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Large parameter cases of the Gauss hypergeometric function

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Large Parameter Cases of the Gauss Hypergeometric Function

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ABSTRACT

We consider the asymptotic behaviour of the Gauss hypergeometric function when several of the parameters a, b, c are large. We indicate which cases are of interest for orthogonal polynomials (Jacobi, but also Meixner, Krawtchouk, etc.), which results are already available and which cases need more attention. We also consider a few examples of ${}_3F_2$ functions of unit argument, to explain which difficulties arise in these cases, when standard integrals or differential equations are not available.

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1. Introduction

The Gauss hypergeometric function

$$ {}_2F_1\left({a,\ b \atop c};\, z\right) = 1 + \frac{ab}{c}z + \frac{a(a+1)\,b(b+1)}{c(c+1)\,2!}z^2 + \ldots = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\ n!}\,z^n, \tag{1.1}$$

where Pochhammer's symbol $(a)_n$ is defined by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n\,n!\binom{-a}{n}, \tag{1.2}$$

and the infinite series in (1.1) is defined for $|z| < 1$ and $c \neq 0, -1, -2, \ldots$.

A simplification occurs when $b = c$:

$$ {}_2F_1\left({a,\ b \atop b};\, z\right) = (1-z)^{-a}. \tag{1.3}$$

When a or b are non-positive integers the series in (1.1) will terminate, and F reduces to a polynomial. We have

$$ {}_2F_1\left({-n,\ b \atop c};\, z\right) = \sum_{k=0}^{n} \frac{(-n)_m\,(b)_m}{(c)_m\ m!}\,z^m = \sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{(b)_m}{(c)_m}\,z^m. \tag{1.4}$$

The value at $z = 1$ is defined when $\Re(c-a-b) > 0$ and is given by

$$ {}_2F_1\left({a,\ b \atop c};\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.5}$$

For the polynomial case we have

$$ {}_2F_1\left({-n,\ b \atop c}\ ;1\right) = \frac{(c-b)_n}{(c)_n}, \quad n = 0,1,2,\ldots\ . \tag{1.6}$$

Generalizing, let $p, q = 0, 1, 2, \ldots$ with $p \le q+1$. Then

$$ {}_pF_q\left({a_1,\ldots,a_p \atop b_1,\ldots,b_q}\ ;z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}. \tag{1.7}$$

This series converges for all z if $p < q+1$ and for $|z| < 1$ if $p = q+1$.

Sometimes we know the value of a terminating function at $z = 1$, such as given by Saalschütz's theorem,

$$ {}_3F_2\left({-n,\ a,\ b \atop a,\ 1+a+b-c-n}\ ;1\right) = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}, \tag{1.8}$$

where $n = 0,1,2,\ldots$.

The behaviour of the Gauss hypergeometric function $F(a,b;c;z)$ for large $|z|$ follows from the transformation formula

$$ \begin{aligned} {}_2F_1\left({a,\ b \atop c}\ ;z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,(-z)^{-a}{}_2F_1\left({a,\ a-c+1 \atop a-b+1}\ ;\frac{1}{z}\right) \\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,(-z)^{-b}{}_2F_1\left({b,\ b-c+1 \atop b-a+1}\ ;\frac{1}{z}\right), \end{aligned} \tag{1.9}$$

where $|\mathrm{ph}(-z)| < \pi$, and from the expansion of the Gauss function given in (1.1).

The asymptotic behaviour of $F(a,b;c;z)$ for the case that one or more of the parameters a, b or c are large is more complicated (except when only c is large). Several contributions in the literature are available for the asymptotic expansions of functions of the type

$$ {}_2F_1\left({a+e_1\lambda,\ b+e_2\lambda \atop c+e_3\lambda}\ ;z\right), \quad e_j = 0, \pm 1, \quad \lambda \to \infty. \tag{1.10}$$

In this paper we give an overview of these results, and we show how this set of 26 cases can be reduced by using several types of transformation formulas.

In particular we indicate which cases are of interest for orthogonal polynomials that can be expressed in terms of the Gauss hypergeometric function. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is an important example, and we mention several cases of this polynomial in which n is a large integer and α and/or β are large. When α or β are negative the zeros of $P_n^{(\alpha,\beta)}(x)$ may be outside the interval $[-1,1]$, and for several cases we give the distribution of the zeros in the complex plane.

We mention a few recent papers where certain uniform expansions of the Gauss function are given (in terms of Bessel functions and Airy functions), from which expansions the distribution of the zeros can be obtained. Many cases need further investigations.

Of a completely different nature is the asymptotics of generalized hypergeometric functions. We consider a few cases of ${}_3F_2$ terminating functions of argument 1 and -1, and show for these cases some ad hoc methods. In general, no standard methods based on integrals or differential equations exist for these quantities.

2. Asymptotics: a first example

Consider the asymptotics of

$$ {}_2F_1\left({a,\ \beta+\lambda \atop \gamma+\lambda}\ ;z\right), \quad \lambda \to \infty. \tag{2.1}$$

We use $\beta+\lambda\sim\gamma+\lambda$ and we try (using (1.3))

$$ {}_2F_1\left({a,\ \beta+\lambda \atop \gamma+\lambda};z\right)\sim{}_2F_1\left({a,\ \beta+\lambda \atop \beta+\lambda};z\right)=(1-z)^{-a}\ ? \tag{2.2}$$

Observe first that (see (1.5)), if $\Re(\gamma-a-\beta)>0$,

$$ {}_2F_1\left({a,\ \beta+\lambda \atop \gamma+\lambda};1\right)=\frac{\Gamma(\gamma+\lambda)\Gamma(\gamma-a-\beta)}{\Gamma(\gamma+\lambda-a)\Gamma(\gamma-\beta)}. \tag{2.3}$$

We see that (2.2) cannot hold for $z\sim 1$ (if $\Re(\gamma-a-\beta)>0$).

