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# Two-point Taylor Expansions of Analytic Functions

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## ABSTRACT

Taylor expansions of analytic functions are considered with respect to two points. Cauchy-type formulas are given for coefficients and remainders in the expansions, and the regions of convergence are indicated. It is explained how these expansions can be used in deriving uniform asymptotic expansions of integrals. The method is also used for obtaining Laurent expansions in two points.

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## 1. Introduction

In deriving uniform asymptotic expansions of a certain class of integrals one encounters the problem of expanding a function, that is analytic in some domain  $\Omega$  of the complex plane, in two points. The first mention of the use of such expansions in asymptotics is given in [1], where Airy-type expansions are derived for integrals having two nearby (or coalescing) saddle points. This reference does not give further details about two-point Taylor expansions, because the coefficients in the Airy-type asymptotic expansion are derived in a different way.

To demonstrate the application in asymptotics we consider the integral

$$F_b(\omega) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\omega(\frac{1}{3}z^3 - b^2z)} f(z) dz, \quad (1)$$

where  $\omega$  is a large positive parameter and  $b$  is a parameter that may assume small values. The contour starts at  $\infty e^{-i\pi/3}$  and terminates at  $\infty e^{i\pi/3}$ , and lies in a domain

where the function  $f$  is analytic. In particular,  $f$  is analytic in a domain that contains the saddle points  $\pm b$  of the exponent in the integrand. One method for obtaining an asymptotic expansion of  $F_b(\omega)$  that holds uniformly for small values of  $b$  is based on expanding  $f$  at the two saddle points:

$$f(z) = \sum_{n=0}^{\infty} A_n(z^2 - b^2)^n + z \sum_{n=0}^{\infty} B_n(z^2 - b^2)^n, \quad (2)$$

and substitute this expansion into (1). When interchanging summation and integration, the result is a formal expansion in two series in terms of functions related with Airy functions. A Maple algorithm for obtaining the coefficients  $A_n$  and  $B_n$ , with applications to Airy-type expansions of parabolic cylinder functions, is given in [4].

In a future paper we use expansions like (2) in order to derive convergent expansions for orthogonal polynomials and hypergeometric functions that also have an asymptotic nature. The purpose of the present paper is to give details on the two-point Taylor expansion (2), in particular on the region of convergence and on representations in terms of Cauchy-type integrals of coefficients and remainders of these expansions. Some information on this type of expansions is also given in [6], p. 149, Exercise 24.

Without referring to applications in asymptotic analysis we include analogous properties of two-point Laurent expansions and of another related type, the two-point Taylor-Laurent expansion.

## 2. Two-point Taylor expansions

We consider the expansion (2) in a more symmetric form and give information on the coefficients and the remainder in the expansion.

**Theorem 1.** *Let  $f(z)$  be an analytic function on an open set  $\Omega \subset \mathcal{C}$  and  $z_1, z_2 \in \Omega$  with  $z_1 \neq z_2$ . Then,  $f(z)$  admits the two-point Taylor expansion*

$$f(z) = \sum_{n=0}^{N-1} [a_n(z_1, z_2)(z - z_1) + a_n(z_2, z_1)(z - z_2)] (z - z_1)^n (z - z_2)^n + r_N(z_1, z_2; z), \quad (3)$$

where the coefficients  $a_n(z_1, z_2)$  and  $a_n(z_2, z_1)$  of the expansion are given by the Cauchy integral

$$a_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\mathcal{C}} \frac{f(w) dw}{(w - z_1)^n (w - z_2)^{n+1}}. \quad (4)$$

The remainder term  $r_N(z_1, z_2; z)$  is given by the Cauchy integral

$$r_N(z_1, z_2; z) \equiv \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w) dw}{(w - z_1)^N (w - z_2)^N (w - z)} (z - z_1)^N (z - z_2)^N. \quad (5)$$

The contour of integration  $\mathcal{C}$  is a simple closed loop which encircles the points  $z_1$  and  $z_2$  (for  $a_n$ ) and  $z$ ,  $z_1$  and  $z_2$  (for  $r_N$ ) in the counterclockwise direction and is contained in  $\Omega$  (see Figure 1 (a)).

The expansion (3) is convergent for  $z$  inside the Cassini oval (see Figure 2)

$$O_{z_1, z_2} \equiv \{z \in \Omega, |(z - z_1)(z - z_2)| < r\}$$

where

$$r \equiv \text{Inf}_{w \in \mathcal{C} \setminus \Omega} \{|(w - z_1)(w - z_2)|\}.$$

In particular, if  $f(z)$  is an entire function ( $\Omega = \mathcal{C}$ ), then the expansion (3) converges  $\forall z \in \mathcal{C}$ .

**Proof.** By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w) dw}{w - z}, \quad (6)$$

where  $\mathcal{C}$  is the contour defined above (Figure 1 (a)). Now we write

$$\frac{1}{w - z} = \frac{z + w - z_1 - z_2}{(w - z_1)(w - z_2)} \frac{1}{1 - u}, \quad (7)$$

where

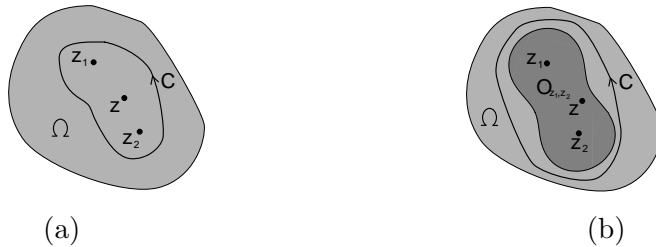
$$u \equiv \frac{(z - z_1)(z - z_2)}{(w - z_1)(w - z_2)}. \quad (8)$$

