quad \text{as} \quad u\to\infty.$$

Following the same argumentation as presented in the proof of Corollary D.1 in [6] (see also [2]) we recall some basic properties of $\mathcal{H}_\eta(\cdot)$ and $\mathcal{H}_{\eta,\eta}(\cdot)$.

Proposition 2.3 *If $\eta(t) \in$ **C1-C2**, then*
(i)$\mathcal{H}_\eta(T) \le T\mathcal{H}_\eta(1)$ for $T \in \mathbb{N}$;
(ii) $\mathcal{H}_{\eta,\eta}(T) \le T^2\mathcal{H}_{\eta,\eta}(1)$ for $T \in \mathbb{N}$;
(iii) $\mathcal{H}_\eta(\cdot)$ is subadditive.

3 Estimates of rate of convergence to $\mathcal{H}_\eta$

In this section we find some estimates of the rate of convergence of $\frac{\mathcal{H}_\eta(T)}{T}$ to $\mathcal{H}_\eta$ as $T\to\infty$. The main theorem of this section is the following upper bound.

Theorem 3.1 *Let $\eta(t) \in$ **C1-C2** and $f:\mathbb{R}^+\to\mathbb{R}^+$ be such that $f(x) < x$ for each $x \ge 1$. If $\Lambda > 0$ is such that $\sup_{t>0}\frac{\sigma_\eta^2(t)}{\sigma_\eta^2(\Lambda t)} \le \frac{1}{2}$, then for each $T \in \mathbb{N}$*

$$\begin{aligned}\left|\mathcal{H}_\eta - \frac{\mathcal{H}_\eta(T)}{T}\right| &\le 2\frac{\mathcal{H}_\eta(f(T))}{T} + 2\frac{\mathcal{H}_{\eta,\eta}(\Lambda^5 T)}{T}\exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) + \\ &\quad + 2\frac{\mathcal{H}_{\eta,\eta}(\Lambda^5 T)}{T}\sum_{i=1}^{\infty}\exp\left(-\frac{\sigma_\eta^2(iT)}{8}\right). \end{aligned} \quad (3.1)$$

Proof. The complete proof of Theorem 3.1 is presented in Section 5.

□

As a consequence of Theorem 3.1 we obtain the following corollary.

Corollary 3.1 *If $\eta(t) \in$ **C1-C2** and $\alpha < 1$, then*

$$\lim_{T\to\infty}\left|\frac{\mathcal{H}_\eta(T)}{T} - \mathcal{H}_\eta\right| T^\alpha = 0. \quad (3.2)$$

Proof. Since $\eta(t) \in$ **C1-C2**, then there exists $\Lambda > 0$ such that (2.1) is satisfied. Let $\alpha < 1$. We take $\epsilon > 0$ such that $\alpha + \epsilon < 1$ and $f(x) = x^{1-\alpha-\epsilon}$. Moreover let T_0 be such that $\min(\Lambda^5 T, f(T)) > 1$ for $T > T_0$. From Proposition 2.3 we have

$$\begin{aligned} \mathcal{H}_{\eta,\eta}(\Lambda^5 T) &\leq \mathcal{H}_{\eta,\eta}(1)(\Lambda^5 T + 1)^2 \\ \mathcal{H}_{\eta}(f(T)) &\leq \mathcal{H}_{\eta}(1)(f(T) + 1) \end{aligned}$$

for $T > T_0$. Thus, using Theorem 3.1, for $T > T_0$

$$\begin{aligned} \left|\mathcal{H}_\eta - \frac{\mathcal{H}_\eta(T)}{T}\right| &\leq 2\mathcal{H}_\eta(1)\frac{T^{1-\alpha-\epsilon}+1}{T} + 2\mathcal{H}_{\eta,\eta}(1)\frac{(\Lambda^5 T+1)^2}{T}\exp\left(-\frac{\sigma_\eta^2(T^{1-\alpha-\epsilon})}{8}\right) + \\ &+2\mathcal{H}_{\eta,\eta}(1)\frac{(\Lambda^5 T+1)^2}{T}\sum_{i=1}^{\infty}\exp\left(-\frac{\sigma_\eta^2(iT)}{8}\right). \end{aligned}$$

Since

$$\frac{(\Lambda^5 T+1)^2}{T}\exp\left(-\frac{\sigma_\eta^2(T^{1-\alpha-\epsilon})}{8}\right) + \frac{(\Lambda^5 T+1)^2}{T}\sum_{i=1}^{\infty}\exp\left(-\frac{\sigma_\eta^2(iT)}{8}\right) = o(T^{-\alpha-\epsilon})$$

as $T \to \infty$, then

$$\left|\mathcal{H}_\eta - \frac{\mathcal{H}_\eta(T)}{T}\right| T^\alpha \to 0$$

as $T \to \infty$. This completes the proof.