However, we can use

$$ {}_2F_1\left({a,\ b \atop c};z\right)=(1-z)^{-a}{}_2F_1\left({a,\ c-b \atop c};\zeta\right),\quad \zeta=\frac{z}{z-1}. \tag{2.4}$$

This gives

$$\begin{aligned} {}_2F_1\left({a,\ \beta+\lambda \atop \gamma+\lambda};z\right)&=(1-z)^{-a}{}_2F_1\left({a,\ \gamma-\beta \atop \gamma+\lambda};\zeta\right)\\ &=(1-z)^{-a}\left[1+\frac{a(\gamma-\beta)}{\gamma+\lambda}\zeta+\frac{a(a+1)(\gamma-\beta)(\gamma-\beta+1)}{(\gamma+\lambda)(\gamma+\lambda+1)\,2!}\zeta^2\cdots\right]\\ &=(1-z)^{-a}\left[1+\frac{a(\gamma-\beta)}{\gamma+\lambda}\zeta+\mathcal{O}\left(\lambda^{-2}\right)\right], \end{aligned}\tag{2.5}$$

as $\lambda\to\infty$ with z fixed. This is the beginning of a complete asymptotic expansion, which converges if $|z/(z-1)|<1$. It is an asymptotic expansion for large values of λ, and all fixed $z, z\neq 1$.

We can also use other transformation formulae:

$$\begin{aligned} {}_2F_1\left({a,\ b \atop c};z\right)&=(1-z)^{-b}{}_2F_1\left({c-a,\ b \atop c};\zeta\right)\\ &=(1-z)^{c-a-b}{}_2F_1\left({c-a,\ c-b \atop c};z\right). \end{aligned}\tag{2.6}$$

These give, with (2.4), the three relations

$$\begin{aligned} {}_2F_1\left({a,\ \beta+\lambda \atop \gamma+\lambda};z\right)&=(1-z)^{-a}{}_2F_1\left({a,\ \gamma-\beta \atop \gamma+\lambda};\zeta\right)\\ &=(1-z)^{-b}{}_2F_1\left({\gamma+\lambda-a,\ \beta+\lambda \atop \gamma+\lambda};\zeta\right)\\ &=(1-z)^{\gamma-a-\beta}{}_2F_1\left({\gamma+\lambda-a,\ \gamma-\beta \atop \gamma+\lambda};z\right) \end{aligned}\tag{2.7}$$

We see in (2.6) that the large parameter λ can be distributed over other parameter places. In the present case only the second form is suitable for giving an asymptotic expansion when using the Gauss series. The third form gives a useless Gauss series (for large λ). In the final form the Gauss series converges at $z=1$ if $\Re(a+\beta-\gamma)>0$, but the series does not have an asymptotic property for large values of λ.

The transformation formulae in (2.6) are an important tool for obtaining asymptotic expansions. We will use these and other formulae for investigating all cases of (1.10).

So far, we mention the power series of (1.1) for obtaining an asymptotic expansion. However, to obtain an asymptotic expansion of (2.1) that holds uniformly if z is close to 1, we need a different approach, as will be explained in later sections.

3. Some history and recent activities

1. Watson studied in [34] the cases

$$ {}_2F_1\left({\alpha+\lambda,\ \beta+\lambda \atop \gamma+2\lambda};z\right),\ {}_2F_1\left({\alpha+\lambda,\ \beta-\lambda \atop c};z\right),\ {}_2F_1\left({a,\ b \atop \gamma+\lambda};z\right). \tag{3.1} $$

   All results are of Poincaré type (i.e., negative powers of the large parameter λ) and hold for large domains of complex parameters and argument. Watson started with contour integrals and evaluated them by the method of steepest descent.

2. Luke summarizes these results in [20] (Vol. I, p. 235), and gives many other results, also for higher ${}_pF_q-$functions. In particular he investigates "extended Jacobi polynomials", which are of the type

$$ {}_{p+2}F_q\left({-n,n+\lambda,a_1,\ldots,a_p \atop b_1,\ldots,b_q};z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}. \tag{3.2} $$

   For $p=0$, $q=1$ and integer n this is the Jacobi polynomial.

3. Jones considers in [15] a uniform expansion of

$$ {}_2F_1\left({\alpha+\lambda,\ \beta-\lambda \atop c};\frac{1}{2}-\frac{1}{2}z\right) \tag{3.3} $$

   and gives a complete asymptotic expansion in term of $I-$Bessel functions with error bounds. Applications are discussed for Legendre functions.

4. Olde Daalhuis gives (see [23] and [24]) new uniform expansions of

$$ {}_2F_1\left({a,\ \beta-\lambda \atop \gamma+\lambda};-z\right) \tag{3.4} $$

   in terms of parabolic cylinder functions, and of

$$ {}_2F_1\left({\alpha+\lambda,\ \beta+2\lambda \atop c};-z\right) \tag{3.5} $$

   in terms of Airy functions.

5. Other recent contributions on uniform expansions for Gauss functions follow from [32](Legendre functions), [4] (Legendre functions), [29] (see later section), [11] (Legendre functions), [7] (conical functions), [8] (Jacobi and Gegenbauer polynomials), [36] (Jacobi polynomials), [27] (Pollaczek), [38] (Jacobi function), [13], (Meixner), [19] (Meixner), [18] (Krawtchouk), [14] (Meixner-Pollaczek).

6. On Bessel polynomials, which are of ${}_2F_0-$type, see [39] and [9]. On Charlier polynomials, which are of ${}_2F_0-$type, see [26]. On Laguerre polynomials (${}_1F_1-$type) see [10] and [30].

4. Large parameter cases

We investigate which cases of the 26 large parameter cases of the form

$$ {}_2F_1\left({a+e_1\lambda,\ b+e_2\lambda \atop c+e_3\lambda};z\right),\quad e_j=0,\pm1,\quad \lambda\to\infty. \tag{4.1}$$

are of interest for asymptotic analysis.

e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
0	0	0	+	0	0	−	0	0
0	0	+	+	0	+	−	0	+
0	0	−	+	0	−	−	0	−
0	+	0	+	+	0	−	+	0
0	+	+	+	+	+	−	+	+
0	+	−	+	+	−	−	+	−
0	−	0	+	−	0	−	−	0
0	−	+	+	−	+	−	−	+
0	−	−	+	−	−	−	−	−

Table 1. All 27 cases of ${}_2F_1\left({a+e_1\lambda,\ b+e_2\lambda \atop c+e_3\lambda};z\right)$, $e_j=0,\pm1$. In the table we write $e_j=0,\pm$.