Now we introduce the expansion

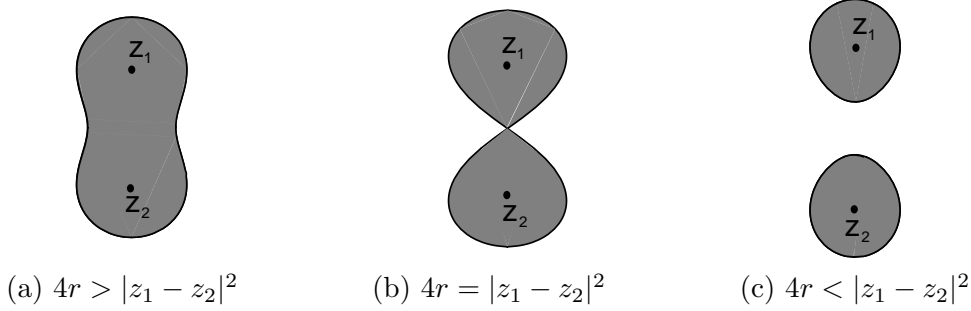
$$\frac{1}{1 - u} = \sum_{n=0}^{N-1} u^n + \frac{u^N}{1 - u} \quad (9)$$

in (7) and this in (6). After straightforward calculations we obtain (3)-(5).

For any  $z \in O_{z_1, z_2}$ , we can take a contour  $\mathcal{C}$  in  $\Omega$  such that  $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)| \forall w \in \mathcal{C}$  (see Figure 1 (b)). In this contour  $|f(w)|$  is bounded by some constant  $C$ :  $|f(w)| \leq C$ . Introducing these two bounds in (5) we see that  $\lim_{N \rightarrow \infty} r_N(z_1, z_2; z) = 0$  and the proof follows.  $\square$



**Figure 1.** (a) Contour  $\mathcal{C}$  in the integrals (3)-(5). (b) For  $z \in O_{z_1, z_2}$ , we can take a contour  $\mathcal{C}$  in  $\Omega$  which contains  $O_{z_1, z_2}$  inside and therefore,  $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)| \forall w \in \mathcal{C}$ .



**Figure 2.** Shape of the Cassini oval depending on the relative size of the parameter  $r$  and the focal distance  $|z_1 - z_2|$ .

### 2.1. An alternative form of the expansion

The present expansion of  $f(z)$  in the form (3) stresses the symmetry of the expansion with respect to  $z_1$  and  $z_2$ . In this representation it is not possible, however, to let  $z_1$  and  $z_2$  coincide, which causes a little inconvenience (the coefficients  $a_n(z_1, z_2)$  become infinitely large as  $z_1 \rightarrow z_2$ ; the remainder  $r_N(z_1, z_2; z)$  remains well-defined). An alternative way is the representation (cf. (2)),

$$f(z) = \sum_{n=0}^{\infty} [A_n(z_1, z_2) + B_n(z_1, z_2)z] (z - z_1)^n (z - z_2)^n,$$

and we have the relations

$$\begin{aligned} A_n(z_1, z_2) &= -z_1 a_n(z_1, z_2) - z_2 a_n(z_2, z_1), \\ B_n(z_1, z_2) &= a_n(z_1, z_2) + a_n(z_2, z_1), \end{aligned}$$

which are regular when  $z_1 \rightarrow z_2$ . In fact we have

$$\begin{aligned} A_n(z_1, z_2) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{w - z_1 - z_2}{[(w - z_1)(w - z_2)]^{n+1}} f(w) dw, \\ B_n(z_1, z_2) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w) dw}{[(w - z_1)(w - z_2)]^{n+1}}. \end{aligned}$$

Letting  $z_1 \rightarrow 0$  and  $z_2 \rightarrow 0$ , we obtain the standard Maclaurin series of  $f(z)$  with even part (the  $A_n$  series) and odd part (the  $B_n$  series).

### 2.2. Explicit forms of the coefficients

Definition (4) is not appropriate for numerical computations. A more practical formula to compute the coefficients of the above two-point Taylor expansion is given in the following proposition.

**Proposition 1.** Coefficients  $a_n(z_1, z_2)$  in the expansion (3) are also given by the formulas:

$$a_0(z_1, z_2) = \frac{f(z_2)}{z_2 - z_1} \quad (10)$$

and, for  $n = 1, 2, 3, \dots$ ,

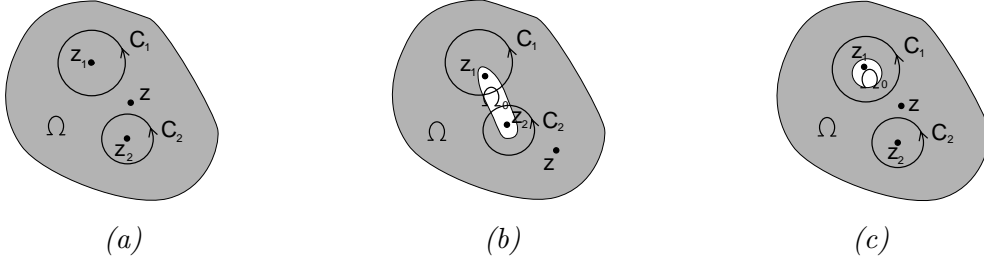
$$a_n(z_1, z_2) = \sum_{k=0}^n \frac{(n+k-1)! (-1)^{n+1} n f^{(n-k)}(z_2) + (-1)^k k f^{(n-k)}(z_1)}{k!(n-k)! n!(z_1 - z_2)^{n+k+1}}. \quad (11)$$