□

4 A convergence theorem

In this section we present theorem that enables us to analyze properties of a sequence of generalized Pickands constants for weakly converging Gaussian processes.

Let $\zeta(t) \in$ **C1-C2** and $\{\zeta_{(n)}(t)\}_{n=1}^{\infty}$ be a sequence of a centered Gaussian processes with stationary increments such that $\zeta_{(n)}(t) \in$ **C1-C2** for each $n \in \mathbb{N}$ and

W1 The sequence $\{\zeta_{(n)}(t)\}_{n=1}^{\infty}$ weakly converges in $C([0,\infty))$ to $\zeta(t)$;

W2 There exists $\Lambda > 0$ such that $\sup_{t>0} \frac{\sigma^2_{\zeta_{(n)}}(t)}{\sigma^2_{\zeta_{(n)}}(\Lambda t)} \leq \frac{1}{2}$ for each $n \in \mathbb{N}$.

In order to short the notation, if $\zeta(t)$ and $\{\zeta_{(n)}(t)\}_{n=1}^{\infty}$ satisfy **W1-W2**, then we will write $(\{\zeta_{(n)}\}, \zeta) \in$ **W1-W2**.

Let

$$\phi_{\zeta,T}(x) = \mathbb{E}\exp\left(x \sup_{t\in[0,T]}(\sqrt{2}\zeta(t) - \sigma_\zeta^2(t))\right)$$

be the moment generating function of random variable $\sup_{t\in[0,T]}(\sqrt{2}\zeta(t) - \sigma_\zeta^2(t))$. Note that $\mathcal{H}_\zeta(T) = \phi_{\zeta,T}(1)$.

Theorem 4.1 *If* $(\{\zeta_{(n)}\}, \zeta) \in$ **W1-W2**, *then*

$$\lim_{n\to\infty}\mathcal{H}_{\zeta_{(n)}} = \mathcal{H}_\zeta.$$

Proof. Since the functional $\sup_{t\in[0,T]} g(t)$ is continuous in uniform metric, then from **W1**

$$\lim_{n\to\infty} \phi_{\zeta_{(n)},T}(1) = \phi_{\zeta,T}(1)$$

for each $T > 0$ and hence

$$\lim_{n\to\infty} \mathcal{H}_{\zeta_{(n)}}(T) = \mathcal{H}_{\zeta}(T) \tag{4.1}$$

for each $T > 0$. Thus in order to complete the proof it suffices to show that $\frac{\mathcal{H}_{\zeta_{(n)}}(T)}{T} \to \mathcal{H}_{\zeta_{(n)}}$, as $T \to \infty$, uniformly with respect to n.

Due to **W2** there exists an universal constant $\Lambda > 1$ such that (2.1) is satisfied for $\zeta_{(n)}(t)$, $n \geq 1$. Hence takeing $f(x) = \sqrt{x}$ in Theorem 3.1 we have

$$\begin{aligned} \left|\mathcal{H}_{\zeta_{(n)}} - \frac{\mathcal{H}_{\zeta_{(n)}}(T)}{T}\right| &\leq 2\frac{\mathcal{H}_{\zeta_{(n)}}(\sqrt{T})}{T} + 2\frac{\mathcal{H}_{\zeta_{(n)},\zeta_{(n)}}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma^2_{\zeta_{(n)}}(\sqrt{T})}{8}\right) + \\ &+2\frac{\mathcal{H}_{\zeta_{(n)},\zeta_{(n)}}(\Lambda^5 T)}{T} \sum_{i=1}^{\infty} \exp\left(-\frac{\sigma^2_{\zeta_{(n)}}(iT)}{8}\right) \end{aligned} \tag{4.2}$$

for $T \in \mathbb{N}$. From Proposition 2.3 we get

$$\begin{aligned} \mathcal{H}_{\zeta_{(n)}}(\sqrt{T}) &\leq (\sqrt{T}+1)\mathcal{H}_{\zeta_{(n)}}(1) \\ \mathcal{H}_{\zeta_{(n)},\zeta_{(n)}}(\Lambda^5 T) &\leq (\Lambda^5 T+1)^2 \mathcal{H}_{\zeta_{(n)},\zeta_{(n)}}(1). \end{aligned}$$