Skipping the dummy (0 0 0), using the symmetry between the a and b parameters:

$$ {}_2F_1\left({a,\ b \atop c};z\right)={}_2F_1\left({b,\ a \atop c};z\right), \tag{4.2}$$

and by using the transformation formulae of (2.6) we reduce the 27 cases to the 8 cases:

	e_1	e_2	e_3
1	0	0	+
2	0	0	−
3	0	+	0
4	0	+	−
5	0	−	+
6	+	+	−
7	+	−	0
8	−	−	+

This set is obtained by using (4.2) and (2.6), that is, only 4 of Kummer's 24 solutions of the hypergeometric differential equation. A further reduction of the set of 8 cases can be obtained by other solutions of the differential equation. Using the connection formula

$$ \begin{aligned} {}_2F_1\left({a,\ b \atop c};z\right) &= \\ &-\frac{\Gamma(c-1)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b)\Gamma(1-c)}z^{1-c}(1-z)^{c-a-b}{}_2F_1\left({1-a,\ 1-b \atop 2-c};z\right) \\ &+\frac{\Gamma(b-c+1)\Gamma(a-c+1)}{\Gamma(a+b-c+1)\Gamma(1-c)}{}_2F_1\left({a,\ b \atop a+b-c+1};1-z\right), \end{aligned} \tag{4.3}$$

we see that the second case (0 0 −) in fact reduces to (0 0 +).

Similarly, we have

$$
{}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z\right)=
$$

$$
e^{a\pi i}\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a+b-c+1)\Gamma(c-a)}\,z^{-a}{}_2F_1\left(\begin{matrix}a,\ a-c+1\\ a+b-c+1\end{matrix};1-\frac{1}{z}\right) \tag{4.4}
$$

$$
+\frac{\Gamma(c)\Gamma(b-c+1)}{\Gamma(a)\Gamma(b-a+1)}z^{a-c}(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}1-a,\ c-a\\ b-a+1\end{matrix};\frac{1}{z}\right),
$$

which reduces the third case (0 + 0) again to (0 0 +).

Finally, we use

$$
{}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z\right)=
$$

$$
e^{c\pi i}\frac{\Gamma(c-1)\Gamma(b-c+1)\Gamma(1-a)}{\Gamma(b)\Gamma(c-a)\Gamma(1-c)}z^{1-c}(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}1-a,\ 1-b\\ 2-c\end{matrix};z\right) \tag{4.5}
$$

$$
+e^{(c-a)\pi i}\frac{\Gamma(1-a)\Gamma(b-c+1)}{\Gamma(1-c)\Gamma(b-a+1)}z^{a-c}(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}1-a,\ c-a\\ b-a+1\end{matrix};\frac{1}{z}\right),
$$

which reduces the forth case (0 + −) to the fifth case (0 − +).

Consequently we have the five remaining cases:

	e_1	e_2	e_3
1	0	0	+
5	0	−	+
6	+	+	−
7	+	−	0
8	−	−	+

These five cannot be reduced further by using relations between the functions

$$
{}_2F_1\left(\begin{matrix}a+e_1\lambda,\ b+e_2\lambda\\ c+e_3\lambda\end{matrix};z\right),\quad e_j=0,\pm1. \tag{4.6}
$$

However ([23]), using

$$
{}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z\right)=
$$

$$
e^{a\pi i}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}z^{-a}{}_2F_1\left(\begin{matrix}a,\ a-c+1\\ a-b+1\end{matrix};\frac{1}{z}\right) \tag{4.7}
$$

$$
+e^{b\pi i}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}z^{-b}{}_2F_1\left(\begin{matrix}b,\ b-c+1\\ b-a+1\end{matrix};\frac{1}{z}\right),
$$

and

$$
{}_2F_1\left({a,\ b \atop c}; z\right) =
$$

$$
e^{(c-b)\pi i}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}z^{b-c}(1-z)^{c-a-b}{}_2F_1\left({1-b,\ c-b \atop a-b+1}; \frac{1}{z}\right) \tag{4.8}
$$

$$
+e^{(c-a)\pi i}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}z^{a-c}(1-z)^{c-a-b}{}_2F_1\left({1-a,\ c-a \atop b-a+1}; \frac{1}{z}\right),
$$

we see that the cases $(+\quad+\quad-)$ and $(-\quad-\quad+)$ both can be handled by the case $(+\quad 2+\quad 0)$. Hence, the smallest set is the following set of four:

	e_1	e_2	e_3
A	0	0	+
B	0	−	+
C	+	−	0
D	+	2+	0

if we accept the special case D:

$$
{}_2F_1\left({a+\lambda,\ b+2\lambda \atop c}; z\right). \tag{4.9}
$$

4.1 Quadratic transformations
For special combinations of the parameters a, b, c other types of relations between ${}_2F_1-$ functions exist. For example, we have the quadratic transformation

$$
{}_2F_1\left({a,\ b \atop 2b}; z\right) = (1-z)^{-a/2}\,{}_2F_1\left({\frac{1}{2}a,\ b-\frac{1}{2}a \atop b+\frac{1}{2}}; \frac{z^2}{4z-4}\right). \tag{4.10}
$$

For large a the left-hand side is of type

$$
(+\ 0\ 0) \equiv (\ 0\ +\ 0\ ) \equiv (\ 0\ 0\ +) = A, \tag{4.11}
$$

whereas the right-hand side is of type $C = (\ +\ -\ 0\ )$.

5. Which special functions are involved?
The next step is to indicate which special functions are associated with the above four cases.

5.1 Legendre functions
We have the Legendre function

$$
P_\nu^\mu(z) = \frac{(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu}}{\Gamma(1-\mu)}{}_2F_1\left({-\nu,\ \nu+1 \atop 1-\mu}; \frac{1}{2}(1-z)\right). \tag{5.1}
$$

Case A : $\quad (\ 0\ 0\ +\ )$
We see that case A occurs when $\mu \to -\infty$. Several relations between the Legendre function and the Gauss function give also the case $\mu \to +\infty$.

Case B : $(\,0\ -\ +\,)$
Because

$$(\,-\ +\ -\,) \equiv (\,0\ +\ -\,) \equiv (\,0\ -\ +\,) = B \tag{5.2}$$

(see the first relation of (2.6) and (4.5)), we see that case B occurs when in $P_\nu^\mu(z)$ both parameters ν and μ tend to $+\infty$.

Case C : $(\,+\ -\ 0\,) \equiv (\,-\ +\ 0\,)$

This also occurs for $P_\nu^\mu(z)$, with ν large and μ fixed.

In the results for Legendre functions in the literature rather flexible conditions on the parameters μ and ν are allowed. For example, in [7] it is assumed that the ratio ν/μ is bounded.