**Proof.** We deform the contour of integration  $\mathcal{C}$  in equation (4) to any contour of the form  $\mathcal{C}_1 \cup \mathcal{C}_2$  also contained in  $\Omega$ , where  $\mathcal{C}_1$  ( $\mathcal{C}_2$ ) is a simple closed loop which encircles the point  $z_1$  ( $z_2$ ) in the counterclockwise direction and does not contain the point  $z_2$  ( $z_1$ ) inside (see Figure 3 (a)). Then,

$$a_n(z_1, z_2) = \frac{1}{2\pi i(z_2 - z_1)} \left\{ \int_{\mathcal{C}_1} \frac{f(w)}{(w - z_2)^{n+1}} \frac{dw}{(w - z_1)^n} + \int_{\mathcal{C}_2} \frac{f(w)}{(w - z_1)^n} \frac{dw}{(w - z_2)^{n+1}} \right\} =$$

$$\frac{1}{(z_2 - z_1)} \left\{ \frac{1}{(n-1)!} \frac{d^{n-1}}{dw^{n-1}} \frac{f(w)}{(w - z_2)^{n+1}} \Big|_{w=z_1} + \frac{1}{n!} \frac{d^n}{dw^n} \frac{f(w)}{(w - z_1)^n} \Big|_{w=z_2} \right\}.$$

From here, equations (10)-(11) follows after straightforward computations.  $\square$



**Figure 3.** (a) The function  $(w - z_2)^{-n-1} f(w)$  is analytic inside  $\mathcal{C}_1$ , whereas  $(w - z_1)^{-n} f(w)$  is analytic inside  $\mathcal{C}_2$ . (b) The function  $(w - z_2)^{-n-1} g_1(w)$  is analytic inside  $\mathcal{C}_1$ , whereas  $(w - z_1)^{-n} g_2(w)$  is analytic inside  $\mathcal{C}_2$ . (c) The function  $(w - z_2)^{-n-1} g(w)$  is analytic inside  $\mathcal{C}_1$ , whereas  $(w - z_1)^{-n} f(w)$  is analytic inside  $\mathcal{C}_2$ .

### 2.3. Two-point Taylor polynomials

Next we can define the two-point Taylor polynomial of the function  $f(z)$  at in the following way:

**Definition 1.** Let  $z$  be a real or complex variable and  $z_1$  and  $z_2$  ( $z_1 \neq z_2$ ) two real or complex numbers. If  $f(z)$  is  $n - 1$ -times differentiable at those two points, we define the two-point Taylor polynomial of  $f(z)$  at  $z_1$  and  $z_2$  and degree  $2n - 1$  as

$$P_n(z_1, z_2; z) \equiv \sum_{k=0}^{n-1} [a_k(z_1, z_2)(z - z_1) + a_k(z_2, z_1)(z - z_2)] (z - z_1)^k (z - z_2)^k,$$

where the coefficients  $a_k(z_1, z_2)$  are given in (10)-(11).

**Proposition 2.** *In the conditions of the above definition, define the remainder of the approximation of  $f(z)$  by  $P_n(z_1, z_2; z)$  at  $z_1$  and  $z_2$  as*

$$r_n(z_1, z_2; z) \equiv f(z) - P_n(z_1, z_2; z).$$

*Then, (i)  $r_n(z_1, z_2; z) = o(z - z_1)^{n-1}$  as  $z \rightarrow z_1$  and  $r_n(z_1, z_2; z) = o(z - z_2)^{n-1}$  as  $z \rightarrow z_2$ . (ii) If  $f(z)$  is  $n$ -times differentiable at  $z_1$  and/or  $z_2$ , then  $r_n(z_1, z_2; z) = \mathcal{O}(z - z_1)^n$  as  $z \rightarrow z_1$  and/or  $r_n(z_1, z_2; z) = \mathcal{O}(z - z_2)^n$  as  $z \rightarrow z_2$ .*

**Proof.** The proof is trivial if  $f(z)$  is analytic at  $z_1$  and  $z_2$  by using (5). In any case, for real or complex variable, the proof follows after straightforward computations by using l'Hôpital's rule and (10)-(11).  $\square$

**Remark 1.** Observe that the Taylor polynomial of  $f(z)$  at  $z_1$  and  $z_2$  and degree  $2n - 1$  is the same as the Hermite's interpolation polynomial of  $f(z)$  at  $z_1$  and  $z_2$  with data  $f(z_i), f'(z_i), \dots, f^{(n-1)}(z_i), i = 1, 2$ .

### 3. Two-point Laurent expansions

In the standard theory for Taylor and Laurent expansions much analogy exists between the two types of expansions. For two-point expansions, we have a similar agreement in the representations of coefficients and remainders.

**Theorem 2.** *Let  $\Omega_0$  and  $\Omega$  be closed and open sets, respectively, of the complex plane, and  $\Omega_0 \subset \Omega \subset \mathcal{C}$ . Let  $f(z)$  be an analytic function on  $\Omega \setminus \Omega_0$  and  $z_1, z_2 \in \Omega_0$  with  $z_1 \neq z_2$ . Then, for any  $z \in \Omega \setminus \Omega_0$ ,  $f(z)$  admits the two-point Laurent expansion*

$$\begin{aligned} f(z) = & \sum_{n=0}^{N-1} [b_n(z_1, z_2)(z - z_1) + b_n(z_2, z_1)(z - z_2)] (z - z_1)^n (z - z_2)^n + \\ & \sum_{n=0}^{N-1} [c_n(z_1, z_2)(z - z_1) + c_n(z_2, z_1)(z - z_2)] (z - z_1)^{-n-1} (z - z_2)^{-n-1} + \\ & r_N(z_1, z_2; z), \end{aligned} \quad (12)$$

where the coefficients  $b_n(z_1, z_2)$ ,  $b_n(z_2, z_1)$ ,  $c_n(z_1, z_2)$  and  $c_n(z_2, z_1)$  of the expansion are given, respectively, by the Cauchy integrals

$$b_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\Gamma_1} \frac{f(w) dw}{(w - z_1)^n (w - z_2)^{n+1}} \quad (13)$$

and

$$c_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\Gamma_2} (w - z_1)^{n+1} (w - z_2)^n f(w) dw. \quad (14)$$