Combining it with the fact that $\lim_{n\to\infty} \mathcal{H}_{\zeta_{(n)}}(1) = \mathcal{H}_{\zeta}(1)$ and $\lim_{n\to\infty} \mathcal{H}_{\zeta_{(n)},\zeta_{(n)}}(1) = \mathcal{H}_{\zeta,\zeta}(1)$ we obtain

$$\mathcal{H}_{\zeta_{(n)}}(\sqrt{T}) \leq 2(\sqrt{T}+1)\mathcal{H}_{\zeta}(1) \tag{4.3}$$

$$\mathcal{H}_{\zeta_{(n)},\zeta_{(n)}}(\Lambda^5 T) \leq 2(\Lambda^5 T+1)^2 \mathcal{H}_{\zeta,\zeta}(1) \tag{4.4}$$

for sufficiently large n. Moreover, following **W2**, we have

$$\sigma^2_{\zeta_{(n)}}(\sqrt{T}) \geq \frac{1}{2} T^{\frac{\log_\Lambda(2)}{2}}\ \sigma^2_{\zeta_{(n)}}(1)$$

and

$$\sigma^2_{\zeta_{(n)}}(iT) \geq \frac{1}{2}(iT)^{\log_\Lambda(2)}\ \sigma^2_{\zeta_{(n)}}(1).$$

Now, using that $\lim_{n\to\infty} \sigma^2_{\zeta_{(n)}}(1) = \sigma^2_{\zeta}(1)$, we obtain that

$$\sigma^2_{\zeta_{(n)}}(\sqrt{T}) \geq T^{\frac{\log_\Lambda(2)}{2}}\ \sigma^2_{\zeta}(1)$$

and

$$\sigma^2_{\zeta_{(n)}}(iT) \geq (iT)^{\log_\Lambda(2)}\ \sigma^2_{\zeta}(1)$$

for sufficiently large n, which implies that

$$\exp\left(-\frac{\sigma^2_{\zeta_{(n)}}(\sqrt{T})}{8}\right) \leq \exp\left(-T^{\frac{\log_\Lambda(2)}{2}}\ \frac{\sigma^2_{\zeta}(1)}{8}\right) \tag{4.5}$$

$$\sum_{i=1}^{\infty} \exp\left(-\frac{\sigma^2_{\zeta_{(n)}}(iT)}{8}\right) \leq \sum_{i=1}^{\infty} \exp\left(-(iT)^{\log_\Lambda(2)}\ \frac{\sigma^2_{\zeta}(1)}{8}\right) \tag{4.6}$$

for sufficiently large n.

Combining (4.2) with (4.3),(4.4),(4.5) and (4.6) we obtain

$$\left|\mathcal{H}_{\zeta_{(n)}} - \frac{\mathcal{H}_{\zeta_{(n)}}(T)}{T}\right| \leq 4\frac{(\sqrt{T}+1)\mathcal{H}_{\zeta}(1)}{T} + 4\frac{(\Lambda^5 T+1)^2\mathcal{H}_{\zeta,\zeta}(1)}{T}\exp\left(-T^{\frac{\log_\Lambda(2)}{2}}\frac{\sigma^2_\zeta(1)}{8}\right) + +4\frac{(\Lambda^5 T+1)^2\mathcal{H}_{\zeta,\zeta}(1)}{T}\sum_{i=1}^{\infty}\exp\left(-(iT)^{\log_\Lambda(2)}\frac{\sigma^2_\zeta(1)}{8}\right)$$

for $T \in \mathbb{N}$ and suffciently large n. Hence we proved that $\frac{\mathcal{H}_{\zeta_{(n)}}(T)}{T} \to \mathcal{H}_{\zeta_{(n)}}$, as $T \to \infty$, uniformly with respect to n. Thus

$$\begin{aligned}\lim_{n\to\infty}\mathcal{H}_{\zeta_{(n)}} &= \lim_{n\to\infty}\lim_{T\to\infty}\frac{\mathcal{H}_{\zeta_{(n)}}(T)}{T} = \\ &= \lim_{T\to\infty}\lim_{n\to\infty}\frac{\mathcal{H}_{\zeta_{(n)}}(T)}{T} \\ &= \lim_{T\to\infty}\frac{\mathcal{H}_{\zeta}(T)}{T} = \mathcal{H}_\zeta.\end{aligned}$$

This completes the proof.