Any hypergeometric function, for which a quadratic transformation exists, can be expressed in terms of Legendre functions. So, many special cases, and mixed versions of the cases A, B, C, D are possible for Legendre functions.

5.2 Hypergeometric polynomials

Several orthogonal polynomials of hypergeometric type (see the Askey scheme in [16]) have representations with Gauss hypergeometric functions. Next to the special cases of the Jacobi polynomials (Gegenbauer, Legendre) we have the following cases.

Pollaczek

$$P_n(\cos\theta; a, b) = e^{n\theta i}\,{}_2F_1\left({-n,\ \tfrac{1}{2}+i\phi(\theta) \atop 1}\,;\, 1-e^{-2i\theta}\right), \tag{5.3}$$

$$\phi(\theta) = \frac{a\cos\theta + b}{\sin\theta}. \tag{5.4}$$

For uniform expansions in terms of Airy functions see [27]. The asymptotics is for $n \to \infty, \theta = t/\sqrt{n}$ with t bounded, and bounded away from zero.

Meixner-Pollaczek

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{n\phi i}\,{}_2F_1\left({-n,\ \lambda+ix \atop 2\lambda}\,;\, 1-e^{2i\phi}\right). \tag{5.5}$$

For uniform expansions in terms of parabolic cylinder functions see [19]. The asymptotics is for $n \to \infty, x = \alpha n$ with α in a compact interval containing the point $\alpha = 0$.

Meixner

$$M_n(x;\beta,c) = {}_2F_1\left({-n,\ -x \atop \beta}\,;\, 1-\frac{1}{c}\right). \tag{5.6}$$

In [13] two uniform expansions in terms of parabolic cylinder functions are given. The asymptotics is for $n \to \infty, x = \alpha n$ with $\alpha > 0$ in compact intervals containing the point $\alpha = 1$.

Krawtchouk

$$K_n(x;p,N) = {}_2F_1\left({-n,\ -x \atop -N}\,;\, \frac{1}{p}\right), \quad n = 0, 1, 2, \ldots, N. \tag{5.7}$$

A uniform expansion in terms of parabolic cylinder functions is given in [18]. The asymptotics is for $n \to \infty, x = \lambda N$ with λ and $\nu = n/N$ in compact intervals of $(0, 1)$.

5.3 Jacobi polynomials
The Jacobi polynomial has the representation:

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left({-n,\ \alpha+\beta+n+1 \atop \alpha+1} ; \frac{1}{2}(1-x)\right), \tag{5.8}$$

and we see that case C : $(+\ -\ 0) \equiv (-\ +\ 0)$ applies if $n \to \infty$. Uniform expansions use Bessel functions for describing the asymptotic behaviour at the points $x = \pm 1$ (Hilb-type formula, see [28] or for complete asymptotic expansions [37]).

A well-known limit is

$$\lim_{\alpha\to\infty} \alpha^{-n/2} P_n^{(a+\alpha,b+\alpha)}\left(\frac{x}{\sqrt{\alpha}}\right) = \frac{H_n(x)}{2^n\, n!}, \tag{5.9}$$

where $H_n(x)$ is the Hermite polynomial, a special case of the parabolic cylinder function. Approximations of $H_n(x)$ are available in terms of Airy functions.

Several other limits are known (see [16]). In [6] and [8] asymptotic expansions are given for large positive parameters α and/or β.

Of particular interest are asymptotic expansions for large n and large negative values of the parameters α, β. See [3] and [21].

With $\alpha = \beta = -n$ we have in (5.8) the case

$$(-\ -\ -) \equiv (0\ 0\ -) \equiv (0\ 0\ +) = A. \tag{5.10}$$

These are non-classical values for the parameters. We have, taking $\alpha = \beta = -\frac{1}{2} - n$,

$$P_n^{(\alpha,\beta)}(z) = \frac{n!}{2^{2n}} C_n^{-n}(x) = (-1)^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n!n!2^{-n-2m}x^{n-2m}}{m!m!(n-2m)!}. \tag{5.11}$$

All zeros are on the imaginary axis.

Other representations can be used:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^{-\beta} {}_2F_1\left({\alpha+n+1,\ -\beta-n \atop \alpha+1} ; \frac{1}{2}(1-x)\right) \\ &= \binom{n+\beta}{n}\left(\frac{1-x}{2}\right)^{-\alpha} (-1)^n {}_2F_1\left({\beta+n+1,\ -\alpha-n \atop \beta+1} ; \frac{1}{2}(1+x)\right), \end{aligned} \tag{5.12}$$

and we see that case D : $(+\ \ 2+\ \ 0)$ applies if $n \to \infty$ and, in the first case, α is constant and β equals $b - 3n$, and similarly in the second case.

See Figure 1 for the zero distribution of the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(z), \quad n = 30, \quad \alpha = \tfrac{1}{2}, \quad \beta = -3n+1. \tag{5.13}$$

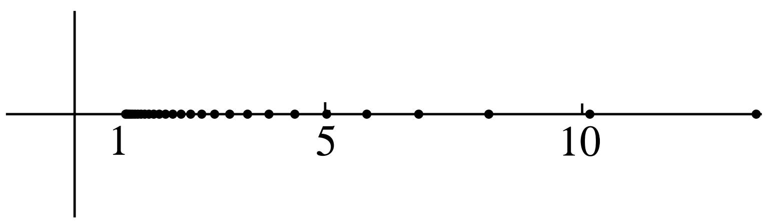

Figure 1. Zeros of $P_n^{(\alpha,\beta)}(z)$, $n = 30, \alpha = \frac{1}{2}$, $\beta = -3n+1$.

The case $B:\ (\ 0\ -\ \ +\ ) \equiv (\ -\ 0\ \ +\ )$ applies in the representations

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) = &\binom{\alpha+\beta+2n}{n}\left(\frac{x-1}{2}\right)^n {}_2F_1\left({-n,\ -\alpha-n \atop -2n-\alpha-\beta};\frac{2}{1-x}\right) \\
= &\binom{\alpha+\beta+2n}{n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left({-n,\ -\beta-n \atop -2n-\alpha-\beta};\frac{2}{1+x}\right),
\end{aligned}
\tag{5.14}
$$

when $\alpha=-a-n, \beta=-b-2n$ (first case) or $\beta=-b-n, \alpha=-a-2n$ (second case), with, again, non-classical parameter values. [24] gives an expansion of the Gauss function for this case in terms of parabolic cylinder functions.