The remainder term  $r_N(z_1, z_2; z)$  is given by the Cauchy integrals

$$\begin{aligned} r_N(z_1, z_2; z) \equiv & \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w) dw}{(w - z_1)^N (w - z_2)^N (w - z)} (z - z_1)^N (z - z_2)^N - \\ & \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(w - z_1)^N (w - z_2)^N f(w) dw}{w - z} \frac{1}{(z - z_1)^N (z - z_2)^N}. \end{aligned} \quad (15)$$



In these integrals, the contours of integration  $\Gamma_1$  and  $\Gamma_2$  are simple closed loops contained in  $\Omega \setminus \Omega_0$  which encircle the points  $z_1$  and  $z_2$  in the counterclockwise direction. Moreover,  $\Gamma_2$  does not contain the point  $z$  inside, whereas  $\Gamma_1$  encircles  $\Gamma_2$  and the point  $z$  (see Figure 4 (a)).

The expansion (12) is convergent for  $z$  inside the Cassini annulus (see Figure 5)

$$A_{z_1, z_2} \equiv \{z \in \Omega \setminus \Omega_0, r_2 < |(z - z_1)(z - z_2)| < r_1\} \quad (16)$$

where

$$r_1 \equiv \text{Inf}_{w \in \mathcal{C} \setminus \Omega} \{|(w - z_1)(w - z_2)|\}, \quad r_2 \equiv \text{Sup}_{w \in \Omega_0} \{|(w - z_1)(w - z_2)|\}.$$

**Proof.** By Cauchy's theorem,

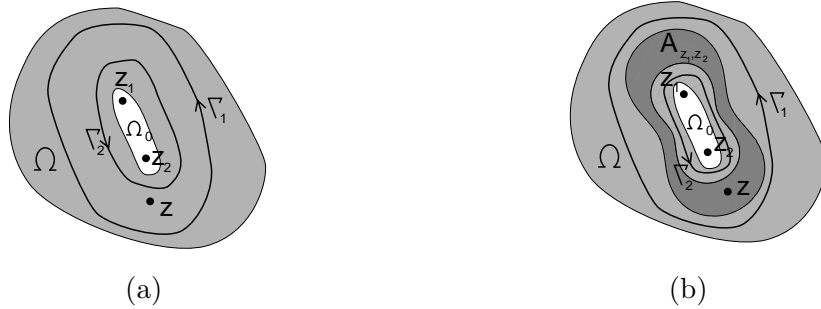
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)dw}{w - z} - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)dw}{w - z}, \quad (17)$$

where  $\Gamma_1$  and  $\Gamma_2$  are the contours defined above. We substitute (7)-(8) into the first integral above and

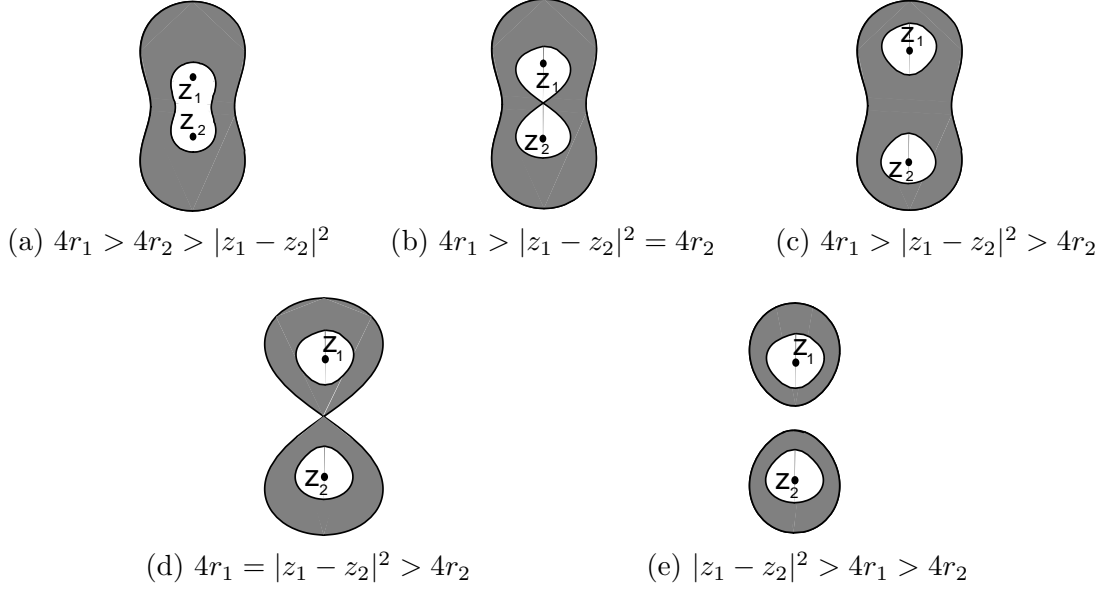
$$\frac{1}{w - z} = \frac{z_1 + z_2 - z - w}{(z - z_1)(z - z_2)} \frac{1}{1 - u}, \quad u \equiv \frac{(w - z_1)(w - z_2)}{(z - z_1)(z - z_2)},$$

into the second one. Now we introduce the expansion (9) of the factor  $(1 - u)^{-1}$  in both integrals in (17). After straightforward calculations we obtain (12)-(15).