□

Theorem 4.1 enables us to obtain the following result, which contradicts Conjecture 1.

Corollary 4.1 *Function* $\Upsilon(\alpha) = \mathcal{H}_{B_{\alpha/2}}$ *is continuous on* $(0,2]$.

Proof. It suffices to prove that if $\alpha_n \to \alpha$, then

$$\lim_{n\to\infty}\mathcal{H}_{B_{\alpha_n/2}} = \mathcal{H}_{B_{\alpha/2}},$$

where $\alpha_n, \alpha \in (0,2]$

Since for sufficiently large n we have $(\{B_{\alpha_n/2}\}, B_{\alpha/2}) \in$ **W1-W2** with $\Lambda = 2^{\frac{2}{\alpha}}$, then applying Theorem 4.1 we obtain the thesis.

□

Corollary 4.2 *If* $\eta(t) = \int_0^t Z(s)ds$*, where* $Z(s)$ *is a stationary centered Gaussian process with a continuous covariance function* $\mathbf{Cov}(Z(s+t), Z(s)) = R(t)$ *and* $\eta(t) \in$ **C1-C2**, *then*

$$\lim_{c\to\infty}\frac{\mathcal{H}_{c\eta}}{c} = \sqrt{\frac{R(0)}{\pi}}.$$

Proof. Let $\eta_{(c)}(t) = \int_0^t Z\left(\frac{s}{c}\right)ds$ and note that for each $c > 0$ we have $c\eta(t) = \eta_{(c)}(ct)$. Thus

$$\begin{aligned}\mathcal{H}_{c\eta} &= \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\exp\left(\max_{t\in[0,T]}\sqrt{2}c\eta(t) - \sigma^2_{c\eta}(t)\right) = \\ &= \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\exp\left(\max_{t\in[0,T]}\sqrt{2}\eta_{(c)}(ct) - \sigma^2_{\eta_{(c)}}(ct)\right) \\ &= c\lim_{T\to\infty}\frac{1}{cT}\mathbb{E}\exp\left(\max_{t\in[0,cT]}\sqrt{2}\eta_{(c)}(t) - \sigma^2_{\eta_{(c)}}(t)\right) \\ &= c\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\exp\left(\max_{t\in[0,T]}\sqrt{2}\eta_{(c)}(t) - \sigma^2_{\eta_{(c)}}(t)\right) = c\mathcal{H}_{\eta_{(c)}}.\end{aligned}$$

On the other hand, using that $\sigma^2_{\eta_{(c)}}(t) = c^2\sigma^2_\eta(\frac{t}{c})$, we have

$$\frac{\sigma^2_{\eta_{(c)}}(t)}{\sigma^2_{\eta_{(c)}}(\Lambda t)} = \frac{c^2\sigma^2_\eta(\frac{t}{c})}{c^2\sigma^2_\eta(\frac{\Lambda t}{c})} = \frac{\sigma^2_\eta(\frac{t}{c})}{\sigma^2_\eta(\frac{\Lambda t}{c})}.$$

Thus, if $\Lambda > 0$ is such that (2.1) is satisfied for $\eta(t)$, then (2.1) is satisfied with the same Λ also for each $\eta_{(c)}(t)$.

Since $\eta_{(c)}(t)$ weakly converges in $C([0,\infty))$ to the process $\sqrt{R(0)}B_1(t) = \sqrt{R(0)}\mathcal{N}t$ as $c \to \infty$, where $\mathcal{N}$ has the standard normal law, then from Theorem 4.1 we conclude that

$$\lim_{c\to\infty} \frac{\mathcal{H}_{c\eta}}{c} = \lim_{c\to\infty} \mathcal{H}_{\eta_{(c)}} = \mathcal{H}_{\sqrt{R(0)}B_1},$$

which combined with the fact that $\mathcal{H}_{\sqrt{R(0)}B_1} = \sqrt{\frac{R(0)}{\pi}}$ completes the proof. □

5 Proof of Theorem 3.1

Before presenting the proof of Theorem 3.1, we need the following lemma, which slightly extends Lemma D.1 from Piterbarg [6], giving the exact form of the constants before the exponent.