See Figure 2 for the zero distribution of the Jacobi polynomial

$$
P_n^{(\alpha,\beta)}(z), \quad n=25, \quad \alpha=-n+\tfrac{1}{2}, \quad \beta=-2n+1. \tag{5.15}
$$

As $n\to\infty$, the zeros approach the curve defined by $\left|1-\left(\frac{3-z}{1+z}\right)^2\right|=1$.

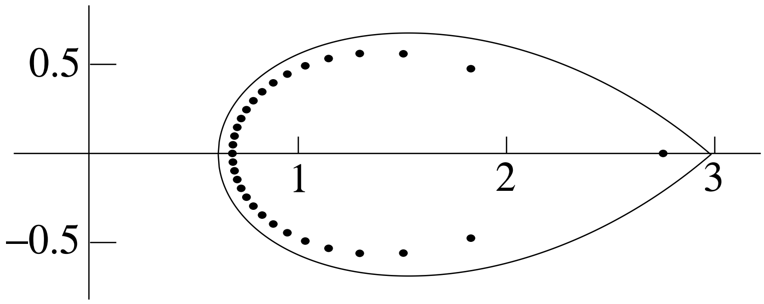

Figure 2. Zeros of $P_n^{(\alpha,\beta)}(z),\ n=25, \alpha=-n+\frac{1}{2},\ \beta=-2n+1$.

Uniform expansions for Jacobi polynomials are available in terms of Bessel functions. Details on uniform expansions of Jacobi polynomials in terms of Bessel functions can be found in [8] and [37], and in cited references in these papers.

5.4 Gegenbauer polynomials

For Gegenbauer (ultraspherical) polynomials we have several representations. For example,

$$
C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left({-n,\ n+2\lambda \atop \lambda+\frac{1}{2}};\frac{1-x}{2}\right), \tag{5.16}
$$

and for $\lambda=-n$, $n\to\infty$, we have

$$
(\ -\ -\ -\ ) \equiv (\ 0\ 0\ -\ ) \equiv (\ 0\ 0\ +\ ) = A. \tag{5.17}
$$

We also have

$$
C_{2n}^\lambda(x) = (-1)^n \frac{(\lambda)_n}{n!}\, {}_2F_1\left({-n,\ n+\lambda \atop \frac{1}{2}};x^2\right), \tag{5.18}
$$

and for $\lambda=-2n$, $n\to\infty$, we have

$$
(\ -\ -\ \ 0\ ) \equiv (\ +\ -\ \ 0\ ) = C. \tag{5.19}
$$

Also in this case a kind of quadratic transformation is used.

*5.5 Asymptotics of the **A**−case*

We give a few steps in deriving the expansion for the case $A = (\ 0\ 0\ +\ )$. As always, we can use the differential equation or an integral representation. We take

$$ {}_2F_1\left({a,\ b \atop c};z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\,dt, \tag{5.20}$$

where $\Re c > \Re b > 0, |\mathrm{ph}(1-z)| < \pi$. A simple transformation gives the Laplace transform representation

$$ {}_2F_1\left({a,\ b \atop c+\lambda};z\right) = \frac{\Gamma(c+\lambda)}{\Gamma(b)\Gamma(c+\lambda-b)}\int_0^\infty t^{b-1}e^{-\lambda t}f(t)\,dt, \tag{5.21}$$

with $\lambda \to \infty$ and where

$$ f(t) = \left(\frac{e^t-1}{t}\right)^{b-1} e^{(1-c)t}\left(1-z+ze^{-t}\right)^{-a}. \tag{5.22}$$

This standard form in asymptotics can be expanded by using Watson's lemma (see [25] or [35]). For fixed a, b, c and z:

$$ {}_2F_1\left({a,\ b \atop c+\lambda};z\right) \sim \frac{\Gamma(c+\lambda)}{\Gamma(c+\lambda-b)}\sum_{s=0}^\infty f_s(z)\frac{(b)_s}{\lambda^{b+s}}, \tag{5.23}$$

where $f_0(z) = 1$ and other coefficients follow from

$$ f(t) = \sum_{s=0}^\infty f_s(z)t^s, \quad |t| < \min(2\pi, |t_0|), \quad t_0 = \ln\frac{z}{z-1}. \tag{5.24}$$

When z is large, the singularity t_0 is close to the origin, and the expansion becomes useless. We can write

$$ {}_2F_1\left({a,\ b \atop c+\lambda};z\right) = \frac{\Gamma(c+\lambda)}{\Gamma(b)\Gamma(c+\lambda-b)}\int_0^\infty \frac{t^{b-1}e^{-\lambda t}g(t)}{(t+\zeta)^a}\,dt, \tag{5.25}$$

where

$$ \zeta = -t_0 = \ln\frac{z-1}{z}, \quad g(t) = (t+\zeta)^a f(t), \tag{5.26}$$

with g regular at $t = t_0$. Expanding $g(t) = \sum_{s=0}^\infty g_s(z)t^s$ gives

$$ {}_2F_1\left({a,\ b \atop c+\lambda};z\right) \sim \frac{\Gamma(c+\lambda)\,\zeta^{b-a}}{\Gamma(c+\lambda-b)}\sum_{s=0}^\infty g_s(z)(b)_s\zeta^s U(b+s, b-a+1+s, \zeta\lambda), \tag{5.27}$$

in which U is the confluent hypergeometric function. This expansion holds if $\lambda \to \infty$, uniformly with respect to small values of ζ, that is, large values of z.

This method can be used to obtain an expansion of the Legendre function $P_\nu^{\pm m}(z)$ for large values of m; see [12], where we used this type of expansion for computing this Legendre function for large m and z.

Other sources with related expansions for this type of Legendre functions are [4], [7], [11], [25] and [32].