For any  $z$  verifying (16), we can take simple closed loops  $\Gamma_1$  and  $\Gamma_2$  in  $\Omega \setminus \Omega_0$  such that  $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)| \forall w \in \Gamma_1$  and  $|(z - z_1)(z - z_2)| > |(w - z_1)(w - z_2)| \forall w \in \Gamma_2$  (see Figure 4 (b)). On these contours  $|f(w)|$  is bounded by some constant  $C$ :  $|f(w)| \leq C$ . Introducing these bounds in (15) we see that  $\lim_{N \rightarrow \infty} r_N(z_1, z_2; z) = 0$  and the proof follows.  $\square$



**Figure 4.** (a) Contours  $\Gamma_1$  and  $\Gamma_2$  in the integrals (12)-(15). (b) For  $z \in A_{z_1, z_2}$ , we can take a contour  $\Gamma_2$  in  $\Omega$  situated between  $\Omega_0$  and  $A_{z_1, z_2}$  and a contour  $\Gamma_1$  in  $\Omega$  which contains  $A_{z_1, z_2}$  inside. Therefore,  $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)| \forall w \in \Gamma_1$  and  $|(w - z_1)(w - z_2)| < |(z - z_1)(z - z_2)| \forall w \in \Gamma_2$ .



**Figure 5.** Shape of the Cassini annulus depending on the relative size of the parameters  $r_1$ ,  $r_2$  and the focal distance  $|z_1 - z_2|$ .

If the only singularities of  $f(z)$  inside  $\Omega_0$  are just poles at  $z_1$  and/or  $z_2$ , then an alternative formula to (13) and (14) to compute the coefficients of the above two-point Laurent expansion is given in the following proposition.

**Proposition 3.** Suppose that  $g_1(z) \equiv (z - z_1)^{m_1} f(z)$  and  $g_2(z) \equiv (z - z_2)^{m_2} f(z)$  are analytic functions in  $\Omega$  for certain  $m_1, m_2 \in \mathbb{N}$ . Then, for  $n = 0, 1, 2, \dots$ , coefficients  $b_n(z_1, z_2)$  and  $c_n(z_1, z_2)$  in the expansion (12) are also given by the formulas:

$$b_n(z_1, z_2) = \sum_{k=0}^{n+m_1-1} \binom{n+m_1-1}{k} \frac{(-1)^{k+1} (n+1)_k g_1^{(n+m_1-k-1)}(z_1)}{(n+m_1-1)! (z_1 - z_2)^{n+k+2}} + \sum_{k=0}^{n+m_2} \binom{n+m_2}{k} \frac{(-1)^k (n)_k g_2^{(n+m_2-k)}(z_2)}{(n+m_2)! (z_2 - z_1)^{n+k+1}}, \quad (18)$$

where  $(n)_k$  denotes the Pochhammer symbol and

$$c_n(z_1, z_2) = - \sum_{k=0}^{m_1-n-2} k! \binom{m_1-n-2}{k} \binom{n}{k} \frac{(z_1 - z_2)^{n-k-1} g_1^{(m_1-n-k-2)}(z_1)}{(m_1-n-2)!} + \sum_{k=0}^{m_2-n-1} k! \binom{m_2-n-1}{k} \binom{n+1}{k} \frac{(z_2 - z_1)^{n-k} g_2^{(m_2-n-k-1)}(z_2)}{(m_2-n-1)!}. \quad (19)$$

In these formulas, empty sums must be understood as zero. Coefficients  $b_n(z_2, z_1)$  and  $c_n(z_2, z_1)$  are given, respectively, by (18) and (19) interchanging  $z_1$ ,  $g_1$  and  $m_1$  by  $z_2$ ,  $g_2$  and  $m_2$  respectively.

**Proof.** We deform both, the contour  $\Gamma_1$  in equation (13) and  $\Gamma_2$  in equation (14), to any contour of the form  $\mathcal{C}_1 \cup \mathcal{C}_2$  contained in  $\Omega$ , where  $\mathcal{C}_1$  ( $\mathcal{C}_2$ ) is a simple closed loop which encircles the point  $z_1$  ( $z_2$ ) in the counterclockwise direction and does not contain the point  $z_2$  ( $z_1$ ) inside (see Figure 3 (b)). Then,

$$\begin{aligned} b_n(z_1, z_2) &= \frac{1}{2\pi i(z_2 - z_1)} \left\{ \int_{\mathcal{C}_1} \frac{g_1(w)}{(w - z_2)^{n+1}} \frac{dw}{(w - z_1)^{n+m_1}} + \right. \\ &\quad \left. \int_{\mathcal{C}_2} \frac{g_2(w)}{(w - z_1)^n} \frac{dw}{(w - z_2)^{n+m_2+1}} \right\} = \\ &\quad \frac{1}{z_2 - z_1} \left\{ \frac{1}{(n + m_1 - 1)!} \frac{d^{n+m_1-1}}{dw^{n+m_1-1}} \frac{g_1(w)}{(w - z_2)^{n+1}} \Big|_{w=z_1} + \right. \\ &\quad \left. \frac{1}{(n + m_2)!} \frac{d^{n+m_2}}{dw^{n+m_2}} \frac{g_2(w)}{(w - z_1)^n} \Big|_{w=z_2} \right\} \end{aligned}$$

and

$$\begin{aligned} c_n(z_1, z_2) &= \frac{1}{2\pi i(z_2 - z_1)} \left\{ \int_{\mathcal{C}_1} \frac{(w - z_2)^n g_1(w)}{(w - z_1)^{m_1-n-1}} dw + \int_{\mathcal{C}_2} \frac{(w - z_1)^{n+1} g_2(w)}{(w - z_2)^{m_2-n}} dw \right\} = \\ &\quad \frac{1}{z_2 - z_1} \left\{ \frac{d^{m_1-n-2}}{dw^{m_1-n-2}} \left[ \frac{(w - z_2)^n g_1(w)}{(m_1 - n - 2)!} \right] \Big|_{w=z_1} + \frac{d^{m_2-n-1}}{dw^{m_2-n-1}} \left[ \frac{(w - z_1)^{n+1} g_2(w)}{(m_2 - n - 1)!} \right] \Big|_{w=z_2} \right\} \end{aligned}$$