Lemma 5.1 *If $\eta(t) \in$ **C1-C2** and $\Lambda > 0$ is such that $\sup_{t>0} \frac{\sigma^2_\eta(t)}{\sigma^2_\eta(\Lambda t)} \le \frac{1}{2}$, then for $\Delta(u)$ such that and $\lim_{u\to\infty} \frac{\Delta(u)}{u} = 0$ and for each $T > 0$, $\delta > 0$*

$$\mathbb{P}\left(\sup_{s\in[0,T]} \overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u); \sup_{t\in[\delta+T,\delta+2T]} \overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right) \le$$
$$\le \mathcal{H}_{\eta,\eta}(\Lambda^5 T)\exp\left(-\frac{\sigma^2_\eta(\delta)}{8}\right)\Psi(\sqrt{2}\sigma_\eta(u))(1+o(1)) \tag{5.1}$$

as $u \to \infty$. Inequality (5.1) holds uniformly with respect to u for $\delta \le \Delta(u) - 2T$.

Proof. Since the idea of the proof is analogous to the proof of Lemma 6.3 in [6] then we present only the main steps of the argumentation. Consider the Gaussian field $\boldsymbol{Y}_u(s,t) = \overline{\eta}_u(s) + \overline{\eta}_u(t)$ and let $A_0 = [0,T]$, $A_{\delta+T} = [\delta+T, \delta+2T]$ for $0 \le \delta \le \Delta(u) - 2T$. We have

$$\mathbb{P}\left(\sup_{t\in[0,T]} \overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{t\in[\delta+T,\delta+2T]} \overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right) \le$$
$$\le \mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \boldsymbol{Y}_u(s,t) > 2\sqrt{2}\sigma_\eta(u)\right). \tag{5.2}$$

Note that for each $s \in A_0$, $t \in A_{\delta+T}$ and sufficiently large u

$$\mathrm{Var}(\boldsymbol{Y}_u(s,t)) \ge 4 - 2\frac{\sigma^2_\eta(|t-s|)}{\sigma^2_\eta(u)} \ge 2 \tag{5.3}$$

and

$$\mathrm{Var}(\boldsymbol{Y}_u(s,t)) \le 4 - \frac{\sigma_\eta^2(|t-s|)}{2\sigma_\eta^2(u)} \le 4 - \frac{\sigma_\eta^2(\delta)}{2\sigma_\eta^2(u)}. \tag{5.4}$$

Let $\overline{\mathbf{Y}}_u(s,t) = \frac{\mathbf{Y}_u(s,t)}{\sqrt{\mathrm{Var}(\mathbf{Y}_u(s,t))}}$ and observe that

$$\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \mathbf{Y}_u(s,t) > \sqrt{2}\sigma_\eta(u)\right) \le \mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \overline{\mathbf{Y}}_u(s,t) > \frac{2\sqrt{2}\sigma_\eta(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{2\sigma_\eta^2(u)}}}\right). \tag{5.5}$$

Moreover for each $s, s_1 \in A_0$ and $t, t_1 \in A_{\delta+T}$

$$\begin{aligned}
\mathbb{E}(\overline{\mathbf{Y}}_u(s,t) - \overline{\mathbf{Y}}_u(s_1,t_1))^2 &\le \frac{4}{\mathrm{Var}(\boldsymbol{Y}_u(s,t))}\mathbb{E}(\boldsymbol{Y}_u(s,t) - \boldsymbol{Y}_u(s_1,t_1))^2 \\
&\le 4(\mathbb{E}(\overline{\eta}_u(s) - \overline{\eta}_u(s_1))^2 + \mathbb{E}(\overline{\eta}_u(t) - \overline{\eta}_u(t_1))^2) \\
&= 8\left((1 - \mathbf{Cov}(\overline{\eta}_u(s), \overline{\eta}_u(s_1))) + (1 - \mathbf{Cov}(\overline{\eta}_u(t), \overline{\eta}_u(t_1)))\right) \\
&\le 16\left(\frac{\sigma_\eta^2(|s-s_1|)}{2\sigma_\eta^2(u)} + \frac{\sigma_\eta^2(|t-t_1|)}{2\sigma_\eta^2(u)}\right) \qquad (5.6) \\
&\le \frac{1}{2}\left(\frac{\sigma_\eta^2(\Lambda_0|s-s_1|)}{2\sigma_\eta^2(u)} + \frac{\sigma_\eta^2(\Lambda_0|t-t_1|)}{2\sigma_\eta^2(u)}\right) \qquad (5.7) \\
&\le ((1 - \mathbf{Cov}(\overline{\eta}_u(\Lambda_0 s), \overline{\eta}_u(\Lambda_0 s_1))) + (1 - \mathbf{Cov}(\overline{\eta}_u(\Lambda_0 t), \overline{\eta}_u(\Lambda_0 t_1)))) \\
&= \frac{1}{2}(\mathbb{E}(\overline{\eta}_u(\Lambda_0 s) - \overline{\eta}_u(\Lambda_0 s_1))^2 + \mathbb{E}(\overline{\eta}_u(\Lambda_0 t) - \overline{\eta}_u(\Lambda_0 t_1))^2),
\end{aligned}$$