*5.6 Another **A**− case*

The case (− 0 0) is very important for all kinds of orthogonal polynomials and can be reduced to the previous case $A = (\ 0\ 0\ +\ )$. It is of interest to give a direct approach of

$$ {}_2F_1\left(\begin{matrix}-n,\ b\\ c\end{matrix};z\right),\quad n\to\infty, \tag{5.28}$$

whether or not n is an integer. First we recall the limit

$$\lim_{b\to\infty} {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z/b\right) = {}_1F_1\left(\begin{matrix}a\\ c\end{matrix};z\right), \tag{5.29}$$

which may be used as a definition for the ${}_1F_1$−function (the Kummer or confluent hypergeometric function). We are interested in the asymptotics behind this limit, and we expect a role of the Kummer function when in (5.28) n becomes large, and z is small. We again can use (5.20), and consider

$$I_n = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^n\,dt, \tag{5.30}$$

where $\Re c > \Re b > 0, |\mathrm{ph}(1-z)| < \pi$. We transform

$$1 - tz = (1-z)^u = e^{u\ln(1-z)}. \tag{5.31}$$

Then we have

$$\begin{aligned}
I_n &= (1-z)^{c-b-1}\left[\frac{\ln(1-z)}{-z}\right]^{c-1} J_n,\\
J_n &= \int_0^1 f(u)u^{b-1}(1-u)^{c-b-1}e^{\omega u}\,du,\\
f(u) &= \left[\frac{1-(1-z)^u}{-u\ln(1-z)}\right]^{b-1}\left[\frac{1-(1-z)^{u-1}}{(1-u)\ln(1-z)}\right]^{c-b-1},\\
\omega &= (n+1)\ln(1-z).
\end{aligned} \tag{5.32}$$

The function $f(u)$ is holomorphic in a neighborhood of $[0,1]$. Singularities occur at

$$u_k = \frac{2k\pi i}{\ln(1-z)},\quad v_m = 1 + \frac{2m\pi i}{\ln(1-z)},\quad k,m \in \mathbb{Z}\setminus\{1\}. \tag{5.33}$$

So, when z ranges through compacta of $\mathbb{C}\setminus\{1\}$, the singularities of f are bounded away from $[0,1]$. If $\Re\omega < 0$ the dominant point in the integral is $u = 0$; if $\Re\omega > 0$, then $u = 1$ is the dominant point. To obtain an asymptotic expansion for large n that combines both cases, and which will give a uniform expansion for all z, $|z-1| > \delta > 0$, f should be expanded at both end-points 0 and 1.

More details can be found in [29], where the results have been applied to a class of polynomials biorthogonal on the unit circle.

*5.7 Asymptotics of the **B**− case*

We consider a simple case:

$$ {}_2F_1\left(\begin{matrix}-n,\ 1\\ n+2\end{matrix};-z\right) = (n+1)\int_0^1 [(1-t)(1+zt)]^n\,dt, \tag{5.34}$$

with z near the point 1.

We write

$$[(1-t)(1+zt)]^n = e^{-n\phi(t)}, \quad \phi(t) = -\ln\left[(1-t)(1+zt)\right], \tag{5.35}$$

and we have

$$\phi'(t) = \frac{-2tz-1+z}{(1-t)(1+zt)}. \tag{5.36}$$

We see that the integrand has a peak value at $t_0 = (z-1)/(2z)$.

So, if $z = 1$ the peak is at $t = 0$, if $z > 1$ then $t_0 \in (0,1)$, and if $z < 1$ then $t_0 < 0$. See Figure 3.

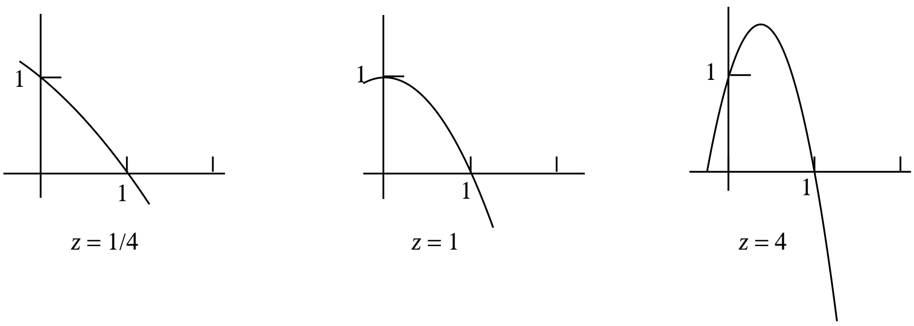

Figure 3. $(1-t)(1+zt)$ for a few values of z.

The same situation occurs for the integral

$$\int_0^\infty e^{-n(\frac{1}{2}w^2-\alpha w)}\,dw \tag{5.37}$$

which has a peak value at $w_0 = \alpha$. This integral is an error function (a special case of the parabolic cylinder functions).

In a uniform expansion of (5.34) that holds for large n and z in a neighborhood of 1 we need an error function. This explains why in Case B (in a more general form than (5.34)) parabolic cylinder functions may occur for special values of z.

We transform the integral in (5.34) by writing

$$(1-t)(1+zt) = e^{-\frac{1}{2}w^2+\alpha w} \tag{5.38}$$

with the conditions

$$t = 0 \Longleftrightarrow w = 0, \quad t = 1 \Longleftrightarrow w = \infty, \quad t = t_0 \Longleftrightarrow w = \alpha. \tag{5.39}$$

The quantity α follows from satisfying the matching of t_0 with α:

$$(1-t_0)(1+zt_0) = e^{-\frac{1}{2}w_0^2+\alpha w_0}, \tag{5.40}$$

giving

$$\tfrac{1}{2}\alpha^2 = -\ln\left[1-\left(\frac{z-1}{z+1}\right)^2\right], \quad \operatorname{sign}(\alpha) = \operatorname{sign}(z-1). \tag{5.41}$$

We obtain

$$ {}_2F_1\left({-n,\ 1 \atop n+2};-z\right) = (n+1)\int_0^\infty e^{-n(\frac12 w^2-\alpha w)}f(w)\,dw, \tag{5.42}$$

where

$$ f(w)=\frac{dt}{dw}=\frac{w-\alpha}{t-t_0}\,\frac{(1-t)(1+zt)}{2z}. \tag{5.43}$$

A first approximation follows by replacing $f(w)$ by

$$ f(\alpha)=\frac{1+z}{2\sqrt{2}\,z}, \tag{5.44}$$

giving

$$ \begin{aligned} {}_2F_1\left({-n,\ 1 \atop n+2};-z\right) &\sim nf(\alpha)\int_0^\infty e^{-n(\frac12 w^2-\alpha w)}\,dw \\ &= \sqrt{\pi n}\frac{1+z}{4z}e^{-\frac12 n\alpha^2}\operatorname{erfc}\left(-\alpha\sqrt{n/2}\right). \end{aligned} \tag{5.45}$$

as $n\to\infty$, uniformly with respect to z in a neighborhood of $z=1$.

This expansion is in agreement with the general case ${}_2F_1(a,b-\lambda;c+\lambda;-z)$ considered in [24].