From here, equations (18) and (19) follow after straightforward computations.  $\square$

**Remark 2.** Let  $z$  be a real or complex variable and  $z_1, z_2$  ( $z_1 \neq z_2$ ) two real or complex numbers. Suppose that  $g_1(z) \equiv (z - z_1)^{m_1} f(z)$  is  $n$ -times differentiable at  $z_1$  and  $g_2(z) \equiv (z - z_2)^{m_2} f(z)$  is  $n$ -times differentiable at  $z_2$ . Define

$$g(z) \equiv f(z) - \sum_{n=0}^{M-1} [c_n(z_1, z_2)(z - z_1) + c_n(z_2, z_1)(z - z_2)] (z - z_1)^{-n-1} (z - z_2)^{-n-1},$$

where  $M \equiv \text{Max}\{m_1, m_2\}$ . Then, the thesis of Proposition 2 holds for  $f(z)$  replaced by  $g(z)$ . Moreover, if  $(z - z_1)^{m_1} (z - z_2)^{m_2} f(z)$  is an analytic function in  $\Omega$ , then the thesis of Theorem 1 applies to  $g(z)$ .

#### 4. Two-point Taylor-Laurent expansions

**Theorem 3.** Let  $\Omega_0$  and  $\Omega$  be closed and open sets, respectively, of the complex plane, and  $\Omega_0 \subset \Omega \subset \mathcal{C}$ . Let  $f(z)$  be an analytic function on  $\Omega \setminus \Omega_0$ ,  $z_1 \in \Omega_0$  and  $z_2 \in \Omega \setminus \Omega_0$ . Then, for  $z \in \Omega \setminus \Omega_0$ ,  $f(z)$  admits the Taylor-Laurent expansion

$$\begin{aligned} f(z) &= \sum_{n=0}^{N-1} [d_n(z_1, z_2)(z - z_1) + d_n(z_2, z_1)(z - z_2)] (z - z_1)^n (z - z_2)^n + \\ &\quad \sum_{n=0}^{N-1} e_n(z_1, z_2)(z - z_2)^n (z - z_1)^{-n-1} + r_N(z_1, z_2; z), \end{aligned} \tag{20}$$

where the coefficients  $d_n(z_1, z_2)$ ,  $d_n(z_2, z_1)$  and  $e_n(z_1, z_2)$  of the expansion are given by the Cauchy integrals

$$d_n(z_1, z_2) \equiv \frac{1}{2\pi i(z_2 - z_1)} \int_{\Gamma_1} \frac{f(w) dw}{(w - z_1)^n (w - z_2)^{n+1}} \quad (21)$$

and

$$e_n(z_1, z_2) \equiv \frac{z_1 - z_2}{2\pi i} \int_{\Gamma_2} \frac{(w - z_1)^n}{(w - z_2)^{n+1}} f(w) dw. \quad (22)$$

The remainder term  $r_N(z_1, z_2; z)$  is given by the Cauchy integrals

$$\begin{aligned} r_N(z_1, z_2; z) \equiv & \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w) dw}{(w - z_1)^N (w - z_2)^N (w - z)} (z - z_1)^N (z - z_2)^N - \\ & \frac{1}{2\pi i} \int_{\Gamma_2} \frac{(w - z_1)^N f(w) dw}{(w - z_2)^N (w - z)} \frac{(z - z_2)^N}{(z - z_1)^N}. \end{aligned} \quad (23)$$

In these integrals, the contours of integration  $\Gamma_1$  and  $\Gamma_2$  are simple closed loops contained in  $\Omega \setminus \Omega_0$  which encircle  $\Omega_0$  in the counterclockwise direction. Moreover,  $\Gamma_2$  does not contain the points  $z$  and  $z_2$  inside, whereas  $\Gamma_1$  encircles  $\Gamma_2$  and the points  $z$  and  $z_2$  (see Figure 6 (a)).

The expansion (20) is convergent in the region (Figure 7)

$$D_{z_1, z_2} \equiv \{z \in \Omega \setminus \Omega_0, |(z - z_1)(z - z_2)| < r_1 \text{ and } |z - z_2| < r_2 |z - z_1|\} \quad (24)$$

where  $r_1 \equiv \text{Inf}_{w \in \mathcal{C} \setminus \Omega} \{|(w - z_1)(w - z_2)|\}$  and  $r_2 \equiv \text{Inf}_{w \in \Omega_0} \{|(w - z_2)(w - z_1)^{-1}|\}$ .