where (5.6) follows from Proposition 2.1 and in (5.7) $\Lambda_0 = \Lambda^5$ and follows from (2.1). Hence for $\overline{\eta}_{(1);u}(t), \overline{\eta}_{(2);u}(t)$ being independent copies of the process $\overline{\eta}_u(t)$, the covariance function of the process $\frac{1}{\sqrt{2}}(\overline{\eta}_{(1);u}(\Lambda_0 s) + \overline{\eta}_{(2);u}(t)(\Lambda_0 t))$ is dominated by the covariance function of $\overline{\mathbf{Y}}_u(s,t)$. Thus from Slepian inequality (see [6], Theorem C.1)

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in A_0\times A_{\delta+T}} \overline{\mathbf{Y}}_u(s,t) > \frac{\sqrt{2}\sigma_\eta(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{2\sigma_\eta^2(u)}}}\right) \le \\
&\qquad \le \mathbb{P}\left(\sup_{(s,t)\in A_0^2} \frac{1}{\sqrt{2}}(\eta_{(1);u}(\Lambda_0 s) + \eta_{(2);u}(\Lambda_0 t)) > \frac{\sqrt{2}\sigma_\eta(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{2\sigma_\eta^2(u)}}}\right) \qquad (5.8) \\
&\qquad = \mathcal{H}_{\eta,\eta}(\Lambda^5 T)\Psi\left(\frac{\sqrt{2}\sigma_\eta(u)}{\sqrt{4-\frac{\sigma_\eta^2(\delta)}{2\sigma_\eta^2(u)}}}\right)(1+o(1)) \qquad (5.9) \\
&= \mathcal{H}_{\eta,\eta}(\Lambda^5 T)\exp\left(-\frac{\sigma_\eta^2(\delta)}{8}\right)\Psi(\sqrt{2}\sigma_\eta(u))(1+o(1)) \qquad (5.10)
\end{aligned}$$

where (5.8) holds uniformly with respect to u for $\delta \le \Delta(u) - 2T$ and (5.9) follows from Proposition 2.2. Thus the assertion of Lemma 5.1 follows by combining (5.2) and (5.5) with (5.10).

□

Proof of Theorem 3.1 The idea of the proof is based on the comparison of upper and lower bound of $\mathbb{P}\left(\sup_{t\in[0,\sigma_\eta(u)]}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right)$.

Let T, S be given. We introduce $A_i = [iS,(i+1)S]$, $B_i = [iT,(i+1)T]$ and let $N_u^{(S)} = \left[\frac{\sigma_\eta(u)}{S}\right]$, $N_u^{(T)} = \left[\frac{\sigma_\eta(u)}{T}\right] - 1$. Note that from Bonferroni inequality and Proposition 2.2

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,\sigma_\eta(u)]}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right) &\le \sum_{i=0}^{N_u^{(S)}}\mathbb{P}\left(\sup_{t\in A_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right)\\ &= N_u^{(S)}\mathcal{H}_\eta(S)\Psi(\sqrt{2}\sigma_\eta(u))(1+o(1))\end{aligned}$$

as $u\to\infty$. Thus for each $S>0$

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[0,\sigma_\eta(u)]}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right)}{\sigma_\eta(u)\Psi(\sqrt{2}\sigma_\eta(u))} \le \frac{\mathcal{H}_\eta(S)}{S}. \tag{5.11}$$