The relation with the Jacobi polynomials is:

$$ {}_2F_1\left({-n,\ 1 \atop n+2};-z\right) = \frac{(n+1)!}{2^{2n}\left(\frac32\right)_n}(1+z)^n P_n^{(n+1,-n-1)}\left(\frac{1-z}{1+z}\right). \tag{5.46}$$

See Figure 4 for the distribution of the zeros of

$$ P_n^{(\alpha,\beta)}(z),\quad n=30,\quad \alpha=n+1,\quad \beta=-n-1. \tag{5.47}$$

As $n\to\infty$, the zeros approach the curve $\left|1-z^2\right|=1$.

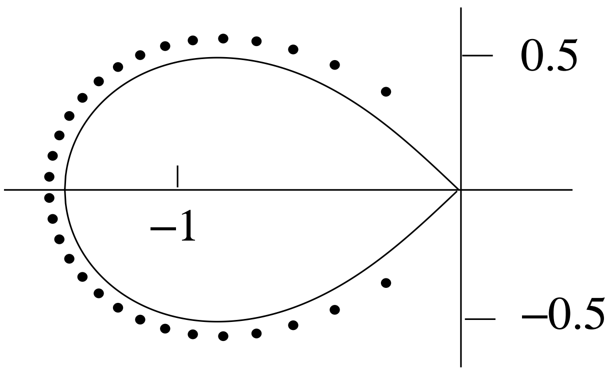

Figure 4. Zeros of $P_n^{(\alpha,\beta)}(z)$, $n=30,\alpha=-\beta=n+1$.

6. Asymptotics of some ${}_3F_2$ polynomials

In the previous examples we used integral representations of the Gauss function. The differential equation can also be used for obtaining asymptotic expansions. As a rule, methods based on differential equations provide more information on the coefficients and remainders in the expansions than methods based on integral representations.

For the generalized hypergeometric functions standard asymptotic expansion are known when the argument z is large, and the other parameters are fixed; see [5] and [20].

For the ${}_pF_q-$functions a differential equation is available of order $\max(p, q+1)$. However, for higher order equations no methods are available for deriving (uniform) expansions for large values of parameters.

The ${}_pF_q-$functions can be written as a Mellin-Barnes contour integral, but, again, no methods are available for deriving (uniform) expansions for large values of parameters from these integrals.

The terminating ${}_pF_q-$functions (one of the parameters a_k is equal to a non-positive integer) and of unit argument ($z=1$) are of great interest in special function theory and in applications.

In many cases recursion relations are available. For the examples to be considered here we use ad hoc methods. It is of interest to investigate recursion methods for the same and other problems.

6.1 A first example

As a first example ([17], consider the asymptotics of

$$f(n) = {}_3F_2\left(\begin{matrix}-n,\ \frac12,\ \frac12\\ \frac12-n,\ \frac12-n\end{matrix};-1\right) \tag{6.1}$$

It was conjectured (by Larcombe) that $\lim_{n\to\infty} f(n) = 2$ and Koornwinder gave a proof. By using (1.7) and replacing a ratio of Pochhammer symbols by an integral:

$$\frac{\left(\frac12\right)_k}{\left(\frac12-n\right)_k} = \frac{(-1)^k n!}{\sqrt{\pi}\,\Gamma(n+\frac12)}\int_0^1 t^{k-\frac12}(1-t)^{n-k-\frac12}\,dt, \tag{6.2}$$

$k = 0, 1, \ldots, n$, we can write $f(n)$ as an integral

$$f(n) = \frac{n!}{\pi\,(\frac12)_n}\int_0^1 t^{-\frac12}(1-t)^{n-\frac12}\,{}_2F_1\left(\begin{matrix}-n,\ \frac12\\ \frac12-n\end{matrix};\frac{t}{1-t}\right)dt, \tag{6.3}$$

and the Gauss function is a Legendre polynomial. We obtain

$$f(n) = \frac{2^{-n}n!n!}{\pi\,(\frac12)_n\,(\frac12}_{n}\int_0^\pi \sin^n\theta\; P_n\left(\frac{1}{\sin\theta}\right)d\theta. \tag{6.4}$$

There is no straightforward way to obtain asymptotics out of this. A few manipulations with (6.4) give the result

$$f(n) = \frac{n!\,n!\,n!}{2^n\,(\frac12)_n\,(\frac12)_n}\;c_n \tag{6.5}$$

where c_n is the coefficient in the Bessel function expansion

$$\left[e^{w/2}\,I_0(w/2)\right]^2 = \sum_{n=0}^{\infty} c_n w^n, \tag{6.6}$$

and now the asymptotics easily follows. We have

$$c_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\left[e^{w/2}\, I_0(w/2)\right]^2}{w^{n+1}}\, dw = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{e^{2w}}{w^{n+1}}\, f(w)\, dw, \tag{6.7}$$

where

$$f(w) = \left[e^{-w/2}\, I_0(w/2)\right]^2 \tag{6.8}$$

and the contour $\mathcal{C}$ is a circle around the origin. The main contribution comes from the saddle point of $e^{2w}w^{-n}$, that is from $w = w_0 = n/2$. Because $f(w_0) \sim 2/(\pi n)$, we have

$$c_n \sim \frac{2}{\pi n}\frac{2^n}{n!}, \quad n \to \infty. \tag{6.9}$$

This gives finally $f(n) \sim 2$. A complete asymptotic expansion follows by using more terms in the expansion of $f(w)$ at $w = w_0$.

6.2 Generalizing Kummer's identity

Kummer's identity reads:

$${}_2F_1\left(\begin{matrix} a,\ b \\ 1+a-b \end{matrix}; -1\right) = \frac{\Gamma(1+a-b)\,\Gamma\left(1+a/2\right)}{\Gamma(1+a)\,\Gamma\left(1+a/2-b\right)}. \tag{6.10}$$

where $1 + a - b \neq 0, -1, -2, \ldots$. A generalization ([33]) is written in the form

$${}_2F_1\left(\begin{matrix} a+n,\ b \\ 1+a-b \end{matrix}; -1\right) = P(n)\frac{\Gamma(a-b)\,\Gamma(a/2+1/2)}{\Gamma(a)\,\Gamma(a/2+1/2-b)} + Q(n)\frac{\Gamma(a-b)\,\Gamma(a/2)}{\Gamma(a)\,\Gamma(a/2-b)}. \tag{6.11}$$