**Proof.** By Cauchy's theorem,

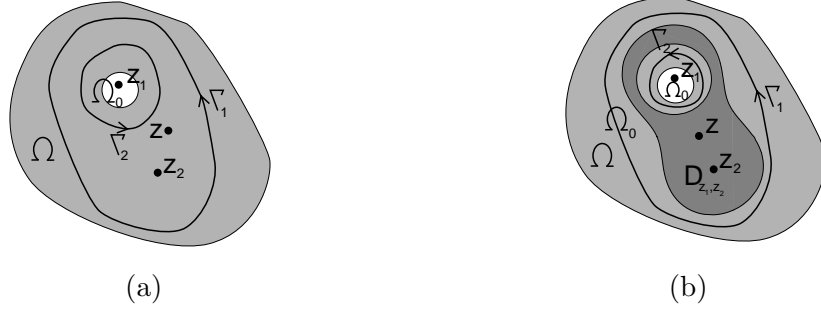
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w) dw}{w - z} - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w) dw}{w - z}, \quad (25)$$

where  $\Gamma_1$  and  $\Gamma_2$  are the contours defined above. We substitute (7)-(8) into the first integral above and

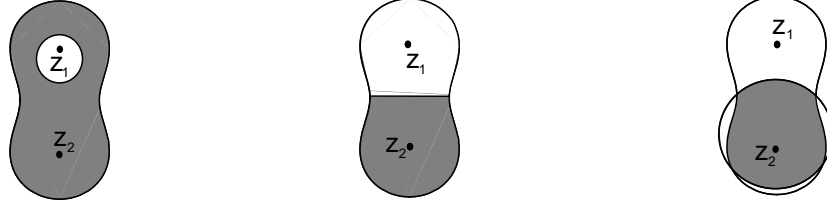
$$\frac{1}{w - z} = \frac{z_2 - z_1}{(z - z_1)(w - z_2)} \frac{1}{1 - u}, \quad u \equiv \frac{(w - z_1)(z - z_2)}{(z - z_1)(w - z_2)} \quad (26)$$

into the second one. Now we introduce the expansion (9) of the factor  $(1 - u)^{-1}$  in both integrals in (25). After straightforward calculations we obtain (20)-(23).

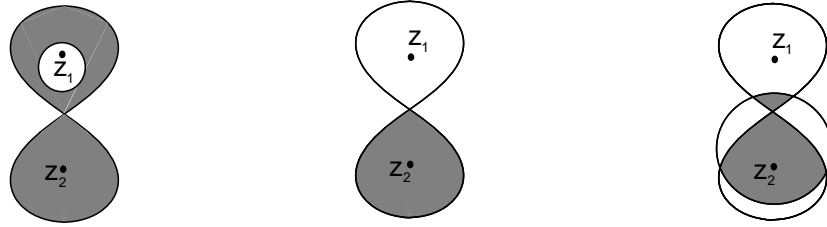
For any  $z$  verifying (24), we can take simple closed loops  $\Gamma_1$  and  $\Gamma_2$  in  $\Omega \setminus \Omega_0$  such that  $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)| \forall w \in \Gamma_1$  and  $|(z - z_1)(w - z_2)| > |(w - z_1)(z - z_2)| \forall w \in \Gamma_2$  (see Figure 6 (b)). On these contours  $|f(w)|$  is bounded by some constant  $C$ :  $|f(w)| \leq C$ . Introducing these bounds in (23) we see that  $\lim_{N \rightarrow \infty} r_N(z_1, z_2; z) = 0$  and the proof follows.  $\square$



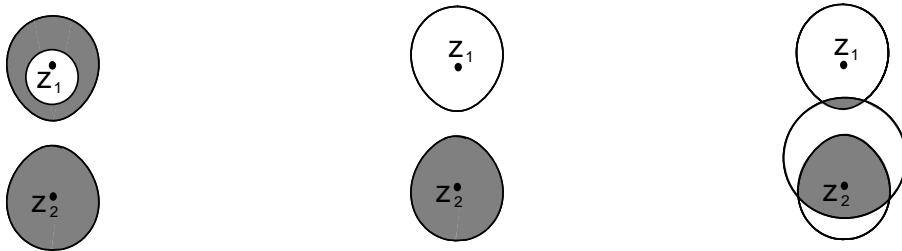
**Figure 6.** (a) Contours  $\Gamma_1$  and  $\Gamma_2$  in the integrals (20)-(23). (b) For  $z \in D_{z_1, z_2}$ , we can take a contour  $\Gamma_2$  situated between  $\Omega_0$  and  $D_{z_1, z_2}$  and a contour  $\Gamma_1$  in  $\Omega$  which contains  $D_{z_1, z_2}$  inside. Therefore,  $|(z - z_1)(z - z_2)| < |(w - z_1)(w - z_2)| \forall w \in \Gamma_1$  and  $|(w - z_1)(z - z_2)| < |(z - z_1)(w - z_2)| \forall w \in \Gamma_2$ .



(a)  $4r_1 > |z_1 - z_2|^2$ ,  $r_2 > 1$  (b)  $4r_1 > |z_1 - z_2|^2$ ,  $r_2 = 1$  (c)  $4r_1 > |z_1 - z_2|^2$ ,  $r_2 < 1$



(d)  $4r_1 = |z_1 - z_2|^2$ ,  $r_2 > 1$  (e)  $4r_1 = |z_1 - z_2|^2$ ,  $r_2 = 1$  (f)  $4r_1 = |z_1 - z_2|^2$ ,  $r_2 < 1$



(g)  $4r_1 < |z_1 - z_2|^2$ ,  $r_2 > 1$  (h)  $4r_1 < |z_1 - z_2|^2$ ,  $r_2 = 1$  (i)  $4r_1 < |z_1 - z_2|^2$ ,  $r_2 < 1$

**Figure 7.** The region  $D_{z_1, z_2}$  defined in Theorem 3 is given by  $D_{z_1, z_2} = D_1 \cap D_2$ , where  $D_1$  is the Cassini oval of focus  $z_1$  and  $z_2$  and parameter  $r_1$ . On the other hand, for  $r_2 < 1$  ( $r_2 > 1$ ),  $D_2$  is the interior (exterior) of the circle of center  $z_1 + (1 - r_2^2)^{-1}(z_2 - z_1) = z_2 + r_2^2(r_2^2 - 1)^{-1}(z_1 - z_2)$  and radius  $|z_1 - z_2|r_2/|r_2^2 - 1|$ . For  $r_2 = 1$ ,  $D_2$  is just the half plane  $|z - z_2| < |z - z_1|$ . The

shape of the Cassini annulus depends on the relative size of the parameters  $\sqrt{r_1}$ ,  $\sqrt{r_2}$  and the focal distance  $|z_1 - z_2|$ .