On the other hand, again using Bonferroni inequality, we have for each $T\ge 1$

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,\sigma_\eta(u)]}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right)\\ &\ge \sum_{i=0}^{N_u^{(T)}}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right)\\ &\quad -2\sum_{0\le i<j\le N_u^{(T)}}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s\in B_j}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right) \qquad (5.12)\end{aligned}$$

We split the sum in line (5.12) on the following sub-sums

$$\begin{aligned}&2\sum_{0\le i<j\le N_u^{(T)}}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s\in B_j}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right) =\\ &= 2\sum_{i+1=j}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s\in B_j}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right) \qquad (5.13)\\ &\quad +2\sum_{1<j-i\le N_u^{(T)}}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s\in B_j}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right). \qquad (5.14)\end{aligned}$$

Note that for the sum in line (5.14), using Lemma 5.1

$$\begin{aligned}&2\sum_{1<j-i\le N_u^{(T)}}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s\in B_j}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right) \le\\ &\le 2N_u^{(T)}\mathcal{H}_{\eta,\eta}(\Lambda^5 T)\sum_{k=1}^{\infty}\exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right)\Psi(\sqrt{2}\sigma_\eta(u))(1+o(1)) \qquad (5.15)\end{aligned}$$

as $u\to\infty$. Moreover for $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(x)<x$ for each $x\ge 1$

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s\in B_{i+1}}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right) \le\\ &\le \mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u); \sup_{s-f(T)\in B_{i+1}}\overline{\eta}_u(s) > \sqrt{2}\sigma_\eta(u)\right)\\ &\quad +\mathbb{P}\left(\sup_{t\in[(i+1)T,(i+1)T+f(T)]}\overline{\eta}_u(t) > \sqrt{2}\sigma_\eta(u)\right).\end{aligned}$$

Thus for the sum in line (5.13) we have

$$\begin{aligned}
2\sum_{i+1=j}\mathbb{P}\left(\sup_{t\in B_i}\overline{\eta}_u(t)>\sqrt{2}\sigma_\eta(u);\sup_{s\in B_j}\overline{\eta}_u(s)>\sqrt{2}\sigma_\eta(u)\right) &\leq \\
\leq\ 2N_u^{(T)}\mathcal{H}_{\eta,\eta}(\Lambda^5T)\exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right)\Psi(\sqrt{2}\sigma_\eta(u))(1+o(1)) & \\
+2N_u^{(T)}\mathcal{H}_\eta(f(T))\Psi(\sqrt{2}\sigma_\eta(u))(1+o(1)). & \qquad (5.16)
\end{aligned}$$

Combining (5.15) with (5.16) we obtain for $T\geq 1$

$$\begin{aligned}
\liminf_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[0,\sigma_\eta(u)]}\overline{\eta}_u(t)>\sqrt{2}\sigma_\eta(u)\right)}{\sigma_\eta(u)\Psi(\sqrt{2}\sigma_\eta(u))} &\geq \\
\geq\ \frac{\mathcal{H}_\eta(T)}{T}-2\frac{\mathcal{H}_{\eta,\eta}(\Lambda^5T)}{T}\exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right)-2\frac{\mathcal{H}_\eta(f(T))}{T} & - \\
-2\frac{\mathcal{H}_{\eta,\eta}(\Lambda^5T)}{T}\sum_{k=1}^\infty\exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right). & \qquad (5.17)
\end{aligned}$$

Hence, using sub-additivity of $\mathcal{H}_\eta(\cdot)$, comparing (5.17) with (5.11) and taking $S\to\infty$ we have that for each $T\geq 1$

$$\begin{aligned}
\mathcal{H}_\eta\ \geq\ & \frac{\mathcal{H}_\eta(T)}{T}-2\frac{\mathcal{H}_{\eta,\eta}(\Lambda^5T)}{T}\exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right)-2\frac{\mathcal{H}_\eta(f(T))}{T}- \\
& -2\frac{\mathcal{H}_{\eta,\eta}(\Lambda^5T)}{T}\sum_{k=1}^\infty\exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right). \qquad (5.18)
\end{aligned}$$

Moreover, due to the fact that $\mathcal{H}_\eta(\cdot)$ is subadditive, we have $\frac{\mathcal{H}_\eta(T)}{T}-\mathcal{H}_\eta\geq 0$, which combined with (5.18) completes the proof.

□

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