Vidunas showed that, for $n = -1, 0, 1, 2, \ldots, (a)_n \neq 0$ and $a - b \neq 0, -1, -2, \ldots$:

$$\begin{aligned}
P(n) &= \frac{1}{2^{n+1}}\ {}_3F_2\left(\begin{matrix} -\frac12 n,\ -\frac12 n - \frac12,\ \frac12 a - b \\ \frac12,\ \frac12 a \end{matrix}; 1\right) \\
&= \frac12\ {}_3F_2\left(\begin{matrix} -\frac12 n,\ -\frac12 n - \frac12,\ b \\ -n,\ \frac{a}{2} \end{matrix}; 1\right), \\
Q(n) &= \frac{n+1}{2^{n+1}}\ {}_3F_2\left(\begin{matrix} -\frac12 n + \frac12,\ -\frac12 n,\ \frac12 a + \frac12 - b \\ \frac32,\ \frac12 a + \frac12 \end{matrix}; 1\right) \\
&= \frac12\ {}_3F_2\left(\begin{matrix} -\frac12 n + \frac12,\ -\frac12 n,\ b \\ -n,\ \frac{a}{2} + \frac12 \end{matrix}; 1\right).
\end{aligned} \tag{6.12}$$

We are interested in the asymptotic behaviour of $P(n)$ and $Q(n)$ for large values of n. Using, if $\Re d > \Re c > 1$,

$$\frac{(c)_k}{(d)_k} = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)}\int_0^1 t^{d-c-1}(1-t)^{c+k-1}\, dt, \tag{6.13}$$

and a few manipulations of the Gauss functions, we obtain the integral representations, for $\Re a > \Re 2b > 0$,

$$\begin{aligned}
P(n) &= \frac{2^{-n}\,\Gamma(\frac12 a)}{\Gamma(b)\,\Gamma(\frac12 a - b)}\int_0^\infty \sinh^{a-2b-1}(t)\, \cosh^{-a-n}(t)\, \cosh(n+1)t\, dt, \\
Q(n) &= \frac{2^{-n}\,\Gamma(\frac12 a + \frac12)}{\Gamma(b)\,\Gamma(\frac12 a + \frac12 - b)}\int_0^\infty \sinh^{a-2b-1}(t)\, \cosh^{-a-n}(t)\, \sinh(n+1)t\, dt,
\end{aligned} \tag{6.14}$$

We can use standard methods for obtaining the asymptotic behaviour of these integrals. First we write the hyperbolic functions $\cosh(n+1)t$ and $\sinh(n+1)t$ as exponential functions.

We obtain for $n \to \infty$

$$
\begin{aligned}
P(n) &\sim \frac{2^{2b-1}\Gamma(\frac12 a)}{\Gamma(b)\,\Gamma(\frac12 a-b)} \sum_{k=0}^{\infty} c_k \frac{\Gamma(b+\frac12 k)}{n^{b+\frac12 k}}, \\
Q(n) &\sim \frac{2^{2b-1}\Gamma(\frac12 a+\frac12)}{\Gamma(b)\,\Gamma(\frac12 a+\frac12-b)} \sum_{k=0}^{\infty} c_k \frac{\Gamma(b+\frac12 k)}{n^{b+\frac12 k}},
\end{aligned}
\tag{6.15}
$$

where

$$
c_0 = 1, \quad c_1 = 0, \quad c_2 = \tfrac12(5b-4a+3). \tag{6.16}
$$

These expansions are not valid for $b = \frac12 a$, $b = \frac12 a + \frac12$, respectively. In these cases we have

$$
P(n) = \frac{1}{2^{n+1}}, \quad Q(n) = \frac{n+1}{2^{n+1}}, \tag{6.17}
$$

as follows from (6.12). Similar for the cases

$$
b = \frac12 a + m, \quad b = \frac12 a + \frac12 + m, \quad m = 0,1,2,\ldots. \tag{6.18}
$$

For negative values of n different integral representations should be used. We have

$$
\begin{aligned}
P(-n-1) &= 2^n \frac{(1-\frac12 a)_n}{(1-b)_n}\, {}_3F_2\left({-\frac12 n,\ -\frac12 n+\frac12,\ \frac12 a-b \atop \frac12,\ \frac12 a-n};1\right), \\
Q(-n-1) &= -n\, 2^n \frac{(\frac12-\frac12 a)_n}{(1-b)_n}\, {}_3F_2\left({-\frac12 n+\frac12,\ -\frac12 n+1,\ \frac12 a+\frac12-b \atop \frac32,\ \frac12 a+\frac12-n};1\right).
\end{aligned}
\tag{6.19}
$$

with the integral representations

$$
\begin{aligned}
P(-n-1) &= \frac{2^{n+1}\,\Gamma(1-b)}{\Gamma(\frac12 a-b)\Gamma(1-\frac12 a)} \int_0^{\frac12\pi} \cos^{n+1-a} t\ \sin^{2c-1} t\ \cos nt\, dt, \\
Q(-n-1) &= -\frac{2^{n+1}\,\Gamma(1-b)}{\Gamma(\frac12 a+\frac12-b)\Gamma(\frac12-\frac12 a)} \int_0^{\frac12\pi} \cos^{n+1-a} t\ \sin^{2c-1} t\ \sin nt\, dt.
\end{aligned}
\tag{6.20}
$$

By integrating on contours $(i\infty,0)\cup(0,\frac12\pi)\cup(\frac12\pi,\frac12\pi+i\infty)$, similar integrals arise as in (6.14), and again standard methods can be used for obtaining the asymptotic behaviour.

7. Concluding remarks

1. The asymptotic analysis of all 26 cases in

$$
{}_2F_1\left({a+e_1\lambda,\ b+e_2\lambda \atop c+e_3\lambda};z\right), \quad e_j = 0,\pm1, \tag{7.1}
$$

can be reduced to the 4 cases

	e_1	e_2	e_3
A	0	0	+
B	0	−	+
C	+	−	0
D	+	2+	0

2. These cases are of interest for orthogonal polynomials and special functions.
3. New recent results on uniform asymptotic expansions have been published or announced for special cases.
4. The distribution of the zeros of Jacobi polynomials for non-classical values of the parameters α and β shows interesting features. New research is needed for describing the asymptotics of these distributions.
5. The asymptotics of ${}_3F_2$ terminating functions with large parameters is quite difficult. Standard methods based on integrals and differential equations are not available. Recursion relations may be explored further.

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