If the only singularities of  $f(z)$  inside  $\Omega_0$  are just poles at  $z_1$ , then an alternative formula to (21)-(22) to compute the coefficients of the above two-point Taylor-Laurent expansion is given in the following proposition.

**Proposition 4.** *Suppose that  $g(z) \equiv (z - z_1)^m f(z)$  is an analytic function in  $\Omega$  for certain  $m \in \mathbb{N}$ . Then, coefficients  $d_n(z_1, z_2)$  and  $d_n(z_2, z_1)$  in the expansion (20) are also given by the formulas:*

$$d_0(z_1, z_2) = \frac{f(z_2)}{z_2 - z_1} - \sum_{k=0}^{m-1} \frac{1}{(m-k-1)!} \frac{g^{(m-k-1)}(z_1)}{(z_2 - z_1)^{k+2}}, \quad (27)$$

$$d_0(z_2, z_1) = \frac{1}{m!} \frac{g^{(m)}(z_1)}{z_1 - z_2},$$

and, for  $n = 1, 2, 3, \dots$ ,

$$d_n(z_1, z_2) = -\frac{(-1)^n}{n!} \left\{ \sum_{k=0}^{m+n-1} \frac{(n+k)!}{k!(m+n-k-1)!} \frac{g^{(m+n-k-1)}(z_1)}{(z_2 - z_1)^{n+k+2}} + n \sum_{k=0}^n \frac{(n+k-1)!}{k!(n-k)!} \frac{f^{(n-k)}(z_2)}{(z_1 - z_2)^{n+k+1}} \right\}, \quad (28)$$

$$d_n(z_2, z_1) = -\frac{(-1)^n}{n!} \left\{ n \sum_{k=0}^{m+n} \frac{(n+k-1)!}{k!(m+n-k)!} \frac{g^{(m+n-k)}(z_1)}{(z_2 - z_1)^{n+k+1}} + \sum_{k=0}^{n-1} \frac{(n+k)!}{k!(n-k-1)!} \frac{f^{(n-k-1)}(z_2)}{(z_1 - z_2)^{n+k+2}} \right\}. \quad (29)$$

For  $n = 0, 1, 2, \dots$ , coefficients  $e_n(z_1, z_2)$  are given by

$$e_n(z_1, z_2) = \frac{(-1)^n}{n!} \sum_{k=0}^{m-n-1} \frac{(n+k)!}{k!(m-n-k-1)!} \frac{g^{(m-n-k-1)}(z_1)}{(z_2 - z_1)^{n+k}}. \quad (30)$$

**Proof.** We deform both, the contour  $\Gamma_1$  in equation (21) and the contour  $\Gamma_2$  in equation (22) to any contour of the form  $\mathcal{C}_1 \cup \mathcal{C}_2$  contained in  $\Omega$ , where  $\mathcal{C}_1$  ( $\mathcal{C}_2$ ) is a simple closed loop which encircles the point  $z_1$  ( $z_2$ ) in the counterclockwise direction and does not

contain the point  $z_2$  ( $z_1$ ) inside (see Figure 3 (c)). Then,

$$d_n(z_1, z_2) = \frac{1}{2\pi i(z_2 - z_1)} \left\{ \int_{C_1} \frac{g(w)}{(w - z_2)^{n+1}} \frac{dw}{(w - z_1)^{n+m}} + \int_{C_2} \frac{f(w)}{(w - z_1)^n} \frac{dw}{(w - z_2)^{n+1}} \right\} =$$

$$\frac{1}{(z_2 - z_1)} \left\{ \frac{1}{(n + m - 1)!} \frac{d^{n+m-1}}{dw^{n+m-1}} \frac{g_1(w)}{(w - z_2)^{n+1}} \Big|_{w=z_1} + \frac{1}{n!} \frac{d^n}{dw^n} \frac{f(w)}{(w - z_1)^n} \Big|_{w=z_2} \right\},$$

an analog formula for  $d_n(z_2, z_1)$ , and

$$e_n(z_1, z_2) = \frac{z_1 - z_2}{2\pi i} \int_{C_1} \frac{g(w)}{(w - z_2)^{n+1}} \frac{dw}{(w - z_1)^{m-n}} =$$

$$(z_1 - z_2) \frac{1}{(m - n - 1)!} \frac{d^{m-n-1}}{dw^{m-n-1}} \frac{g(w)}{(w - z_2)^{n+1}} \Big|_{w=z_1}.$$

From here, equations (27)-(30) follow after straightforward computations.  $\square$

**Remark 3.** Let  $z$  be a real or complex variable and  $z_1$  and  $z_2$  ( $z_1 \neq z_2$ ) two real or complex numbers. Suppose that  $(z - z_1)^m f(z)$  is  $n$ -times differentiable at  $z_1$  for certain  $m \in \mathbb{N}$  and  $f(z)$  is  $n$ -times differentiable at  $z_2$ . Define

$$g(z) \equiv f(z) - \sum_{n=0}^{m-1} e_n(z_1, z_2)(z - z_1)^{-n-1}(z - z_2)^n.$$

Then, the thesis of Proposition 2 holds for  $g(z)$ . If moreover,  $(z - z_1)^m f(z)$  is an analytic function in  $\Omega$ , then the thesis of Theorem 1 applies to  $g(z)$ .

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